Application of Geometric Sequences and Series: Compound Interest and Annuities

Example: Brandon's Problem

Brandon, who is now sixteen, would like to be a poker champion some day. At the age of twenty-one, he would like to make a name for himself in the world of poker by entering and winning a big-time poker tournament in Las Vegas. The only problem is that he needs a lot of cash to realize this dream. Besides paying for the \$5000 tournament entry fee, Brandon also needs to save an additional \$10000 to cover travelling costs and living expenses. After learning about compound interest and annuities in Mr. Nolfi's class, Brandon decided it might be a good idea to invest some money. He considers several possibilities as outlined below.

(a) \$10000 Canada Savings Bond that pays interest at a rate of 4% per annum, ***not*** compounded (***Simple Interest***)

(b) \$10000 Canada Savings Bond that pays interest at a rate of 4% per annum, ***compounded*** annually (***Compound Interest***)

(c) Brandon then considers making monthly deposits to an annuity. To achieve his goal by the age of twenty-one, how much would he need to deposit at the ***end of each month*** if the annuity pays interest at the rate of 6% annum ***compounded monthly***?

An investment in which deposits/payments are made at regular intervals is called an ***Annuity***. If the deposit/payment is made at the ***end*** of the payment period, the annuity is called an ***Ordinary Annuity***. If instead the deposits/payments are made at the ***beginning*** of the payment period, the annuity is called an ***Annuity Due***. This course will deal only with ordinary annuities.

Which of these investments will allow Brandon to realize his dream?

Solution

Because the given information is incomplete, we need to make a ***reasonable assumption*** before we attempt to solve this problem. Although we are told that Brandon is now sixteen, we do not know his exact age. He could have just turned sixteen, he may have turned sixteen a few months ago or he may be about to turn seventeen. To avoid this confusion, let's assume that he has just turned sixteen and that he would like to have the money by his twenty-first birthday. This gives him ***exactly five years*** to save the money.

(a) ***Simple Interest:*** Interest is paid directly to the investor, that is, it is ***not*** added to the value of the investment. This means that ***interest payments remain constant over time***.

P = principal = original amount invested = \$10000.00
r = annual rate of interest = 4% = 0.04
t = time in years = 5

By doing a few simple calculations, we can easily see that Brandon will fall far short of his goal!

Interest Payment #	Interest Payment	Account Balance	Total Interest Paid	How Account Balance is Calculated
		\$10,000.00		
1	\$400.00	**\$10,000.00**	\$400.00	The account balance does not change because interest payments are
2	\$400.00	**\$10,000.00**	\$800.00	made to the investor, ***not*** to the account!
3	\$400.00	**\$10,000.00**	\$1,200.00	
4	\$400.00	**\$10,000.00**	\$1,600.00	Total value of the investment = \$10000.00 + \$2000.00 < \$15000.00
5	\$400.00	**\$10,000.00**	\$2,000.00	

Easier way to perform the calculations:

I = total interest paid after t years

$I = Prt = \$10000.00(0.04)(5) = \$2000.00$,

A = total value of investment after t years

$A = P(1+rt) = \$10000.00[1+0.04(5)] = \$12000.00$

(b) *Compound Interest:* Interest paid to the account, which means that ***interest payments increase with time***.

P = principal = original amount invested = \$10000.00

r = annual rate of interest = 4% = 0.04

n = total number of compounding periods
= total number of times interest is calculated and paid to the account
$=5(1)=5$

i = periodic rate of interest
= interest rate per compounding period
= r ÷ (# of compounding periods per year)
$=\dfrac{0.04}{1}=0.04$

Interest Payment #	Interest Payment	Account Balance	Total Interest Paid	How Account Balance is Calculated
		\$10,000.00		
1	\$400.00	**\$10,400.00**	\$400.00	\$10,000 × 1.04
2	\$416.00	**\$10,816.00**	\$816.00	\$10,400 × 1.04 = (\$10,000 × 1.04) × 1.04 = \$10,000 × 1.04^2
3	\$432.64	**\$11,248.64**	\$1,248.64	\$10,816 × 1.04 = (\$10,000 × 1.04^2) × 1.04 = \$10,000 × 1.04^3
4	\$449.95	**\$11,698.59**	\$1,698.59	\$11,248.64 × 1.04 = (\$10,000 × 1.04^3) × 1.04 = \$10,000 × 1.04^4
5	\$467.94	**\$12,166.53**	\$2,166.53	\$11,698.59 × 1.04 = (\$10,000 × 1.04^4) × 1.04 = \$10,000 × 1.04^5

