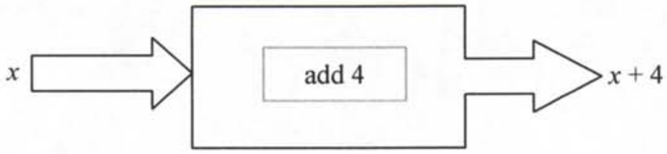
UNIT 1 – ACTIVITY SOLUTIONS

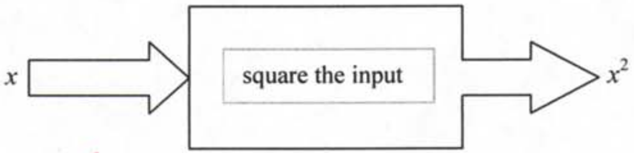
Pages 6→11

For example, consider the function machine that adds 4 to the input to produce the output.



Exercise

Consider the function machine that takes an input x and outputs x^2.



What is the output if the input is: 4 16, 0 0 -4 16 ?

Is it possible for a specific input to have more than one output? Explain. For a function, this is NOT possible. Each input must have ONE AND ONLY ONE OUTPUT.

What is the input if the output is 9? 3 OR −3

Is it possible for a specific output to have more than one input in this case? Explain. It is possible in this case because the definition of a function is not violated. Different inputs are allowed to have the same output but not vice versa.

The following ordered pairs show the inputs and outputs of a function machine. What is the operation?

(a) (10, 5), (3, −2), (−1, −6) operation: $y = x - 5$

(b) (10, 32), (−10, −28), (−3, −7), (1, 5), (0, 2) operation: $y = 3x + 2$

Mapping Diagram Perspective

In addition to function machines, functions can be represented using *mapping diagrams*, *tables of values*, *graphs* and *equations*. You may not know it but you already have a great deal of experience with functions from previous math courses.

A mapping diagram uses arrows to map each element of the input to its corresponding output value. Remember that a function has only one output for each input.

Can a function have more than one input for a particular output? Explain. See

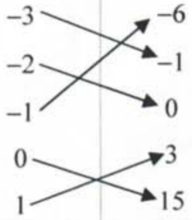
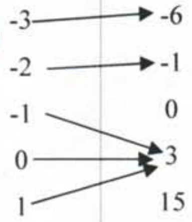
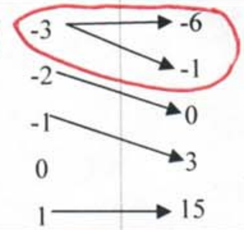
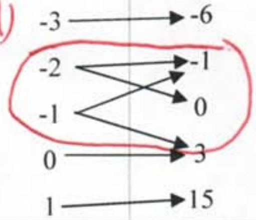
Domain and Range

The set of all possible input values of a function is referred to as the *domain*. That is, if D represents the domain of a function f, then $D = \{x : (x, y) \in f\}$.

The set of all possible output values of a function is referred to as the *range*. That is, if R represents the domain of a function f, then $R = \{y : (x, y) \in f\}$.

Exercise 1

In the table, the relations are shown using a mapping diagram. The domain (or inputs) on the left have arrows pointing to the range (or output) on the right. Explain whether the relation is a function or not and justify your answer. The first one is done for you.

Relation as a Mapping Diagram	*Is it a function? Why or why not?*	*Set of ordered pairs*
Domain *Range*		
(a) 	Since every element of the domain has only one corresponding element in the range, this relation is a function. This function has a *one-to-one mapping*.	{(−3, −1), (−2,0), (−1, −6), (0, 15), (1, 3)}
(b) 	This is a function because every element of the domain has only one corresponding element in the range. This function has a many-to-one mapping.	{(-3,-6), (-2,-1), (-1,3), (0,3), (1,15)}
(c) 	This is NOT a function because the element -3 in the domain is mapped to two different elements in the range.	{(-3,-6), (-3,-1), (-2,0), (-1,3), (1,15)}
(d) 	This is NOT a function because two elements in the domain are each mapped to two different elements in the range.	{(-3,-6), (-2,-1), (-2,0), (-1,-1), (-1,3), (0,3), (1,15)}

A *one-to-one mapping* is explained above. Which relation above has

a *many-to-one mapping*? (b)

a *one-to-many mapping*? (c)

a *many-to-many mapping*? (d)

Exercise 2

Use the Internet to find definitions of the following terms:

one-to-one mapping, one-to-many mapping, many-to-many mapping, surjection, injection, bijection

also called an ONTO mapping

injection → a one-to-one mapping

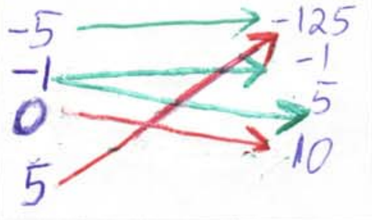
surjection → a mapping in which every element of the range has a corresponding element in the domain

bijection → a mapping that is BOTH a surjection and an injection

Numerical Perspective – Tables of Values

Consider the following relations expressed in table form.

(a) Which relations are functions? Justify your answers.
(b) Draw a mapping diagram for each relation.
(c) Write the set of ordered pairs for each relation.

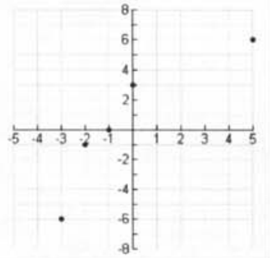
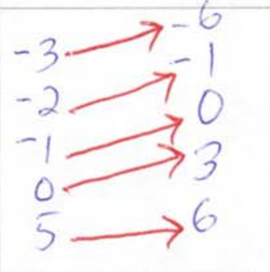
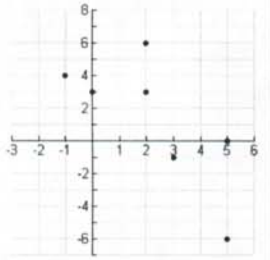
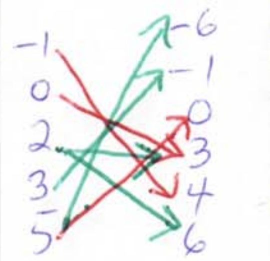
Relation as a Table of Values		*Is it a function? Why or why not?*	*Mapping Diagram*	*Set of Ordered Pairs*
x	*y*	This is a function because each element of the domain is mapped to a single element in the range		{(-3,9), (-1,1), (1,1), (3,9)}
−3	9			
−1	1			
1	1			
3	9			
x	*y*	This is not a function because -1 is mapped to two different values.		{(-5,-125), (-1,-1), (-1,5), (0,10), (5,-125)}
−5	−125			
−1	−1			
−1	5			
0	10			
5	−125			

Geometric Perspective – Graphs of Functions

Discrete Relations

A *discrete relation* either has a *finite number* of ordered pairs *OR* the ordered pairs can be *numbered* using integers. Graphs of discrete relations consist of either a *finite* or an *infinite* number of "disconnected" points, much like a "connect-the-dots" picture *before* the dots are connected.

Examine the following graphs of *discrete relations* and then complete the table. *DO NOT CONNECT THE DOTS!*

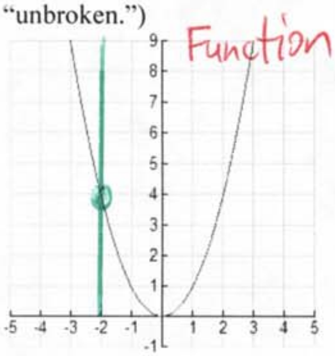
Relation in Graphical Form	*Is it a function? Why or why not?*	*Table of Values*	*Mapping Diagram*	*Set of Ordered Pairs*
	Yes. Each element of the domain is mapped to a single element of the range	x y -3 -6 -2 -1 0 3 -1 0 5 6		{(-3,-6), (-2,-1), (-1,0), (0,3), (5,6)}
	No! The element 2 is mapped to two different values in the range (so is 5)!	x y -1 4 0 3 2 3 2 6 3 -1 5 0 5 -6		{(-1,4), (0,3), (2,3), (2,6), (3,-1), (5,0), (5,-6)}

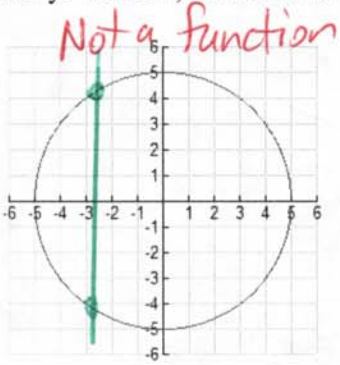
Definition of "Discrete" (from www.dictionary.com)

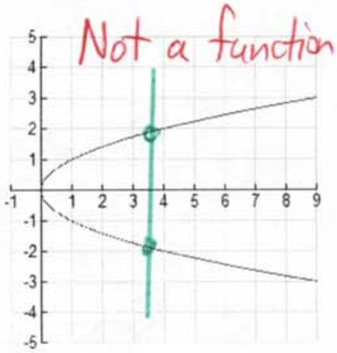
1. apart or detached from others; separate; distinct: *six discrete parts.*
2. consisting of or characterized by distinct or individual parts; discontinuous.
3. *Mathematics.*
   a. (of a topology or topological space) having the property that every subset is an open set.
   b. defined only for an isolated set of points: *a discrete variable.*
   c. using only arithmetic and algebra; not involving calculus: *discrete methods.*

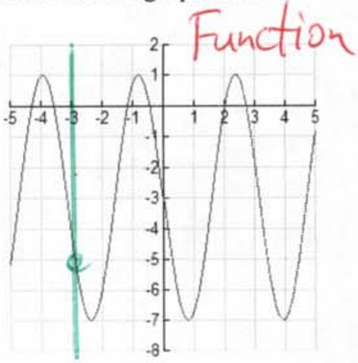
Continuous Relations

Unlike the graphs on the previous page, the following graphs *do not represent discrete relations*. They are called *continuous relations* because their graphs do not consist of disconnected points. (You need to study calculus to learn a more precise definition of continuity. For now, it suffices to think of continuous relations as those whose graphs are "unbroken.")


Function


Not a function


Not a function


Function

Examine the graphs and decide whether they represent functions.

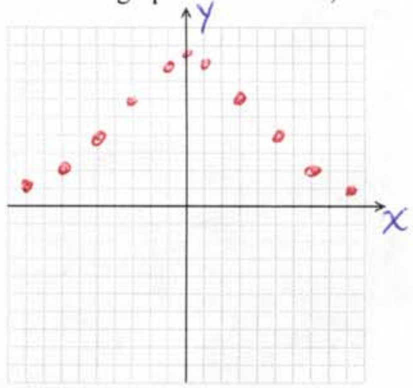
Summary – Vertical Line Test

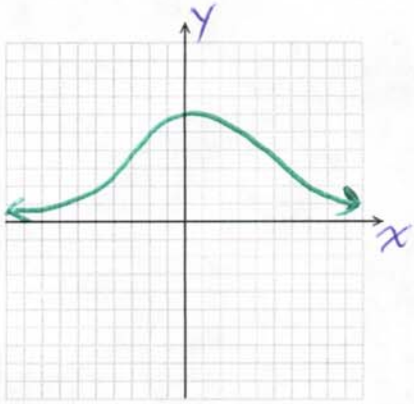
When you look at a graph, how can you decide whether it represents a function?

A relation is a function IF AND ONLY IF any vertical line passes through (intersects) the curve at no more than a single point.

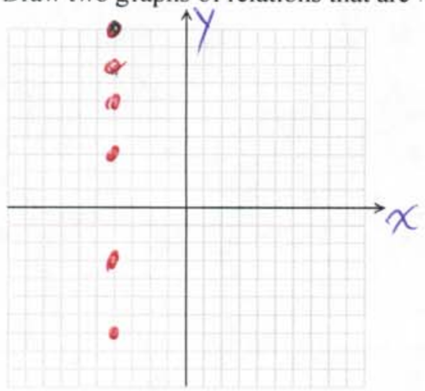
Exercise

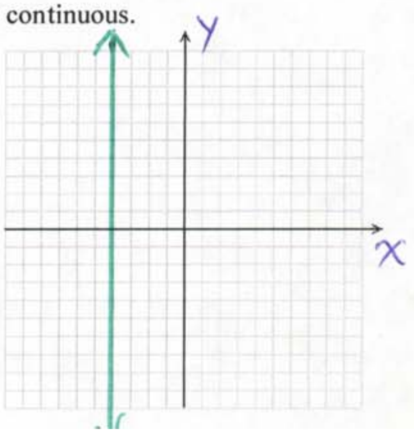
Draw two graphs of functions, one that is discrete and one that is continuous.





Draw two graphs of relations that are *NOT* functions, one that is discrete and one that is continuous.



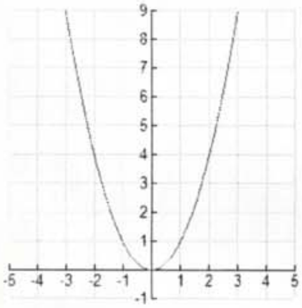
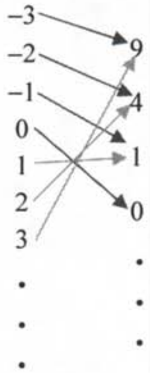
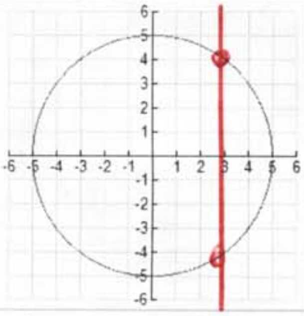
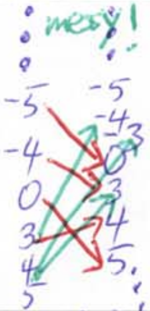
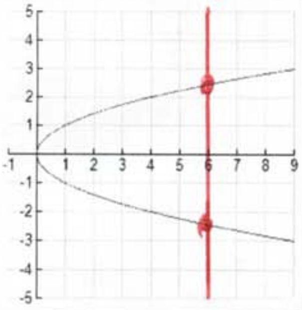
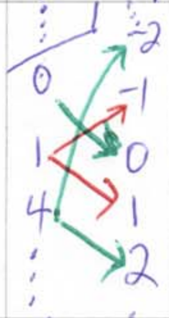
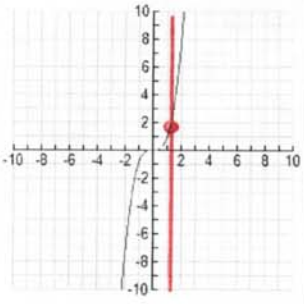
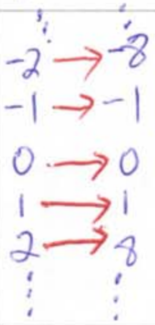


Algebraic Perspective – Equations of Relations

The perspectives given above help us to understand the properties of relations and functions. However, when it comes time to computing with relations and functions, then equations become an indispensable tool!

For each of the following (the first is done for you)

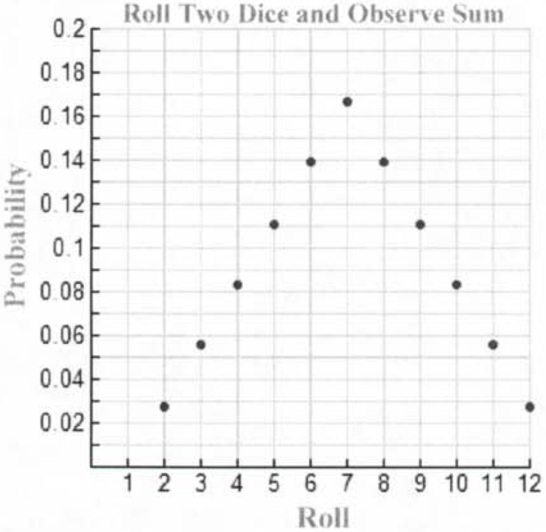
(a) determine whether it is a function.
(b) state an equation that describes the relation.
(c) complete a table of values and a mapping diagram for each relation. Note that the given relations are all *continuous*, which means that it is impossible to create a complete table of values or mapping diagram. (Why?)
(d) state the domain and range and the type of mapping.

Relation in Graphical Form	*Is it a function? Explain.*	*Equation*	*Table of Values*	*Mapping Diagram*	*Domain and Range (D & R)*	*Type of Mapping*
	This is a function because any vertical line will intersect at only a single point.	$f(x)=x^2$	x $f(x)$ −2 4 −1 1 0 0 1 1 2 4 • • • • • •		The domain is the set of all real numbers, that is, $D=\mathbb{R}$. The range is the set of all real numbers greater than or equal to zero, that is, $R=\{y\in\mathbb{R}: y\geq 0\}$	many-to-one
	This is NOT a function because the vertical line test fails	$x^2+y^2=25$	x y -5 0 -4 3 -4 -3 0 5 0 -5 3 4 3 -4		$D=\{x\in\mathbb{R} \mid -5\leq x\leq 5\}$ $R=\{y\in\mathbb{R} \mid -5\leq y\leq 5\}$ $\therefore D=R$	one-to-many
	This is NOT a function because the vertical line test fails	$x=y^2$ OR $y=\pm\sqrt{x}$	x y 0 0 1 1 1 -1 4 2 4 -2		$D=\{x\in\mathbb{R} \mid x\geq 0\}$ $R=\mathbb{R}$	one-to-many
	This is a function. The vertical line test does not fail.	$f(x)=x^3$	x y -2 -8 -1 -1 0 0 1 1 2 8		$D=\mathbb{R}$ $R=\mathbb{R}$ $\therefore D=R$	one-to-one

Examples of Discrete and Continuous Relations

(a) *Discrete Function with a Finite Number of Ordered Pairs – Roll Two Dice and Observe the Sum*

Consider rolling two dice and observing the sum. Clearly, there are only eleven possible outcomes, the whole values from 2 to 12 inclusive. It's also clear that some outcomes are more likely than others. If you have played board games that involve dice rolling, you surely have noticed that rolls like "2" and "12" occur infrequently while "7" occurs much more often. The reason for this is that there is only *one way* of obtaining "2," for example, but there are *six ways* of obtaining "7." By working out all the possibilities, we obtain the probability of each outcome as shown in the table at the right.

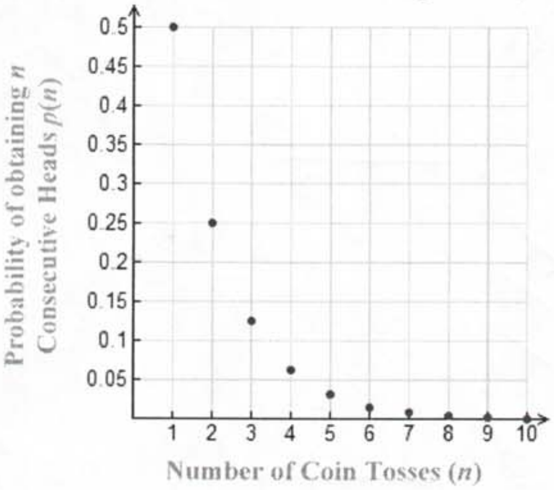


The main point to remember here is that that we have a *discrete function*. In this case, there are only a finite number of ordered pairs in this function. Furthermore, they are "disconnected" from each other, as the graph clearly shows. This disconnectedness is a natural consequence of the physical nature of rolling a pair of dice. It is possible to roll a "2" or a "3" but it is *not possible* to roll 3.141592654 or any number that is not a whole number between 2 and 12 inclusive!

Possible Outcomes	*Probability*
2	$\frac{1}{36} \doteq 0.02778$
3	$\frac{2}{36} \doteq 0.05556$
4	$\frac{3}{36} \doteq 0.08333$
5	$\frac{4}{36} \doteq 0.11111$
6	$\frac{5}{36} \doteq 0.13889$
7	$\frac{6}{36} \doteq 0.16667$
8	$\frac{5}{36} \doteq 0.13889$
9	$\frac{4}{36} \doteq 0.11111$
10	$\frac{3}{36} \doteq 0.08333$
11	$\frac{2}{36} \doteq 0.05556$
12	$\frac{1}{36} \doteq 0.02778$

(b) *Discrete Function with an Infinite Number of Points – Probability of "n" Consecutive "Heads" in "n" Coin Tosses*

Once again, it makes no sense to "connect the dots" in this case. The number of coin tosses *must be a positive whole number*. It is difficult to imagine how one could make 3.75, 4.99 or $\sqrt{5}$ coin tosses! Once again then, we have a discrete function. The difference in this case is that there are an infinite number of possibilities. There is no limit to the number of coin tosses that one can make. Moreover, there is no bound on the number of consecutive heads that can be obtained. Although it is extremely unlikely to obtain a very large number of consecutive heads, it is *not* impossible!



