Annuities Software Assignment – Table of Contents

Annuities Software Assignment

What is an Annuity?

An ***annuity*** is a ***financial instrument that involves making payments or deposits at regular intervals***. Annuities can be used for both saving and borrowing. If an annuity is used for ***saving***, ***deposits*** are made at regular intervals. If an annuity is used for ***borrowing***, ***payments*** are made at regular intervals. Since this description may be somewhat difficult to understand at first, it's best to investigate a few simple examples.

Example 1

I have designed an Excel spreadsheet (**Annuity Exploration.xls**) to help you review/learn the following concepts:

- Simple Interest
- Compound Interest
- Annual Interest Rate
- Periodic Interest Rate
- Ordinary Simple Annuities
- Ordinary General Annuities (including Mortgages)

The spreadsheet is available from **I:\Out\Ics3u0\Annuities** and can also be downloaded from www.misternolfi.com.

- Open the spreadsheet
- Study it carefully
- Experiment with different values (**e.g.** change annual rate of interest, number of payments per year, etc)
- Answer the following questions

Questions

1. What is the difference between an ***annual rate of interest*** and a ***periodic rate of interest***?

2. In what ways does ***simple interest*** differ from ***compound interest***? Which one allows an investment to grow more quickly?

3. What is the difference between an ***ordinary simple annuity*** and an ***ordinary general annuity***?

4. What is the effect of increasing the number of compounding periods per year? What is the effect of decreasing the number of compounding periods per year?

Example 2 – Brandon's Problem

Brandon, who is now sixteen, would like to be a poker champion some day. At the age of twenty-one, he would like to make a name for himself in the world of poker by entering and winning a big-time poker tournament in Las Vegas. The only problem is that he needs a lot of cash to realize this dream. Besides paying for the $5000 tournament entry fee, Brandon also needs to save an additional $10000 to cover travelling costs and living expenses. He considers various investment possibilities as outlined below.

(a) First, Brandon considers buying a $10000 Canada Savings Bond that pays interest at a rate of 4% per annum compounded annually. Would this allow him to end up with enough money by the age of twenty-one?

(b) Brandon then considers making monthly deposits to an annuity. To achieve his goal by the age of twenty-one, how much would he need to deposit at the ***end of each month*** if the annuity pays interest at the rate of 6% per annum compounded monthly? Would the amount differ if the deposit were made at the ***beginning of each month*** instead of the end of each month?

Solution

Because the given information is incomplete, we need to make a ***reasonable assumption*** before we attempt to solve this problem. Although we are told that Brandon is now sixteen, we do not know his exact age. He could have just turned sixteen, he may have turned sixteen a few months ago or he may be about to turn seventeen. To avoid this confusion, let's assume that he has just turned sixteen and that he would like to have the money by his twenty-first birthday. This gives him ***exactly five years*** to save the money.

(a) By using the "Annuity Exploration" spreadsheet, we see that Brandon will fall far short of his goal!

Compound Interest 1 (Compounding Frequency must be the same as Payment Frequency)

Annual Rate of Interest	4.00%	Present Value	$10,000.00	# Payments Per Year	1	# Compounding Periods Per Year	1	Amortization Period	5.00

i = 4.000000% n = 5

Interest Payment #	Interest Payment	Account Balance	Total Interest Paid
		$10,000.00	
1	$400.00	**$10,400.00**	$400.00
2	$416.00	**$10,816.00**	$816.00
3	$432.64	**$11,248.64**	$1,248.64
4	$449.95	**$11,698.59**	$1,698.59
5	$467.94	**$12,166.53**	$2,166.53

How would we solve this problem without the help of the "Annuity Exploration" spreadsheet? Before we can investigate this issue, we need to learn some important terminology.

Terminology	*Value in this Example*	*Explanation*
P = Present Value	$10,000.00	Amount of money with which Brandon started
F = Future Value	$12,166.53	The value of the Canada Savings Bond after 5 years
n = Total Number of Compounding Periods	5	This is the total number of compounding periods over the course of the investment (or loan).
r = Annual Interest Rate	4% = 0.04	Rate of interest payable per year
i = Periodic Interest Rate	4% = 0.04	Rate of interest payable per compounding period. In this example, $i = r$ because the interest compounds annually.
p = Payments per Year (Payment Frequency)	1	The number of payments/deposits made per year.
c = Compounding Periods per Year (Compounding Frequency)	1	The number of times per year that the interest compounds. Often, $p = c$ but this does not need to be the case.

The table on the next page should help you understand how to calculate F without the help of a spreadsheet.