We can see that the account balance is calculated by multiplying the previous balance by 1.04. The final balance of \$12,166.53 (i.e. the future value) is calculated by multiplying the original amount of \$10,000.00 (i.e. the present value) by 1.04, five times. In other words,

\$12,166.53 = \$10,000.00 (1.04)(1.04)(1.04)(1.04)(1.04) = \$10,000.00 $(1.04)^5$

Using the above example as a guide, we can make the following observation:

- P represents the ***principal*** (original amount invested)
- A represents the ***value of the investment after n compounding periods***
- i represents the ***periodic interest rate***, then

$$A=P(1+i)^n$$

This equation is known as the "***compound interest formula***."

Substituting into this formula we obtain

$$\begin{aligned}A&=P(1+i)^n\\&=10000(1+.04)^5\\&=10000(1.04)^5\\&=12166.53\end{aligned}$$

Very Important Observation

Notice that the compound interest formula is an example of the ***general term of a geometric sequence***:

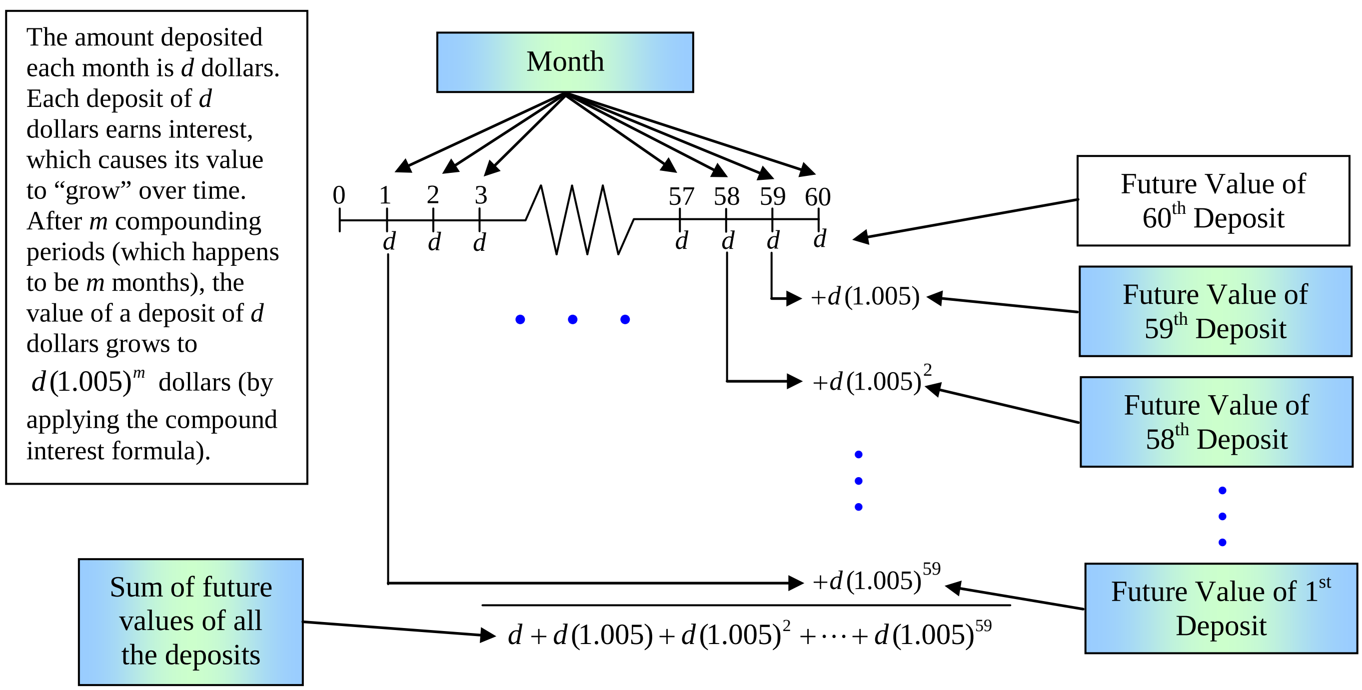
$t_n=ar^{n-1} \quad \rightarrow \quad t_{n+1}=ar^n \qquad$ Since $A=P(1+i)^n$, $\quad A=t_{n+1}$, $\quad P=a$, $\quad 1+i=r$