Number of Tosses (n)	*Probability of Obtaining n Consecutive Heads*
1	$\frac{1}{2} = 0.5 = 50\%$
2	$\frac{1}{4} = 0.25 = 25\%$
3	$\frac{1}{8} = 0.125 = 12.5\%$
4	$\frac{1}{16} = 0.0625 = 6.25\%$
5	$\frac{1}{32} = 0.03125 = 3.125\%$
• • •	The number of ordered pairs is infinite in this case because (at least in principle) any whole number of consecutive heads is possible.

We can take this example one step further and write an equation. Let $p(n)$ represent the probability of obtaining n consecutive heads when a coin is tossed n times. Then,

$$p(n) = \left(\frac{1}{2}\right)^n = \frac{1}{2^n} = 2^{-n},\ n \in \mathbb{N}.$$

Note that "$n \in \mathbb{N}$" means that n is an element of the set of *natural numbers*, that is, n is a positive whole number.

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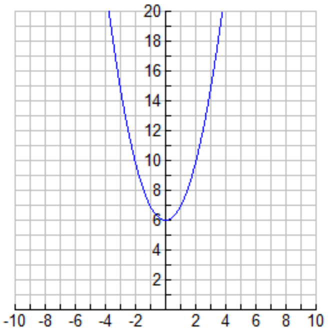
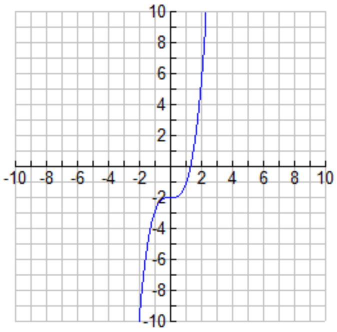
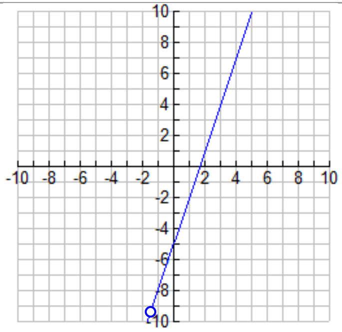
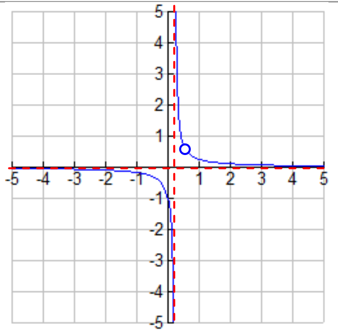
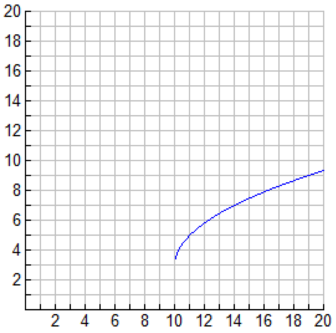
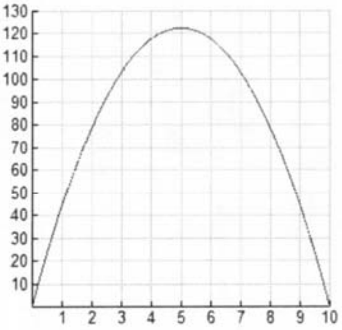
Equation of Function	***Graph***	***Domain and Range***	***Explanation***
$f(x)=x^2+6$		$D=\mathbb{R}$ $R=\{y\in\mathbb{R}:y\geq 6\}$	The graph of $y=x^2$ is shifted up by 6 units because the constant "6" is added to the "*y*-value" x^2.
$g(x)=x^3-2$		$D=\mathbb{R}$ $R=\mathbb{R}$	The graph of $y=x^3$ is shifted down by 2 units because the constant "2" is subtracted from the "*y*-value" x^3.
$h(x)=\dfrac{6x^2-x-15}{2x+3}$		$D=\left\{x\in\mathbb{R}:x\neq-\dfrac{3}{2}\right\}$ $R=\left\{y\in\mathbb{R}:y\neq-\dfrac{19}{2}\right\}$	$h(x)=\dfrac{6x^2-x-15}{2x+3},\ 2x+3\neq 0$ $=\dfrac{(3x-5)(2x+3)}{2x+3},\ 2x+3\neq 0$ $=3x-5,\ x\neq-\dfrac{3}{2}$
$p(u)=\dfrac{2u-1}{10u^2-7u+1}$		$D=\left\{x\in\mathbb{R}:x\neq\dfrac{1}{5},x\neq\dfrac{1}{2}\right\}$ $R=\left\{y\in\mathbb{R}:y\neq 0,\ y\neq\dfrac{2}{3}\right\}$	$p(u)=\dfrac{2u-1}{10u^2-7u+1}$ $=\dfrac{2u-1}{(2u-1)(5u-1)},\ 2u-1\neq 0,\ 5u\text{ -}1\neq 0$ $=\dfrac{1}{5u-1},\ u\neq\dfrac{1}{2},\ u\neq\dfrac{1}{5}$
$f(u)=2\sqrt{u-10}+3$		$D=\{x\in\mathbb{R}:x\geq 10\}$ $R=\{y\in\mathbb{R}:y\geq 3\}$	The graph of $y=\sqrt{x}$ is shifted 10 units to the right and 3 units up.

Physical Perspective – Applying Functions to a Physical Situation

A cannonball is fired vertically into the air with an initial speed of 49 m/s. Its height above the ground in metres, at a time t seconds after it is fired, is given by the function h defined by $h(t)=49t-4.9t^2$. Because this function is used to *model a physical situation*, we must keep in mind that not all values of t make sense. For example, it is nonsensical to consider negative values of t because the timing begins at t=0 when the cannonball is fired. Similarly, there is no point in considering values of t greater than the time that it takes to fall back to the earth. This is summarized in the table given below.

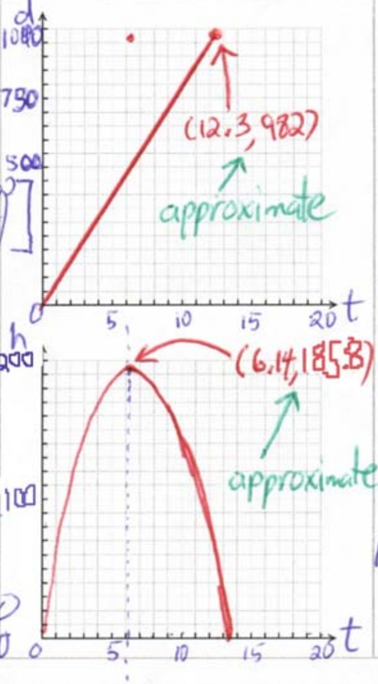
Physical Situation	*Algebraic Support*	*Graph*	*Domain and Range*
A cannonball is fired *vertically* into the air with an initial speed of 49 m/s. Its height above the ground in metres, at a time t seconds after it is fired, is given by the function h defined by $h(t)=49t-4.9t^2$.	$h(t)=49t-4.9t^2$ $=-4.9t^2+49t$ $=-4.9(t^2+10t)$ $=-4.9(t^2+10t+5^2-5^2)$ $=-4.9(t+5)^2+4.9(5)^2$ $=-4.9(t+5)^2+122.5$ $49t-4.9t^2=0$ $\therefore 4.9t(10-t)=0$ $\therefore t=0$ or $10-t=0$ $\therefore t=0$ or $t=10$		$D=\{x\in\mathbb{R}:0\le x\le 10\}$ $R=\{y\in\mathbb{R}:0\le y\le 122.5\}$

Since the function $h(t)=49t-4.9t^2$ is used to model a physical situation, its domain and range are *restricted*.

Exercise

Complete the following table.

Note that the given initial speed of 100 m/s is extremely unrealistic. Even the most powerful MLB (Major League Baseball) batters rarely hit beyond 500 feet (about 150 m). An initial speed of 35 m/s to 40 m/s would have been far more realistic.

Physical Situation	*Algebraic Support*	*Graphs*	*Domain and Range*
A baseball is hit from a point 1 m above the ground, at an angle of 37° to the ground and with an initial velocity of 100 m/s. The *horizontal distance* travelled by the ball is given by the function $d(t)=(100\cos 37°)t$ and the *vertical distance* travelled by the ball is given by the function $h(t)=-4.9t^2+(100\sin 37°)t+1$. How far did the ball travel? What was the maximum height reached by the ball? *Note that time is measured in seconds and distance is measured in metres.*	$h(t)=-4.9t^2+100\sin 37^\circ t+1$ $=-4.9\left(t^2+\frac{100\sin 37^\circ}{-4.9}t\right)+1$ $=-4.9\left[t^2-\frac{100\sin 37^\circ}{4.9}t+\left(\frac{100\sin 37^\circ}{9.8}\right)^2-\left(\frac{100\sin 37^\circ}{9.8}\right)^2\right]$ $+1$ $=-4.9\left(t-\frac{100\sin 37^\circ}{9.8}\right)^2$ $+4.9\times\left(\frac{100\sin 37^\circ}{9.8}\right)^2+1$ $\doteq -4.9(t-6.14)^2+185.8$ when ball hits ground, height = 0 $\therefore -4.9t^2+100\sin 37^\circ t+1=0$ $\therefore t\doteq -0.17$ OR $t\doteq 12.30$	 	Given domain and range are approximate Domain of both d and h $D=\{t\in\mathbb{R}\mid 0\le t\le 12.3\}$ Range of d $R=\{d\in\mathbb{R}\mid 0\le d\le 982\}$ Range of h $R=\{h\in\mathbb{R}\mid 0\le h\le 185.8\}$

using quadratic formula

Homework

pp. 180 – 181. #16, 19, 20. 22. 23. 24, 25, 26, 28, 29. 31. 32

Note: Instead of completing the square, we could have found the vertex by noting that the axis of symmetry must be the vertical line (approximately)

$$t=\frac{12.3-(-0.17)}{2}\doteq 6.2$$ (very close to answer of 6.14)

TRANSFORMATIONS OF THE BASE FUNCTIONS

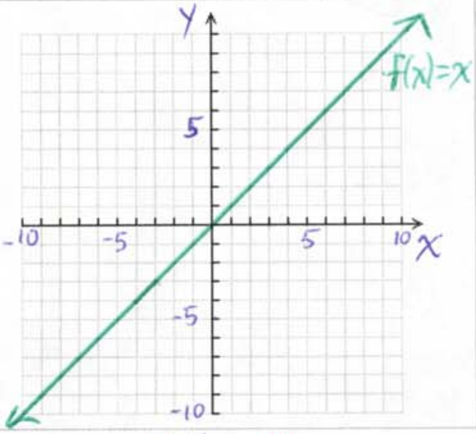
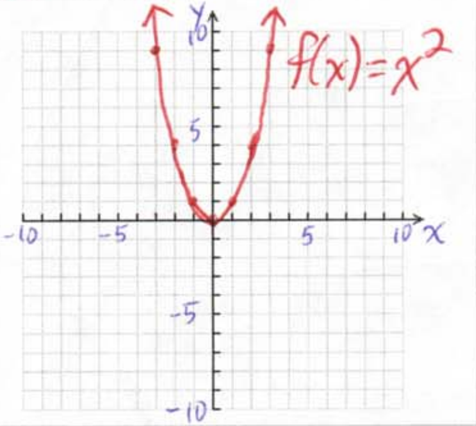
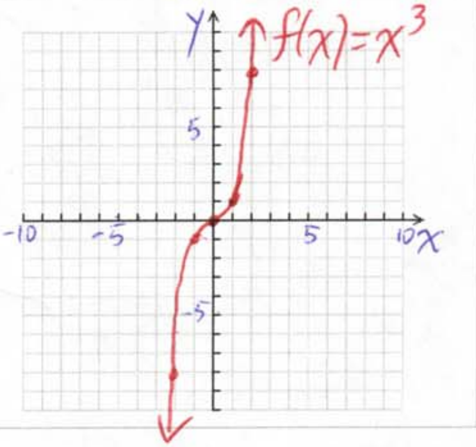
The Base Functions

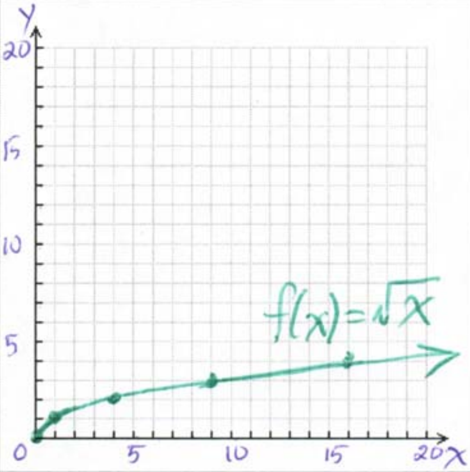
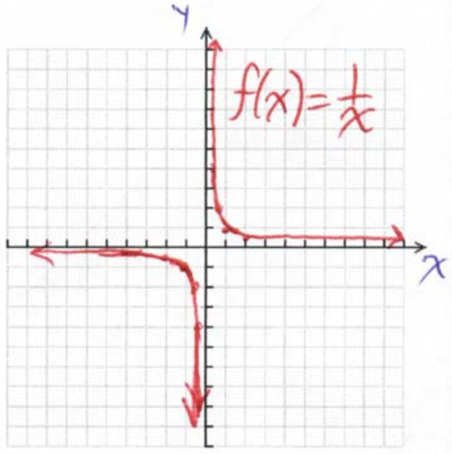
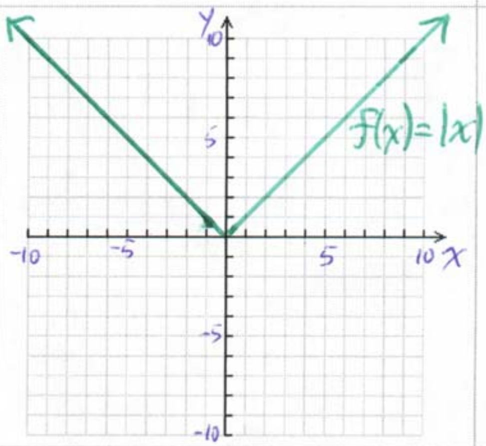
Mathematicians study mathematical objects such as functions to learn about their properties. An important objective of such research is to be able to describe the properties of the objects of study as *concisely* as possible. By keeping the basic set of principles as small as possible, this approach greatly simplifies the daunting task of understanding the behaviour of mathematical structures.

Take, for example, the goal of understanding all quadratic functions. Instead of attempting to understand such functions all in one fell swoop, mathematicians divide the process into two simpler steps as shown below.

1. First, understand the *base function* $f(x)=x^2$ completely.
2. Then, learn how to *transform* the *base function* $f(x)=x^2$ into any other quadratic.

Complete the following table to ensure that you understand the base functions of a few common *families of functions*.

Family of Functions	*Base Function*	*Domain and Range*	*Table of Values*		*Graph*	*Intercepts*
			x	$f(x)$		
Linear	$f(x)=x$	$D=R=\mathbb{R}$	-3	-3		x-int: 0 y-int: 0
			-2	-2		
			-1	-1		
			$-\frac{1}{2}$	$-\frac{1}{2}$		
			0	0		
			$\frac{1}{2}$	$\frac{1}{2}$		
			1	1		
			2	2		
			3	3		
			x	$f(x)$		
Quadratic	$f(x)=x^2$	$D=\mathbb{R}$, $R=\{y\in\mathbb{R} \mid y\geq 0\}$	-3	9		x-int: 0 y-int: 0
			-2	4		
			-1	1		
			$-\frac{1}{2}$	$\frac{1}{4}$		
			0	0		
			$\frac{1}{2}$	$\frac{1}{4}$		
			1	1		
			2	4		
			3	9		
			x	$f(x)$		
Cubic	$f(x)=x^3$	$D=R$ $=\mathbb{R}$	-3	-27		x-int: 0 y-int: 0
			-2	-8		
			-1	-1		
			$-\frac{1}{2}$	$-\frac{1}{8}$		
			0	0		
			$\frac{1}{2}$	$\frac{1}{8}$		
			1	1		
			2	8		
			3	27		

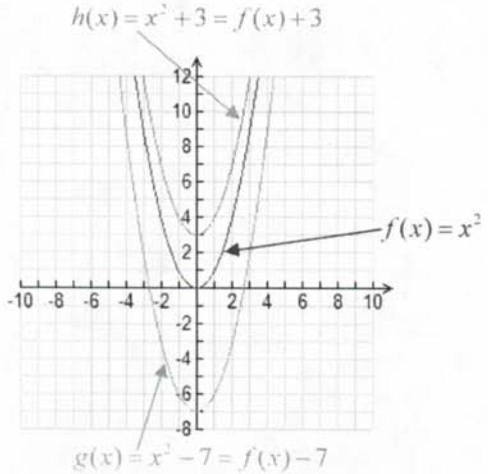
<table>
<tr><th>Family of Functions</th><th>Base Function</th><th>Domain and Range</th><th>Table of Values</th><th>Graph</th><th>Intercepts</th></tr>
<tr><td>Square Root</td><td>$f(x)=\sqrt{x}$</td><td>$D=\{x\in\mathbb{R} \mid x\geq 0\}$
$R=\{y\in\mathbb{R} \mid y\geq 0\}$
$\therefore D=R$</td><td><table>
<tr><th>x</th><th>$f(x)$</th></tr>
<tr><td>-2</td><td>und.</td></tr>
<tr><td>-1</td><td>und.</td></tr>
<tr><td>$-\frac{1}{2}$</td><td>und.</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>4</td><td>2</td></tr>
<tr><td>6</td><td>$\sqrt{6}$</td></tr>
<tr><td>9</td><td>3</td></tr>
<tr><td>12</td><td>$\sqrt{12}$</td></tr>
<tr><td>14</td><td>$\sqrt{14}$</td></tr>
<tr><td>16</td><td>4</td></tr>
</table></td><td> </td><td>x-int: 0
y-int: 0</td></tr>
<tr><td>Reciprocal</td><td>$f(x)=\frac{1}{x}$</td><td>$D=\{x\in\mathbb{R} \mid x\neq 0\}$
$R=\{y\in\mathbb{R} \mid y\neq 0\}$
~~$\therefore D\neq\mathbb{R}$~~
$\therefore D=R$</td><td><table>
<tr><th>x</th><th>$f(x)$</th></tr>
<tr><td>-6</td><td>$-\frac{1}{6}$</td></tr>
<tr><td>-4</td><td>$-\frac{1}{4}$</td></tr>
<tr><td>-2</td><td>$-\frac{1}{2}$</td></tr>
<tr><td>-1</td><td>-1</td></tr>
<tr><td>$-\frac{1}{2}$</td><td>-2</td></tr>
<tr><td>$-\frac{1}{4}$</td><td>-4</td></tr>
<tr><td>0</td><td>und.</td></tr>
<tr><td>$\frac{1}{4}$</td><td>4</td></tr>
<tr><td>$\frac{1}{2}$</td><td>2</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>$\frac{1}{2}$</td></tr>
<tr><td>4</td><td>$\frac{1}{4}$</td></tr>
<tr><td>6</td><td>$\frac{1}{6}$</td></tr>
</table></td><td> </td><td>No intercepts</td></tr>
<tr><td>Absolute Value</td><td>$f(x)=|x|$</td><td>$D=\mathbb{R}$
$R=\{y\in\mathbb{R} \mid y\geq 0\}$</td><td><table>
<tr><th>x</th><th>$f(x)$</th></tr>
<tr><td>-7</td><td>7</td></tr>
<tr><td>-4</td><td>4</td></tr>
<tr><td>-3</td><td>3</td></tr>
<tr><td>-2</td><td>2</td></tr>
<tr><td>-1</td><td>1</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>2</td></tr>
<tr><td>3</td><td>3</td></tr>
<tr><td>4</td><td>4</td></tr>
<tr><td>7</td><td>7</td></tr>
</table></td><td> </td><td>x-int: 0
y-int: 0</td></tr>
</table>

TRANSFORMATION #1 – TRANSLATIONS OF FUNCTIONS

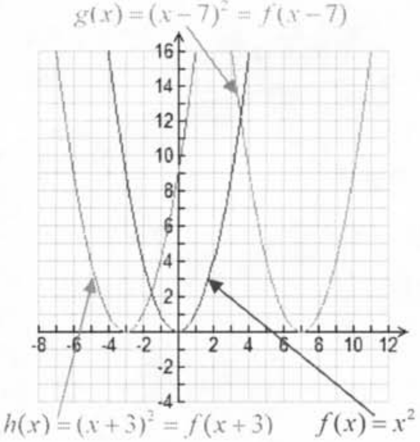
What is a Translation?

