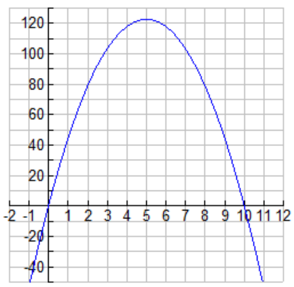
Table of Contents – Unit 0 – Review of Critically Important Mathematical Concepts

MCR3U0 – Diagnostic Test and Mastery Test Topics

What is Mathematics?

- In a nutshell, ***mathematics*** is the ***investigation*** of ***axiomatically defined abstract structures*** using ***logic*** and ***mathematical notation***. Accordingly, mathematics can be seen as an ***extension*** of ***spoken*** and ***written natural languages***, with an extremely precisely defined vocabulary and grammar, for the purpose of ***describing and exploring physical and conceptual RELATIONSHIPS.*** (Math is like a ***dating service***. It's all about ***relationships***! The lonely and very simple "Mr. x" is looking for a lovely but perhaps somewhat complex "Miss y.")
- Mathematical relationships can be viewed from a variety of different perspectives as shown below:

<table>
<tr><th>Algebraic</th><th>Geometric</th><th>Physical</th><th>Verbal</th><th>Numerical</th></tr>
<tr>
<td>$h = 49t - 4.9t^2$</td>
<td>
</td>
<td>A cannonball is fired vertically into the air with an initial speed of 49 m/s. Its height above the ground at a given time t is quadratically related to t.</td>
<td>The value of h is equal to the product of 49 and t reduced by the product of 4.9 and the square of t.</td>
<td><table>
<tr><th>x</th><th>y1(x)
49x-4.9x^2</th></tr>
<tr><td>-2</td><td>-117.6</td></tr>
<tr><td>-1</td><td>-53.9</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>44.1</td></tr>
<tr><td>2</td><td>78.4</td></tr>
<tr><td>3</td><td>102.9</td></tr>
<tr><td>4</td><td>117.6</td></tr>
<tr><td>5</td><td>122.5</td></tr>
<tr><td>6</td><td>117.6</td></tr>
<tr><td>7</td><td>102.9</td></tr>
<tr><td>8</td><td>78.4</td></tr>
<tr><td>9</td><td>44.1</td></tr>
<tr><td>10</td><td>-5.68E-14</td></tr>
<tr><td>11</td><td>-53.9</td></tr>
</table></td>
</tr>
</table>

Each of these perspectives has an important role to play in the understanding of mathematical relationships.

Linear Relationships

- These are the ***simplest*** of all mathematical relationships. They are very easy to analyze mathematically and are completely understood (i.e. there are no unresolved problems regarding quadratics).
- The graphs of linear relationships are ***straight lines***.
- If y is linearly related to x, then the ***rate of change of y is constant***. That is, for a given Δx , Δy is always the same. Another way of expressing this is that the ***first differences are constant***.
- Linear relationships can be used to ***model*** any quantity that ***changes at a constant rate***. For example, for a car that is travelling at a constant speed, distance travelled is linearly related to time elapsed. That is, the graph of distance versus time would be a straight line.
- The general equation of a linear relationship is $Ax + By + C = 0$. This equation can be rewritten in the form $y = -\frac{A}{B}x - \frac{C}{B}$. By comparing this to the slope-intercept form a linear equation, we see that slope $= -\frac{A}{B}$ and y-intercept $= -\frac{C}{B}$. By comparing to the slope-intercept form of a linear relationship we find $m = -\frac{A}{B}$ and $b = -\frac{C}{B}$.
- Any equation involving only linear terms can be solved by using the ***balancing method***. For example, an equation such as $\frac{2}{3}(5x-7) - \frac{1}{2} = \frac{7}{9}x + 10$ can be solved by balancing.
- While using the balancing method, it is helpful to remember the "dressing/undressing" analogy. Remember that as long as you perform the ***same operation to both sides of any equation***, you will obtain an equivalent equation (i.e. having the same solutions).

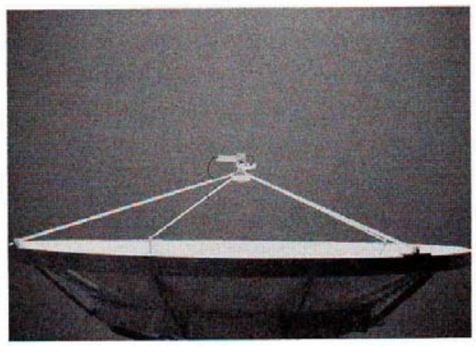
Quadratic Relationships

- These are not as simple as linear relationships but are still quite easy to analyze mathematically. Like linear relationships, quadratics are completely understood (i.e. there are no unresolved problems regarding quadratics).
- The graphs of quadratic relationships are ***parabolas***.
- If y is quadratically related to x, then the ***rate of change of y is NOT constant***. That is, for a given Δx , Δy is ***NOT*** always the same. For quadratic relationships, the ***first differences change linearly*** and ***the second differences are constant***.
- Quadratic relationships can be used to ***model*** many different quantities. For example, the position of any object moving solely under the influence of gravity (close to the surface of the Earth) changes quadratically. (See cannonball example in the "What is Mathematics?" section.

- The general equation of a quadratic relationship is $y = ax^2 + bx + c$. By ***completing the square***, this equation can be rewritten in the more convenient form $y = a(x-h)^2 + k$, where (h,k) is the vertex of the parabola.
- An equation involving quadratic terms ***cannot*** be solved entirely by using the ***balancing method***. Quadratic equations can be solved by ***factoring***, ***completing the square*** and ***the quadratic formula***.
- The roots of a quadratic equation are completely characterized by the discriminant, which equals $b^2 - 4ac$. This expression appears under the square root sign in the quadratic formula, which is what allows us to decide the nature of the roots.
- Once we know the nature of the roots of a quadratic, we can deduce whether its associated parabola lies entirely above the x-axis, entirely below the x-axis, crosses the x-axis at two points or just "touches" it at one point.
- Contrary to Jonathan Coulson's claims, quadratics have a wide variety of applications. An interesting one is given below. Satellite dishes are really just antennas that are designed to receive signals originating from satellites in geostationary orbits around the Earth. Most satellite dishes have a parabolic cross-section.

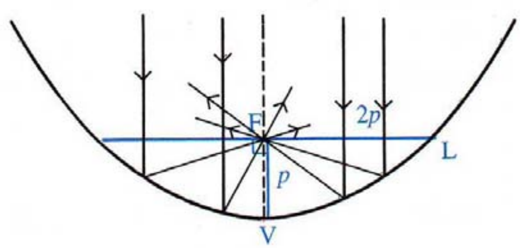
Reflector Property of the Parabola

We have all seen dish antennas for receiving TV signals from satellites. These antennas have parabolic cross sections. When the antenna is aimed at a satellite, the signals entering the antenna are reflected to the receiver, which is placed at the focus of the antenna.



Every parabola has a *focus*, which is a particular point on the axis of symmetry. The position of the focus can be defined as follows.

For any parabola, the *focus* is the point on the axis of symmetry which is half as far from the vertex as it is from the parabola, measured along a line perpendicular to the axis of symmetry. For example, in the diagram, FV = p, and FL = $2p$. That is, F is half as far from V as from L. Hence, F is the focus of the parabola. Every parabola has one and only one focus.



You can illustrate the reflector property of the parabola by completing the questions below.

QUESTIONS

1. a) Use a table of values to construct an accurate graph of the parabola defined by $y = \frac{1}{8}x^2$ for values of x between -8 and 8.
   b) Mark the point F(0,2) on the graph. Verify that F satisfies the above definition of the focus.
2. a) Mark any point P on the parabola you constructed in *Question 1*. Join PF, and draw a line PM parallel to the axis of symmetry. By estimation, draw a tangent to the parabola at P. Verify that PF and PM form equal angles with the tangent.
   b) Repeat part a) for other points P on the parabola.
3. Use the above definition of the focus to prove that the coordinates of the focus of the parabola defined by $y = ax^2$ are $\left(0, \frac{1}{4a}\right)$.

Terminology

- By this point in your mathematics education, you must understand and use the following terminology correctly: ***equation***, ***expression***, ***term***, ***factor***, ***polynomial***, ***monomial***, ***binomial***, ***trinomial***, ***factor***, ***expand***, ***solve***, ***simplify***, ***evaluate***, ***roots***, ***discriminant***, ***intercept***, ***intersect***, ***vertex***, ***axis of symmetry***, ***(simultaneous) system of equations***, ***relation***, ***relationship***, ***rate of change***, ***distance***, ***length***, ***area***, ***volume***, ***speed***, ***surface area***

Measurement

- Pythagorean Theorem and distance between two points (length of a line segment)
- Midpoint of a line segment
- Area of rectangle, parallelogram, triangle, trapezoid, circle
- Surface area of cylinder, cone, sphere, prism
- Volume of cylinder, cone, sphere, prism

Systems of Linear Equations (Two Linear Equations in Two Unknowns)

- Solve by using the method of substitution
- Solve by using the method of elimination
- Solve graphically

Notes

- Include any relevant notes in the space provided below.

Critically Important Logical Principles and Mathematical Terminology

Logic

The study of the ***principles of reasoning***, especially of the ***structure*** of propositions as distinguished from their ***content*** and of ***method***, and ***validity*** in deductive reasoning.

Premise, Conclusion, Logical Implication

- A ***premise*** is a statement that is known or assumed to be true and from which a ***conclusion*** can be drawn.
- A ***conclusion*** is a position, opinion or judgment reached after consideration.
- A ***logical implication*** or ***conditional statement*** is a statement that takes the form "If ***premise*** then ***conclusion***." In such a statement, the truth of the premise ***guarantees*** the truth of the conclusion.