(c) To understand this part of the example, it's important to comprehend the terminology in the following table:

Terminology	***Value in this Example***	***Explanation***
PV = Present Value	\$0.00	Amount of money with which Brandon started.
FV = Future Value	\$15,000.00	The amount of money Brandon wants to have at the end of the five year period.
n = Total Number of Compounding Periods	12×5 = 60	This is the total number of compounding periods over the course of the investment (or loan).
r = Annual Interest Rate	6% = 0.06	Rate of interest payable per year
i = Periodic Interest Rate	$\frac{r}{c}=\frac{0.06}{12}=0.005$	Rate of interest payable per compounding period.
d = Deposit/Payment made at Regular Intervals	Unknown	The amount deposited/paid at regular intervals.
p = Payments per Year (Payment Frequency)	12	The number of payments/deposits made per year.
c = Compounding Periods per Year (Compounding Frequency)	12	The number of times per year that the interest compounds.

Note: For the sake of simplicity, in this course it will be assumed that $p = c$ in all cases. However, this need not be the case. For example, for all mortgages in Canada, $c = 2$ (i.e. interest must be compounded twice per year or ***semi-annually***) but p can have many different values such as 12 (***monthly***), 24 (***semi-monthly***) and 52 (***weekly***).

The following diagram, which is called a "time-line," illustrates this situation. Although this seems somewhat more complicated than the scenario in part **(b)**, in reality, it is merely a simple application of the ideas in **(b)**. The only difference is that in this case, the method of part **(b)** needs to be applied multiple times.



So it appears that the ***crux*** of this problem is to figure out how to add up the future values of each deposit of d dollars. Since we know that Brandon needs to have \$15,000.00 by the time he is 21, the sum of the future values of each deposit of d dollars will have to be \$15,000.00. Knowing this will allow us to solve for d and finally determine exactly how much money Brandon needs to pay to the annuity each month.

Gladly, we already know how to find such a sum. Careful examination of the expression reveals that it is nothing more than a ***geometric series***.

In this geometric series, $a = d$, $r = 1.005$ and $n = 60$. Therefore,

$$d + d(1.005) + d(1.005)^2 + \cdots + d(1.005)^{59} = \frac{d(1.005^{60} - 1)}{1.005 - 1} = \frac{d(1.005^{60} - 1)}{0.005}.$$

Also recall that Brandon needs a future value of $15,000 to realize his dream of becoming a poker champion. Therefore,

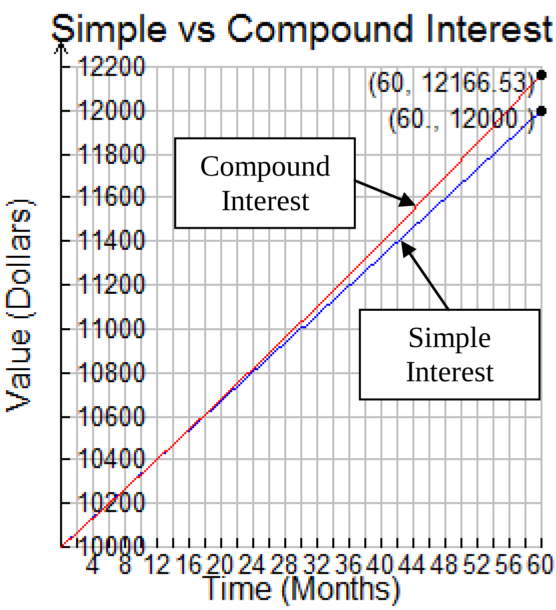
$$\frac{d(1.005^{60} - 1)}{0.005} = 15000$$