Interest Payment #	Interest Payment	Account Balance	Total Interest Paid	How Account Balance is Calculated
		$10,000.00		
1	$400.00	**$10,400.00**	$400.00	$\$10,000 \times 1.04$
2	$416.00	**$10,816.00**	$816.00	$\$10,400 \times 1.04 = (\$10,000 \times 1.04) \times 1.04 = \$10,000 \times 1.04^2$
3	$432.64	**$11,248.64**	$1,248.64	$\$10,816 \times 1.04 = (\$10,000 \times 1.04^2) \times 1.04 = \$10,000 \times 1.04^3$
4	$449.95	**$11,698.59**	$1,698.59	$\$11,248.64 \times 1.04 = (\$10,000 \times 1.04^3) \times 1.04 = \$10,000 \times 1.04^4$
5	$467.94	**$12,166.53**	$2,166.53	$\$11,698.59 \times 1.04 = (\$10,000 \times 1.04^4) \times 1.04 = \$10,000 \times 1.04^5$

We can see that the account balance is calculated by multiplying the previous balance by 1.04. The final balance of $12,166.53 (i.e. the future value) is calculated by multiplying the original amount of $10,000.00 (i.e. the present value) by 1.04, five times. In other words,

$$\$12,166.53 = \$10,000.00\,(1.04)(1.04)(1.04)(1.04)(1.04) = \$10,000.00\,(1.04)^5$$

Using the above example as a guide, we can make the following observation. If *P* represents the present value, *F* represents the future value, *i* represents the periodic rate and *n* represents the number of compounding periods, then

$$F = P(1+i)^n$$

This formula is known as the "***compound interest formula***" and is often written $A = P(1+i)^n$.

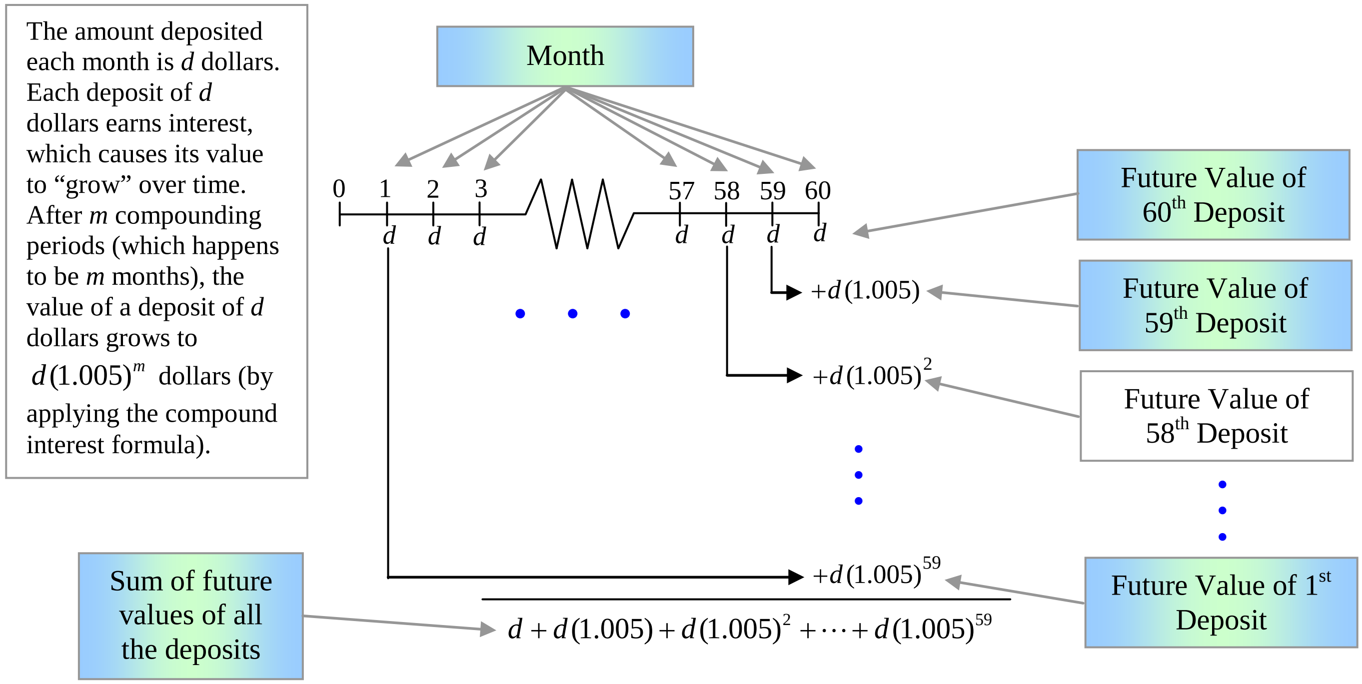
Substituting into this formula we obtain

$$\begin{aligned} F &= P(1+i)^n \\ &= 10000(1+.04)^5 \\ &= 10000(1.04)^5 \\ &= 12166.53 \end{aligned}$$

(b) To understand the this part of the example, it's important to comprehend the terminology in the following table:

Terminology	*Value in this Example*	*Explanation*
P = Present Value	$0.00	Amount of money with which Brandon started.
F = Future Value	$15,000.00	The amount of money Brandon wants to have at the end of the five year period.
n = Total Number of Compounding Periods	$12 \times 5 = 60$	This is the total number of compounding periods over the course of the investment (or loan).
r = Annual Interest Rate	6% = 0.06	Rate of interest payable per year
i = Periodic Interest Rate	$\frac{r}{c} = \frac{0.06}{12} = 0.005$	Rate of interest payable per compounding period.
d = Deposit/Payment made at Regular Intervals	Unknown	The amount deposited/paid at regular intervals.
p = Payments per Year (Payment Frequency)	12	The number of payments/deposits made per year.
c = Compounding Periods per Year (Compounding Frequency)	12	The number of times per year that the interest compounds.