Example

Premise → "If he has been injured"
Conclusion → "then he will not be able to play in tonight's hockey game"

Logical Implication → "If he has been injured, then he will not be able to play in tonight's hockey game."

Extremely Important Logical Implications in Mathematics

If $x = y$ then $x + a = y + a$, $x - a = y - a$, $ax = ay$, $\frac{x}{a} = \frac{y}{a}$ (if $a \neq 0$), $x^2 = y^2$, $\sqrt{x} = \sqrt{y}$, $\sin x = \sin y$, etc If the same operation is performed to both sides of an equation, then equality is preserved. ***Example*** $2x - 7 = 9$ $\therefore 2x - 7 + 7 = 9 + 7$ $\therefore 2x = 16$ $\therefore \frac{2x}{2} = \frac{16}{2}$ $\therefore x = 8$	If $a = b$ ***and*** $a = c$, then $b = c$. If two quantities are each equal to the same quantity, then they are equal to each other. ***Example*** $h(t) = -4.9t^2 + 49t$ and $h(t) = 0$ $\therefore -4.9t^2 + 49t = 0$	If $xy = 0$, then $x = 0$ or $y = 0$. If the product of two numbers is equal to zero, then at least one of the numbers must be zero. ***Example*** $(x-3)(x+7) = 0$ $\therefore x - 3 = 0$ or $x + 7 = 0$

Logical Fallacy: Ad Hominem Argument – An Example of Studying the Structure of an Argument

Basic Structure of an Ad Hominem Argument	***What is wrong with an ad hominem argument?***
A (fallacious) ad hominem argument has the basic form: Person A makes claim X There is something objectionable about Person A Therefore claim X is false	

Deductive Reasoning

Deductive arguments take the form "If ***Cause*** Then ***Effect***" or "If ***Premise*** Then ***Conclusion***"

In a deductive argument, we know that a ***cause*** (premise) produces a certain ***effect*** (conclusion). If we observe the cause, we can ***deduce*** (conclude by reasoning) that the effect ***must*** occur. These arguments always produce ***definitive*** conclusions; ***general principles*** are applied to reach specific conclusions. We must keep in mind, however, that a false premise can lead to a false result and an inconclusive premise can yield an inconclusive conclusion.

Examples of Deductive Reasoning from Everyday Life

1. If I spill my drink on the floor, the floor will get wet.
2. When students "forget" to do homework, Mr. Nolfi gets angry!
3. If a student is caught cheating, Mr. Nolfi will assign a mark of zero to him/her, ridicule the student publicly, turn red in the face and yell like a raving madman whose underwear are on fire!
4. Drinking too much alcohol causes drunkenness.

Exercise

Rephrase examples 2 and 4 in "If … then" form.

The Meaning of π – An Example of Deductive Reasoning

The following is an example of a typical conversation between Mr. Nolfi and a student who blindly memorizes formulas:

Student: Sir, I can't remember whether the area of a circle is πr^2 or $2\pi r$. Which one is it?

Mr. Nolfi: If you remember the meaning of π, you should be able to figure it out.

Student: How can 3.14 help me make this decision? It's only a number!

Mr. Nolfi: How dare you say something so disrespectful about one of the most revered numbers in the mathematical lexicon! (Just kidding. I wouldn't really say that.) It's true that the number 3.14 is an approximate value of π. But I asked you for its ***meaning***, not its value.

Student: I didn't know that π has a meaning. I thought that it was just a "magic" number.

Mr. Nolfi: Leave magic to the magicians. In mathematics, every term (except for primitive terms) has a very precise definition. Read the following carefully and you'll never need to ask your original question ever again!

In ***any*** circle, the ***ratio*** of the ***circumference*** to the ***diameter*** is equal to a ***constant*** value that we call π. That is,

$$C : d = \pi .$$

Alternatively, this may be written as

$$\frac{C}{d} = \pi$$

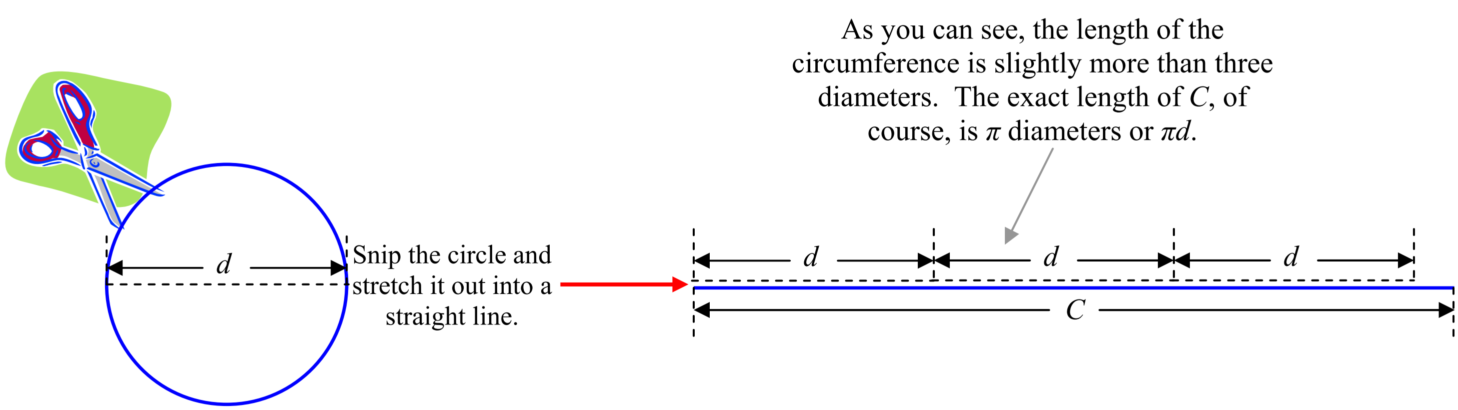
or in the more familiar form

$$C = \pi d .$$

If we recall that $d = 2r$, then we finally arrive at the most common form of this ***relationship***,

$$C = 2\pi r .$$

> This is an example of a ***deductive argument***. Each statement ***follows logically*** from the previous statement.
>
> That is, the argument takes the form "If P is true then Q must also be true" or more concisely, "P implies Q."



Mr. Nolfi: So you see, by understanding the meaning of π, you can ***deduce*** that $C = 2\pi r$. Therefore, the formula for the area must be $A = \pi r^2$. Furthermore, it is not possible for the expression $2\pi r$ to yield units of area. The number 2π is dimensionless and r is measured in units of distance such as metres. Therefore, the expression $2\pi r$ must result in a value measured in units of distance. On the other hand, the expression πr^2 must give a value measured in units of area because $r^2 = r(r)$, which involves multiplying a value measured in units of distance by itself. Therefore, by considering units alone, we are drawn to the inescapable conclusion that the area of a circle must be πr^2 and ***not*** $2\pi r$!

Examples

$2\pi r \doteq 2(3.14)(3.6\text{ cm}) = 22.608\text{ cm} \rightarrow$ This answer cannot possibly measure area because cm is a unit of distance.

Therefore, πr^2 must be the correct expression for calculating the area of a circle.

$\pi r^2 \doteq 3.14(3.6\text{ cm})^2 = 3.14(3.6\text{ cm})(3.6\text{ cm}) = 40.6944\text{ cm}^2 \rightarrow$ Notice that the unit cm^2 is appropriate for area.

Extremely Important Terminology

Term	*Meaning*	*Example(s)*
Expression	A combination of constants, operators, and variables representing numbers or quantities	**1.** $3(5)^2-4(5)(-1)+(-1)^2$ **2.** $3x^2-4xy+y^2$
Equation	A mathematical statement asserting that two expressions have the same value	$2(6z-1)(z-1)=-6(z-1)(2z-5)+3z+25$
Evaluate	Ascertain the numerical value of an expression	If $x=5$ and $y=-1$, ***evaluate*** $3x^2-4xy+y^2$. $3x^2-4xy+y^2$ $=3(5)^2-4(5)(-1)+(-1)^2$ $=3(25)+20+1$ $=96$
Simplify	Convert a mathematical expression to a simpler form	***Simplify*** $5(x-1)(x-2)-(5x-7)(x+10)$ $5(x-1)(x-2)-(5x-7)(x+10)$ $=5(x^2-3x+2)-(5x^2+43x-70)$ $=5x^2-15x+10-5x^2-43x+70$ $=5x^2-5x^2-15x-43x+10+70$ $=-48x+80$
Term	A mathematical expression that is associated to another through the operation of addition.	The algebraic expression $3x^2-4xy+y^2$ can be rewritten as $3x^2+(-4xy)+y^2$. Therefore, its ***terms*** are $3x^2$, $-4xy$ and y^2.
Factor	***Noun:*** One of two or more numbers or quantities that can be multiplied together to give a particular number or quantity. **e.g.** The factors of 15 are 1, 3, 5 and 15. ***Verb:*** To determine the factors of a number or expression.	***Factor*** $3n^2-2n-5$ $3n^2-2n-5$ $=3n^2-5n+3n-5$ $=(3n^2-5n)+(3n-5)$ $=n(3n-5)+1(3n-5)$ $=(3n-5)(n+1)$
Solve	Work out the solution to an equation	***Solve*** $3n^2-2n-5=0$ $3n^2-2n-5=0$ $\therefore 3n^2-5n+3n-5=0$ $\therefore (3n^2-5n)+(3n-5)=0$ $\therefore n(3n-5)+1(3n-5)=0$ $\therefore (3n-5)(n+1)=0$ $\therefore 3n-5=0$ or $n+1=0$ $\therefore n=\frac{5}{3}$ or $n=-1$
Relationship or Relation	A property of association shared by ordered pairs of terms or objects	The equation $y=x^2$ expresses a ***relationship*** between the x-co-ordinate and the y-co-ordinate of any point that lies on the upward opening parabola with vertex (0, 0) and vertical stretch factor 1.