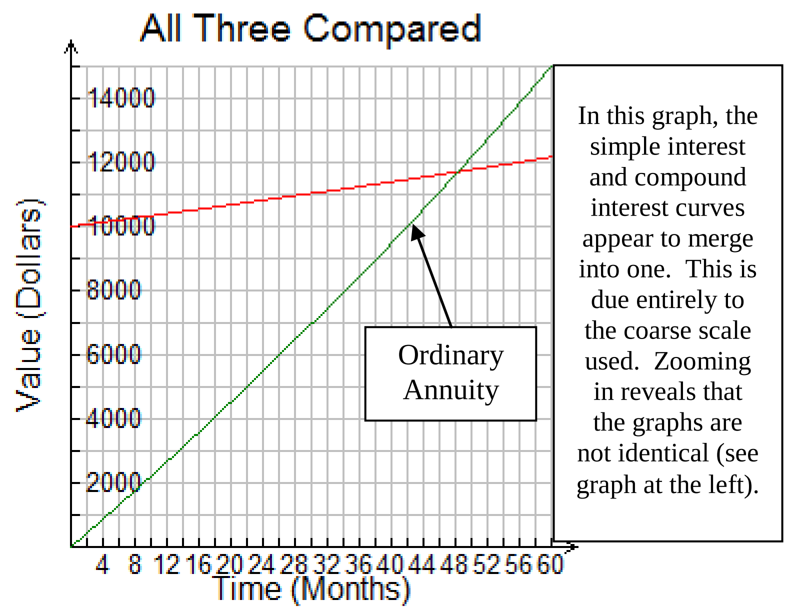
$$\therefore d = \frac{15000(0.005)}{1.005^{60} - 1} \doteq 214.99$$

Therefore, Brandon must deposit $214.99 ***at the end of each month***.

Summary of Brandon's Investment Options

The following is a graphical summary of the three investments considered by Brandon. Clearly, the annuity is the only investment that can satisfy his needs. In addition to the faster growth rate, the annuity has the advantage of not requiring a large amount of cash to be invested up front. All Brandon needs to do is to be able to put away $214.99 each month.





Summary of Investment Equations

Type of Investment	***Equations***		***Meaning of Variables***
Simple Interest	$I = Prt$	$A = P(1+rt)$	r = annual rate of interest i = periodic rate of interest = r ÷ (# compounding periods per year) n = total number of compounding periods t = time in years (simple interest only) R = amount deposited/paid at regular intervals PV = present value FV = future value I = amount of interest earned/paid after t years (simple interest) A = value of investment after t years (simple interest) ***or*** value of investment after n compounding periods (compound interest)
Compound Interest	$A = P(1+i)^n$	$P = \frac{A}{(1+i)^n} = A(1+i)^{-n}$	
Ordinary Annuity	$FV = R\left[\frac{(1+i)^n - 1}{i}\right]$	$PV = R\left[\frac{1-(1+i)^{-n}}{i}\right]$	

More Examples

1. Simple Interest

Philip borrows \$540 for 85 days by taking a cash advance on his credit card. The interest rate is 26%/a simple interest. How much will he need to pay back at the end of the loan period, and how much interest will he have paid?

Lara's Solution

$$t = \frac{85}{365}$$

$$P = \$540$$

$$r = 26\% = 0.26$$

Philip isn't borrowing the money for a full year, so I expressed the time as a fraction of 365 days in a year. I knew the principal, P, and I wrote the interest rate, r, as a decimal.

$$A = P(1 + rt)$$

$$= (\$540)\left(1 + 0.26 \times \frac{85}{365}\right)$$

To calculate the amount at the end of the loan period, I substituted the values of P, r, and t into the formula.

$$\doteq (\$540)(1.061)$$

$$= \$572.70$$

I rounded to the nearest cent.

Of the \$572.70 that Philip has to pay back, \$32.70 is interest.

I subtracted the principal to get the interest.

2. Compound Interest

On her 15th birthday, Trudy invests \$10 000 at 8%/a compounded monthly. When Lina turns 45, she invests \$10 000 at 8%/a compounded monthly. If both women leave their investments until they are 65, how much more will Trudy's investment be worth?

Henry's Solution

$$P = \$10\,000$$

$$i = \frac{0.08}{12}$$

To calculate how much Trudy's investment will be worth when she turns 65, I first determined the interest rate per month as a fraction.

$$n = (65 - 15) \times 12$$

$$= 600$$

Since interest is compounded monthly and she is investing for 50 years, there will be $50 \times 12 = 600$ compounding periods.

$$A = 10\,000\left(1 + \frac{0.08}{12}\right)^{600}$$

$$\doteq \$538\,781.94$$

Trudy's investment will be worth \$538 781.94 when she turns 65.