The following diagram, called a "time-line" illustrates this situation, which seems somewhat more complicated than the scenario in part (a). In reality, however, this situation is almost exactly the same as that in (a). The only difference is that in this case, the method of part (a) needs to be applied multiple times.



So it appears that the ***crux*** of this problem is to figure out how to add up the future values of each deposit of d dollars. Since we know that Brandon needs to have \$15,000.00 by the time he is 21, the sum of the future values of each deposit of d dollars will have to be \$15,000.00. Knowing this will allow us to solve for d and finally determine exactly how much money Brandon needs to pay to the annuity each month.

At this point, however, we need to digress for a short time and learn how to find the sum of the future values. This requires learning about geometric series, which is the subject of the next section.

Geometric Series

A series is simply a sum of one or more numbers. Examples of series are shown below:

$1+2+3+\cdots+99+100$

$1+11+21+\cdots+91+101$

These are examples of ***arithmetic series***. In an arithmetic series the terms are evenly "spaced out." That is, the ***difference*** of any two consecutive terms is constant.

$1+4+9+\cdots+576+625=1^2+2^2+3^2+\cdots+24^2+25^2$

$30+21+3-5-1099-21100$

There is no discernible pattern in this series.

$$\begin{aligned}&6+18+54+162+486\\&=6+6(3)+6(9)+6(27)+6(81)\\&=6(3^0)+6(3^1)+6(3^2)+6(3^3)+6(3^4)\end{aligned}$$

This is an example of a ***geometric series***. In a geometric series, the terms are ***not*** evenly "spaced out." However, the ***ratio*** of any two consecutive terms is constant. To obtain the "next" term in ***this example***, simply multiply the "previous" term by three.

In general, the letter a is usually used to represent the ***first term*** of a series. Therefore, in the above geometric series, a = 6. In addition, the letter r is usually used to represent the ***common ratio of a geometric series***. In the example above, r = 3 because any term in the series can be generated by multiplying the previous term by 3.

A ***geometric series*** with n terms, first term a and common ratio r is of the form

$$a+ar+ar^2+\cdots+ar^{n-1}$$

Sum of a Geometric Series

Let $S = a + ar + ar^2 + \cdots + ar^{n-1}$ (call this equation **(1)**).

By multiplying both sides of equation **(1)** by r we obtain
$rS = r(a + ar + ar^2 + \cdots + ar^{n-1})$.

$\therefore rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$ (call this equation **(2)**)

By subtracting equation **(1)** from equation **(2)** we obtain

$$\begin{aligned} rS - S &= ar + ar^2 + \cdots + ar^{n-1} + ar^n - (a + ar + ar^2 + \cdots + ar^{n-1}) \\ &= ar + ar^2 + \cdots + ar^{n-1} + ar^n - ar^0 - ar^1 - ar^2 - \cdots - ar^{n-1} \\ &= ar^n + ar - ar + ar^2 - ar^2 + \cdots + ar^{n-1} - ar^{n-1} - a \\ &= ar^n - a \end{aligned}$$

$$\begin{array}{l} \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ \underline{a + ar + ar^2 + \cdots + ar^{n-1} \qquad\quad} \\ ar^n - a \end{array}$$

When the expression in the second line is subtracted from the expression in the first line, all terms "cancel" except for ar^n and a.

$\therefore rS - S = ar^n - a$

$\therefore S(r-1) = a(r^n - 1)$

$\therefore S = \dfrac{a(r^n - 1)}{r - 1}$

$\therefore S = \dfrac{a(1 - r^n)}{1 - r}$ (by multiplying both numerator and denominator of the right-hand side by –1).

The sum of a geometric series of the form $a + ar + ar^2 + ar^3 + \cdots ar^{n-1}$ is given by

$$S = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$$

Back to Brandon's Problem

Recall that we left off with the sum of the future values $d + d(1.005) + d(1.005)^2 + \cdots + d(1.005)^{59}$. Now we have the theoretical support to simplify this expression. In this geometric series, $a = d$, $r = 1.005$ and $n = 60$. Therefore,

$$d + d(1.005) + d(1.005)^2 + \cdots + d(1.005)^{59} = \frac{d(1.005^{60} - 1)}{1.005 - 1} = \frac{d(1.005^{60} - 1)}{0.005}.$$

Also recall that Brandon needs a future value of \$15,000 to realize his dream of becoming a poker champion. Therefore,