- A *vertical translation* of a function is obtained by *adding a constant value to each value of the dependent variable* of the given function. This results in the graph of the function "sliding" up or down, depending on whether the constant is positive or negative.
- A *horizontal translation* of a function is obtained by *adding a constant value to each value of the independent variable* of the given function. This results in the graph of the function "sliding" left or right, depending on whether the constant is positive or negative.

Example Vertical Translations



Example Horizontal Translations

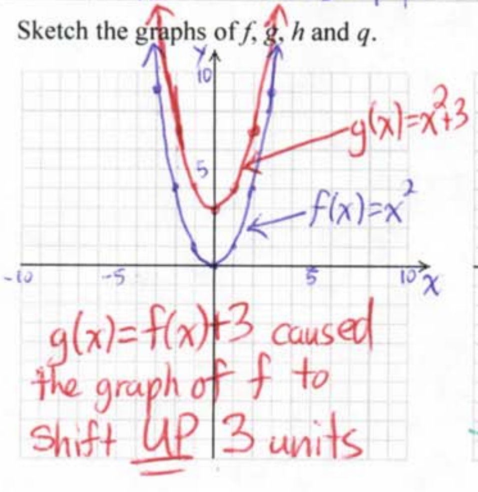


Understanding Translations of Functions

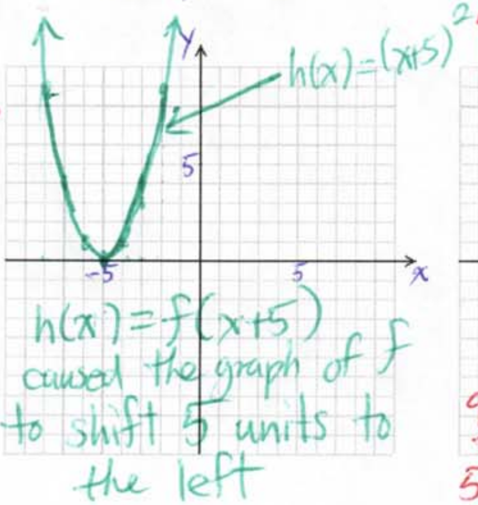
Complete the following table of values.

x	$f(x)=x^2$	$g(x)=x^2+3=f(x)+3$	$h(x)=(x+5)^2=f(x+5)$	$q(x)=(x+5)^2+3=f(x+5)+3$
−4	16	16+3= 19	$(-4+5)^2=1^2=1$	4
−3	9	9+3= 12	$(-3+5)^2=2^2=4$	7
−2	4	4+3= 7	$(-2+5)^2=3^2=9$	12
−1	1	1+3= 4	$(-1+5)^2=4^2=16$	19
0	0	0+3= 3	$(0+5)^2=25$	28
1	1	1+3= 4	$(1+5)^2=36$	39
2	4	4+3= 7	$(2+5)^2=49$	52
3	9	9+3= 12	$(3+5)^2=64$	67
4	16	16+3= 19	$(4+5)^2=81$	84

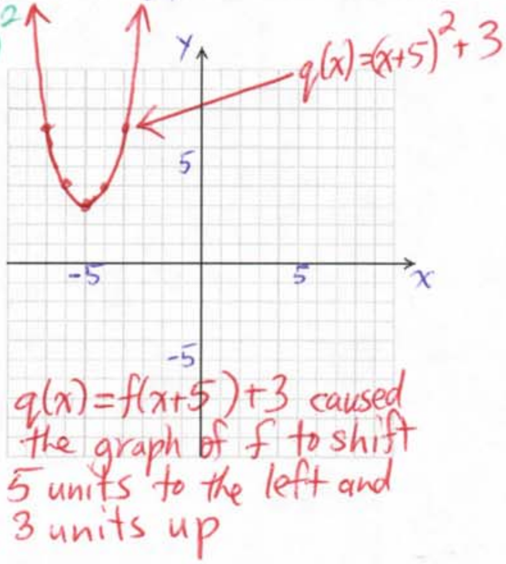
Sketch the graphs of *f*, *g*, *h* and *q*.



$g(x)=f(x)+3$ caused the graph of f to shift UP 3 units



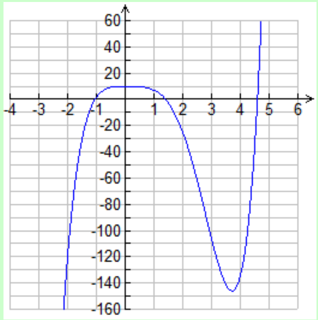
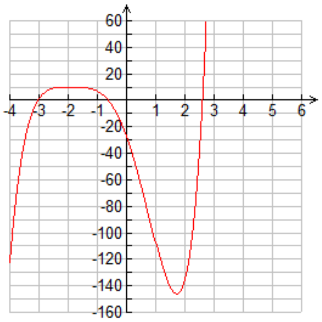
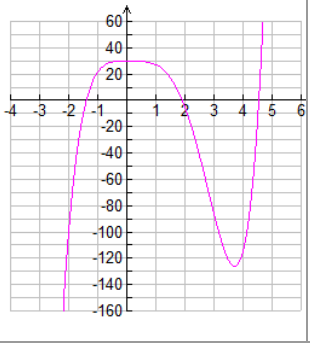
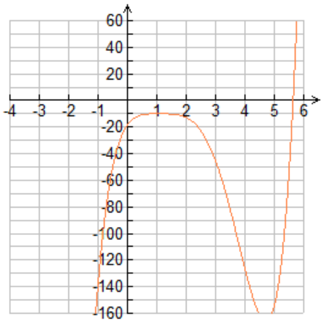
$h(x)=f(x+5)$ caused the graph of f to shift 5 units to the left



$q(x)=f(x+5)+3$ caused the graph of f to shift 5 units to the left and 3 units up

Analysis and Conclusions

Now carefully study the graphs and the tables on the two previous pages. Then complete the following table.

Base Function $y = f(x)$	***Translations of Base Function***		
	$y = f(x-h),\ h \in \mathbb{R}$	$y = f(x)+k,\ k \in \mathbb{R}$	$y = f(x-h)+k,\ h \in \mathbb{R},\ k \in \mathbb{R}$
Description of Translations	**If *h* > 0, the graph of *f* shifts *h* units to the right.** **If *h* < 0, the graph of *f* shifts *h* units to the left.** $(x,y) \to (x+h, y)$	**If *k* > 0, the graph of *f* shifts *k* units up.** **If *k* < 0, the graph of *f* shifts *k* units to the down.** $(x,y) \to (x, y+k)$	**Shift of *h* units left or right** **AND** ***k* units up or down** **(depending on the signs of *h* & *k*)** $(x,y) \to (x+h, y+k)$
$y = f(x)$	$y = f(x+2)$ **The graph of *f* is shifted 2 units to the left.**	$y = f(x)+20$ **The graph of *f* is shifted 20 units up.**	$y = f(x-1)-10$ **The graph of *f* is shifted 1 unit right and 10 units down.**
			

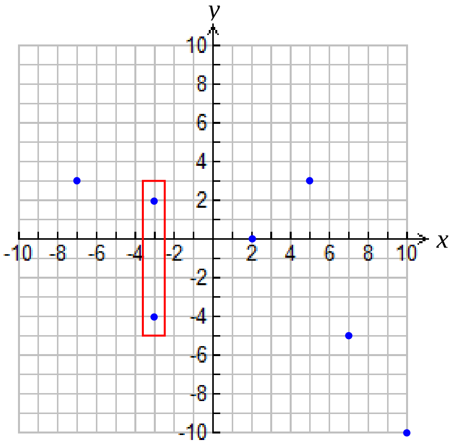
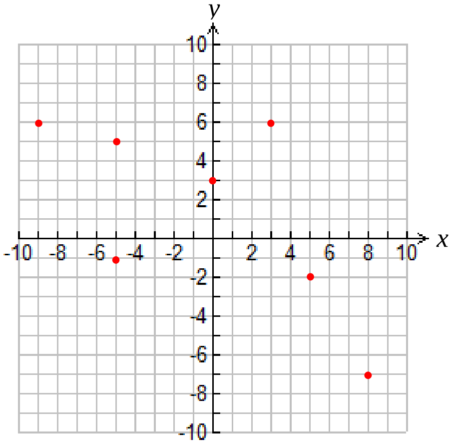
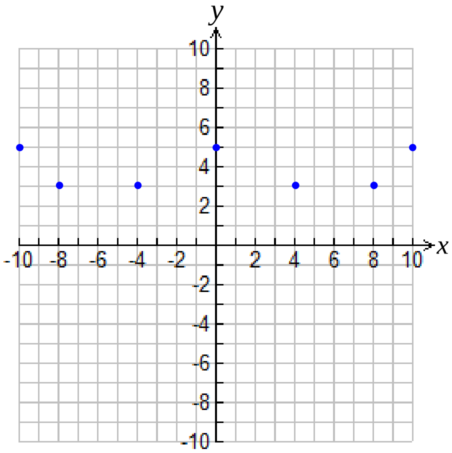
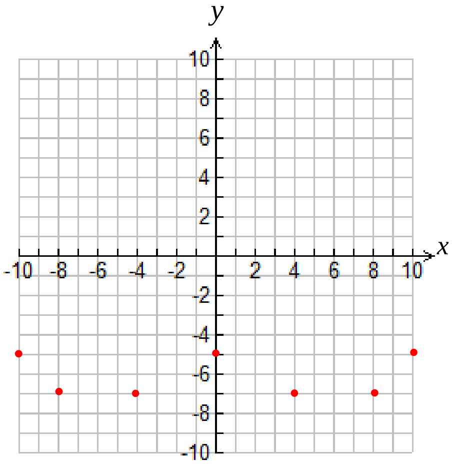
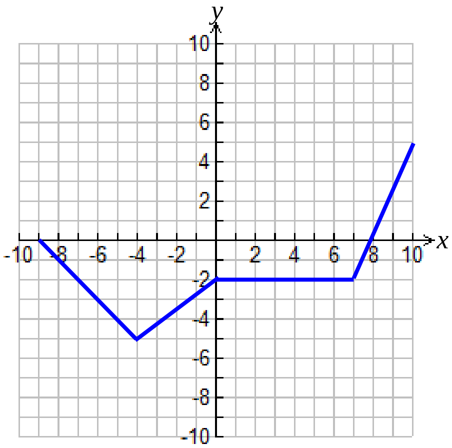
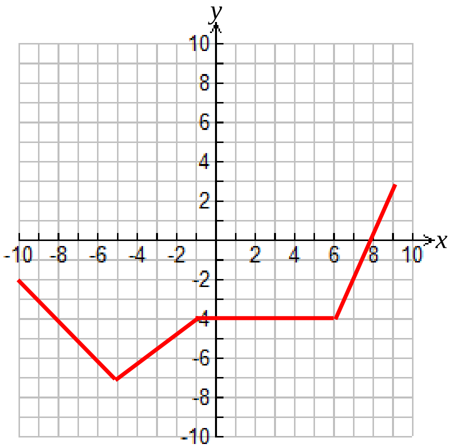
Extremely Important Questions and Exercises…

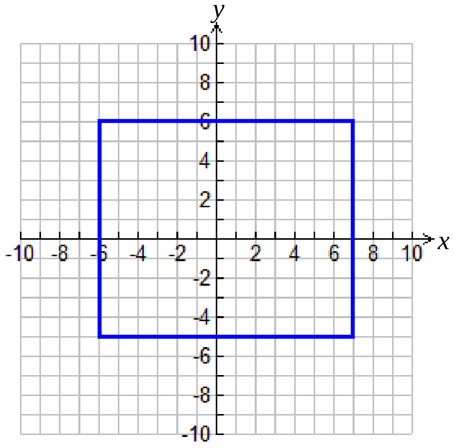
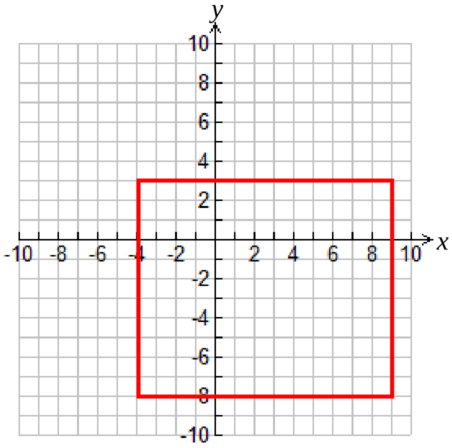
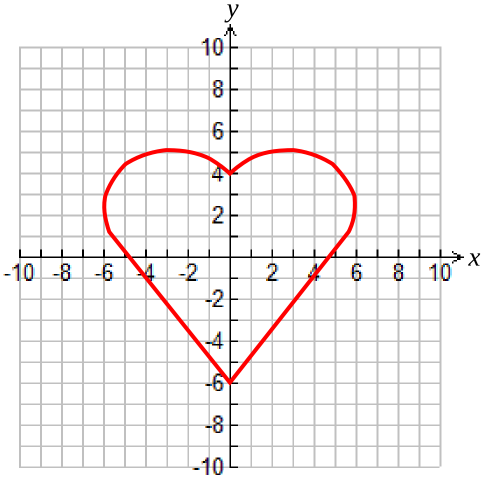
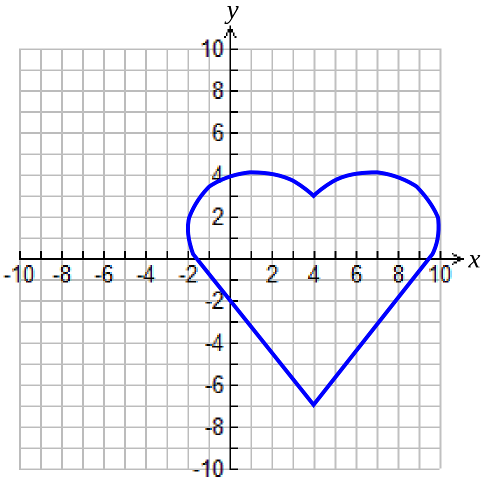
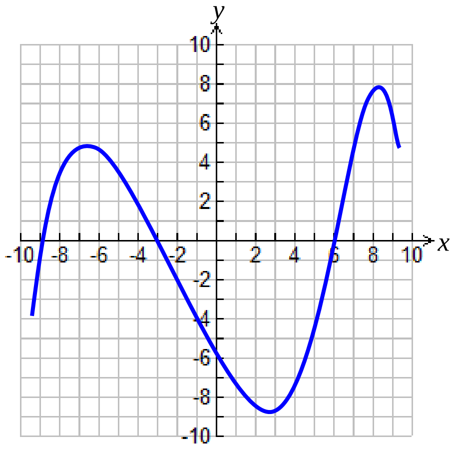
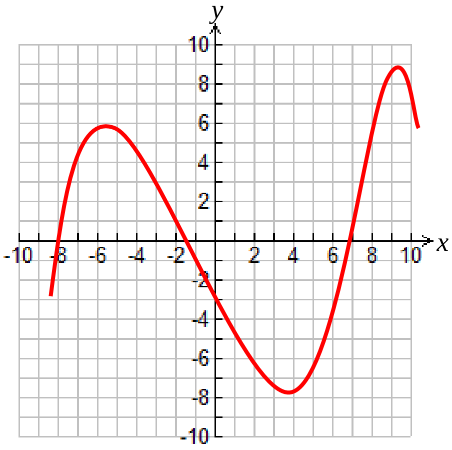
1. So far we have considered both *horizontal* and *vertical* translations of functions. Does it matter in which order these translations are performed? It does not matter. Vertical and horizontal translations are independent of each other

2. Complete the following table.

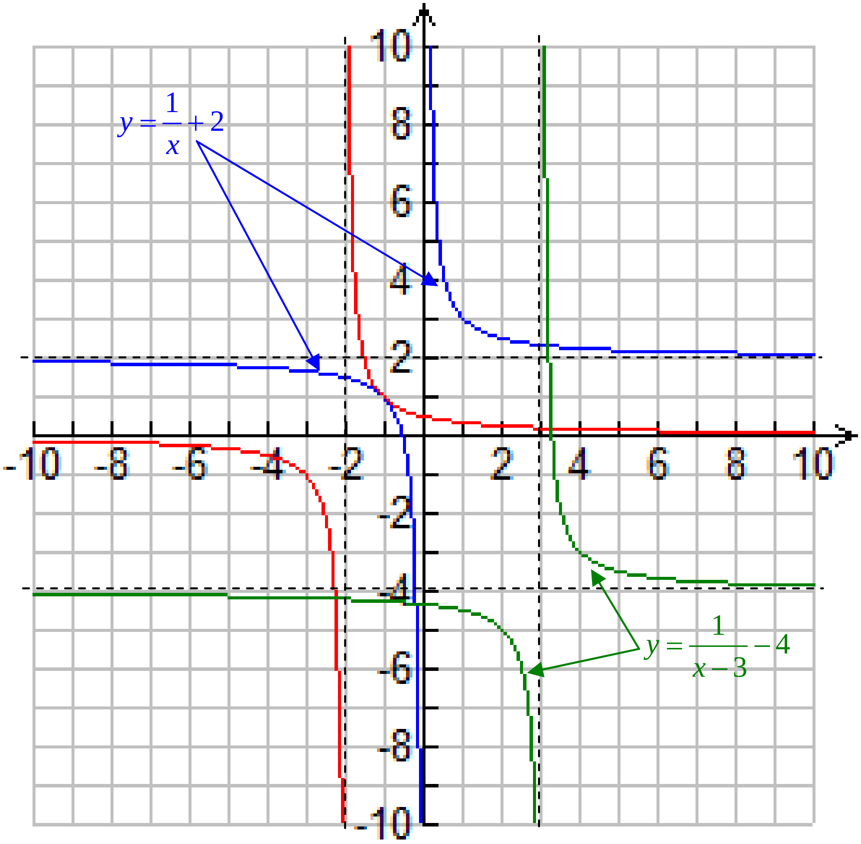
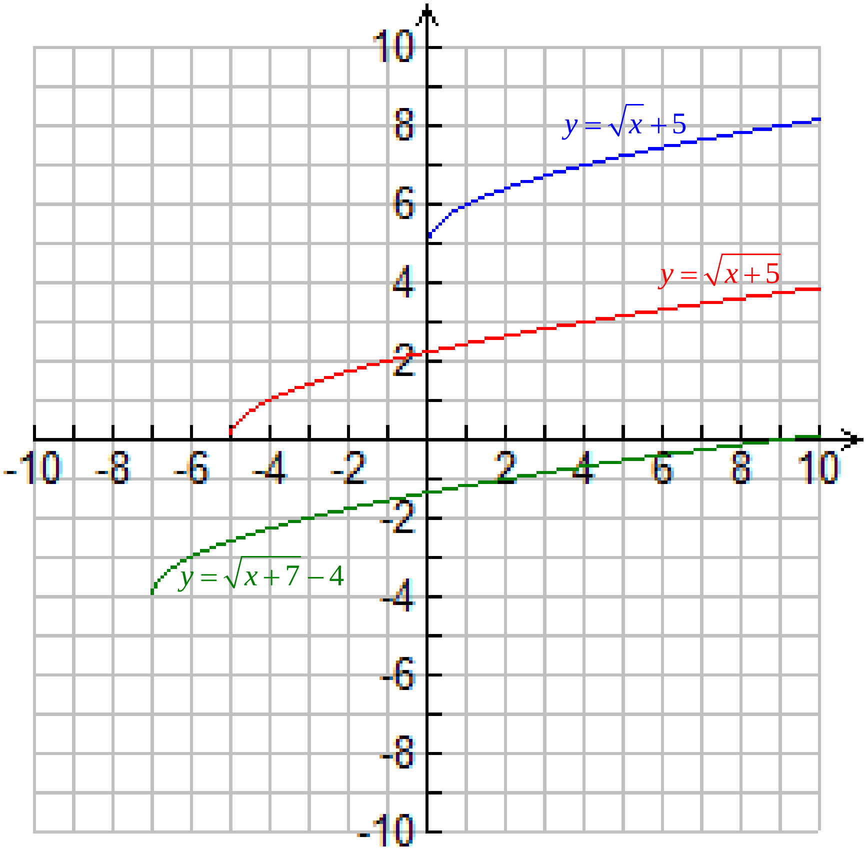
Function	*Base Function*	*Translation(s) of Base Function Required to obtain Function*
(a) $f(x) = x+16$	$f(x) = x$	There are two correct answers for this one. The base function is translated 16 units up OR 16 units to the left
(b) $g(x) = x^2 - 5x + 6$	$f(x) = x^2$	$x^2-5x+6 = x^2-5x+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+6$ $= \left(x-\frac{5}{2}\right)^2 - \frac{1}{4}$ The base function is translated $\frac{5}{2}$ units right and $\frac{1}{4}$ units down
(c) $h(x) = \sqrt{x-6} + 5$	$f(x) = \sqrt{x}$	The base function is translated 6 units right and 5 units up
(d) $p(x) = \dfrac{1}{x+2} - 5$	$f(x) = \frac{1}{x}$	The base function is translated 2 units left and 5 units down.
(e) $q(x) = x^3 + 3x^2 + 3x + 1$ Graph using TI-Interactive	$f(x) = x^3$	$q(x) = x^3+3x^2+3x+1 = (x+1)^3$ The base function is shifted 1 unit to the left expand this to check