Critically Important Prerequisite Skills for MCR3U0

Operator Precedence (Order of Operations)

Standard Order (no parentheses)	*Notes*	*Example*	*Example with Parentheses*
1. Exponents		$12 \div 4 \times 3 - 2 \times 4^3 \div 32$	$12 \div 4 \times (3-2) \times 64 \div 32$
2. Multiplication and Division	Performed in order of occurrence from left to right when both of these operations occur in a given term.	$= 3 \times 3 - 2 \times 64 \div 32$ $= 9 - 128 \div 32$	$= 3 \times 1 \times 64 \div 32$ $= 3 \times 64 \div 32$
3. Addition and Subtraction	Performed in order of occurrence from left to right when both of these operations occur in a given expression.	$= 9 - 4$ $= 5$	$= 192 \div 32$ $= 6$

Parentheses are used to ***override*** the standard order of precedence. When a departure from the standard order is required (for example when subtraction needs to be performed before multiplication) parentheses must be used.

Operating with Integers

Adding and Subtracting Integers

- Movements on a number line
- Moving from one floor to another using an elevator
- Loss/gain of yards in football
- Loss/gain of money in bank account or stock market

Add a Positive Value or Subtract a Negative Value

$+(+\ )$ *or* $-(-\ ) \rightarrow$ ***GAIN*** (move ***up*** or ***right***)

Add a Negative Value or Subtract a Positive Value

$+(-\ )$ *or* $-(+\ ) \rightarrow$ ***LOSS*** (move ***down*** or ***left***)

Multiplying and Dividing Integers

- Multiplication is ***repeated addition***
  e.g. $5(-2) = 5$ groups of -2
  $= (-2)+(-2)+(-2)+(-2)+(-2) = -10$
- Division is the ***opposite of*** multiplication
  e.g. $-10 \div (-2) =$ How many groups of -2 in -10?
  $= 5$

Multiply or Divide Two Numbers of Like Sign

$(+\ )(+\ )$ *or* $(-\ )(-\ ) \rightarrow$ ***POSITIVE RESULT***

Multiply or Divide Two Numbers of Unlike Sign

$(+\ )(-\ )$ *or* $(-\ )(+\ ) \rightarrow$ ***NEGATIVE RESULT***

Operating with Fractions

Adding and Subtracting Rational Numbers (Fractions)

- Express each fraction with a ***common denominator***
- Use rules for operating with integers
- $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$

Example

$$\frac{-3}{6} + \left(\frac{-5}{9}\right) = \frac{-9}{18} + \left(\frac{-10}{18}\right) = \frac{-9+(-10)}{18} = -\frac{19}{18}$$

Multiplying and Dividing Rational Numbers (Fractions)

- ***NO*** common denominator
- $\frac{a}{b}\left(\frac{c}{d}\right) = \frac{ac}{bd}$ and $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

Example

$$\frac{-3}{6} \div \left(\frac{-5}{9}\right) = \frac{-3}{6} \times \left(\frac{9}{-5}\right) = \frac{-3}{2} \times \left(\frac{3}{-5}\right) = \frac{9}{10}$$

Simplifying Algebraic Expressions

Adding and Subtracting TERMS

- Collect ***like terms***
- Use rules for adding/subtracting integers

Example

$$-3x^2y + 5xy - 6x^2y - 13xy = -9x^2y - 8xy$$

Multiplying FACTORS

- Use rules for multiplying integers, laws of exponents and the distributive law

Examples

1. $-3a^2(5a^6)(-6b^7) = (-3)(5)(-6)a^2a^6b^7 = 90a^8b^7$

2. $(2x-7)(3x-8) = 2x(3x) - 2x(8) - 7(3x) - 7(-8) = 6x^2 - 37x + 56$

Dividing Algebraic Expressions

- Use rules for dividing integers
- Use laws of exponents

Examples

1. $\frac{-6a^5b^3}{-18a^3b^5} = \frac{a^2}{3b^2}$, **2.** $\frac{7m^3n - 14m^6n^3}{-2m^2n^7} = \frac{7m^3n}{-2m^2n^7} - \frac{14m^6n^3}{-2m^2n^7} = -\frac{7m}{2n^6} + \frac{7m^4}{n^4}$

Factoring

An ***expression*** is ***factored*** if it is written as a ***product***.

Common Factoring	***Factor Simple Trinomial***	***Factor Complex Trinomial***	***Difference of Squares***
Example $-42m^3n^2+13mn^2p-39m^4n^3q$ $=-13mn^2(4m^2-p+3m^3nq)$	***Example*** $n^2-20n+91$ $=(n-7)(n-13)$ ***Rough Work*** $(-7)(-13)=91$ $-7+(-13)=-20$	***Example*** $10x^2-x-21$ $=(10x^2-15x)+(14x-21)$ $=5x(2x-3)+7(2x-3)$ $=(2x-3)(5x+7)$ ***Rough Work*** $(10)(-21)=-210$, $(-15)(14)=-210$ $-15+14=-1$	***Example*** $98x^2-50y^2$ $=2(49x^2-25y^2)$ $=2((7x)^2-(5y)^2)$ $=2(7x-5y)(7x+5y)$

Solving Equations

- An ***equation*** has an ***expression*** on the left-hand side, an ***expression*** on the right-hand side and an ***equals sign*** between the left and right sides.
- If an equation needs to be ***solved***, then we must find a value of the unknown that ***satisfies*** the equation. That is, when a solution is substituted into the equation, the left side ***must*** equal the right side.
- It is also very important to understand ***graphical solutions*** of equations.
- While using the balancing method, it is helpful to remember the "dressing/undressing" analogy. Remember that as long as you perform the ***same operation to both sides of any equation***, you will obtain an ***equivalent equation*** (i.e. an equation that has the same solutions as the original).

Solving Linear Equations

- Use the "balancing" method.
- Eliminate fractions by multiplying both sides of the equation by the ***least common multiple of all denominators***.

Example

$\frac{2}{3}(5x-7)-\frac{1}{2}=\frac{7}{9}x+10$

$\therefore 18\left[\frac{2}{3}(5x-7)-\frac{1}{2}\right]=18\left[\frac{7}{9}x+10\right]$

$\therefore 12(5x-7)-9=14x+180$

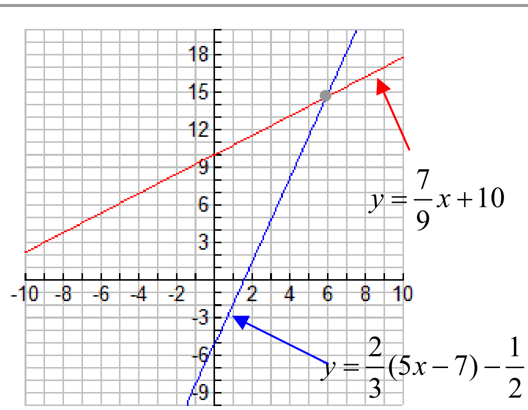
$\therefore 60x-84-9=14x+180$

$\therefore 60x-93=14x+180$

$\therefore 60x-93-14x+93=14x+180-14x+93$

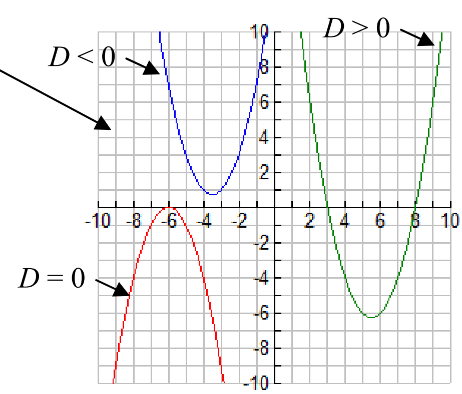
$\therefore 46x=273$

$\therefore x=\frac{273}{46}$



Solving Quadratic Equations

- First write the quadratic equation in the form $ax^2+bx+c=0$.
- Try ***factoring first***.
- If the quadratic does not factor, use the ***quadratic formula***.
- Only use the method of "completing the square" if you are asked to!
- The ***nature of the roots*** can be determined by calculating the ***discriminant*** $D=b^2-4ac$.