$$\frac{d(1.005^{60} - 1)}{0.005} = 15000$$

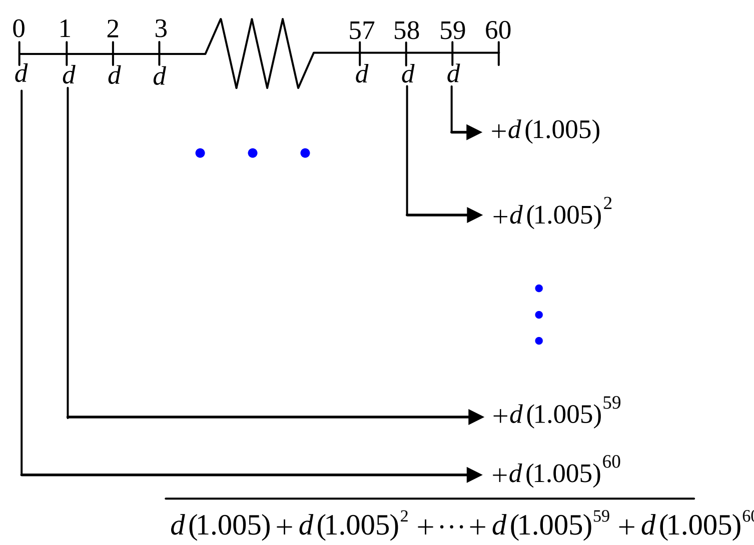
$$\therefore d = \frac{15000(0.005)}{1.005^{60} - 1} \doteq 214.99$$

Therefore, Brandon must deposit \$214.99 ***at the end of each month***.

What if Brandon makes Deposits at the Beginning of each Month?

When deposits are made at the end of each month, the very last deposit earns no interest at all because it is made on the last day of the life of the annuity. If deposits are made at the beginning of each month, however, then the final deposit is made on the first day of the final month. This means that every deposit earns interest.

The timeline at the right shows how making payments at the beginning of each month affects the sum of the future values. We can see clearly that only one value changes, that of a. The values of r and n remain the same. Summarizing, we see that $a = d(1.005)$, $r = 1.005$ and $n = 60$.



By substituting these values into the formula for the sum of a geometric series we obtain

$$d(1.005)+d(1.005)^2+\cdots+d(1.005)^{60}=\frac{d(1.005)(1.005^{60}-1)}{1.005-1}=\frac{d(1.005^{60}-1)}{0.005}$$

Again, since Brandon needs a future value of $15,000 to realize his dream of becoming a poker champion,

$$\frac{d(1.005)(1.005^{60}-1)}{0.005}=15000$$

$$\therefore d=\frac{15000(0.005)}{1.005(1.005^{60}-1)}\doteq 213.92$$

Therefore, if Brandon makes deposits at the beginning of each month, he only needs to deposit $213.92.

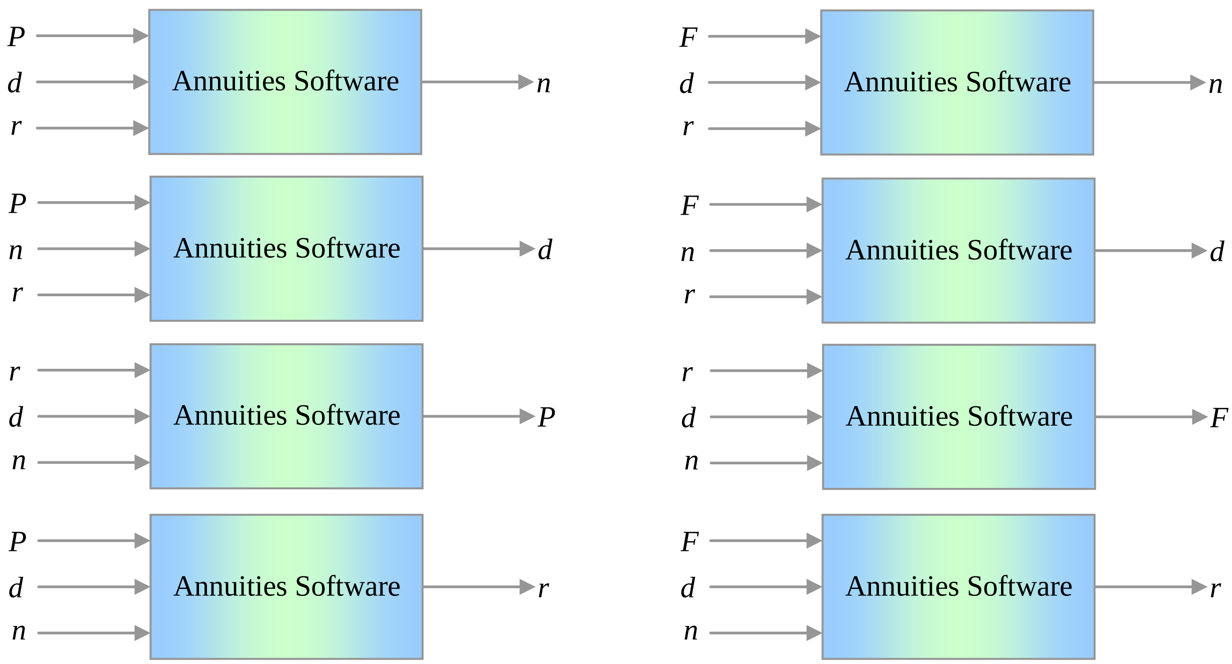
Example 3 – Interest Rate Calculator Software

The software that you develop will allow the user to calculate values related to annuities as shown in the diagram given below.

P = Present Value, F = Future Value, d = payment/deposit, n = Number of Payments, r = Annual Interest Rate, i = Periodic Interest Rate

Note

To simplify the software that you need to develop, we shall assume that P and F cannot both be non-zero. That is, we shall assume that if $P > 0$, then $F = 0$ and if $F > 0$, then $P = 0$.