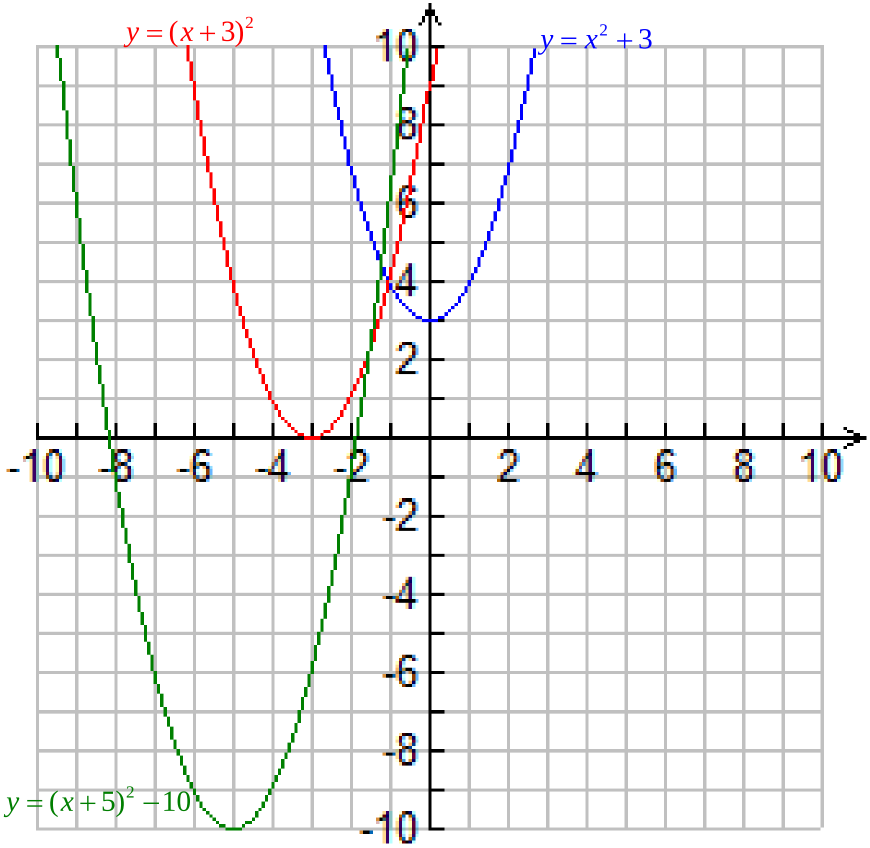
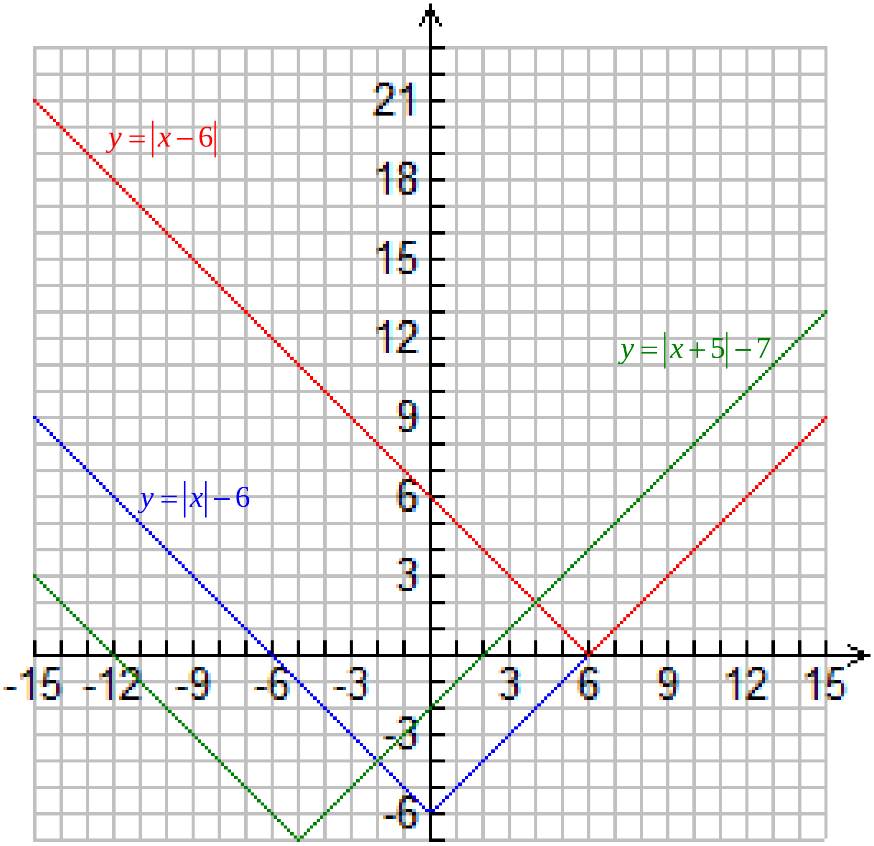
3. Complete the following table.

Graph of Relation	*Relation or Function?*	*Discrete or Continuous?*	*Graph of Translated Relation*
	**Not a Function**	**Discrete**	Translate 3 units up and 2 to the left.  If the given relation is a function $y = f(x)$, then this is ______________
	**Function**	**Discrete**	Translate 10 units down.  If the given relation is a function $y = f(x)$, then this is $y = f(x) - 10$
	**Function**	**Continuous**	Translate 2 units down and 1 to the left.  If the given relation is a function $y = f(x)$, then this is $y = f(x+1) - 2$

Graph of Relation	*Relation or Function?*	*Discrete or Continuous?*	*Graph of Translated Relation*
	**Not a Function**	**Continuous**	Translate 3 units down and 2 to the right.  If the given relation is a function $y = f(x)$ then this is ________________
	**Not a Function**	**Continuous**	Translate 1 unit down and 4 to the right.  If the given relation is a function $y = f(x)$ then this is ________________
	**Function**	**Continuous**	Translate 2 units up and 1 to the right.  If the given relation is a function $y = f(x)$ then this is $y = f(x+1)+2$

4. Without using a table of values, graph the following functions on the same grid. In addition, state the domain and range of each function.

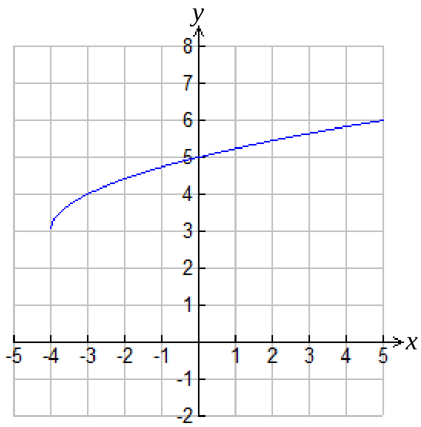
Functions	***Graphs***	***Domain and Rage***
$y=\frac{1}{x}+2$ $y=\frac{1}{x+2}$ $y=\frac{1}{x-3}-4$		
$y=\sqrt{x}+5$ $y=\sqrt{x+5}$ $y=\sqrt{x+7}-4$		

Functions	***Graphs***	***Domain and Range***
$y = x^2 + 3$ $y = (x+3)^2$ $y = (x+5)^2 - 10$		
$y = \|x\| - 6$ $y = \|x - 6\|$ $y = \|x + 5\| - 7$		

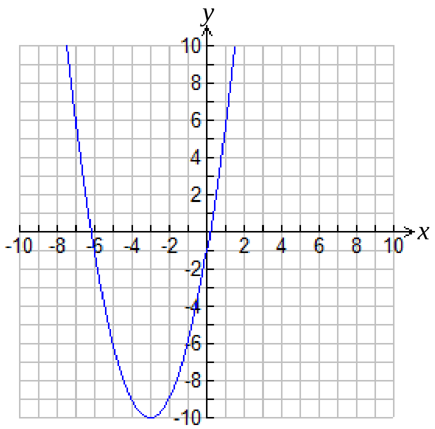
5. Without graphing, state the domain, range, intercepts (if any) and asymptotes (if any) of each given function.

Function	***Domain and Range***	***Intercepts***	***Asymptotes***
$f(x)=x^2+5$	$D=\mathbb{R}$ $R=\{y\in\mathbb{R}:y\geq 5\}$	x-intercept: none y-intercept: 5	none
$g(x)=\sqrt{x}-10$	$D=\{x\in\mathbb{R}:x\geq 0\}$ $R=\{y\in\mathbb{R}:y\geq -10\}$	x-intercept: 100 y-intercept: –10	none
$h(x)=x^3-15$	$D=\mathbb{R}$ $R=\mathbb{R}$	x-intercept: $\sqrt[3]{15}$ y-intercept: –15	none
$s(t)=\dfrac{1}{t+3}-15$	$D=\{x\in\mathbb{R}:x\neq -3\}$ $R=\{y\in\mathbb{R}:y\neq -15\}$	t-intercept: $-\frac{44}{15}$ s-intercept: $-\frac{44}{3}$	$t=-3$ (vertical) $s(t)=-15$ (horizontal)
$p(y)=\lvert y-8\rvert+15$	$D=\mathbb{R}$ $R=\{y\in\mathbb{R}:y\geq 15\}$	y-intercept: none p-intercept: 23	none

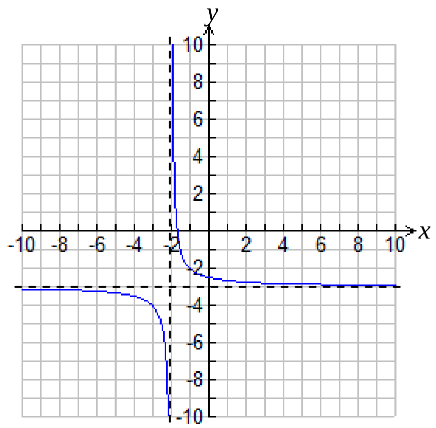
6. For each graph, state an equation that best describes it.



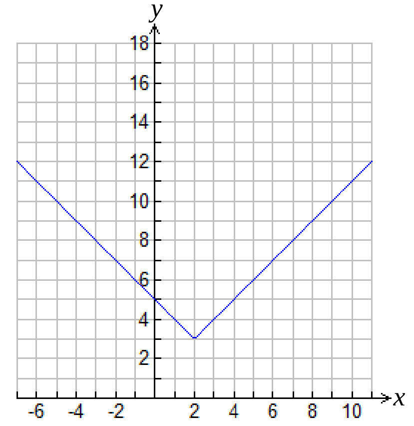
$f(x)=\sqrt{x+4}+3$



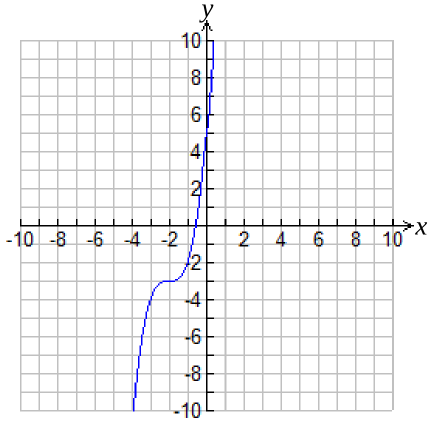
$g(x)=(x+3)^2-10$



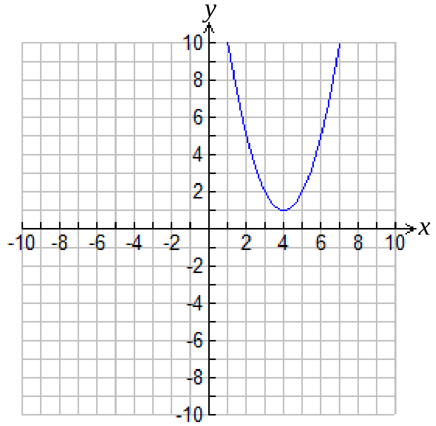
$h(x)=\dfrac{1}{x+2}-3$



$p(x)=\lvert x-2\rvert+3$



$g(x)=(x+2)^3-3$



$g(x)=(x-4)^2+1$

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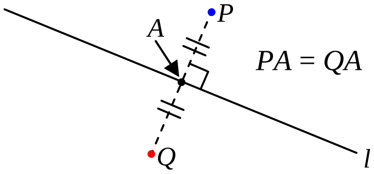


1. Suppose that the point Q is the reflection of the point P in the line l. Suppose further that the line segment PQ intersects the line l at the point A. What can you conclude about the lengths of the line segments PA and QA? Draw a diagram to illustrate your answer.

2. Is the reflection of a function in the x-axis also a function? Explain.

The reflection in the x-axis of any function must also be a function. When reflecting in the x-axis, the pre-image point (x, y) is mapped to the image $(x, -y)$. Thus, each x-value still has only one corresponding y-value, which is equal to the "negative" of the pre-image point's y-value.

3. Is the reflection of a function in the y-axis also a function? Explain.

The reflection in the y-axis of any function must also be a function. When reflecting in the y-axis, the pre-image point (x, y) is mapped to the image $(-x, y)$. Thus, although all x-values change sign, each x-value still has only one corresponding y-value

4. Is the reflection of a function in the line $y = x$ also a function? Explain.

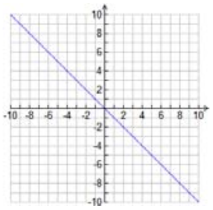
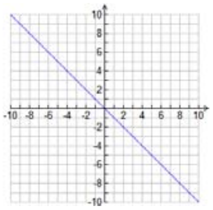
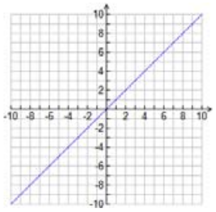
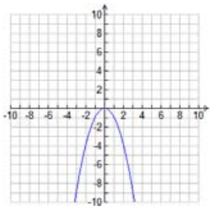
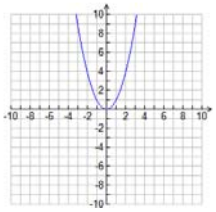
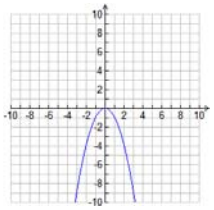
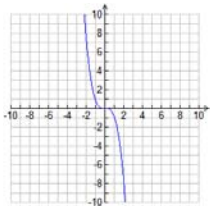
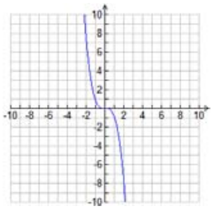
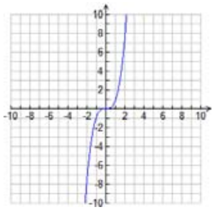
If the function being reflected is one-to-one, then the reflection in the line $y = x$ will also be a function. However, this is not the case for many-to-one functions. For instance, when $f(x) = x^2$ is reflected in the line $y = x$, the resulting relation $y = \pm\sqrt{x}$ (or $y^2 = x$) is not a function.

5. Suppose that R represents a relation that is ***not*** a function. Can a reflection of R be a function? If so, give examples.

A reflection of a general relation need not be a function. However, in certain cases such a reflection can be a function. For example, when $y^2 = x$ is reflected in the line $y = x$, $y = x^2$ is obtained, which is a function.

Investigation

You may use a graphing calculator or graphing software such as TI-Interactive to complete the following table.

Function	***Equation of*** $-f(x)$	***Equation of*** $f(-x)$	***Equation of*** $-f(-x)$	***Graph of*** $y = -f(x)$	***Graph of*** $y = f(-x)$	***Graph of*** $y = -f(-x)$
$f(x) = x$	$g(x) = -f(x)$ $= -x$	$h(x) = f(-x)$ $= -x$	$p(x) = -f(-x)$ $= -(-x)$ $= x$			
$f(x) = x^2$	$g(x) = -f(x)$ $= -x^2$	$h(x) = f(-x)$ $= (-x)^2$ $= x^2$	$p(x) = -f(-x)$ $= -(-x)^2$ $= -x^2$			
$f(x) = x^3$	$g(x) = -f(x)$ $= -x^3$	$h(x) = f(-x)$ $= (-x)^3$ $= -x^3$	$p(x) = -f(-x)$ $= -(-x)^3$ $= -(-x^3)$ $= x^3$			

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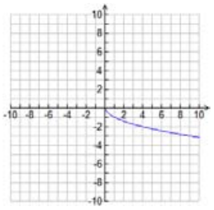
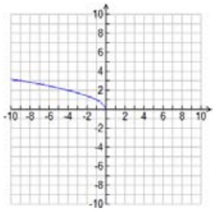
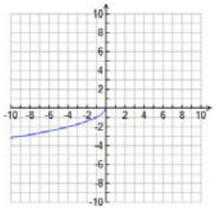
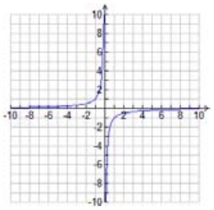
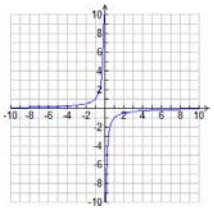
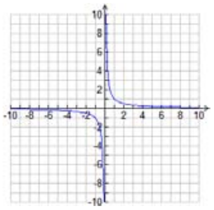
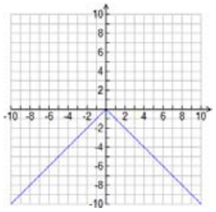
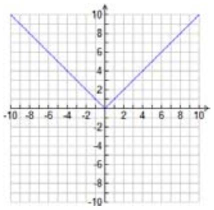
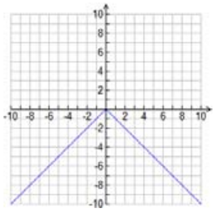
Function	***Equation of*** $-f(x)$	***Equation of*** $f(-x)$	***Equation of*** $-f(-x)$	***Graph of*** $y=-f(x)$	***Graph of*** $y=f(-x)$	***Graph of*** $y=-f(-x)$
$f(x)=\sqrt{x}$	$g(x)=-f(x)$ $=-\sqrt{x}$	$h(x)=f(-x)$ $=\sqrt{-x}$	$p(x)=-f(-x)$ $=-\sqrt{-x}$			
$f(x)=\dfrac{1}{x}$	$g(x)=-f(x)$ $=-\dfrac{1}{x}$	$h(x)=f(-x)$ $=\dfrac{1}{-x}$ $=-\dfrac{1}{x}$	$p(x)=-f(-x)$ $=-\left(\dfrac{1}{-x}\right)$ $=\dfrac{1}{x}$			
$f(x)=\lvert x\rvert$	$g(x)=-f(x)$ $=-\lvert x\rvert$	$h(x)=f(-x)$ $=\lvert -x\rvert$ $=\lvert x\rvert$	$p(x)=-f(-x)$ $=-\lvert -x\rvert$ $=-\lvert x\rvert$			

Given a function *f*,

- the graph of $y=-f(x)$ is the **reflection** of the graph of $y=f(x)$ in the ***x*-axis**. $(x,y)\rightarrow(x,-y)$
- the graph of $y=f(-x)$ is the **reflection** of the graph of $y=f(x)$ in the ***y*-axis**. $(x,y)\rightarrow(-x,y)$
- the graph of $y=-f(-x)$ is the **reflection** of the graph of $y=f(x)$ in the ***x*-axis** followed by the **reflection** of $y=-f(x)$ in the ***y*-axis**. (OR, the graph of $y=-f(-x)$ is the **reflection** of the graph of $y=f(x)$ in the ***y*-axis** followed by the **reflection** of $y=f(-x)$ in the ***x*-axis**.) $(x,y)\rightarrow(-x,-y)$

- Suppose that a transformation is applied to a point *P* to obtain the point *Q*. Then, *P* is called the ***pre-image of Q*** and *Q* is called the ***image of P***.
- If *Q* is the image of *P* under some transformation and *P* = *Q*, then *P* is said to be ***invariant*** under the transformation.