Examples

$3n^2-2n-5=0$

$\therefore 3n^2-5n+3n-5=0$

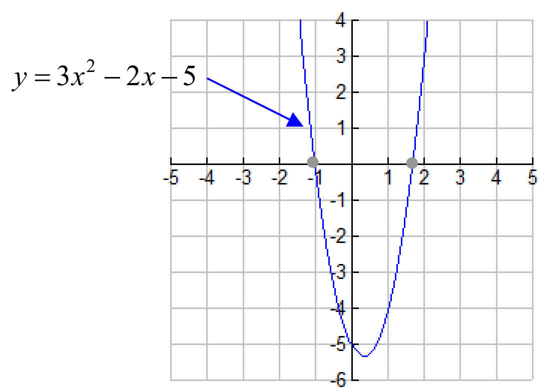
$\therefore (3n^2-5n)+(3n-5)=0$

$\therefore n(3n-5)+1(3n-5)=0$

$\therefore (3n-5)(n+1)=0$

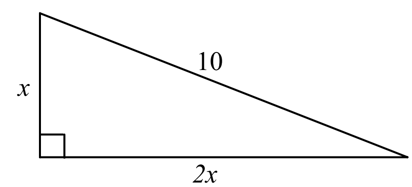
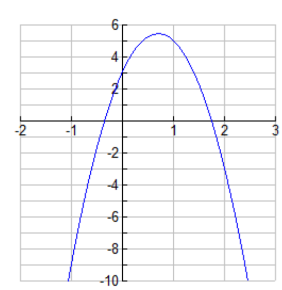
$\therefore 3n-5=0$ or $n+1=0$

$\therefore n=\frac{5}{3}$ or $n=-1$



Mathematical Relationships

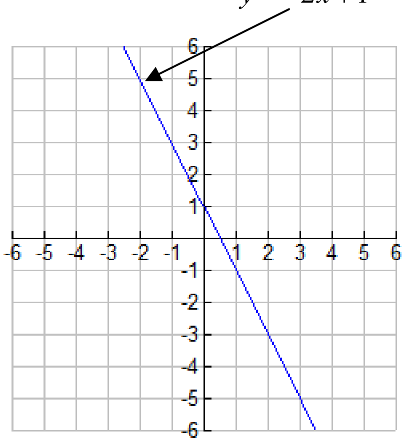
A ***formula*** is an ***equation*** that expresses a ***mathematical relationship*** between ***two or more unknowns***. ***All formulas are equations but not all equations are formulas!***

Example Formulas	*Relationship Expressed*	*Example*
$c^2 = a^2 + b^2$ This is the famous Pythagorean Theorem	This formula expresses the relationship among the sides of a right triangle.	**The hypotenuse of a right triangle has a length of 10 m and the lengths of other sides are in the ratio 2:1. Find the lengths of the other sides.** $x^2 + (2x)^2 = 10^2$ $\therefore x^2 + 4x^2 = 100$ $\therefore 5x^2 = 100$ $\therefore x^2 = 20$ $\therefore x = \sqrt{20} = 2\sqrt{5}$  The lengths of the other two sides are $2\sqrt{5}$ m and $4\sqrt{5}$ m.
$Ax + By + C = 0$ This is the so-called ***standard form*** of a linear equation.	This formula expresses the relationship between the x-co-ordinate and the y-co-ordinate of any point lying on a straight line with slope equal to $-\frac{A}{B}$ and y-intercept equal to $-\frac{C}{B}$.	**Find the slope and y-intercept of a straight line with equation $Ax + By + C = 0$.** First solve for y. Then compare to the $y = mx + b$ form. $Ax + By + C = 0$ $\therefore By = -Ax - C$ $\therefore y = -\frac{A}{B}x - \frac{C}{B}$ By comparing to the $y = mx + b$ form, we see that $m = -\frac{A}{B}$ and $b = -\frac{C}{B}$.
$y = ax^2 + bx + c$ This is the so-called ***standard form*** of a quadratic equation.	This formula expresses the relationship between the x-co-ordinate and the y-co-ordinate of any point lying on a parabola with vertex $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ and vertical stretch factor a. If $a > 0$ then the parabola opens ***upward***. Otherwise, if $a < 0$ then the parabola opens ***downward***.	**Find the vertex, vertical stretch factor and direction of opening of a parabola with equation $y = -5x^2 + 7x + 3$.** $y = -5x^2 + 7x + 3$ $\therefore y = -5\left(x^2 - \frac{7}{5}x\right) + 3$ $\therefore y = -5\left(x^2 - \frac{7}{5}x + \left(\frac{7}{10}\right)^2 - \left(\frac{7}{10}\right)^2\right) + 3$ $\therefore y = -5\left(x - \frac{7}{10}\right)^2 + 5\left(\frac{7}{10}\right)^2 + 3$ $\therefore y = -5\left(x - \frac{7}{10}\right)^2 + \frac{109}{20}$  Therefore, the vertex is $\left(\frac{7}{10}, \frac{109}{20}\right)$, the direction of opening is downward and the vertical stretch factor is 5.

Rate of Change (How fast a Quantity Changes relative to another Quantity)

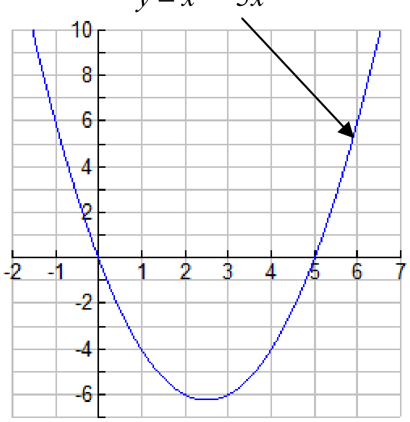
- If y is ***linearly related*** to x, then the ***rate of change of y is constant relative to x***. That is, for a given Δx, Δy is always the same. Another way of expressing this is that the ***first differences are constant***.
- If y is quadratically related to x, then the ***rate of change of y is NOT constant relative to x***. That is, for a given Δx, Δy is ***NOT*** always the same. For quadratic relationships, the ***first differences change linearly*** and ***the second differences are constant***.

$y=-2x+1$



x	Δx	$y=-2x+1$	First Differences Δy	Second Differences $\Delta(\Delta y)$
-1		3		
0	1	1	-2	
1	1	-1	-2	0
2	1	-3	-2	0
3	1	-5	-2	0
4	1	-7	-2	0
5	1	-9	-2	0
6	1	-11	-2	0

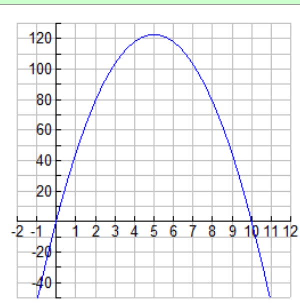
$y=x^2-5x$



x	Δx	$y=x^2-5x$	First Differences Δy	Second Differences $\Delta(\Delta y)$
-1		6		
0	1	0	-6	
1	1	-4	-4	2
2	1	-6	-2	2
3	1	-6	0	2
4	1	-4	2	2
5	1	0	4	2
6	1	6	6	2

What is Mathematics?

- In a nutshell, ***mathematics*** is the ***investigation*** of ***axiomatically defined abstract structures*** using ***logic*** and ***mathematical notation***. Accordingly, mathematics can be seen as an ***extension*** of ***spoken*** and ***written natural languages***, with an extremely precisely defined vocabulary and grammar, for the purpose of ***describing and exploring physical and conceptual RELATIONSHIPS.*** (Math is like a ***dating service***. It's all about ***relationships***! The lonely and very simple "Mr. x" is looking for a lovely but perhaps somewhat complex "Miss y.")
- Mathematical relationships can be viewed from a variety of different perspectives as shown below:

Algebraic	*Geometric*	*Physical*	*Verbal*	*Numerical*
$h=49t-4.9t^2$		A cannonball is fired vertically into the air with an initial speed of 49 m/s. Its height above the ground at a given time t is quadratically related to t.	The value of h is equal to the product of 49 and t reduced by the product of 4.9 and the square of t.	x | y1(x) 49x-4.9x^2 -2 | -117.6 -1 | -53.9 0 | 0 1 | 44.1 2 | 78.4 3 | 102.9 4 | 117.6 5 | 122.5 6 | 117.6 7 | 102.9 8 | 78.4 9 | 44.1 10 | -5.68E-14 11 | -53.9

Each of these perspectives has an important role to play in the understanding of mathematical relationships.

Simultaneous Systems of Linear Equations

Solving Simultaneous Systems of Linear Equations

- Use ***substitution*** or ***elimination***.
- The solutions ***must satisfy both equations***.

Example

$x-2y=-5$ (1)

$5x+6y=7$ (2)

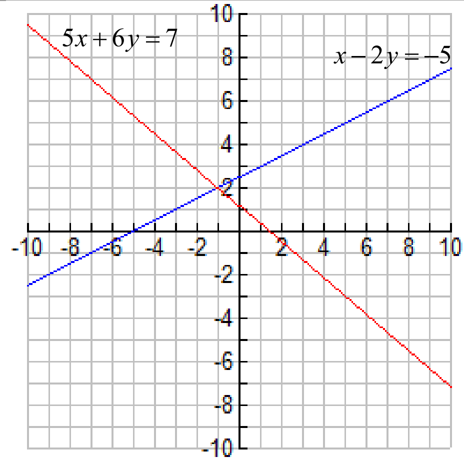
$(1)\times 3,\quad 3x-6y=-15$ (3)

$(2)+(3),\ 8x=-8$

$\therefore x=-1$

$\therefore y=2$

Graphical Solution



George Polya's Four Steps of Problem Solving

1. Understand the Problem
2. Choose a Strategy
3. Carry out the Strategy
4. Check the Solution

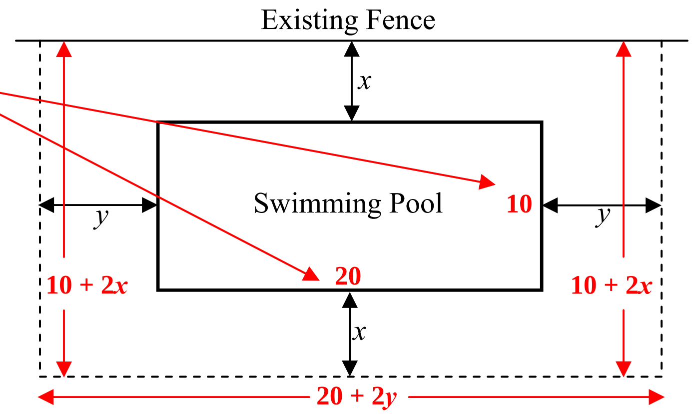
STEP 1 IS OF CRITICAL IMPORTANCE!
To complete step 1, you must ***understand*** the ***terminology*** used in the problem statement!

- ***equation*** → L.H.S. = R.H.S. → a complete mathematical "***sentence***"
- ***expression*** → ***not*** a complete mathematical "sentence" → more like a ***phrase***

Solving the so-called "word problems" that you are given in school is usually just a matter of ***translating English sentences into mathematical equations***.