What happens if the Compounding Frequency does not equal the Payment Frequency?

The following table summarizes what we have learned thus far. In all of the following, ***it is assumed that the number of compounding periods per year (compounding frequency) equals the number of payments per year (payment frequency)***.

Financial Concept	***Important Terminology***	***Relationships***
Simple Interest *	P, F and r are the same as for compound interest. t = Time in Years ***Slow, linear growth of money.***	$F = P(1+rt)$
Compound Interest	P = Present Value = Current Value of Money F = Future Value = Value of Money once all Interest is Paid r = Annual Interest Rate n = Total Number of Compounding Periods i = Periodic Interest Rate = Rate of Interest per Compounding Period $= \frac{r}{n}$ ***Fast, exponential growth of money.***	$F = P(1+i)^n$
Sum of a Geometric Series	A ***geometric series*** has the form $a + ar + ar^2 + \cdots + ar^{n-1}$. a = First Term of the Series r = Common Ratio n = Total Number of Terms	$S = \frac{a(r^n - 1)}{r-1} = \frac{a(1-r^n)}{1-r}$
Annuities	P, F, r and i are the same as for compound interest. d = Deposit/Payment made at Regular Intervals ***Fast, exponential growth of money.***	**See pages 10 - 13**

* The notes on simple interest are included for reference purposes only. Simple interest formulas are not required for annuity calculations.

Calculating the Periodic Rate when Payment Frequency equals Compounding Frequency

As shown in the following examples, calculating the periodic rate is quite easy when the number of compounding periods per year equals the number of payments per year.

Annual Rate (r)	*Payments per Year (p)*	*Compounding Periods per Year (c)*	*Periodic Rate (i)*
$r = 12\% = 0.12$	$p = 12$ (monthly)	$c = 12$	$i = \frac{12\%}{12} = 1\% = 0.01$
$r = 6\% = 0.06$	$p = 2$ (semi-annually)	$c = 2$	$i = \frac{6\%}{2} = 3\% = 0.03$
$r = 3\% = 0.03$	$p = 4$ (quarterly)	$c = 4$	$i = \frac{3\%}{4} = 0.75\% = 0.0075$
$r = 2.6\% = 0.026$	$p = 52$ (weekly)	$c = 52$	$i = \frac{2.6\%}{52} = 0.05\% = 0.0005$

Calculating the Periodic Rate when Payment Frequency does not equal Compounding Frequency

When the payment frequency is not equal to the compounding frequency, a more detailed analysis is needed to calculate the periodic rate. As shown in the table below, it's not immediately obvious how to calculate the periodic rate!

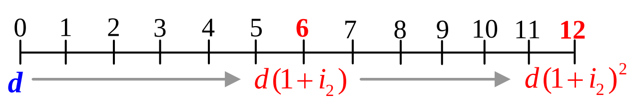
Annual Rate (r)	*Payments per Year (p)*	*Compounding Periods per Year (c)*	*Periodic Rate (i)*
$r = 12\% = 0.12$	$p = 12$ (monthly)	$c = 2$	$i = ?$
$r = 6\% = 0.06$	$p = 2$ (semi-annually)	$c = 1$	$i = ?$
$r = 3\% = 0.03$	$p = 4$ (quarterly)	$c = 3$	$i = ?$
$r = 2.6\% = 0.026$	$p = 52$ (weekly)	$c = 4$	$i = ?$

Since it's very difficult to understand what is meant by having a compounding frequency that is not equal to the payment frequency (i.e. $p \neq c$), perhaps we should find a way to convert this problem into one in which the two values ***are*** equal (i.e. $p = c$). The following example shows how this can be done.

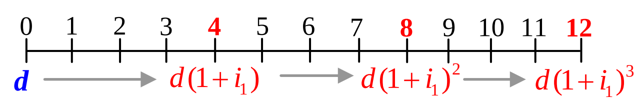
Example

Suppose that an annuity has ***interest compounding three times per year*** but ***payments are made twice per year*** (i.e. $p = 2$, $c = 3$). If we "adjust" the interest rate, we can convert this annuity into one that has the same future value but which compounds at the same time as the payments are made (i.e. $p = c = 2$).

Future Value of d Dollars after 12 Months with Interest Compounding Twice per Year (Periodic Rate i_2)



Future Value of d Dollars after 12 Months with Interest Compounding Three Times per Year (Periodic Rate i_1)



As long as $d(1+i_2)^2 = d(1+i_1)^3$, then the original investment of d dollars will grow to the same future value after 12 months regardless of whether the interest compounds twice per year or three times per year.

$$d(1+i_2)^2 = d(1+i_1)^3$$
$$\therefore (1+i_2)^2 = (1+i_1)^3$$
$$\therefore \left[(1+i_2)^2\right]^{\frac{1}{2}} = \left[(1+i_1)^3\right]^{\frac{1}{2}}$$
$$\therefore 1+i_2 = (1+i_1)^{\frac{3}{2}}$$
$$\therefore i_2 = (1+i_1)^{\frac{3}{2}} - 1$$

Therefore, an annuity in which payments are made twice per year with interest compounding three times per year at periodic rate i_1 is equivalent to an annuity in which payments are made twice per year with interest compounding twice per year at periodic rate $i_2 = (1+i_1)^{\frac{3}{2}} - 1$.