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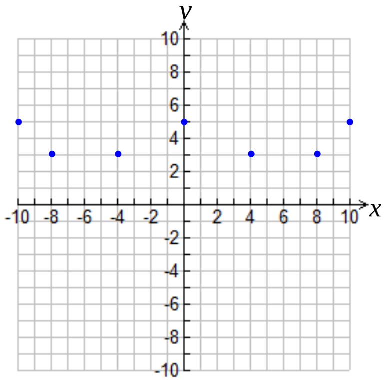
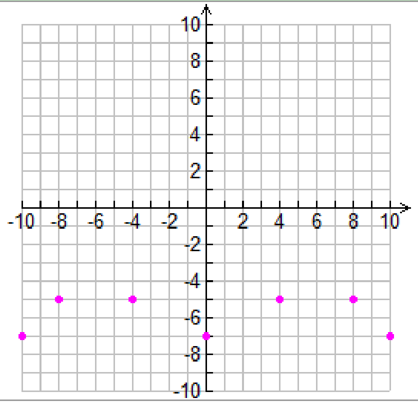
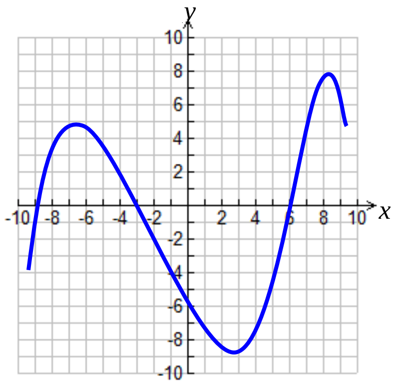
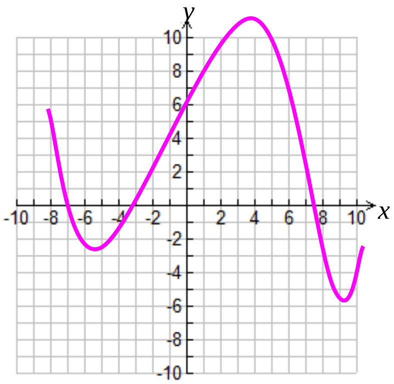
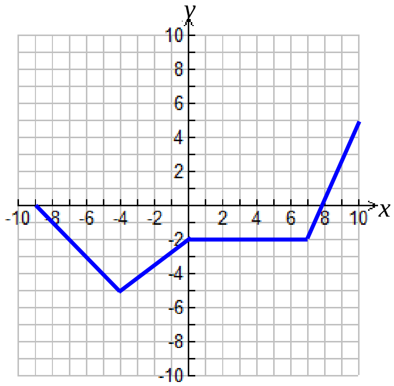
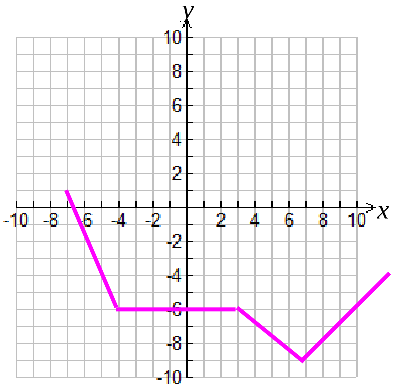
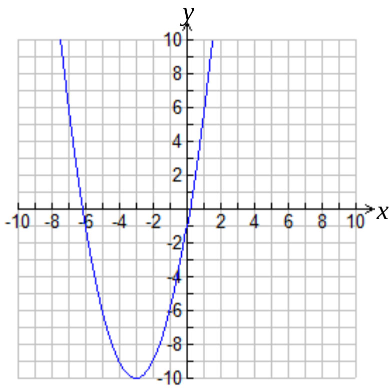
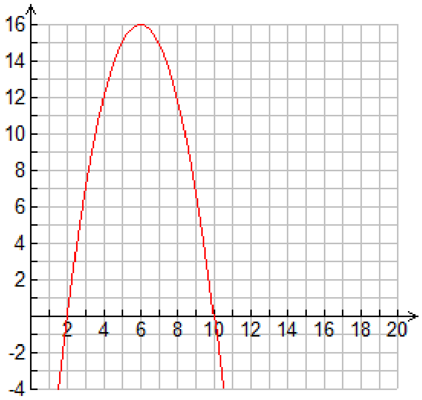
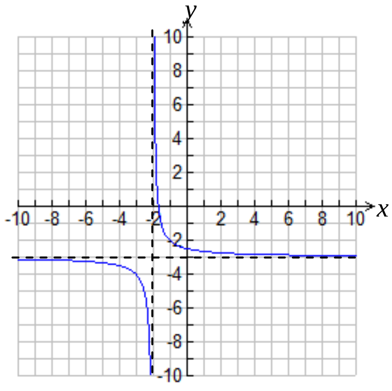
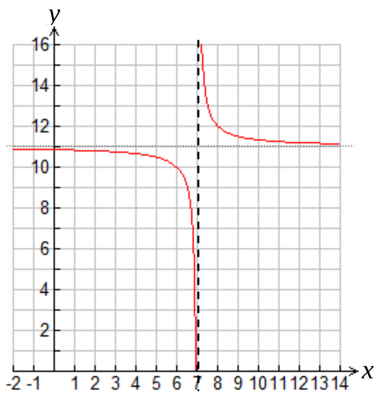
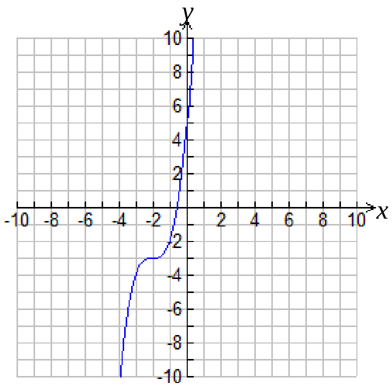
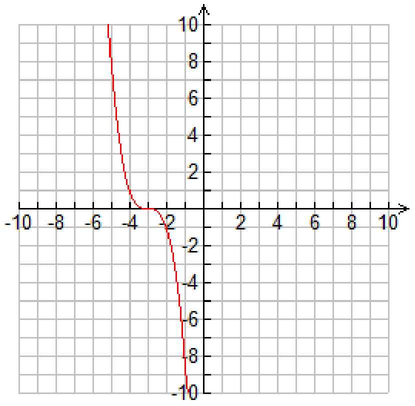
Perform the specified transformations on the graph of each given function.

Graph of Function $y = f(x)$	***Transformation of f***	***Graph of Transformed Function $y = g(x)$***
	$g(x) = -f(x) - 2$ Explain this transformation of *f* in words. 1. Reflect in the *x*-axis. 2. Translate down 2 units.	
	$g(x) = -f(x-1) + 2$ Explain this transformation of *f* in words. ***Horizontal*** 1. Translate right 1 unit. ***Vertical*** 1. Reflect in the *x*-axis. 2. Translate up 2 units.	
	$g(x) = f(-(x+3)) - 4$ Explain this transformation of *f* in words. ***Horizontal*** 1. Reflect in the *y*-axis. 2. Translate right 3 units. ***Vertical*** 1. Translate down 4 units.	
	$g(x) = -f(-(x-3)) + 6$ Explain this transformation of *f* in words. ***Horizontal:*** reflect in *y*-axis then shift 3 units right ***Vertical:*** Reflect in *x*-axis then shift 6 units up Possible equations of *f* and *g*. $f(x) = (x+3)^2 - 10$ $g(x) = -(x-6)^2 + 16$	

	$g(x)=-f(-(x-5))+8$ Explain this transformation of *f* in words. ***Horizontal:*** reflect in *y*-axis then shift 5 units right ***Vertical:*** Reflect in *x*-axis then shift 8 units up Possible equations of *f* and *g*. $f(x)=\frac{1}{x+2}-3$ $g(x)=\frac{1}{x-7}+11$	
	$g(x)=f(-x-5)+3$ $=f(-(x+5))+3$ Explain this transformation of *f* in words. ***Horizontal:*** reflect in *y*-axis then shift 5 units left ***Vertical:*** shift 3 units up Possible equations of *f* and *g*. $f(x)=(x+2)^3-3$ $g(x)=f(-x-5)+3$ $=-(x+3)^3$ ***Alternative Solution*** Instead of factoring $-x-5$, we can interpret it directly. Simply reverse the operations in the order opposite the order of operations. ***Horizontal:*** Translate 5 units right then reflect in *y*-axis. ***Vertical:*** same as above	

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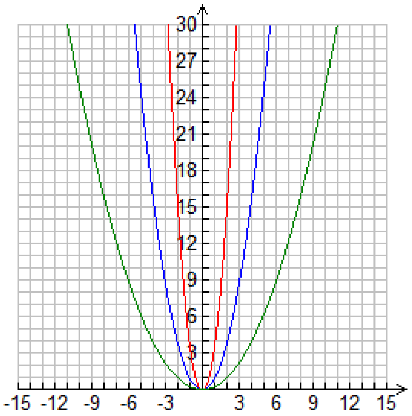
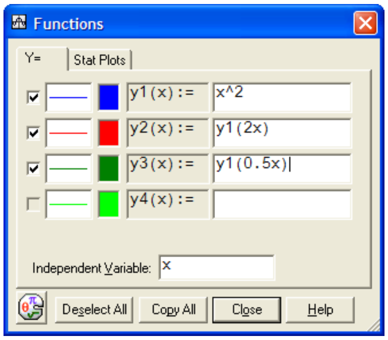
Investigation

1. Sketch $f(x)=x^2$, $g(x)=f(2x)=(2x)^2=4x^2$ and $h(x)=f(\frac{1}{2}x)=(\frac{1}{2}x)^2=\frac{1}{4}x^2$ on the same grid. Describe how the graphs of *g* and *h* are related to the graph of *f*.





☑ y1(x) := x^2

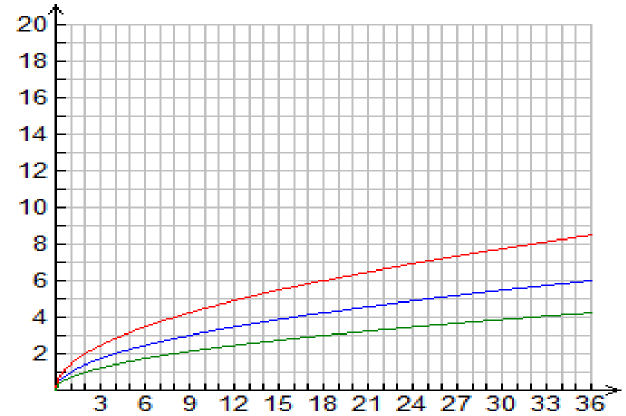
☑ y2(x) := y1(2x)

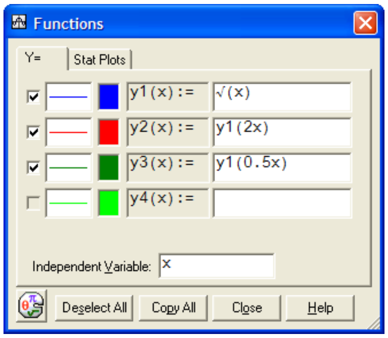
☑ y3(x) := y1(0.5x)|

☐ y4(x) :=

Independent Variable: x

2. Sketch $f(x)=\sqrt{x}$, $g(x)=f(2x)=\sqrt{2x}$ and $h(x)=f(\frac{1}{2}x)=\sqrt{\frac{1}{2}x}$ on the same grid. Describe how the graphs of *g* and *h* are related to the graph of *f*.





☑ y1(x) := √(x)

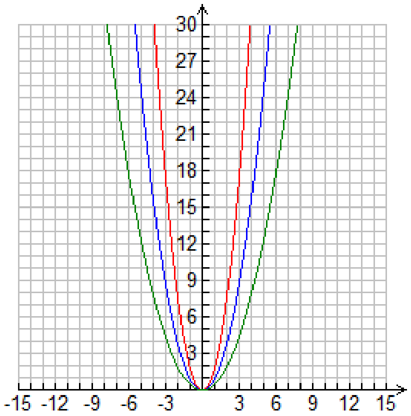
☑ y2(x) := y1(2x)

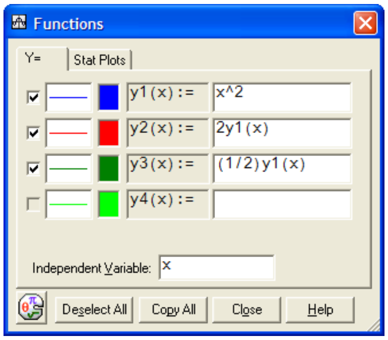
☑ y3(x) := y1(0.5x)

☐ y4(x) :=

Independent Variable: x

3. Sketch $f(x)=x^2$, $g(x)=2f(x)=2x^2$ and $h(x)=\frac{1}{2}f(x)=\frac{1}{2}x^2$ on the same grid. Describe how the graphs of *g* and *h* are related to the graph of *f*.





☑ y1(x) := x^2

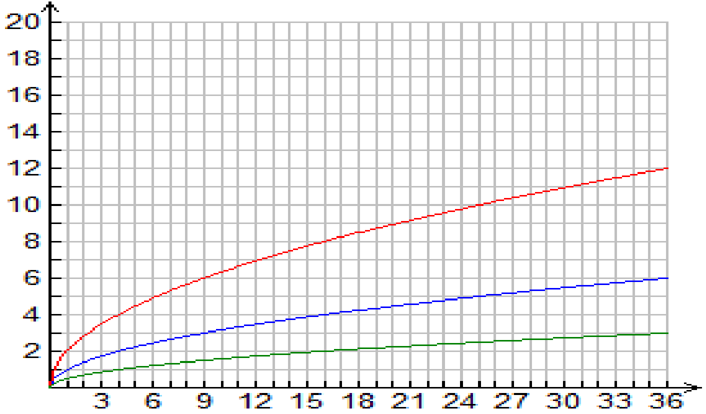
☑ y2(x) := 2y1(x)

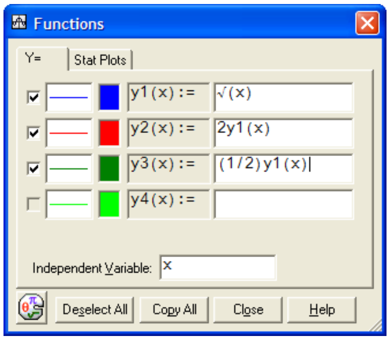
☑ y3(x) := (1/2)y1(x)

☐ y4(x) :=

Independent Variable: x

4. Sketch $f(x)=\sqrt{x}$, $g(x)=2f(x)=2\sqrt{x}$ and $h(x)=\frac{1}{2}f(x)=\frac{1}{2}\sqrt{x}$ on the same grid. Describe how the graphs of *g* and *h* are related to the graph of *f*.





☑ y1(x) := √(x)

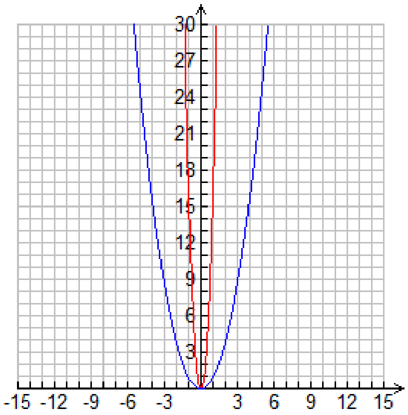
☑ y2(x) := 2y1(x)

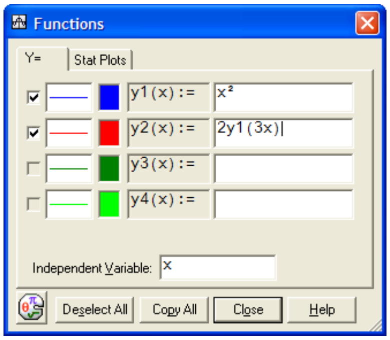
☑ y3(x) := (1/2)y1(x)|

☐ y4(x) :=

Independent Variable: x

5. Sketch $f(x)=x^2$ and $g(x)=2f(3x)=18x^2$. Describe how the graph of *g* is related to the graph of *f*.





☑ y1(x) := x²

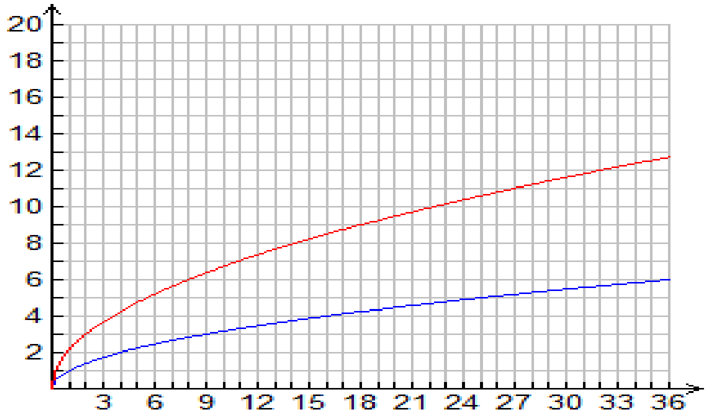
☑ y2(x) := 2y1(3x)

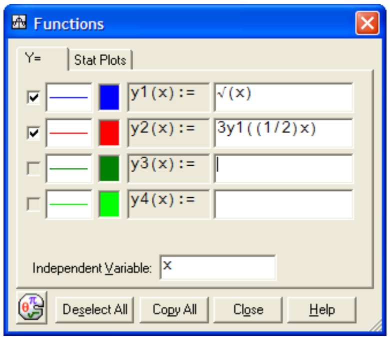
☐ y3(x) :=

☐ y4(x) :=

Independent Variable: x

6. Sketch $f(x)=\sqrt{x}$ and $g(x)=3f(\tfrac{1}{2}x)=3\sqrt{\tfrac{1}{2}x}$. Describe how the graph of *g* is related to the graph of *f*.





☑ y1(x) := √(x)

☑ y2(x) := 3y1((1/2)x)

☐ y3(x) :=

☐ y4(x) :=

Independent Variable: x

Summary

- To obtain the graph of $y=g(x)=af(x), a\in\mathbb{R}$ from the graph of $y=f(x)$ ***stretch*** (or ***compress***) ***vertically*** by a factor of *a*. This means that the ordered pairs of the function *g* are obtained by taking each ordered pair belonging to the function *f* and keeping the *x*-co-ordinate the same while multiplying the *y*-co-ordinate by *a*. Symbolically, we can write this as $(x,y)\rightarrow(x,ay)$ (i.e. pre-image → image).
- To obtain the graph of $y=g(x)=f(bx), b\in\mathbb{R}$ from the graph of $y=f(x)$ ***stretch*** (or ***compress***) ***horizontally*** by a factor of $\frac{1}{b}=b^{-1}$. This means that the ordered pairs of the function *g* are obtained by taking each ordered pair belonging to the function *f* and keeping the *y*-co-ordinate the same while multiplying the *x*-co-ordinate by $\frac{1}{b}=b^{-1}$. Symbolically, we can write this as $(x,y)\rightarrow(b^{-1}x,y)$ (i.e. pre-image → image).
- To obtain the graph of $y=g(x)=af(bx), a\in\mathbb{R}, b\in\mathbb{R}$ from the graph of $y=f(x)$ ***stretch*** (or ***compress***) ***vertically*** by a factor of *a* **and** ***stretch*** (or ***compress***) ***horizontally*** by a factor of $\frac{1}{b}=b^{-1}$. Symbolically, we can write this as $(x,y)\rightarrow(b^{-1}x,ay)$ (i.e. pre-image → image).

Page 38 – Important Exercise on Inverses of Functions

Complete the following table. The first two rows are done for you.

Hint: To find the inverse of each given function, apply the ***inverse operations*** in the ***reverse order***.