Example

To help prevent drowning accidents, a protective fence is to be erected around a **pool** whose dimensions are **20 m by 10 m**. Since there is an existing fence on one side, new fencing is only required around ***three*** sides of the pool (see diagram). In addition, the gap between the fence and the edge of the pool must be uniform on opposite sides of the pool (see diagram). If **100 m of fencing material is available**, what is the **maximum area that can be enclosed by the fencing**?



100 m of fencing material is available

↓

Length of fence is 100 m

$10+2x+20+2y+10+2x=100$

$\therefore 4x+2y+40=100$

$\therefore 4x+2y=60$

$\therefore 2x+y=30$

$\therefore 2x=30-y$

area that can be enclosed by the fencing

↓

area of fence = length × width

$A=(20+2y)(10+2x)$

$\therefore A=(20+2y)(10+30-y)$

$\therefore A=(20+2y)(40-y)$

Understanding how to avoid Blind Memorization by Reviewing Cartesian (Analytic, Co-ordinate) Geometry

The only Equation you need to know to find an Equation of a Line

Slope = Slope

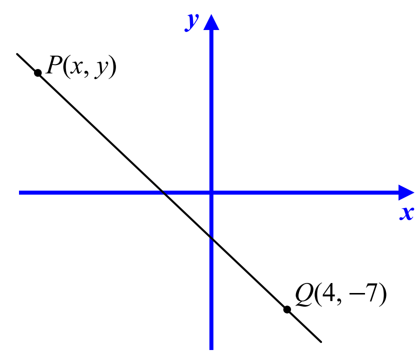
Example

Find an equation of the line with slope $-\frac{2}{3}$ and passing through the point (4, –7).

Solution

Let $P(x, y)$ be any point on the line other than (4, -7).

Now since



$$\text{slope} = \text{slope},$$
$$\therefore \frac{\Delta y}{\Delta x} = -\frac{2}{3}$$
$$\therefore \frac{y-(-7)}{x-4} = -\frac{2}{3}$$
$$\therefore y+7 = -\frac{2}{3}(x-4)$$
$$\therefore y = -\frac{2}{3}x - \frac{13}{3}$$

Therefore, $y = -\frac{2}{3}x - \frac{13}{3}$ is ***an*** (not "the") equation of the required line. //

Review of finding the Distance between Two Points

The only Equation you need to know to find the Distance between Two Points

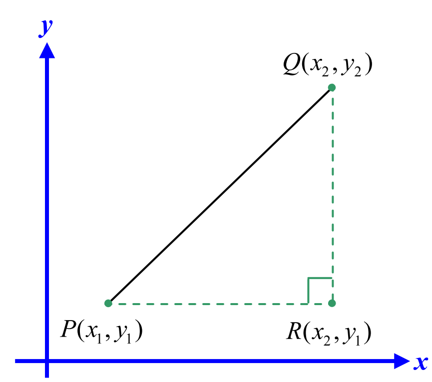
The Pythagorean Theorem

Example

Find the distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Solution

Let P, Q and R be as shown in the diagram.



Clearly, $PR = |x_2 - x_1|$ and $QR = |y_2 - y_1|$ Therefore, by the Pythagorean Theorem,

$$PQ^2 = PR^2 + QR^2$$
$$\therefore PQ^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$\therefore PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$\therefore PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Therefore, finding the distance between two points is a simple application of the Pythagorean Theorem. //

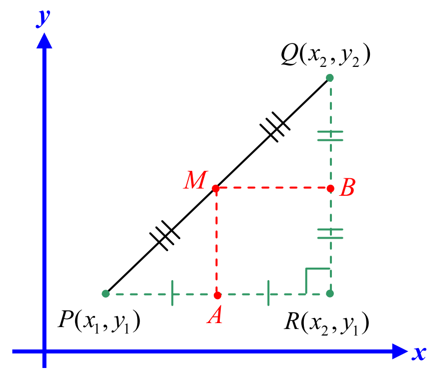
Review of finding the Midpoint of a Line Segment

The only Equation you need to know to find the Midpoint of a Line Segment

$$\text{The average of two numbers } a \text{ and } b \text{ is } \frac{a+b}{2}$$

Example

Find the midpoint of the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.



Solution

As shown in the diagram, let *M*, *A* and *B* represent the midpoints of the line segments *PQ*, *PR* and *QR* respectively. Since *A* lies on *PR* and *PR* is parallel to the *x*-axis, the *y*-co-ordinate of *A* must be y_1. Also, since *A* lies exactly half way between *P* and *R*, its *x*-co-ordinate must be the average of the *x*-co-ordinates of *P* and *R*. Therefore, the co-ordinates of *A* must be $\left(\frac{x_1+x_2}{2}, y_1\right)$. Using similar reasoning, the co-ordinates of *B* must be $\left(x_2, \frac{y_1+y_2}{2}\right)$.

Since *MA* is parallel to the *y*-axis, the *x*-co-ordinate of *M* must equal that of *A*. Since *MB* is parallel to the *x*-axis, the *y*-co-ordinate of *M* must equal that of *B*. Therefore, the co-ordinates of *M* must be

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Therefore, the midpoint *M* of *PQ* must have co-ordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. //

Important Exercises

1. Using a diagram and an argument similar to those given above, explain why parallel lines have equal slope.

2. Using a diagram and an argument similar to those given above, explain why any line perpendicular to a line with slope m must have slope $-\frac{1}{m}$.

3. Give a physical interpretation of slope.

4. Given the points $P(-3,5)$ and $Q(11,11)$, find an equation of the line passing through the midpoint of the line segment *PQ* and having slope $-\frac{2}{7}$.

5. Given the points $P(-3,5)$ and $Q(11,11)$, find an equation of the line perpendicular to *PQ* and passing through *Q*.

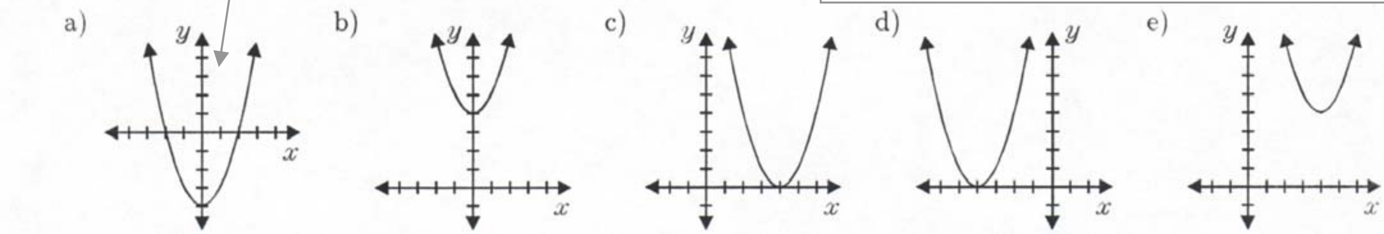
6. Find the distance from the point $P(-3,5)$ to the line $y = 2x + 3$. (If you hope to solve this problem, a diagram is a must!)

Review of Quadratic Relations – Diagnostic Test Solutions

This picture is the ***graphical*** or ***geometric representation*** of the parabola ***described by*** the equation $y = x^2 - 4$.

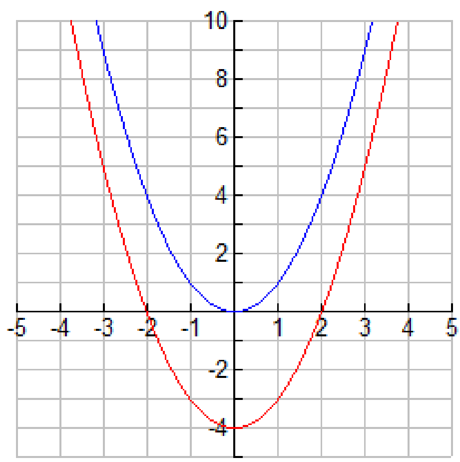
The equation $y = x^2 - 4$ is the ***algebraic representation*** of a particular parabola. The equation tells us the ***relationship*** between x and y. It is used to find ***points*** that lie on the curve.

1. Which one of the following could be the graph of $y = x^2 - 4$?



Question 1 – Solution and Explanation

The graph of the equation $y = x^2 - 4$ should be closely related to the graph of $y = x^2$, which describes a parabola that opens upward and whose vertex is at the origin. We should ask ourselves what effect subtracting 4 would have on the graph of $y = x^2$. By comparing the two equations carefully, we should eventually notice that for a given value of x, the y-co-ordinate of a point lying on $y = x^2 - 4$ will be ***4 less than*** the y-co-ordinate of $y = x^2$. Geometrically, this results in the graph of $y = x^2$ shifting down by 4 units. Therefore, the correct answer must be "a)." This is much easier to understand through a graph and a table of values as shown below.



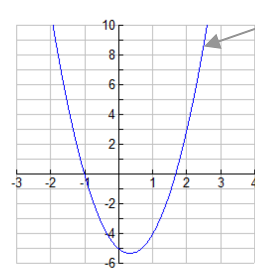
x	y1(x) x^2	y2(x) x^2-4
-5	25	21
-4	16	12
-3	9	5
-2	4	0
-1	1	-3
0	0	-4
1	1	-3
2	4	0
3	9	5
4	16	12
5	25	21

Note – General Form of a Quadratic

The general form of a quadratic equation is $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$. While this form of a quadratic is easy to understand, it is not very convenient because it makes graphing difficult. By using the method called ***completing the square***, however, the form $y = ax^2 + bx + c$ can be transformed to the equivalent form $y = a(x-h)^2 + k$, which is extremely easy to graph.

$$y = a(x-h)^2 + k$$

- (h, k) is the ***vertex*** of the parabola
- The equation of the ***axis of symmetry*** is $x = h$.
- If $a > 0$, the parabola ***opens upward***. Since $a(x-h)^2 \geq 0$ for all possible values of x, the ***absolute minimum*** value of y is k.
- If $a < 0$, the parabola ***opens downward***. Since $a(x-h)^2 \leq 0$ for all possible values of x, the ***absolute maximum*** value of y is k.