How to Convert a General Annuity into a Simple Annuity

Simple Annuity: p must equal c

General Annuity: p does not need to equal c

Suppose that you are given a ***general annuity*** with p payments per year and c compounding periods per year. We would like to convert this to a ***simple annuity*** with p payments per year and p compounding periods per year. To do this, we shall let $i_1 = \frac{r}{c}$ represent the periodic rate that would be used if the interest were paid and compounded c times per year and let i_2 represent the (unknown) rate to be paid if the interest were compounded and paid p times per year. Then consider the following very simple equation:

$$\begin{array}{c}\text{Future Value of } d \text{ Dollars after One Year with Interest} \\ \text{Compounding } p \text{ times per Year (Periodic Rate } i_2\text{)}\end{array} = \begin{array}{c}\text{Future Value of } d \text{ Dollars after One Year with Interest} \\ \text{Compounding } c \text{ times per Year (Periodic Rate } i_1\text{)}\end{array}$$

$$\therefore d(1+i_2)^p = d(1+i_1)^c$$

$$\therefore (1+i_2)^p = (1+i_1)^c$$

$$\therefore \left[(1+i_2)^p\right]^{\frac{1}{p}} = \left[(1+i_1)^c\right]^{\frac{1}{p}}$$

$$\therefore 1+i_2 = (1+i_1)^{\frac{c}{p}}$$

$$\therefore i_2 = (1+i_1)^{\frac{c}{p}} - 1$$

$$\therefore i_2 = (1+\frac{r}{c})^{\frac{c}{p}} - 1$$

Summary

A ***general annuity*** with p payments per year, c compounding periods per year and ***annual rate r***

is equivalent to

a ***simple annuity*** with p payments per year, p compounding periods per year and ***periodic rate***

$$i = (1+\frac{r}{c})^{\frac{c}{p}} - 1.$$

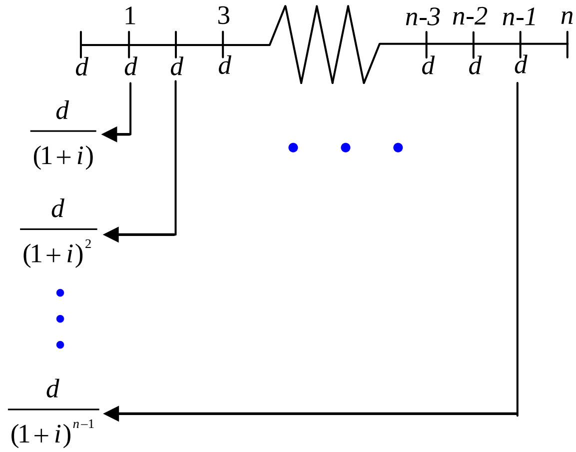
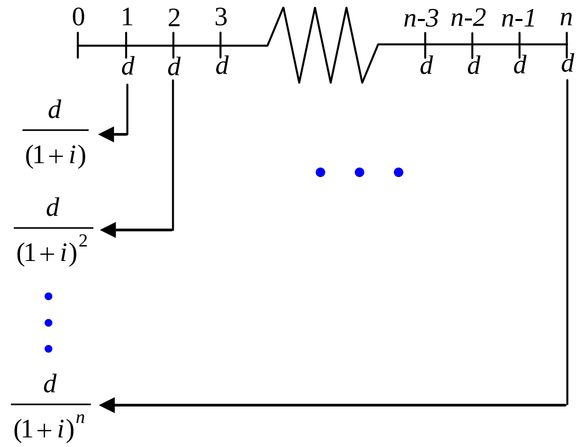
Solving any Annuity in which either P = 0 or F = 0

To simplify the problem of solving annuities, we shall assume that either $P = 0$ or $F= 0$. If $P = 0$, we can deduce that the annuity is used for ***saving*** because we begin with no money but end up having money at some point in the future. If $F = 0$, we can deduce that the annuity is used for ***borrowing*** because we begin with some money but end up with no money after we have repaid the debt.

n = # payments = # compounding periods, d = deposit/payment amount, P = present value (principal), F = future value, i = periodic interest rate $= (1+\frac{r}{c})^{\frac{c}{p}} - 1$, r = annual interest rate, p = #payments per year, c = #compound periods per year.

Given	*Payment made at Beginning of Interval (Annuity Due)*	*Payment made at End of Interval (Ordinary Annuity)*
n, i, P	Find d. (In this case, $F = 0$.)	Find d. (In this case, $F = 0$.)
n, i, F	Find d. (In this case, $P = 0$.)	Find d. (In this case, $P = 0$.)
n, d, P	Find i. (In this case, $F = 0$.)	Find i. (In this case, $F = 0$.)
n, d, F	Find i. (In this case, $P = 0$.)	Find i. (In this case, $P = 0$.)
n, i, d	Find P or find F.	Find P or find F.
P, i, d	Find n. (In this case, $F = 0$.)	Find n. (In this case, $F = 0$.)
F, i, d	Find n. (In this case, $P = 0$.)	Find n. (In this case, $P = 0$.)