Function		*One-to-One or Many-to-One?*	*Inverse Function – State any Restrictions to Domain of f*		*Domain and Range of f*	*Domain and Range of* f^{-1}
Function Notation	*Mapping Notation*		*Function Notation*	*Mapping Notation*		
$f(x)=x^3$	$x\mapsto x^3$	one-to-one	$f^{-1}(x)=\sqrt[3]{x}$	$x\mapsto \sqrt[3]{x}$	$D=\mathbb{R}$ $R=\mathbb{R}$	$D=\mathbb{R}$ $R=\mathbb{R}$
$f(x)=x^2$	$x\mapsto x^2$	many-to-one	$f^{-1}(x)=\sqrt{x}$, provided that $x\geq 0$	$x\mapsto\sqrt{x}$	$D=\{x\in\mathbb{R}: x\geq 0\}$ $R=\{y\in\mathbb{R}: y\geq 0\}$	$D=\{x\in\mathbb{R}: x\geq 0\}$ $R=\{y\in\mathbb{R}: y\geq 0\}$
$f(x)=2x+1$	$x\mapsto 2x+1$	one-to-one	$f^{-1}(x)=\frac{x-1}{2}$	$x\mapsto\frac{x-1}{2}$	$D=\mathbb{R}$ $R=\mathbb{R}$	$D=\mathbb{R}$ $R=\mathbb{R}$
$f(x)=2x^3-7$	$x\mapsto 2x^3-7$	one-to-one	$f^{-1}(x)=\sqrt[3]{\frac{x+7}{2}}$	$x\mapsto\sqrt[3]{\frac{x+7}{2}}$	$D=\mathbb{R}$ $R=\mathbb{R}$	$D=\mathbb{R}$ $R=\mathbb{R}$
$f(x)=\frac{1}{x}$	$x\mapsto\frac{1}{x}$	one-to-one	$f^{-1}(x)=\frac{1}{x}$, provided that $x\neq 0$	$x\mapsto\frac{1}{x}$	$D=\{x\in\mathbb{R}: x\neq 0\}$ $R=\{y\in\mathbb{R}: y\neq 0\}$	$D=\{x\in\mathbb{R}: x\neq 0\}$ $R=\{y\in\mathbb{R}: y\neq 0\}$
$f(x)=x^2-4$	$x\mapsto x^2-4$	many-to-one	$f^{-1}(x)=\sqrt{x+4}$, provided that $x\geq -4$	$x\mapsto\sqrt{x+4}$	$D=\mathbb{R}$ $R=\{y\in\mathbb{R}: y >= -4\}$	$D=\{x\in\mathbb{R}: x >= -4\}$ $R=\mathbb{R}$
$f(x)=x^2+10x+1$	$x\mapsto x^2+10x+1$	many-to-one	$f(x)=x^2+10x+1$ $=x^2+10x+5^2-5^2+1$ $=(x+5)^2-24$ $f^{-1}(x)=\sqrt{x+24}-5$, provided that $x\geq -24$	$x\mapsto\sqrt{x+24}-5$	$D=\mathbb{R}$ $R=\{y\in\mathbb{R}: y >= -24\}$	$D=\{x\in\mathbb{R}: x >= -24\}$ $R=\mathbb{R}$

Page 41 – Extremely Important Follow-up Questions

1. If $f(x)=x^3$, we have discovered that $f^{-1}(x)=\sqrt[3]{x}$. Evaluate $f(f^{-1}(x))$ and $f^{-1}(f(x))$.

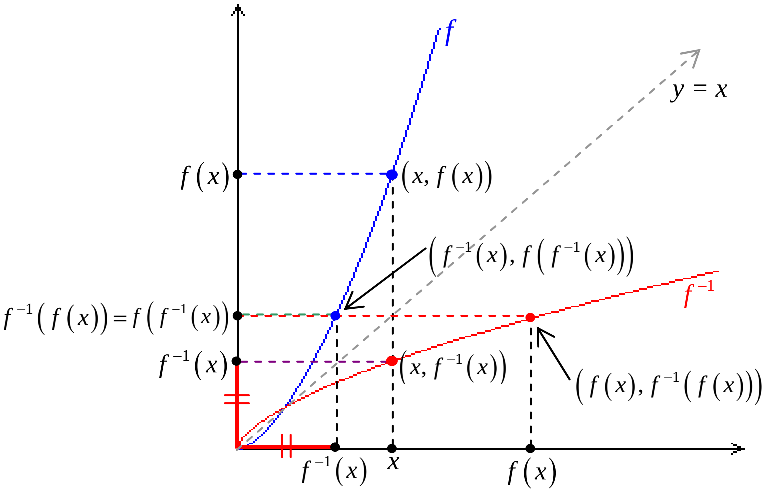
Solution

$$f\left(f^{-1}(x)\right)=f\left(\sqrt[3]{x}\right)=\left(\sqrt[3]{x}\right)^3=\left(x^{\frac{1}{3}}\right)^3=x \text{ and } f^{-1}\left(f(x)\right)=f^{-1}\left(x^3\right)=\sqrt[3]{x^3}=\left(x^3\right)^{\frac{1}{3}}=x$$

2. For any function *f*, evaluate $f(f^{-1}(x))$ and $f^{-1}(f(x))$. (**Hint:** It does not matter what *f* is. All that matters is that f^{-1} is defined at *x* and at $f(x)$. In addition, keep in mind that an inverse of a function ***undoes*** the function.)

Solution

Since *f* and f^{-1} undo each other, it follows that $f^{-1}\left(f(x)\right)=f\left(f^{-1}(x)\right)=x$ no matter what *f* is. You can convince yourself that this true by studying the diagram at the right. Keep in mind that *f* and f^{-1} are reflections of each other in the line *y* = *x*. It's clear from the graph that $f^{-1}\left(f(x)\right)=f\left(f^{-1}(x)\right)$. It's also clear that the points $\left(x, f(x)\right)$ and $\left(f(x), f^{-1}\left(f(x)\right)\right)$ are reflections of each other in the line *y* = *x* as are the points $\left(x, f^{-1}(x)\right)$ and $\left(f(x), f\left(f^{-1}(x)\right)\right)$. Therefore, $f^{-1}\left(f(x)\right)=f\left(f^{-1}(x)\right)=x$. If you need further convincing, use a ruler to measure the distance from the origin to the point marked "*x*" on the *x*-axis. Then measure the distance from the origin to the point marked "$f^{-1}\left(f(x)\right)=f\left(f^{-1}(x)\right)$" on the *y*-axis. You should find that the distances are exactly the same!



3. The slope of $f(x)=mx+b$ is *m*. What is the slope of f^{-1}?

Solution

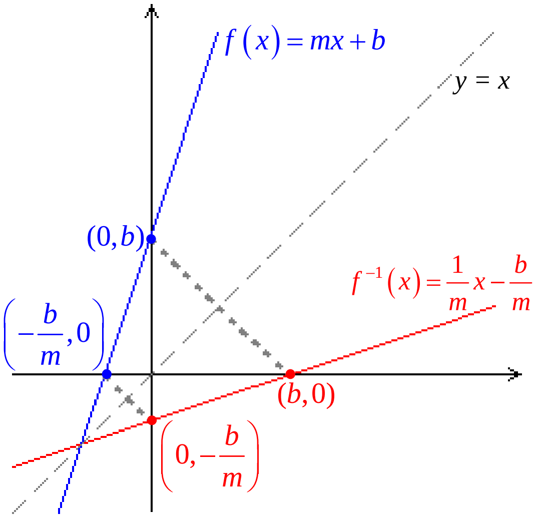
The equation of f^{-1} can be found in a variety of ways:

(a) The method of interchanging *x* and *y* and solving for *y* can be used if $f(x)=mx+b$ is rewritten in the form $y=mx+b$.

(b) The flowchart method can be used (i.e. apply the inverse operations in the reverse order).

(c) The most revealing method is to apply the transformation $(x,y)\to(y,x)$ (reflection in the line *y* = *x*). As you can see from the graph at the right, the images of the *x*-intercept and *y*-intercept of *f* respectively are the *y*-intercept and *x*-intercept of f^{-1}. From the information in the graph it is easy to see that the slope of f^{-1} must be $\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}=\frac{0-(-\frac{b}{m})}{b-0}=\frac{(\frac{b}{m})}{b}=\frac{b}{m}\left(\frac{1}{b}\right)=\frac{1}{m}$.

Consequently, since the *y*-intercept of f^{-1} is $-\frac{b}{m}$, its equation must be $f^{-1}(x)=\frac{1}{m}x-\frac{b}{m}$.



Pages 42→47 – Selected Answers to Problem Solving Activity

1. The word “function” is often used to express relationships that are not mathematical. Explain the meaning of the following statements:

(a) Crime is a function of socioeconomic status.

Solution

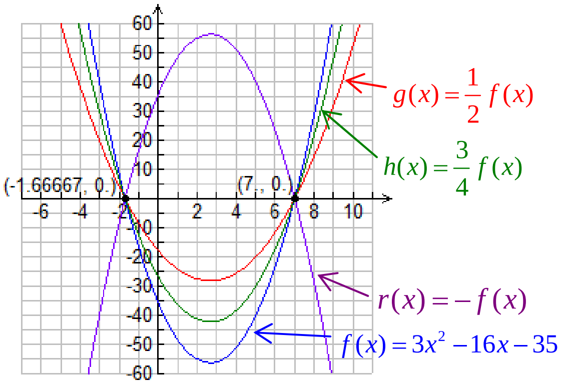
When used in non-mathematical contexts, “a function of” still means “depends on.” Just as “*f*(*x*)” means “the value of *f* depends on *x*,” this means that the crime rate of a region depends on the socioeconomic status of its residents.

2. Investigate the transformations that affect the ***number of roots*** of the following quadratic equations.

(d) $3x^2-16x-35=0$

Graphical

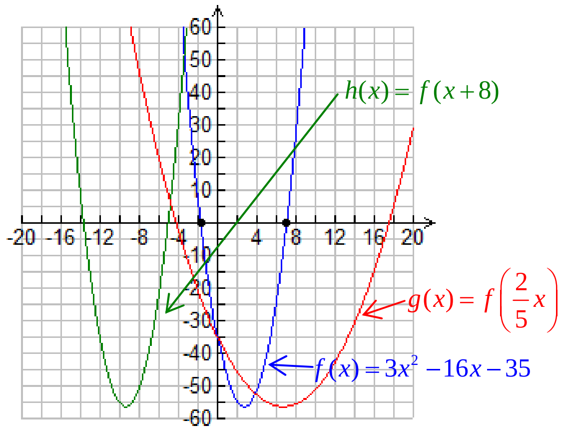
Vertical Stretches



As you can see from the graph, the roots of *f* are invariant under a vertical stretch by any factor (including a reflection in the *x*-axis).

Horizontal Translations and Stretches

As shown in the following examples, horizontal translations and stretches of a quadratic function will generally change the roots but not the number of roots.



Algebraic

If $f(x)=3x^2-16x-35$ then the roots of $f(x)=0$ are $r_1=-\dfrac{5}{3}$ and $r_2=7$. That is, $f(x)=(x-7)\left(x+\frac{5}{3}\right)$.

Vertical Stretches

If we define $g(x)=af(x)$ for any non-zero value $a\in\mathbb{R}$, then $g(x)=a(x-7)\left(x+\frac{5}{3}\right)$. Clearly, $g(x)=0$ has the same roots as $f(x)=0$. Thus, the zeros of $f(x)=3x^2-16x-35$ are invariant under any vertical stretch.

In general, as shown below, for any function *f*, the zeros of $g(x)=af(x)$ will be exactly the same as the zeros of *f*.

If *r* is a zero of *f*, then $f(r)=0$. In addition, $g(r)=af(r)=a(0)=0$. Therefore, *r* is also a zero of *g*.

This shows that the roots of any function are invariant under a vertical stretch (or compression). Note that vertical reflections are special vertical stretches in which *a* = –1. Therefore, the number of roots is unaffected by a vertical stretch.

Horizontal Translations and Stretches

Let *f* be any function and *r* be any root of *f*. Then *f*(*r*) = 0, which means that $(r,0)$ lies on the graph of *f*. What happens to this point under a horizontal translation or stretch?

Stretch: $(x,y)\to(ax,y)$ so $(r,0)\to(ar,0)$

Translation: $(x,y)\to(x+h,y)$ so $(r,0)\to(r+h,0)$

Therefore, the image of any point lying on the *x*-axis will be a point on the *x*-axis. **This means that the number of roots is unaffected by a horizontal stretch or compression.**

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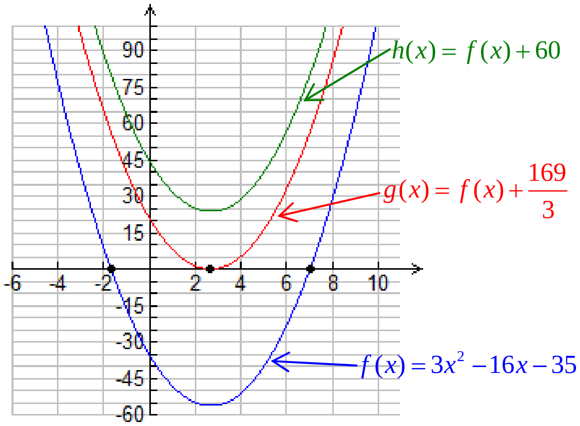
Vertical Translations

As can be seen from the following examples, vertical translations ***can affect*** the number of roots.



A detailed algebraic analysis of the effect of vertical translations can get a little messy. For our purposes, it suffices to note the following:

Parabola Opens Upward

If the parabola has two roots, it can be shifted up by an amount less than the absolute value of the *y*-co-ordinate of its vertex and still have two roots.

However, if it is shifted up by exactly this amount, the number of roots changes to 1.

If shifted up by more than this amount, the parabola will have no real roots.

Parabola Opens Downward

If the parabola has two roots, it can be shifted down by an amount less than the absolute value of the *y*-co-ordinate of its vertex and still have two roots.

However, if it is shifted down by exactly this amount, the number of roots changes to 1.

If shifted down by more than this amount, the parabola will have no real roots.

3. Although the method of completing the square is very powerful, it can involve a great deal of work. If your goal is to find the maximum or minimum of a quadratic function, you can use a simpler method called ***partial factoring***. Explain how partially factoring $f(x) = 3x^2 - 6x + 5$ into the form $f(x) = 3x(x-2) + 5$ helps you determine the minimum of the function.

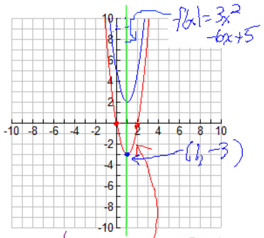
Solution

$g(x) = 3x^2 - 6x$

$\therefore g(x) = 3x(x-2)$

easy to figure out

to avoid completing the square



① opens upward ⟶ min at vertex

② zeros: 0 and 2 ⟶ axis of symmetry is half way between 0 and 2

$3x(x-2) = 0$

$\therefore 3x = 0$ or $x - 2 = 0$

$\therefore x = 0$ or $x = 2$

i.e. $\frac{0+2}{2} = 1$

$\therefore$ x-coordinate of vertex is 1

$\therefore$ y-coordinate of vertex is $g(1) = 3(1)(1-2) = -3$

$$\text{But } f(x) = 3x^2 - 6x + 5 = 3x(x-2) + 5 = g(x) + 5$$

$\therefore$ f is g shifted up 5 units

$\therefore$ min value of f must be

$$\underbrace{-3}_{\text{min of } g} + \underbrace{5}_{\text{shift}} = \underbrace{2}_{\text{min of } f}$$

4. Find the ***maximum or minimum*** value of each quadratic function using ***three different algebraic methods***. Check your answers by using a graphical (geometric) method.

(a) $h(t) = -4.9t^2 + 4.9t + 274.4$ **(b)** $g(y) = 6y^2 - 5y - 25$

Answers

The three different methods are

- Completing the square
- Partial Factoring
- Finding the roots

Regardless of the method used, you should obtain the following answers:

(a) Maximum value is $\frac{2205}{8}$=275.625 (occurs at $t = \frac{1}{2}$)

(b) Minimum value is $-\frac{625}{24}$ (occurs at $y = \frac{5}{12}$)

5. Given the quadratic function $f(x) = (x-3)^2 - 2$ and $g(x) = af(b(x-h)) + k$, describe the effect of each of the following. If you need assistance, you can use a "slider control" in TI-Interactive (to be demonstrated in class).

(a) $a = 0$
(b) $b = 0$
(c) $a > 1$
(d) $a < -1$
(e) $0 < a < 1$
(f) $-1 < a < 0$
(g) $b > 1$
(h) $b < -1$
(i) $0 < b < 1$
(j) $-1 < b < 0$
(k) k is increased
(l) k is decreased
(m) h is increased
(n) h is decreased
(o) $a = 1, b = 1, h = 0, k = 0$
(p) $a = -1, b = -1, h = 0, k = 0$

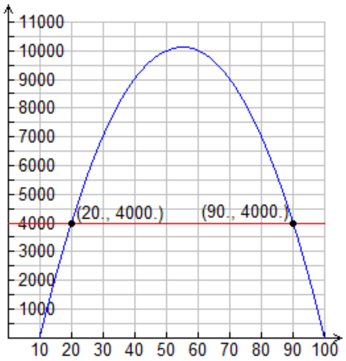
Solution

Use the TI-Interactive file "Sliders.tii" found on the "I:" drive and on www.misternolfi.com. (On the Web site, the required file is called "Sliders.zip." Download the "zip" file to your computer and then extract "Sliders.tii." "Sliders.zip" can be "unzipped" using utilities built into Windows XP and Vista, or by using third party applications such as "WinZip," "WinRar" and "AlZip."

6. The profit, $P(x)$, of a video company, in ***thousands of dollars***, is given by $P(x) = -5x^2 + 550x - 5000$, where x is the amount spent on advertising, in thousands of dollars. Determine the maximum profit that the company can make and the amounts spent on advertising that will result in a profit of at least $4000000.

Solution

First you must recall that x is the independent variable and that $P(x)$, the profit, depends on x, the amount spent on advertising. When you sketch the graph of P, which is a parabola that opens downward, you will note that the maximum value occurs at the vertex. The maximum profit can be found by determining the y-co-ordinate of the vertex. **Answer**: (i) max profit = $10,125,000 (ii) for profit of at least $4,000,000, the amount spent on advertising must be ≥$20,000 and ≤$90,000. This answer can be obtained by carefully sketching and inspecting the graph or by solving the inequality $-5x^2 + 550x - 5000 \ge 4000$.



7. Suppose that a quadratic function $f(x) = ax^2 + bx + c$ has x-intercepts r_1 and r_2. Describe transformations of f that produce quadratic functions with the same x-intercepts. That is, describe transformations of f under which the points $(r_1, 0)$ and $(r_2, 0)$ are ***invariant***.

Solution

See #2.

8. Determine the equation of the quadratic function that passes through (2, 5) if the roots of the corresponding quadratic equation are $1-\sqrt{5}$ and $1+\sqrt{5}$.

Solution

Let f represent the required function. Then, for some non-zero real number a (the vertical stretch factor),

$$f(x) = a\left(x-\left(1-\sqrt{5}\right)\right)\left(x-\left(1+\sqrt{5}\right)\right) = a\left((x-1)+\sqrt{5}\right)\left((x-1)-\sqrt{5}\right) = a\left((x-1)^2 - 5\right)$$

Since the point $(2,5)$ lies on the parabola, $f(2) = 5$. Therefore,

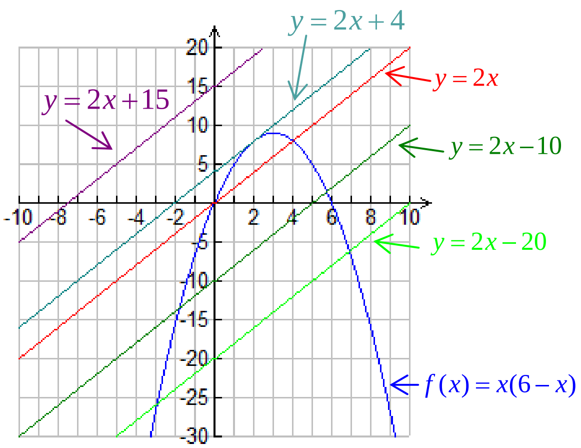
$$a\left((2-1)^2 - 5\right) = 5$$
$$\therefore -4a = 5$$
$$\therefore a = -\frac{5}{4}$$

Thus, $f(x) = -\frac{5}{4}\left((x-1)^2 - 5\right) = -\frac{5}{4}(x-1)^2 + \frac{25}{4}$.

9. Determine, through investigation, the equations of the lines that have a slope of 2 and that intersect the quadratic function $f(x) = x(6-x)$

(a) once **(b)** twice **(c)** never

<table>
<tr><th>Graphical</th><th>Algebraic</th></tr>
<tr><td>



The equation $y = 2x + b$, where $b \in \mathbb{R}$, can be used to describe the family of all lines having a slope of 2. From the graph, ***it appears that*** the following is true:

- one point of intersection: $b = 4$
- two points of intersection: $b < 4$
- no points of intersection: $b > 4$

Remember that graphs do not qualify as mathematical proof. Nonetheless, they are very valuable tools that give us a geometric viewpoint as well as a very good idea of what answers to expect.