$y = 3x^2 - 2x - 5$

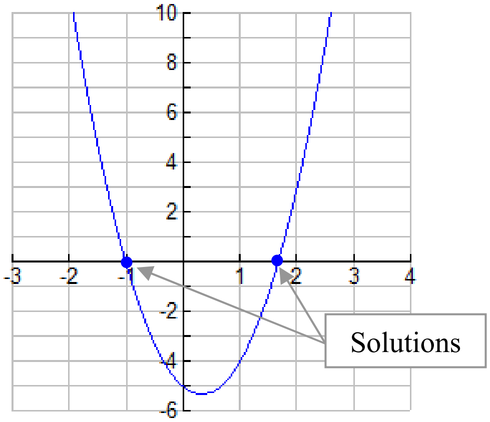
By completing the square, we find the equivalent equation

$$y = 3\left(x + \frac{1}{3}\right)^2 - \frac{16}{3}$$

2. Solve: $3n^2 - 2n - 5 = 0$

Question 2 – Solution and Explanation

Whenever you encounter what appears to be a purely algebraic situation, it is always wise to consider whether there is a way to ***visualize*** the problem. That is, it is important to consider ***geometric*** or ***physical*** (real-world) interpretations. In this case, it is possible to interpret this quadratic equation geometrically by considering the graph of $y = 3n^2 - 2n - 5$ and inquiring into what value(s) of n would make the "y-value" equal to zero. Clearly, the y-co-ordinate of a point can be zero only if the point lies on the x-axis. Therefore, solving the quadratic equation $3n^2 - 2n - 5 = 0$ is equivalent to determining the x-intercepts of the graph of $y = 3n^2 - 2n - 5$ (or equivalently, the graph of $y = 3x^2 - 2x - 5$).



- Solving $3n^2 - 2n - 5 = 0$ is equivalent to finding the points at which $y = 3x^2 - 2x - 5$ intersects the x-axis (i.e. the x-intercepts)
- To solve a quadratic equation, try ***factoring*** first:
  simple trinomial $\rightarrow$ $x^2 + bx + c$
  general (complex) trinomial $\rightarrow$ $ax^2 + bx + c$ (factor by ***decomposition***)
  difference of squares $\rightarrow$ $x^2 - a^2$
- If the quadratic does not factor, then use the ***quadratic formula***:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$3n^2 - 2n - 5 = 0$ ← 3(–5) = –15 → look for two numbers whose sum is –2 and whose product is –15

$\therefore 3n^2 - 5n + 3n - 5 = 0$ ← Rewrite the middle term –2*n* in ***decomposed*** form: –2*n* = –5*n*+3*n* or –2*n* =3*n*–5*n*

$\therefore (3n^2 - 5n) + (3n - 5) = 0$ ← Group the resulting four term expression as a pair of binomials

$\therefore n(3n - 5) + 1(3n - 5) = 0$ ← Factor out common factors in each binomial

$\therefore (3n - 5)(n + 1) = 0$ ← Factor again

$\therefore 3n - 5 = 0$ or $n + 1 = 0$ ← Now we have a product that equals zero. This can only happen if one of the factors is zero.

$\therefore n = \frac{5}{3}$ or $n = -1$

3. Solve for x: $4x^2 - 36 = 160$

Question 3 – Solution

$4x^2 - 36 = 160$

$\therefore 4x^2 - 36 - 160 = 0$

$\therefore 4x^2 - 196 = 0$

$\therefore 4(x^2 - 49) = 0$

$\therefore 4(x - 7)(x + 7) = 0$

$\therefore x - 7 = 0$ or $x + 7 = 0$

$\therefore x = 7$ or $x = -7$

OR

$4x^2 - 36 = 160$

$\therefore 4x^2 = 160 + 36$

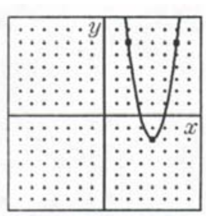
$\therefore 4x^2 = 196$

$\therefore x^2 = 49$

$\therefore x = \pm 7$

4. What is the equation of the graph shown?

Question 4 – Solution



Vertex: (4, –2)

$\therefore$ an equation of the parabola is of the form $y = a(x - 4)^2 - 2$, where a is some real number.

Other points on parabola: (2, 6), (6, 6)

Since (2, 6) lies on the parabola, it must satisfy the equation. Therefore,

$6 = a(2 - 4)^2 - 2$

$\therefore 4a = 8$

$\therefore a = 2$

5. The trinomial $x^2 - 10kx + R$ is a perfect square. Find the value of R.

Question 5 – Solution

$(x - a)^2 = x^2 - 10kx + R$

$\therefore x^2 - 2ax + a^2 = x^2 - 10kx + R$

By comparing the left and right sides of the equation, we conclude that $2a = 10k$ or $a = 5k$. Also, $R = a^2 = (5k)^2 = 25k^2$.

6. Put the following in the form $a(x-h)^2+k$:

a) $y = 2x^2 - 20x + 41$

b) $y = -x^2 + 14x - 46$

c) $y = 3x^2 - 2x + \frac{22}{3}$

Question 6 – Solution to c)

$$y = 3x^2 - 2x + \tfrac{22}{3}$$

$$\therefore y = 3(x^2 - \tfrac{2}{3}x) + \tfrac{22}{3}$$

$$\therefore y = 3(x^2 - \tfrac{2}{3}x + (\tfrac{1}{3})^2 - (\tfrac{1}{3})^2) + \tfrac{22}{3}$$

$$\therefore y = 3(x - \tfrac{1}{3})^2 - 3(\tfrac{1}{3})^2 + \tfrac{22}{3}$$

$$\therefore y = 3(x - \tfrac{1}{3})^2 + 7$$

7. A 25 m by 20 m rug is placed in a convention room so that the floor not covered by the rug forms a border of uniform width around the rug. The area of the rug is equal to the area of the floor that is uncovered. Which of the following would be a suitable equation for finding the width w of the border?

a) $(w+25)(w+20) = 1000$ b) $(25-w)(20-w) = 500$ c) $(25-2w)(20-2w) = 500$

d) $(2w+25)(2w+20) = 1000$ e) $(2w+25)(2w+20) = 500$

Question 7 – Solution

Area of rug = 25(20) = 500

Width of border = unknown = w

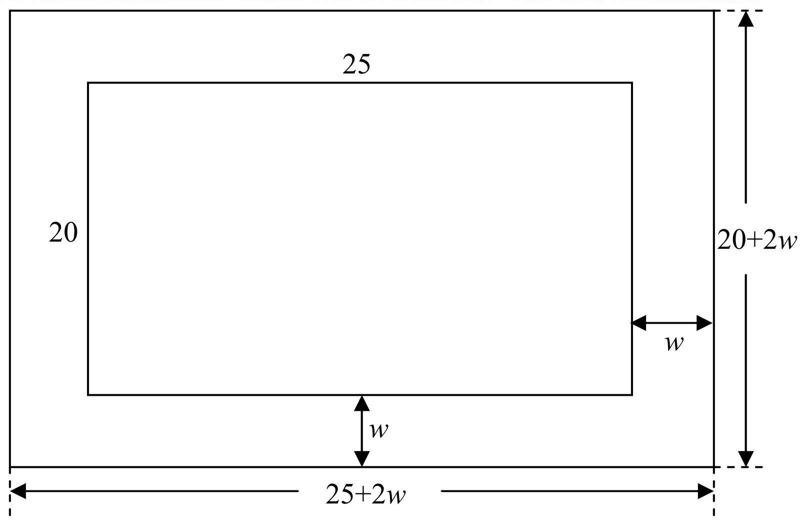
Length of room = 25+2w

Width of room = 20+2w

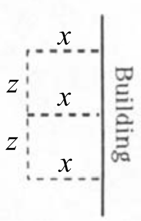
Area of room = (25+2w)(20+2w)
Area of room = area of rug + area of border
= 500 + 500
= 1000

Area of room = Area of room
∴(25+2w)(20+2w) = 1000

The correct answer is "d)."



8. A piece of land is to be fenced and then divided by an inner fence as shown in the diagram. A building forms one side of the total fenced area. If the total length of fencing available for both the outer fencing and the partition down the middle is 24 m, what is the maximum area that can be enclosed by the fencing?



Question 8 – Solution

Total length of fence = 24 m
Total length of fence = $3x + 2z$

Therefore, $3x + 2z = 24 \rightarrow 2z = 24 - 3x$

Area enclosed by fence = $x(2z) = x(24-3x) = -3x^2 + 24x$

The graph of $y = -3x^2 + 24x$ is a parabola that opens downward, which means that it has an absolute maximum at its vertex. To find the vertex, we complete the square:

$$y = -3x^2 + 24x$$

$$\therefore y = -3(x^2 - 8x)$$

$$\therefore y = -3(x^2 - 8x + 4^2 - 4^2)$$

$$\therefore y = -3(x^2 - 8x + 4^2) - (-3)(4^2)$$

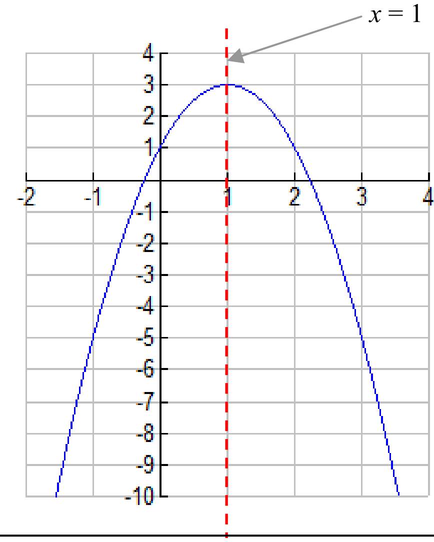
$$\therefore y = -3(x-4)^2 + 48$$

The co-ordinates of the vertex of the parabola are (4, 48), which is where it reaches its absolute maximum value of 48. Therefore, the maximum area that can be enclosed is 48 m^2.