Derivation of Annuity Formulas

	Payment made at Beginning of Interval (Annuity Due)	*Payment made at End of Interval (Ordinary Annuity)*
Present Value	 The ***present value P*** is given by the sum of the following geometric series: $P = d + \frac{d}{1+i} + \frac{d}{(1+i)^2} + \cdots + \frac{d}{(1+i)^{n-1}}$ $= d + d(1+i)^{-1} + d(1+i)^{-2} \cdots + d(1+i)^{-(n-1)}$ In this series, $a = d$ and $r = \frac{1}{i+1} = (1+i)^{-1}$. Therefore, $P = \frac{a(1-r^n)}{1-r}$ $= \frac{d\left(1-\left[(1+i)^{-1}\right]^n\right)}{1-(1+i)^{-1}}$ $= d\left(\frac{1-(1+i)^{-n}}{i}\right)(1+i)$ Solving for d we obtain, $d = P\left(\frac{i}{1-(1+i)^{-n}}\right)\left(\frac{1}{1+i}\right).$ Solving for n we obtain, $n = -\frac{\ln\left(1-\frac{Pi}{d(1+i)}\right)}{\ln(1+i)}$	 The ***present value P*** is given by the sum of the following geometric series: $P = \frac{d}{1+i} + \frac{d}{(1+i)^2} + \cdots + \frac{d}{(1+i)^n}$ In this series, $a = \frac{d}{i+1} = d(1+i)^{-1}$ and $r = \frac{1}{i+1} = (1+i)^{-1}$. Using a process similar to that in the left column, we obtain $P = d\left(\frac{1-(1+i)^{-n}}{i}\right).$ Solving for d we obtain, $d = P\left(\frac{i}{1-(1+i)^{-n}}\right).$ Solving for n we obtain, $n = -\frac{\ln\left(1-\frac{Pi}{d}\right)}{\ln(1+i)}$

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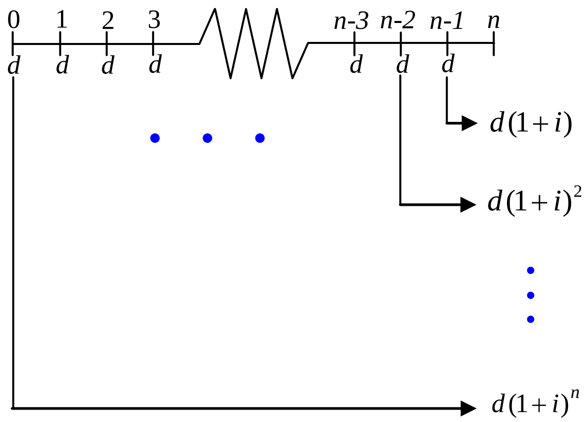
<table>
<tr><th></th><th>Payment made at Beginning of Interval (Annuity Due)</th><th>Payment made at End of Interval (Ordinary Annuity)</th></tr>
<tr><td>Future Value</td>
<td>



The ***future value F*** is given by the sum of the following geometric series:

$$F = d(1+i) + d(1+i)^2 + \cdots + d(1+i)^n$$

In this series, $a = d(1 + i)$ and $r = 1 + i$. Therefore,

$$F = d\left(\frac{(1+i)^n - 1}{i}\right)(1+i)\ .$$

Solving for d we obtain,

$$d = F\left(\frac{i}{(1+i)^n - 1}\right)\left(\frac{1}{1+i}\right).$$

Solving for n we obtain,

$$n = \frac{\ln\left(\frac{Fi}{d(1+i)} + 1\right)}{\ln(1+i)}$$

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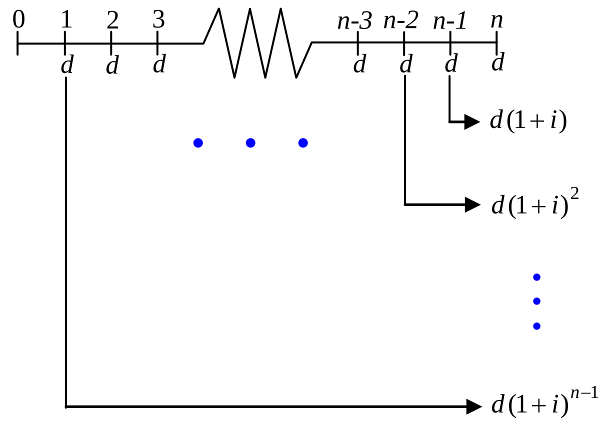

The ***future value F*** is given by the sum of the following geometric series:

$$F = d + d(1+i) + d(1+i)^2 + \cdots + d(1+i)^{n-1}$$

In this series, $a = d$ and $r = 1 + i$. Therefore,

$$F = d\left(\frac{(1+i)^n - 1}{i}\right).$$

Solving for d we obtain,

$$d = F\left(\frac{i}{(1+i)^n - 1}\right).$$

Solving for n we obtain,

$$n = \frac{\ln\left(\frac{Fi}{d} + 1\right)}{\ln(1+i)}$$

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</table>

Note: Unfortunately, it is not possible to solve for i algebraically. We need to use ***methods of approximation*** to calculate i.