</td><td>

To find the point(s) of intersection of $f(x) = x(6-x)$ and $y = 2x + b$, we must solve the equation $x(6-x) = 2x + b$:

$$x(6-x) = 2x + b$$
$$\therefore -x^2 + 6x = 2x + b$$
$$\therefore -x^2 + 4x - b = 0$$
$$\therefore x^2 - 4x + b = 0$$

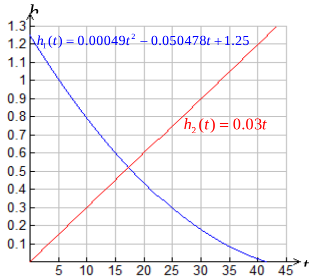
Since we are only interested in determining the number of points of intersection, it suffices to calculate the ***discriminant*** of the quadratic equation $x^2 - 4x + b = 0$.

$$D = (-4)^2 - 4(1)(b) = 16 - 4b$$

- one point of intersection → one real root → $D = 0$
  $\therefore 16 - 4b = 0$
  $\therefore b = 4$
- two points of intersection → two real roots → $D > 0$
  $\therefore 16 - 4b > 0$
  $\therefore -4b > -16$
  $\therefore \frac{-4b}{-4} < \frac{-16}{-4}$
  $\therefore b < 4$
- no points of intersection → no real roots → D < 0
  $\therefore 16 - 4b < 0$
  $\therefore -4b < -16$
  $\therefore \frac{-4b}{-4} > \frac{-16}{-4}$
  $\therefore b > 4$

</td></tr>
</table>

10. At a wild party, some inquisitive MCR3U0 students performed an interesting experiment. They obtained two containers, one of which was filled to the brim with a popular party beverage while the other was empty. The full container was placed on a table and the empty container was placed on the floor right next to it. Then, a hole was poked near the bottom of the first container, which caused the party "liquid" to drain out of the first container and into the other. By carefully collecting and analyzing data, the students determined two functions that modelled how the heights of the liquids (in metres) varied with time (in seconds): $h_1(t) = 0.00049t^2 - 0.050478t + 1.25$ and $h_2(t) = 0.03t$



(a) Which function applies to the container with the hole? Explain.

Solution

Since h_1 is a decreasing function, it must describe the height of the liquid in the container that is being drained.

(b) At what time were the heights of the liquids equal?

Solution

The heights of the liquids are the same when $h_1(t) = h_2(t)$.Therefore,

$\therefore 0.00049t^2 - 0.050478t + 1.25 = 0.03t$

$\therefore 0.00049t^2 - 0.080478t + 1.25 = 0$

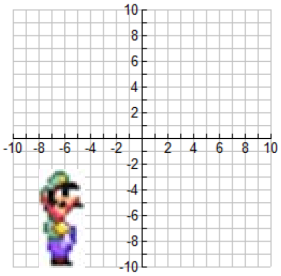
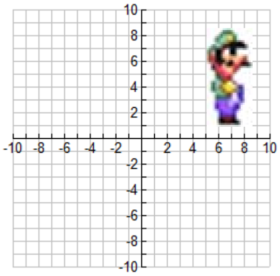
$\therefore t = \dfrac{0.080478 \pm \sqrt{0.080478^2 - 4(0.00049)(1.25)}}{2(0.00049)}$

$\therefore t \doteq 17.4$ or $t \doteq 146.8$

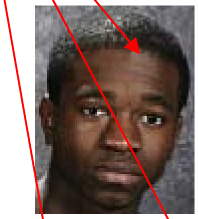
Therefore, the heights of the liquids are the same at about 17.4s.

(c) Explain why the height of the liquid in one container varied linearly with time while the height in the other container varied quadratically with time.

11. Video games depend heavily on transformations. The following table gives a few examples of simple transformations you might see while playing Super Mario Bros. Complete the table.

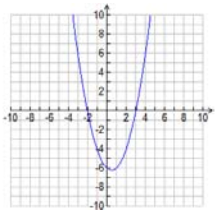
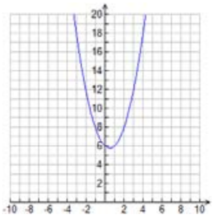
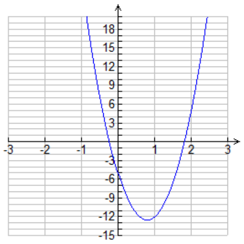
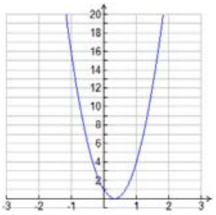
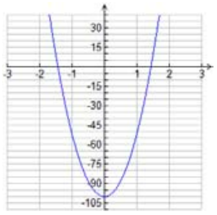
Before Transformation	*After Transformation*	*Nature of the Transformation*	*Use in the Video Game*
		Reflection in the vertical line that divides Luigi's body in half	Change direction from left to right
		Reflection in the ***horizontal line*** that divides green creature's body in half	Kill the character. (Other answers are possible)
		Combination of some horizontal translation and some vertical translation. (Other answers are also possible but this is the simplest.)	Move from one place to another.

12. Complete the following table.

Before Transformation	*After Transformation*	*Nature of the Transformation*
		Horizontal compression by a factor of ½.
		Vertical compression by a factor of ½. Reflection in the *y*-axis.
		Vertical stretch by a factor of 3/2. Reflection in the *x*-axis. Horizontal compression by a factor of 1/3.
		Vertical compression by a factor of ½. Reflection in the *x*-axis. Reflection in the *y*-axis.

Found Guilty of being Physically Present but Mentally Absent!

13. For each quadratic function, determine the number of zeros (*x*-intercepts) using the methods listed in the table.

Quadratic Function	*Graph*	*Discriminant* $D = b^2 - 4ac$	*Factored Form*	*Conclusion – Number of Zeros*
$f(x) = x^2 - x - 6$		$a = 1, b = -1, c = -6$ $\therefore D = b^2 - 4ac$ $= (-1)^2 - 4(1)(-6)$ $= 1 + 24$ $= 25$	$x^2 - x - 6 = 0$ $\therefore (x-3)(x+2) = 0$ $\therefore x - 3 = 0$ or $x + 2 = 0$ $\therefore x = 3$ or $x = -2$	- Since there are 2 *x*-intercepts, *f* has two zeros. - Since $D > 0$, *f* has two zeros. - Since $x^2 - x - 6 = 0$ has two solutions, *f* has two zeros.
$g(t) = t^2 - t + 6$		$a = 1, b = -1, c = 6$ $\therefore D = b^2 - 4ac$ $= (-1)^2 - 4(1)(6)$ $= 1 - 24$ $= -23$	Since *D* is not a perfect square, this quadratic does not factor over the integers.	- Since there are 2 *x*-intercepts, *g* has two zeros. - Since $D < 0$, *g* has no zeros.
$P(t) = 12t^2 - 19t - 5$		$a = 12, b = -19, c = -5$ $\therefore D = b^2 - 4ac$ $= (-19)^2 - 4(12)(-5)$ $= 361 + 240$ $= 601$	Since *D* is not a perfect square, this quadratic does not factor over the integers.	- Since there are 2 *x*-intercepts, *P* has two zeros. - Since $D > 0$, *P* has two zeros.
$q(z) = 9z^2 - 6z + 1$		$a = 9, b = -6, c = 1$ $\therefore D = b^2 - 4ac$ $= (-6)^2 - 4(9)(1)$ $= 36 - 36$ $= 0$	$9z^2 - 6z + 1 = 0$ $\therefore (3z-1)(3z-1) = 0$ $\therefore (3z-1)^2 = 0$ $\therefore 3z - 1 = 0$ $\therefore z = \frac{1}{3}$	- Since there is 1 *x*-intercept, *f* has one root. - Since $D > 0$, *f* has two zeros. - Since $9z^2 - 6z + 1 = 0$ has one solution, *q* has one zero.
$f(x) = 49x^2 - 100$		$a = 49, b = 0, c = -100$ $\therefore D = b^2 - 4ac$ $= (0)^2 - 4(49)(-100)$ $= 19600$	$49x^2 - 100 = 0$ $\therefore (7x-10)(7x+10) = 0$ $\therefore 7x - 10 = 0$ or $7x + 10 = 0$ $\therefore x = \frac{7}{10}$ or $x = -\frac{7}{10}$	- Since there are 2 *x*-intercepts, *f* has two zeros. - Since $D > 0$, *f* has two zeros. - Since $49x^2 - 100 = 0$ has two solutions, *f* has two zeros.

14. Use the quadratic formula to explain how the discriminant allows us to predict the number of roots of a quadratic equation.

Solution

If $ax^2+bx+c=0$, then by completing the square we can deduce that $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

(a) If $b^2-4ac<0$, then $\sqrt{b^2-4ac}$ ***is not*** a real number because it evaluates to the square root of a negative number. Therefore, if $b^2-4ac<0$, there are ***no real roots***.

(b) If $b^2-4ac=0$, then $x=\dfrac{-b\pm\sqrt{0}}{2a}=\dfrac{-b\pm\sqrt{0}}{2a}=\dfrac{-b\pm 0}{2a}=\dfrac{-b}{2a}$. Thus, there is ***one real root***, $x=\dfrac{-b}{2a}$.

(c) If $b^2-4ac>0$, then $\sqrt{b^2-4ac}$ ***is*** a real number. Therefore, there are ***two real roots***, $x=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $x=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

15. How can the discriminant be used to predict whether a quadratic function can be factored? Explain.

Solution

If $D=b^2-4ac$ is a ***perfect square whole number*** (i.e. 1, 4, 9, 16, 25, 36, 49, …) then $\sqrt{b^2-4ac}$ will be a whole number. This means that both roots of the quadratic function will be ***rational*** (i.e. either an integer or a fraction). In this case, the quadratic will factor over the integers.

e.g. If $f(x)=6x^2+11x-35$, then $D=b^2-4ac=11^2-4(6)(-35)=961=31^2$. Since D is a perfect square, f will factor over the integers. If we solve the equation $f(x)=0$, we obtain $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{-11\pm\sqrt{961}}{2(6)}=\dfrac{-11\pm 31}{2(6)}$.

Thus, $x=\dfrac{-11+31}{2(6)}=\dfrac{20}{12}=\dfrac{5}{3}$ or $x=\dfrac{-11-31}{2(6)}=-\dfrac{42}{12}=-\dfrac{7}{2}$. Using the roots we can determine that the factored form is $f(x)=6x^2+11x-35=(3x-5)(2x+7)$.

Note that if $D=b^2-4ac$ is a ***perfect square rational number*** (e.g. $\left(\dfrac{1}{2}\right)^2=\dfrac{1}{4}$, $\left(\dfrac{2}{3}\right)^2=\dfrac{4}{9}$,…), the above argument can be repeated. Therefore, a quadratic function can be factored over the integers if and only if the discriminant is a perfect square rational number.

16. The following table lists the approximate accelerations due to gravity near the surface of the Earth, moon and sun.

Earth	***Jupiter***	***Saturn***
9.87 m/s^2	25.95 m/s^2	11.08 m/s^2

The data in the above table lead to the following equations for the height of an object dropped near the surface of each of the celestial bodies given above. In each case, $h(t)$ represents the height, in metres, of an object above the surface of the body t seconds after it is dropped from an initial height h_0.

Earth	***Jupiter***	***Saturn***
$h(t)=-4.94t^2+h_0$	$h(t)=-12.97t^2+h_0$	$h(t)=-5.54t^2+h_0$

In questions (a) to (d), use an initial height of 100 m for the Earth, 200 m for Saturn and 300 m for Jupiter.

(a) On the same grid, sketch each function.

(b) Explain how the "Jupiter function" can be transformed into the "Saturn function."

(c) Consider the graphs for Jupiter and Saturn. Explain the ***physical meaning*** of the point(s) of intersection of the two graphs.

(d) State the domain and range of each function. Keep in mind that each function is used to ***model*** a physical situation, which means that the allowable values of t are restricted.

Solution

See "Super Skills Review" for a similar problem.

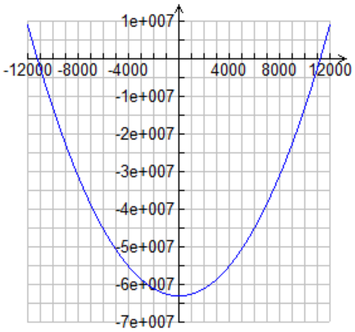
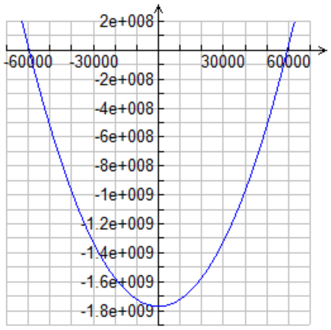
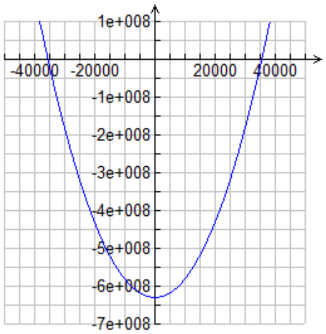
17. Have you ever wondered why an object that is thrown up into the air always falls back to the ground? Essentially, this happens because the kinetic energy of the object (energy of motion) is less than its potential energy (the energy that the Earth's gravitational field imparts to the object). As long as an object's kinetic energy is less than its potential energy, it will either fall back to the ground or remain bound in a closed orbit around the Earth. On the other hand, if an object's kinetic energy exceeds its potential energy, then it can break free from the Earth's gravitational field and escape into space.

The function E_Δ, defined by the equation $E_\Delta(v)=\frac{mv^2}{2}-\frac{GMm}{r}=m\left(\frac{v^2}{2}-\frac{GM}{r}\right)$, gives the difference between the kinetic energy and potential energy of an object of mass *m* moving with velocity *v* in the gravitational field of a body of mass *M* and radius *r*. In addition, *G* represents the universal gravitational constant and is equal to 6.67429×10^{-11} m^3 kg^{-1} s^{-2}.

(a) Use the data in the table to determine E_Δ for the Earth, Jupiter and Saturn for an object with a mass of 1 kg.

Planet	*Radius (m)*	*Mass (kg)*	E_Δ
Earth	6.38×10^6	5.98×10^{24}	$E_\Delta(v)=\frac{1}{2}v^2-6.23\times10^7$
Jupiter	7.15×10^7	1.90×10^{27}	$E_\Delta(v)=\frac{1}{2}v^2-1.77\times10^9$
Saturn	6.03×10^7	5.68×10^{26}	$E_\Delta(v)=\frac{1}{2}v^2-6.29\times10^8$

(b) Sketch the graph of E_Δ for the Earth, Jupiter and Saturn (for an object of a mass of 1 kg).

Earth	*Jupiter*	*Saturn*
		

(c) An object can escape a body's gravitational field if its kinetic energy exceeds its potential energy. Using the graphs from part (b), determine the escape velocity for each of the given planets.

Solution

For an object to escape the gravitational field of a body, its kinetic energy must be greater than or equal to its potential energy. Therefore, we must find out when $E_\Delta(v)>=0$. By solving $E_\Delta(v)=0$ and observing the graphs, we can determine the values of *v* that satisfy this condition.

$\frac{1}{2}v^2-6.23\times10^7=0$	$\frac{1}{2}v^2-1.77\times10^9=0$	$\frac{1}{2}v^2-6.29\times10^8=0$
$\therefore\frac{1}{2}v^2=6.23\times10^7$	$\therefore\frac{1}{2}v^2=1.77\times10^9$	$\therefore\frac{1}{2}v^2=6.29\times10^8$
$\therefore v^2=1.246\times10^8$	$\therefore v^2=3.54\times10^9$	$\therefore v^2=1.258\times10^9$
$\therefore v=\sqrt{1.246\times10^8}$	$\therefore v=\sqrt{3.54\times10^9}$	$\therefore v=\sqrt{1.258\times10^9}$
$\therefore v\doteq11200$	$\therefore v\doteq59500$	$\therefore v\doteq35500$

Earth: 11200 m/s = 11.2 km/s **Jupiter**: 59500 m/s = 59.5 km/s **Saturn**: 35500 m/s = 35.5 km/s

(d) Does the escape velocity of an object depend on its mass?

When solving $E_\Delta(v)=m\left(\frac{v^2}{2}-\frac{GM}{r}\right)=0$, the roots are independent of *m*. Therefore, escape velocity does not depend on the mass of the object. Here, *m* plays the role of the vertical stretch factor, which as we know, does not affect the roots.

18. Suppose that $f(x)=x^2-5x$ and that $g(x)=2f^{-1}(\tfrac{1}{3}x-3)+1=2f^{-1}(\tfrac{1}{3}(x-9))+1$.

(a) The following table lists the transformations, in mapping notation, applied to *f* to obtain *g*. Give a verbal description of each transformation.

	Mapping Notation	***Verbal Description***
Vertical	$(x,y)\to(x,2y+1)$	Vertical stretch by a factor of 2 followed by a shift up 1 unit.
Horizontal	$(x,y)\to(3x+9,y)$	Horizontal stretch by a factor of 3 followed by a shift right 9 units.
Other	$(x,y)\to(y,x)$	Reflection in the line *y* = *x*.

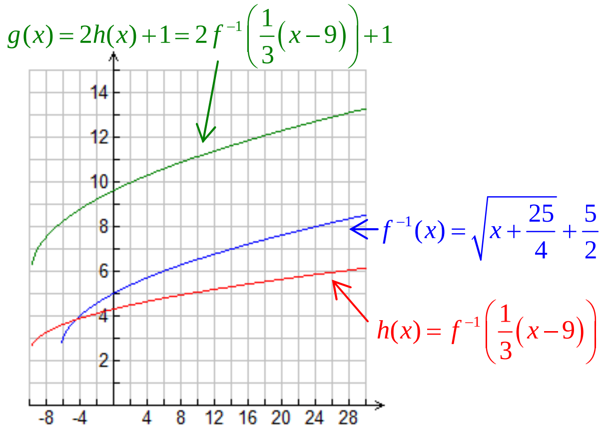
(b) As you already know, horizontal and vertical transformations are independent of each other, which means that the results are the same regardless of the order in which they are applied. Does it matter at what point in the process the transformation $(x,y)\to(y,x)$ is applied? Explain.

Explanation

The horizontal and vertical transformations are applied to f^{-1} not *f*. Therefore the reflection in the line *y* = *x*, $(x,y)\to(y,x)$, should be performed ***first***. After that, the others can be performed in any order. In other words, we need to find the equation of f^{-1}. By applying an appropriate method, it can be shown that

$$f^{-1}(x)=\sqrt{x+\frac{25}{4}}+\frac{5}{2}.$$

(c) Sketch the graph of *g*.



(d) State an equation of *g*.