9. The graph of the parabola $y = -2(x-1)^2 + 3$ is symmetric about a line. What is the equation of that line?

Question 9 – Solution

Every parabola whose equation is of the form $y = a(x-h)^2 + k$ is symmetric about a vertical line with equation $x = h$. Such a line is called the ***axis of symmetry*** of the parabola. Therefore it is easy to see that $x = 1$ is a correct solution to this question.



Questions that Deepen your Understanding

1. Describe how the graph of $y = x^2$ can be transformed to obtain the graph of $y = -3(x+2)^2 - 5$.
2. No solutions were given for 6 a) and 6 b). Use the method of completing the square to rewrite the equations in the more convenient form of $y = a(x-h)^2 + k$. Graph each parabola.
3. Complete the following table.

Type of Polynomial	*General Equation*	*Example*	*Graph of Example*	*Important Points*
Linear	$y = mx + b$	$y = -2x - 7$		A linear relation is completely determined by two points ***or*** one point and a slope
Quadratic				
Cubic				
Quartic				
Quintic				

4. Why is so much time in school devoted to studying linear and quadratic relations? Why do we not devote much time to studying polynomial equations of higher degree?
5. Describe a physical phenomenon that behaves linearly. Use a graph to illustrate your answer.
6. Describe a physical phenomenon that behaves quadratically. Use a graph to illustrate your answer.
7. Use the method of completing the square to ***derive*** the quadratic formula. Why is this derivation important?
   Hint: Begin with the general quadratic equation $ax^2 + bx + c = 0$ and then apply the method of completing the square.

REVIEW EXERCISES

Simplifying, Factoring, Solving Equations, Inequalities

1. Simplifying expressions Expand and simplify.

a) $3(4t-8)+6(2t-1)$
b) $7(3w-4)-5(5w-3)$
c) $6(m+3)+2(m-11)-4(3m-9)$
d) $5(3y-4)-2(y+7)-(3y-8)$
e) $4(3x^2-2x+5)-6(x^2-2x-1)$
f) $6(x-y)-2(2x+7y)-(3x-2y)$
g) $3(x^2-2xy+2y^2)-5(2x^2-2xy-y^2)$

2. Solving linear equations Solve and check.

a) $2(2r-1)+4=5(r+1)$
b) $5(x-3)-2x=-6$
c) $7-2(1-3x)+16=8x+11$
d) $4y-(3y-1)-3+6(y-2)=0$
e) $4(w-5)-2(w+1)=3(1-w)$
f) $0=2(t-6)+8+4(t+7)$
g) $4(y-2)=3(y+1)+1-3y$

3. Solving linear equations Solve and check.

a) $\dfrac{x}{3}+\dfrac{1}{2}=0$
b) $\dfrac{y-1}{3}=6$
c) $\dfrac{x}{3}-\dfrac{1}{2}=\dfrac{1}{4}$
d) $\dfrac{m+2}{2}=\dfrac{m-1}{3}$
e) $\dfrac{w+1}{2}+\dfrac{w+1}{3}=5$
f) $\dfrac{2x+1}{3}-\dfrac{x+1}{4}=3$
g) $0.4(c-8)+3=4$
h) $0.5x-0.1(x-3)=4$
i) $1.5(a-3)-2(a-0.5)=10$
j) $1.2(10x-5)-2(4x+7)=8$

4. Common factors Factor.

a) $7t^2-14t^3$
b) $36x^7+24x^5$
c) $4xy-2xz+10x$
d) $8x^3-16x^2+4x$
e) $9x^2y+6xy-3xy^2$
f) $10a^2b+5ab-15a$

5. Factoring ax^2+bx+c, $a=1$ Factor.

a) $x^2+7x+12$ **b)** y^2-2y-8
c) $d^2+3d-10$ **d)** $x^2-8x+15$
e) w^2-81 **f)** t^2-4t
g) $y^2-10y+25$ **h)** $x^2-3x-40$

6. Factoring ax^2+bx+c, $a\neq 1$ Factor.

a) $2x^2+7x+3$ **b)** $2x^2-3x+1$
c) $3t^2-11t-20$ **d)** $2y^2-7y+5$
e) $6x^2+x-1$ **f)** $4x^2+12x+9$
g) $9a^2-16$ **h)** $6s^2-7s-3$
i) $2u^2+7u+6$ **j)** $9x^2-6x+1$
k) $3x^2+7x-20$ **l)** $4v^2+10v$

7. Solving quadratic equations by factoring Solve by factoring. Check your solutions.

a) $x^2-x-2=0$
b) $y^2-9=0$
c) $n^2-7n=0$
) $x^2-4x=-4$
) $6x+8=-x^2$
$z^2+12=-z$
g) $2x^2-5x+2=0$
h) $2y^2+7y+3=0$

8. Inequalities Graph the following integers on a number line.

a) $x>-2$ **b)** $x<3$
c) $x\geq 0$ **d)** $x\leq -1$

Solving Quadratic Equations, Graphing Quadratic Relations, Physical Interpretations of Quadratics

1. Solve:
   a) $(x+7)(x-3)=(x+7)(5-x)$
   b) $(3x-9)(x+2)=(x-3)(2x+1)$
   c) $(x+4)(x-4)=-9(x+1)(x-1)$
   d) $(x+5)(2x-3)=(x+3)(x+4)$
2. Calculate the radius of a circle that has an area of:
   a) 169 cm^2; b) 1772 mm^2; c) $16\pi\text{ km}^2$.
3. Solve graphically:
   a) $2x^2+11x-6=0$ b) $2x^2-5x-12=0$
   c) $4x^2-25=0$ d) $16x^2+8x-143=0$
4. Write a quadratic equation with roots:
   a) $7, -1$; b) $0, \frac{11}{2}$; c) $\frac{4}{3}, -\frac{3}{4}$; d) $1.125, -5.875$.
5. Solve.
   a) $x^2-5x-14=0$ b) $m^2+4m-32=0$
   c) $3v^2-2v-1=0$ d) $6t^2-11t-10=0$
6. Solve:
   a) $x^2-3x-22=4(x-1)$ b) $7v(v-1)=5(v^2-1.2)$
   c) $2(x-3)(x+3)+5x=0$
   d) $(z-4)(3z+2)=(z-5)(2z+1)-1$
7. One side of a right triangle is 2 cm shorter than the hypotenuse and 7 cm longer than the third side. Find the lengths of the sides of the triangle.
8. The height, h, in metres, of an infield fly ball t seconds after being hit is given by the formula: $h=30t-5t^2$. How long is the ball in the air?
9. The length of a rectangular picture is 5 cm greater than the width. Find the dimensions of the picture if its area is:
   a) 150 cm^2; b) 300 cm^2.
10. Solve by completing the square:
    a) $x^2-8x-30=0$ b) $x^2+6x-90=0$
    c) $x^2-5x+2=0$ d) $x^2+15x+25=0$
11. Solve by completing the square:
    a) $2x^2+9x+3=0$ b) $6x^2+2x-5=0$
    c) $7x^2-16x+5=0$ d) $10x^2+7x-10=0$
12. Solve:
    a) $5x^2+11x-12=0$ b) $3x^2+10x-32=0$
    c) $5x^2-15x+11=0$ d) $9x^2-6x-143=0$
    e) $12x^2-29x+14=0$ f) $20x^2+x-12=0$
13. The surface area, A, of a closed cylinder of radius r is given by the formula: $A=6.28r^2+92.1r$. Find the radius of the cylinder if the surface area is:
    a) 1138.72 cm^2; b) 1772.98 cm^2.

2. The zeros of a quadratic relation are -3 and 9. The second differences are positive.
   (a) Explain whether the optimal value will be a maximum or a minimum.
   (b) What value of the independent variable will produce the optimal value?
   (c) Explain whether the optimal value is a negative or positive value.

3. **(a)** Points $(-9, 0)$ and $(19, 0)$ lie on the curve of a parabola. What is the axis of symmetry for the parabola?
   (b) What are the zeros of the parabola?
   (c) The optimal value of the parabola is -28. Write the algebraic expression of the parabola in standard form.

4. Sketch each graph, using the x-intercepts and the optimal value as reference points. Clearly identify the x-intercepts and vertex.
   (a) $y=(x-6)(x+2)$ **(b)** $y=(4+x)(6-x)$

5. Expand and simplify each expression.
   (a) $(2x-3)(5x+2)$ **(b)** $(5a+2b)(3a-4b)$ **(c)** $-5(x-6)(2x+7)$

6. Factor each expression.
   (a) $x^2-9x+14$ **(b)** $16x^2-25$
   (c) $6x^2+5x-4$ **(d)** $2x^2+10x+12$

Simultaneous Systems of Linear Equations and their Applications

1. Solve graphically:

a) $2x - y = 3$
$3x - 5y = 15$

b) $-2x + y = 3$
$x + y = 6$

c) $3x - y = -4$
$x - y = -8$

d) $2y - 5x = 15$
$3y - 4x = 12$

2. Solve graphically:

a) $x + 3y = 2$
$3x + 9y = 8$

b) $3x + 7y = 27$
$5x + 2y = 16$

c) $4x - 6y = -10$
$-6x + 9y = 15$

d) $4x - 3y = 0$
$5x - 6y = -18$

3. Given: lines L_1: $5x + 2y + 10 = 0$, L_2: $x + 3y - 11 = 0$. On the same grid, graph L_1 and L_2, and the equations represented by:
a) $L_1 + L_2$; b) $L_1 - L_2$; c) $3L_1 - 2L_2$; d) $L_1 + 3L_2$.