Annuity Algorithm

1. The user enters ***three of the five values*** P, F, r, d, n. (If P is entered, $F = 0$. If F is entered, $P = 0$.)

2. The user must also enter the values of p and c.

3. If the user has entered the value of r (i.e. the annual rate), it ***must be converted*** to an equivalent periodic rate using the formula $i = (1+\frac{r}{c})^{\frac{c}{p}} - 1$.

4. Based on the information entered by the user, one of the unknown values is calculated, as outlined in the table below.

Given	*Payment made at Beginning of Interval (Annuity Due)*	*Payment made at End of Interval (Ordinary Annuity)*
n, i, P ($F = 0$)	Calculate d using $d = P\left(\frac{i}{1-(1+i)^{-n}}\right)\left(\frac{1}{1+i}\right)$.	Calculate d using $d = P\left(\frac{i}{1-(1+i)^{-n}}\right)$.
n, i, F ($P = 0$)	Calculate d using $d = F\left(\frac{i}{(1+i)^{n}-1}\right)\left(\frac{1}{1+i}\right)$.	Calculate d using $d = F\left(\frac{i}{(1+i)^{n}-1}\right)$.
n, d, P ($F = 0$)	Calculate an approximation for i using a sufficient number of iterations of the recursive formula $i_{m+1} = i_m - \frac{d(1+i_m)^{1-n} + (P-d)i_m - d}{d(1-n)(1+i_m)^{-n} + P - d}$.	Calculate an approximation for i using a sufficient number of iterations of the recursive formula $i_{m+1} = i_m - \frac{d(1+i_m)^{-n} + Pi_m - d}{P - dn(1+i_m)^{-n-1}}$.
n, d, F ($P = 0$)	Calculate an approximation for i using a sufficient number of iterations of the recursive formula $i_{m+1} = i_m - \frac{d(1+i_m)^{n+1} - (F+d)i_m - d}{d(n+1)(1+i_m)^{n} - F - d}$.	Calculate an approximation for i using a sufficient number of iterations of the recursive formula $i_{m+1} = i_m - \frac{d(1+i_m)^{n} - Fi_m - d}{dn(1+i_m)^{n-1} - F}$.
n, i, d	Calculate P using $P = d\left(\frac{1-(1+i)^{-n}}{i}\right)(1+i)$ and/or calculate F using $F = d\left(\frac{(1+i)^{n}-1}{i}\right)(1+i)$.	Calculate P using $P = d\left(\frac{1-(1+i)^{-n}}{i}\right)$ and/or calculate F using $F = d\left(\frac{(1+i)^{n}-1}{i}\right)$.
P, i, d	Calculate n using $n = -\frac{\ln\left(1-\frac{Pi}{d(1+i)}\right)}{\ln(1+i)}$.	Calculate n using $n = -\frac{\ln\left(1-\frac{Pi}{d}\right)}{\ln(1+i)}$.
F, i, d	Calculate n using $n = \frac{\ln\left(\frac{Fi}{d(1+i)}+1\right)}{\ln(1+i)}$.	Calculate n using $n = \frac{\ln\left(\frac{Fi}{d}+1\right)}{\ln(1+i)}$.

5. The calculated value is displayed. (If the user asks for the annual rate to be calculated, an additional step is necessary. The algorithm shown in the table will calculate the periodic rate i, not the annual rate r. Therefore, before displaying the final result, the periodic rate must be converted to an annual rate using the formula $r = c[(1+i)^{\frac{p}{c}} - 1]$.)

Practice Exercises

1. Carol invested $1000 at 6% per annum, compounded annually. Without using the compound interest formula, calculate the future value of the investment after 7 years (use the given table).

Year	***Interest ($)***	***Value of Investment ($)***
0	$0.00	$1000.00
1		
2		
3		
4		
5		
6		
7		

2. For each, state the values of P, n, and i.

(a) a $600 loan at 8%, compounded quarterly for 5 years

(b) a $2000 investment at 2.4%, compounded monthly for 2 years

3. Tara borrowed $2000 at 5.9%, compounded semi-annually for a 2-year term. She promises to repay principal and interest at the end of the two year period. How much will Tara have to pay back altogether?

4. Determine whether the following are geometric series. If they are, calculate the sum.

(a) $2 + 4 + 6 + 8$

(b) $5 + 10 + 20 + 40$

(c) $20 + 15 + 10$

(d) $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4}$

(e) $3 + 12 + 27 + 48$

(f) $2000 + 2000(1.02) + 2000(1.02)^2 + 2000(1.02)^3$

5. Allison deposits $100 at the ***end of each month*** into a savings account that pays interest at 3%, compounded monthly. What will her savings be worth in 1 year?

6. Ryan makes deposits of $2000 semi-annually into an ***annuity due*** that pays 4% interest, compounded semi-annually. How much will the annuity be worth after the 5-year term?

7. Andre bought a new sound system. He was offered a payment plan that consists of $200 payments made at the end of every 3 months, for 2 years. The plan is really a loan that involves interest that is calculated at 8%, compounded quarterly. What is the actual cost of the sound system if he pays for it right away?

8. John had $7000 cash for a down payment on a car. He took out a loan at 8%, compounded monthly, to pay for the rest. His monthly loan payments will be $500.00 for the next 3 years. What is the price of the car?