3. Ordinary Annuity (Future Value)

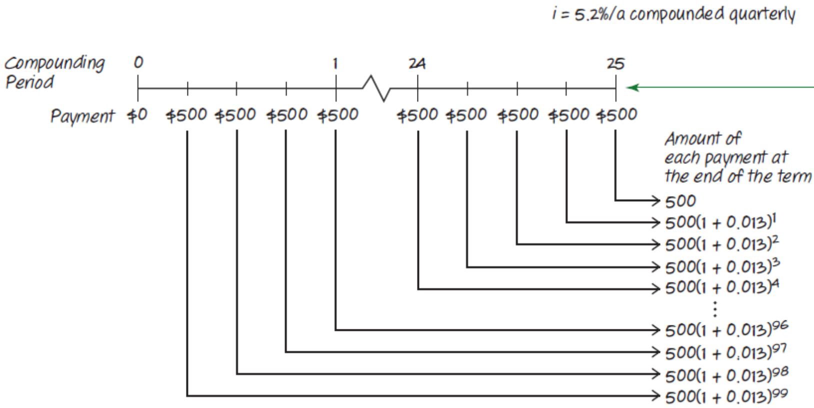
Chie puts away \$500 every 3 months at 5.2%/a compounded quarterly. How much will her annuity be worth in 25 years?

Kew's Solution: Using a Geometric Series

$$i = \frac{0.052}{4} = 0.013$$

$$n = 25 \times 4 = 100$$

First I calculated the interest rate per compounding period and the number of compounding periods.



I drew a timeline of the investments for each quarter to show the amounts of each investment. I calculated the value of each investment at the end of 25 years.

$$500, 500(1.013), 500(1.013)^2, \ldots, 500(1.013)^{98}, 500(1.013)^{99}$$

The values form a geometric sequence with first term \$500 and common ratio 1.013.

$$S_{100} = 500 + 500(1.013) + 500(1.013)^2 + \ldots + 500(1.013)^{98} + 500(1.013)^{99}$$

The total amount of all these investments is the first 100 terms of a geometric series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{100} = \frac{500(1.013^{100} - 1)}{1.013 - 1}$$

I used the formula for the sum of a geometric series to calculate the total amount of all of Chie's investments.

$$\doteq \$101\,487.91$$

I rounded to the nearest cent.

The total amount of all of Chie's investments at the end of 25 years will be \$101 487.91.

Tina's Solution: Using the Formula for the Future Value of an Annuity

$$R = \$500$$

$$i = \frac{0.052}{4} = 0.013$$

$$n = 25 \times 4 = 100$$

I calculated the interest rate per compounding period and the number of compounding periods.

$$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$$

$$= 500 \times \left(\frac{(1 + 0.013)^{100} - 1}{0.013}\right)$$

I substituted the values of R, i, and n into the formula for the future value of a simple, ordinary annuity.

$$\doteq \$101\,487.91$$

I rounded to the nearest cent.

The future value of Chie's annuity will be \$101 487.91.

4. Ordinary Annuity (Present Value)

Len borrowed $200 000 from the bank to purchase a yacht. If the bank charges 6.6%/a compounded monthly, he will take 20 years to pay off the loan.

a) How much will each monthly payment be?
b) How much interest will he have paid over the term of the loan?

Jasmine's Solution: Using the Formula

a)

$$i = \frac{0.066}{12} = 0.0055$$

$$n = 20 \times 12 = 240$$

I calculated the interest rate per compounding period and the number of compounding periods.

$$PV = \$\,200\,000$$

$$PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$$

$$200\,000 = R \times \left(\frac{1 - (1 + 0.0055)^{-240}}{0.0055}\right)$$

I substituted the values of *PV*, *i*, and *n* into the formula for the present value of an annuity.

$$200\,000 \doteq R \times 133.072$$

$$\frac{200\,000}{133.072} = R \times \frac{133.072}{133.072}$$

To solve for *R*, I divided both sides of the equation by 133.072.

$$R \doteq 1502.94$$

I rounded to the nearest cent.

Len will have to pay $1502.94 per month for 20 years to pay off the loan.

b)

$$A = 1502.94 \times 240$$
$$= \$\,360\,706.60$$

I calculated the total amount that Len will have paid over the 20-year term.

$$I = A - PV$$
$$= \$\,360\,706.60 - \$\,200\,000$$
$$= \$\,160\,706.60$$

I determined the interest by subtracting the present value from the total amount that Len will have paid.

Over the 20-year term of the loan, Len will have paid $160 706.60 in interest.

Terms you Need to Know

Term	***Deposit/Payment Frequency***	
Daily	Every day:	365 times per year
Weekly	Every week:	52 times per year
Semi-Monthly	Twice per month:	24 times per year
Monthly	Every month:	12 times per year
Bi-Monthly	Every two months:	6 times per year
Quarterly	Every three months:	4 times per year
Semi-Annually	Every six months:	2 times per year
Annually	Every year:	1 time per year