Solution

$$f^{-1}(x)=\sqrt{x+\frac{25}{4}}+\frac{5}{2}$$

$$\therefore g(x)=2f^{-1}\left(\frac{1}{3}(x-9)\right)+1$$

$$=2\left(\sqrt{\frac{1}{3}(x-9)+\frac{25}{4}}+\frac{5}{2}\right)+1$$

$$=2\left(\sqrt{\frac{1}{3}x+\frac{13}{4}}+\frac{5}{2}\right)+1$$

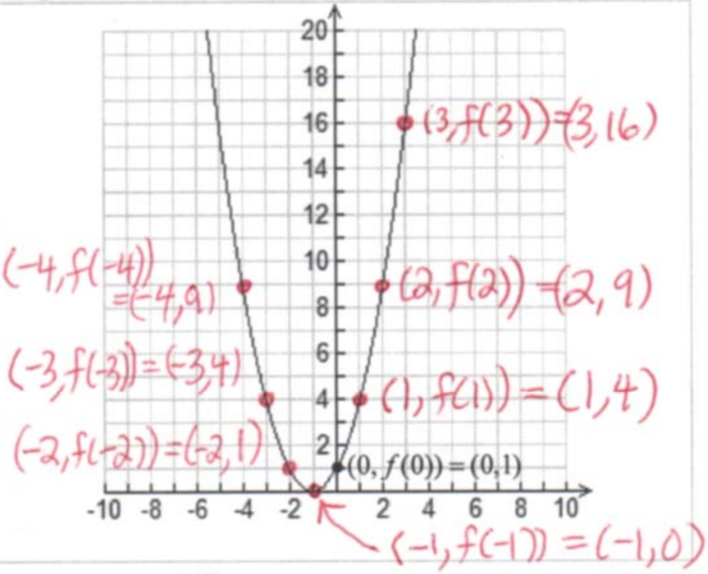
$$=2\sqrt{\frac{1}{3}x+\frac{13}{4}}+6$$

Summary of Problem Solving Strategies used in this Section

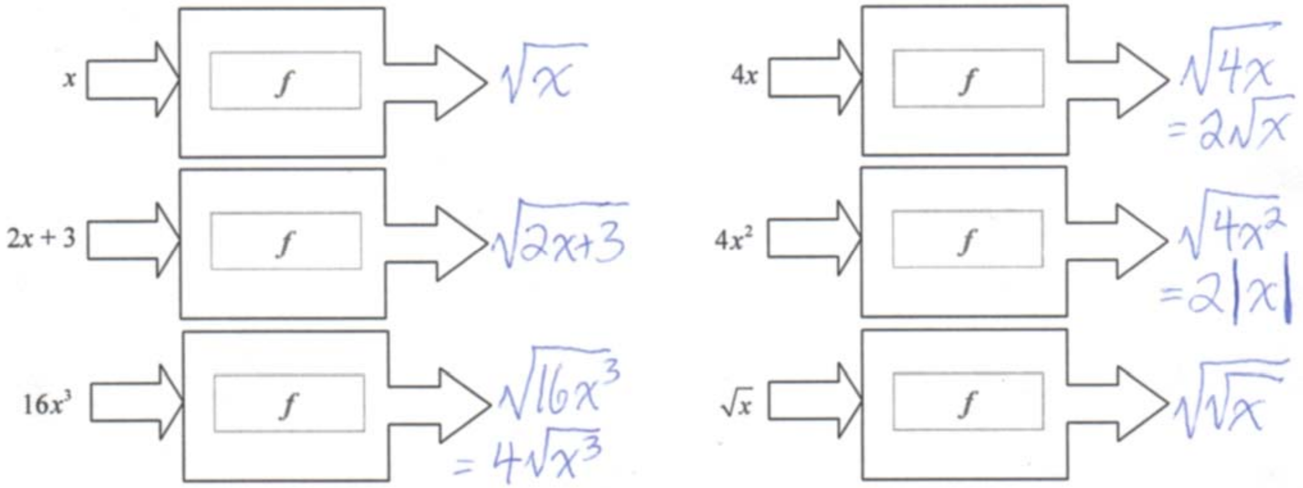
Problem	*Strategies*
Determine the number of zeros of a quadratic function. (This is equivalent to finding the number of *x*-intercepts.)	**1.** Factor **2.** Quadratic Formula **3.** Complete the Square
Predict whether a quadratic function can be factored.	The discriminant must be a perfect square
Find the maximum or minimum value of a quadratic function. (This is equivalent to finding the vertex of the corresponding parabola.)	**1.** Partial Factoring **2.** Complete the Square **3.** Evaluate the average of the roots, then use this "*x*-value" to find the "*y*-value."
Find the equation of a quadratic function that passes through a given point and whose *x*-intercepts are the same as the roots of a corresponding quadratic equation.	Suppose that the roots of the quadratic function *f* are r_1 and r_2 and the point (m,n) lies on the corresponding parabola. Therefore, for some non-zero real number *a* (the vertical stretch factor) $f(x)=a(x-r_1)(x-r_2)$ and $f(m)=n$. Consequently, $a(m-r_1)(m-r_2)=n$. Then solve for *a*.
Intersection of a Linear Function and a Quadratic Function	Quadratic Function: $y=ax^2+b_1x+c$ Linear Function: $y=mx+b$ Solve the equation $ax^2+b_1x+c=mx+b$ Note that the symbol b_1 is used to distinguish the coefficient of *x* in the quadratic function from the *y*-intercept of the linear function.

Super Skills Review

1. Complete the following table for $f(x)=(x+1)^2$. The first row is done for you.

Evaluate	*Ordered Pair* $(x, f(x))$	*Graph – Mark the Ordered Pairs on the Graph*
$f(0)=(0+1)^2=1$	$(0,f(0))=(0,1)$	
$f(-1)=(-1+1)^2=0$	$(-1,f(-1))=(-1,0)$	
$f(1)=(1+1)^2=4$	$(1,f(1))=(1,4)$	
$f(-2)=(-2+1)^2=1$	$(-2,f(-2))=(-2,1)$	
$f(2)=(2+1)^2=9$	$(2,f(2))=(2,9)$	
$f(-3)=(-3+1)^2=4$	$(-3,f(-3))=(-3,4)$	
$f(3)=(3+1)^2=16$	$(3,f(3))=(3,16)$	
$f(-4)=(-4+1)^2=9$	$(-4,f(4))=(-4,9)$	
$f(4)=(4+1)^2=25$	$(4,f(4))=(4,25)$	

2. Complete the following function machine diagrams for the function $f(x)=\sqrt{x}$. Simplify if possible.



x → f → $\sqrt{x}$

$4x$ → f → $\sqrt{4x}$
$=2\sqrt{x}$

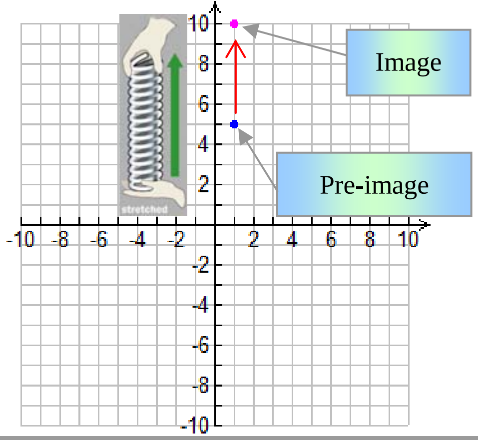
$2x+3$ → f → $\sqrt{2x+3}$

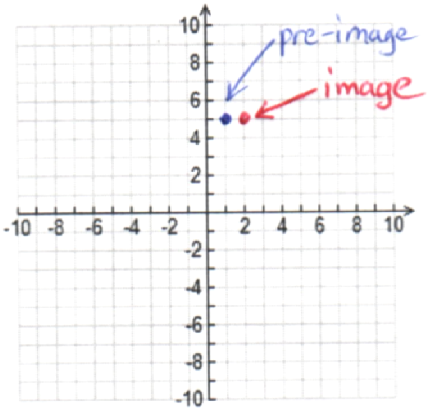
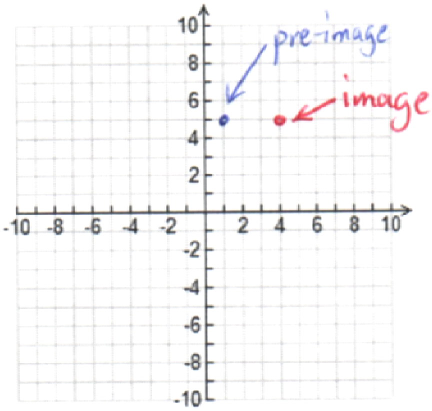
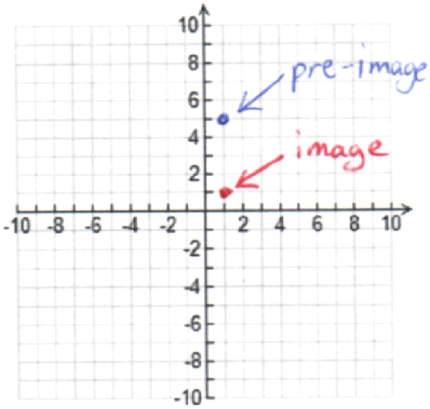
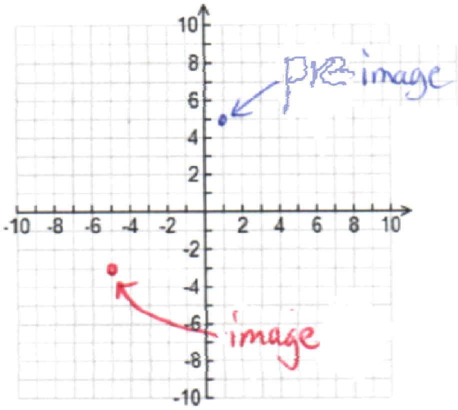
$4x^2$ → f → $\sqrt{4x^2}$
$=2|x|$

$16x^3$ → f → $\sqrt{16x^3}$
$=4\sqrt{x^3}$

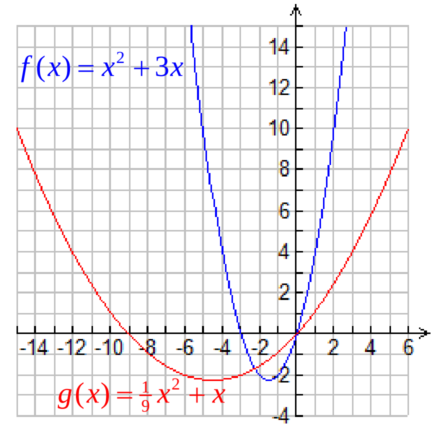
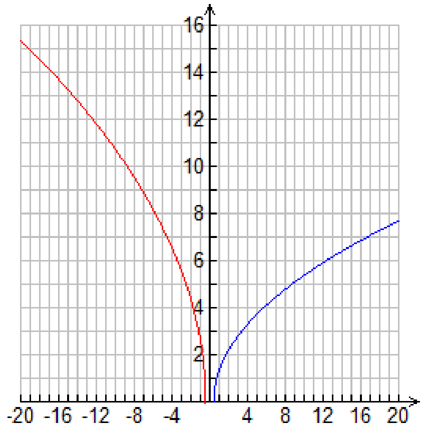
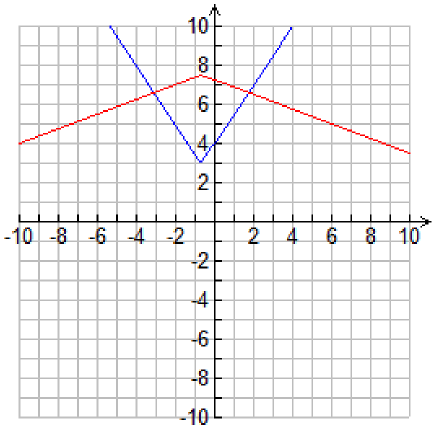
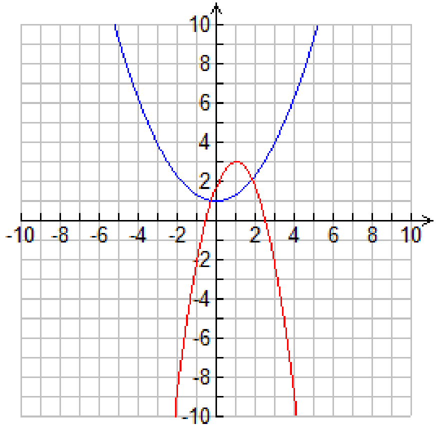
$\sqrt{x}$ → f → $\sqrt{\sqrt{x}}$

3. Complete the following table. The first row is done for you.

Pre-image	***Transformation***	***Transformation in Mapping Notation***	***Image***	***Graph***
$(1,5)$	Stretch vertically by a factor of 2.	$(x,y)\rightarrow(x,2y)$	$(1,10)$	

Pre-image	*Transformation*	*Transformation in Mapping Notation*	*Image*	*Graph*
(1,5)	Stretch horizontally by a factor of 2.	$(x,y) \to (2x,y)$	(2,5)	
(1,5)	Translate to the right 3 units.	$(x,y) \to (x+3,y)$	(4,5)	
(1,5)	Translate down 4 units.	$(x,y) \to (x,y-4)$	(1,1)	
(1,5)	*Horizontal* 1. Stretch by a factor of –2. 2. Translate 3 units left *Vertical* 1. Reflect in the *x*-axis. 2. Translate up 2 units.	$(x,y) \to (-2x-3,-y+2)$	(-5,-3)	

4. Complete the following table. The first row is done for you.

Equation of Pre-image Function	*Transformation*	*Equation of Image Function*	*Graph*
$f(x)=x^2+3x$	***Verbal*** Stretch horizontally by a factor of 3. ***Function Notation*** $g(x)=f(\frac{1}{3}x)$ ***Mapping Notation*** $(x,y)\rightarrow(3x,y)$	$g(x)=f(\frac{1}{3}x)$ $=(\frac{1}{3}x)^2+3(\frac{1}{3}x)$ $=\frac{1}{9}x^2+x$	
$f(x)=\sqrt{3x-1}$	***Verbal*** Horizontal: Reflect in y-axis Vertical: Stretch by a factor of 2 ***Function Notation*** $g(x)=2f(-x)$ ***Mapping Notation*** $(x,y)\rightarrow(-x,2y)$	$g(x)=2f(-x)$ $=2\sqrt{3(-x)-1}$ $=2\sqrt{-3x-1}$	
$f(x)=\lvert\frac{3}{2}x+1\rvert+3$	***Verbal*** Compress vertically by a factor of $\frac{1}{2}$ and reflect in the x-axis then shift up 9 units. Stretch horizontally by a factor of 2, reflect in y-axis, shift left 2 units ***Function Notation*** $g(x)=-\frac{1}{2}f(-\frac{1}{2}x-1)+9$ $=-\frac{1}{2}f(-\frac{1}{2}(x+2))+9$ ***Mapping Notation*** $(x,y)\rightarrow(-2x-2,-\frac{1}{2}y+9)$	$g(x)=-\frac{1}{2}f(-\frac{1}{2}x-1)+9$ $=-\frac{1}{2}(\lvert\frac{3}{2}(-\frac{1}{2}x-1)+1\rvert+3)+9$ $=-\frac{1}{2}\lvert-\frac{3}{4}x-\frac{3}{2}+1\rvert-\frac{3}{2}+9$ $=-\frac{1}{2}\lvert-\frac{3}{4}x-\frac{1}{2}\rvert+\frac{15}{2}$	
$f(x)=\frac{1}{3}x^2+1$	***Verbal*** Reflect in the x-axis, then shift up 4 units. Compress horizontally by a factor of 0.5, then shift right 1 unit. ***Function Notation*** $g(x)=-f(2(x-1))+4$ ***Mapping Notation*** $(x,y)\rightarrow(0.5x+1,-1y+4)$	$g(x)=-f(2(x-1))+4$ $=-[\frac{1}{3}[2(x-1)]^2+1]+4$ $=-[\frac{4}{3}(x-1)^2+1]+4$ $=-\frac{4}{3}(x-1)^2+3$	

5. Complete the following table. The first row is done for you.

Pre-image Function	***Image Function***	***Horizontal Stretch or Compression that produces the Image function***	***Vertical Stretch or Compression that produces the Image function***
$f(x)=x^2$	$g(x)=\frac{1}{3}x^2$	Horizontal stretch by a factor of $\sqrt{3}$. $g(x)=f\left(\frac{1}{\sqrt{3}}x\right)$ $(x,y)\rightarrow(\sqrt{3}x,y)$	Vertical compression by a factor of 1/3. $g(x)=\frac{1}{3}f(x)$ $(x,y)\rightarrow(x,\frac{1}{3}y)$
$f(x)=\lvert x\rvert$	$g(x)=\lvert 3x\rvert$	Horizontal Compression by a factor of $\frac{1}{3}$ $g(x)=f(3x)$ $(x,y)\longrightarrow(\frac{1}{3}x,y)$	Vertical Stretch by a Factor of 3 $g(x)=\lvert 3x\rvert=3\lvert x\rvert=3f(x)$ $(x,y)\longrightarrow(x,3y)$
$f(x)=x^3$	$g(x)=125x^3$	Horizontal compression by a factor of $\frac{1}{5}$ $g(x)=125x^3=(5x)^3=f(5x)$ $(x,y)\longrightarrow(\frac{1}{5}x,y)$	Vertical stretch by a factor of 125 $g(x)=125f(x)$ $(x,y)\rightarrow(x,125y)$
$f(x)=\sqrt{x}$	$g(x)=\sqrt{10x}$	Horizontal compression by a factor of $\frac{1}{10}$ $g(x)=f(10x)$ $(x,y)\rightarrow(\frac{1}{10}x,y)$	Vertical stretch by a factor of $\sqrt{10}$ $g(x)=\sqrt{10x}=\sqrt{10}\sqrt{x}=\sqrt{10}f(x)$ $(x,y)\rightarrow(x,\sqrt{10}y)$

6. What conclusions can you draw from the table in question 5?

We can conclude that in each case considered, it was possible to produce the image function by using both vertical and horizontal stretches or compression. In addition, it appears that vertical stretches can also be viewed as horizontal compressions and vice versa

7. The following table lists the approximate accelerations due to gravity near the surface of the Earth, moon and sun.

Earth	***Moon***	***Sun***
9.87 m/s^2	1.62 m/s^2	2.74×10^2 m/s^2

The data in the above table lead to the following equations for the height of an object dropped near the surface of each of the celestial bodies given above. In each case, $h(t)$ represents the height, in metres, of an object above the surface of the body t seconds after it is dropped from an initial height h_0.

Earth	***Moon***	***Sun***
$h(t)=-4.94t^2+h_0$	$h(t)=-0.81t^2+h_0$	$h(t)=-1.37\times10^2t^2+h_0$

In questions (a) to (d), use an initial height of 100 m for the Earth, 25 m for the moon and 1000 m for the sun.

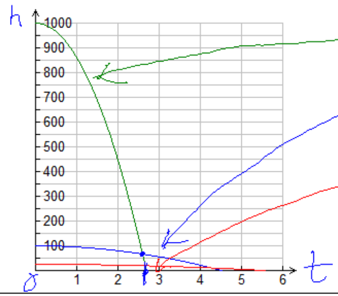
(a) On the same grid, sketch each function.

(b) Explain how the "moon function" can be transformed into the "Earth function."

(c) Consider the graphs for the Earth and the sun. Explain the ***physical meaning*** of the point(s) of intersection of the two graphs.

(d) State the domain and range of each function. Keep in mind that each function is used to ***model*** a physical situation, which means that the allowable values of t are highly restricted.

(a)



$h_1(t) = -1.37\times10^2 t^2 + 1000$

$h_2(t) = -4.94t^2 + 100$

$h_3(t) = -0.81t^2 + 25$

$\frac{-4.94}{-0.81} \doteq 6.1$

(b) $h_3 \longrightarrow h_2$

$\boxed{a \doteq 6.1 \\ k \doteq -52.5}$ Conjecture (Guess)

Stretch h_3 by a factor of 6.1

$6.1\,h_3(t) -52.5 \doteq 6.1(-0.81t^2 + 25) - 52.5$

$= -4.94t^2 + 152.5 - 52.5$

$= -4.94t^2 + 100$

$$\boxed{h_2(t) \doteq 6.1h_3(t) - 52.5}$$

(c) The point of intersection tells us the time at which both objects have the same height.

(d) Sun: $h(t) = -1.37\times10^2 t^2 + 1000$

$D \doteq \{t \in \mathbb{R} \mid 0 \leq t \leq 2.7\}$

$R = \{h \in \mathbb{R} \mid 0 \leq h \leq 1000\}$

For domain, we need to find t at which $h = 0$

$\therefore -1.37\times10^2 t^2 + 1000 = 0$

$\therefore -1.37\times10^2 t^2 = -1000$

$\therefore t^2 = \frac{+1000}{+1.37\times10^2}$

$\therefore t = \sqrt{\frac{+1000}{+1.37\times10^2}} \doteq 2.7$

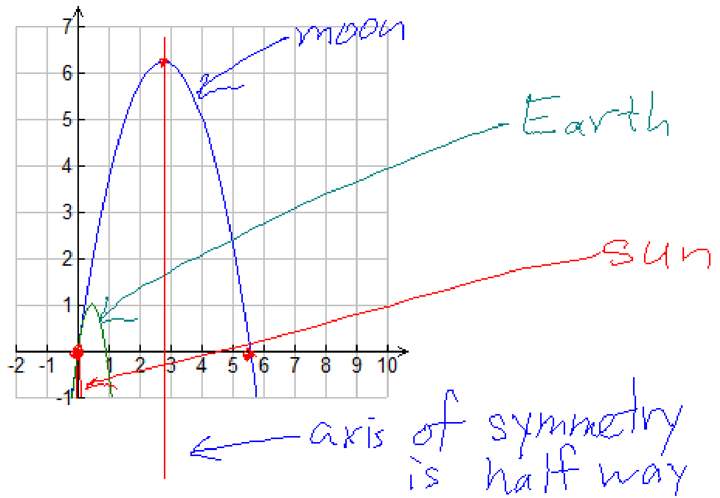
(e) Let a represent acceleration due to gravity, v_0 represent initial velocity, t represent time and $h(t)$ represent height above the "ground" at time t. Then, the function $h(t) = -\frac{a}{2}t^2 + v_0 t$ can be used to describe the height above the "ground" of a person who jumps up from the surface of a body, at a time t after the initial jump. A typical human can jump vertically with an initial velocity of about 4.5 m/s, which on the Earth would result in a jump about 1 m high. How high would a typical human be able to jump on the moon? If the surface of the sun were solid, how high would a typical human be able to jump on the sun?

$$h_m(t) = -0.81t^2 + 4.5t$$

$$h_s(t) = -1.37\times10^2 t^2 + 4.5t$$

m = moon

s = sun



axis of symmetry is half way between roots

Moon

To find the max height reached

Method 1

Find vertex by completing the square

Method 2

(a) Find roots

(b) Calculate value half way between roots ("t-value" for axis of symmetry)

(c) Substitute this value into the equation