4. Solve algebraically:

a) $2x + y = 3$
$4x - y = -6$

b) $y = 3x - 7$
$y = 5 - x$

c) $5x - 7y = 0$
$7x + 5y = 74$

d) $3x + y = 6$
$2x - 3y - 9 = 0$

5. Find the solution correct to two decimal places:

a) $5x - 3y = 9$
$2x + 5y = 8$

b) $10x - 3y = 1$
$y = -2x + 7$

c) $8x + 7y = 3$
$2x - 6y - 1 = 0$

d) $3x + 8y + 2 = 0$
$2x - y + 1 = 0$

6. The lines $2x + y = 10$ and $7x + 8y = 53$ intersect at A, and the lines $2x - y = 12$ and $x + 3y = 27$ intersect at B. Find the equation of the line through A and B.

7. A triangle has vertices (−1, 3), (4, −2), (8, 6). It is intersected by the line $3x - y - 10 = 0$ at P and Q. Find:
a) the coordinates of P and Q; b) the length of PQ.

8. An office is equipped with two card-sorting machines, A and B. If A is operated for 2 min and B for 5 min, 20 500 cards can be sorted. If A is operated for 5 min and B for 2 min, 25 000 can be sorted. What are the sorting rates of the machines?

9. A motorist travels 400 km partly at 100 km/h and partly at 80 km/h. If he had travelled at 80 km/h instead of 100 km/h and 100 km/h instead of 80 km/h, the journey would have taken 0.5 h longer. Find his time for the trip.

10. Solve graphically:

a) $x^2 + y^2 = 17$
$y = 4x$

b) $x^2 + y^2 = 18$
$y = 2x + 3$

c) $y = x^2$
$x - y + 6 = 0$

d) $y = x^2 - 3x + 5$
$x + y = 8$

1. (a) Determine graphically the point of intersection between the lines defined by $y = -2x + 6$ and $8 = 5x - y$.
   (b) Verify that you determined the correct point by solving the system of equations in part (a) algebraically.

2. Solve by substitution.

   (a) $3x + y = 5$, $x - 2y = 11$
   (b) $5x - 2y = -16$, $-2x + y = 7$
   (c) $4x - 3y = 10$, $2x + 3y = 4$

3. Solve by elimination.

   (a) $a - 15b = 3$, $3b + a = 21$
   (b) $2x + 5y = 19$, $3x - y = 3$
   (c) $3x - 2y = -8$, $3y - 21 = 9x$

4. Confirm or deny: The ordered pair $(3, -5)$ is the solution to the linear system defined by $2x + 5y = -19$ and $6y - 8x = -54$. Justify your answer.

5. Jeff is a cashier at the grocery store. He has a total of \$580 in bills. He has 76 bills, consisting of \$5 bills and \$10 bills. How many of each type does he have?

6. A traffic helicopter pilot finds that with a tailwind her 120 km trip away from the airport takes 30 min. On her return trip to the airport, into the wind, she finds that her trip is 10 min longer. What is the speed of the helicopter? What is the speed of the wind?

7. Rani is comparing the monthly costs from two Internet service providers. Netaxes charges a flat monthly fee of \$10, plus \$0.75 per hour spent on-line. Webz charges a flat monthly fee of \$5, plus \$1 per hour.
   (a) Determine when the monthly costs are the same.
   (b) Rani plans to use the Internet for at least 30 h each month. Which provider should she choose? Explain.

8. Premium gasoline sells for 78.9¢/L. Regular gas sells for 71.9¢/L. To boost sales, a middle octane gasoline is formed by mixing premium and regular. If 1000 L of this middle octane gas is produced, and is sold at 73.9¢/L, then how much of each type of gasoline can you assume was used in the mixture?

9. Graph a linear system with no solution. Determine two possible equations that could represent both lines in your graph.

10. Solve.

   (a) $12(x - 2) - (2y - 1) = 14$, $5(x - 1) + 2(1 - 2y) = 14$
   (b) $\frac{x-2}{3} - \frac{y+5}{2} = -3$, $3x - \frac{2y}{3} = 13$

More Quadratics

1. Find the vertex, axis of symmetry, and direction of opening of the parabola. Use this information to sketch the graph.

(a) $y = x^2 - 7$ **(b)** $y = (x - 3)^2$ **(c)** $y = -(x + 1)^2 + 10$

(d) $y = 3x^2 - 12$ **(e)** $y = -\frac{1}{2}(x + 2)^2 - 3$ **(f)** $y = -2(x - 5)^2 + 6$

2. Describe how the graph of $y = x^2$ can be transformed, step by step, to obtain the graph of each quadratic relation in question 1.

3. Find the quadratic relation in vertex form that

(a) has its vertex at $(-6, 0)$ and passes through $(-3, 27)$

(b) has its vertex at $(3, 7)$ and passes through $(-1, -17)$

(c) has its vertex at $(-4, -2)$ and has a y-intercept of -8

(d) has zeros -3 and 5 and passes through $(3, 6)$

4. Express each relation you found in (3) in standard form $y = ax^2 + bx + c$.

5. Without graphing, tell how many zeros (x-intercepts) the quadratic relation has. Justify your answers.

(a) $y = x^2 + 3$ **(b)** $y = -3(x + 5)^2$ **(c)** $y = \frac{2}{3}(x - 2)^2 - 7$

6. A concrete bridge over a river has an underside in the shape of a parabolic arch. At the water level, the arch is 30 m wide. It has a maximum height of 10 m above the water. The minimum vertical thickness of the concrete is 1.5 m.

(a) Find an algebraic relation that represents the shape of the arch.

(b) What is the vertical thickness of the concrete 3 m from the centre of the arch?

(c) If the water level rises 2 m, how wide will the arch be at this new level?

7. A baseball is hit into the air by the Blue Jays' batting coach. Its height h, in metres, after t seconds is $h = -4.9(t - 2.8)^2 + 39$.

(a) How high off the ground was the ball when it was hit?

(b) What is the maximum height of the ball?

(c) What is the height of the ball after 2.5 s? Is it on the way up or down? Justify your answer.

(d) Is the ball still in the air after 6 s? Explain how you know.

(e) To one decimal place, when does the ball hit the ground?

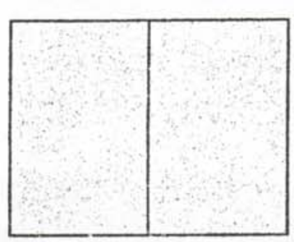
8. Find the coordinates of the vertex of the quadratic relation using the most appropriate method. List two other points that lie on the graph of the relation. Express the relation in vertex form.

(a) $y = 2(x - 3)(x + 7)$ **(b)** $y = x(2x + 6) - 11$

(c) $y = -3x^2 + 12x + 15$ **(d)** $y = x^2 - 6x - 4$

(e) $y = 4x^2 - 11x - 55$ **(f)** $y = 5x^2 + 20x - 11$

9. A farmer has \$2400 to spend to fence two rectangular pastures as shown in the diagram. The local contractor will build the fence at a cost of \$6.25/m. What is the largest total area that the farmer can have fenced for that price?



Mixed Review of Algebra

Write an equation to model each problem. Then solve.

1. The product of 3 and a number is 27.	2. Eight less than ten times a number is 62.
3. One-third of Natalie's age is 16 years less than her age 20 years ago. How old is Natalie?	4. Nine cards and four boxes of candy cost $24.73. Ten cards and five boxes of candy cost $29.20. How much do two cards cost?

Solve. In each case, sketch a diagram so that you also have a *geometric viewpoint*.

5. Write an equation of a line that passes through the point (1, 29) and is parallel to the following equation: $19x + 8y = 251$	6. Write an equation of a line that passes through the point (40, 4) and is perpendicular to the line with the following equation: $\frac{7x+2}{2} = 144$	7. Write an equation of a line that passes through the point (-38, -34) and is perpendicular to the line with the following equation: $y = \frac{-1}{9}x - \frac{344}{9}$

Simplify and write in lowest terms.

8. $\frac{4h}{6} + \frac{7h}{3}$	9. $\frac{8d}{28} + \frac{9d}{7}$	10. $\frac{9}{b} + \frac{2}{p}$

Expand and simplify.

11. $(5x^7 - 10x^6 - 10x)(-11x^2 + 12x + 4)$	12. $(3x^2 - 11x - 9)(11x + 10)$	13. $(-2x^7 + 12x - 2)(-9x^2 + 8x + 9)$

Write an equation for a parabola that passes through the given points. (Sketch a graph of each parabola before using algebra.)

14. (6, –55), (0, –7), (4, –31), (–1, -6)	15. (–2, –7), (6, –87), (2, –31)

Solve each equation by factoring.

16. $15x^2 + 99x = 168$	17. $-120x^2 + 390x - 315 = 0$	18. $\frac{2}{5}x^2 + 3x = 10$

Simplify.

19. $(-4h^{-5})(-4h^4t^{-4})$	20. $(f^{-5})(3f^3v^3)$	21. $(-9q^{-5}y^5)(10q^2)$