Table of Contents - Unit 2 – Trigonometric Functions

Essential Concepts of Trigonometry

The Canadarm2 robotic manipulator on the International Space Station is operated by controlling the angles of its joints. Calculating the final position of the astronaut at the end of the arm requires repeated use of the trigonometric functions of those angles.

What is Trigonometry?

Trigonometry (Greek *trigōnon* "triangle" + *metron* "measure") is a branch of mathematics that deals with the relationships among the interior angles and side lengths of triangles, as well as with the study of trigonometric functions. Although the word "trigonometry" emerged in the mathematical literature only about 500 years ago, the origins of the subject can be traced back more than 4000 years to the ancient civilizations of Egypt, Mesopotamia and the Indus Valley. Trigonometry has evolved into its present form through important contributions made by, among others, the Greek, Chinese, Indian, Sinhalese, Persian and European civilizations.

Why Triangles?

Triangles are the basic building blocks from which any shape (with straight boundaries) can be constructed. A square, pentagon or any other polygon can be divided into triangles, for instance, using straight lines that radiate from one vertex to all the others.

Examples of Problems that can be solved using Trigonometry

- ☺ How tall is Mount Everest? How tall is the CN Tower?
- ☺ What is the distance from the Earth to the sun? How far is the Alpha Centauri star system from the Earth?
- ☺ What is the diameter of Mars? What is the diameter of the sun?
- ☺ At what times of the day will the tide come in?

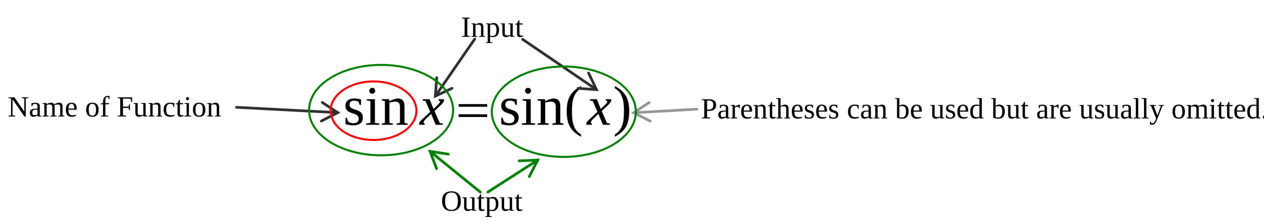
General Applications of Trigonometric Functions

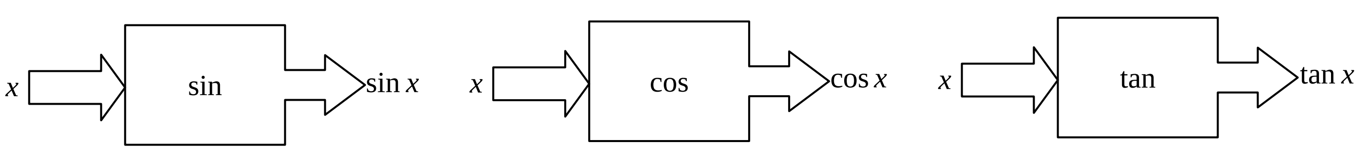
Trigonometry is one of the most widely applied branches of mathematics. The following is just a small sample of its myriad uses.

Application	*Examples*
Modelling of ***cyclic*** (periodic) processes	Orbits (of planets, moons, etc.) Hours of Daylight Tides
Measurement	Navigation Engineering Construction Surveying
Electronics	Circuit Analysis (Modelling of Voltage Versus Time in AC Circuits, Fourier Analysis, etc)

Extremely Important Prerequisite Knowledge

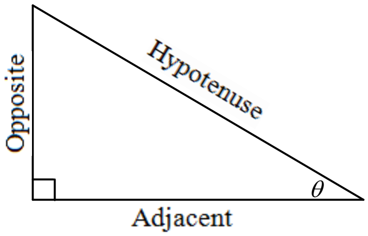
Before tackling the details of trigonometric functions, it is important to review the trigonometry that you learned in grade 10. However, now that you have a solid foundation in functions, the following should be noted.





Trigonometry of Right Triangles – Trigonometric Ratios of Acute Angles

Right triangles can be used to define the trigonometric ratios of ***acute*** angles (angles that measure less than 90º).



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

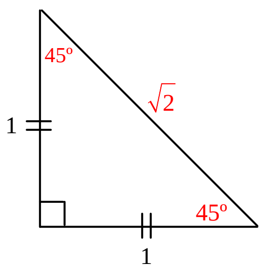
SOH CAH TOA

"Shout Out Hey" "Canadians Are Hot" "Tight Oiled Abs"

Have fun by creating your own mnemonic!

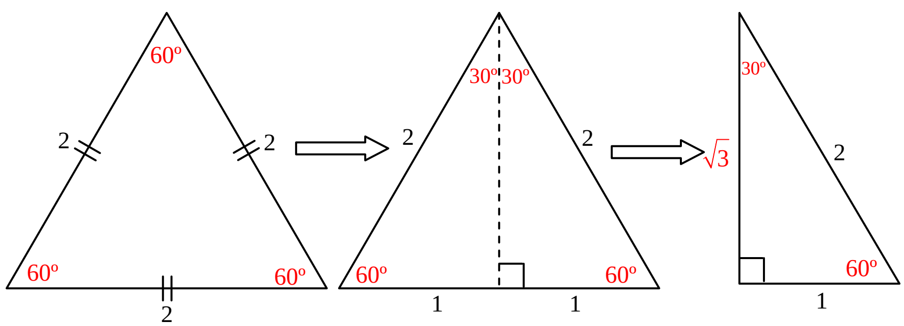
The Special Triangles – Trigonometric Ratios of Special Angles

For certain ***special angles***, it is possible to calculate the ***exact value*** of the trigonometric ratios. As I have mentioned on many occasions, it is not advisable to memorize blindly! Instead, you can deduce the values that you need to calculate the trig ratios by ***understanding*** the following triangles!



- Isosceles Right Triangle
- Let the length of the equal sides be 1 unit.
- By the Pythagorean Theorem, the length of the hypotenuse must be $\sqrt{2}$.

$\sin 45° = \frac{1}{\sqrt{2}}$ $\cos 45° = \frac{1}{\sqrt{2}}$ $\tan 45° = \frac{1}{1} = 1$



- Begin with an equilateral triangle having sides of length 2 units. Then cut it in half to form a 30º-60º-90º right triangle.
- Use the Pythagorean Theorem to calculate the height of the triangle. ($\sqrt{3}$)

$\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}}$

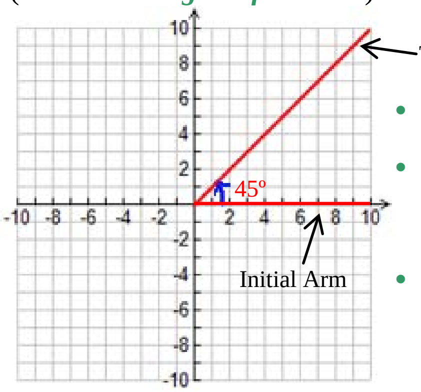
$\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$ $\tan 60° = \sqrt{3}$

Homework

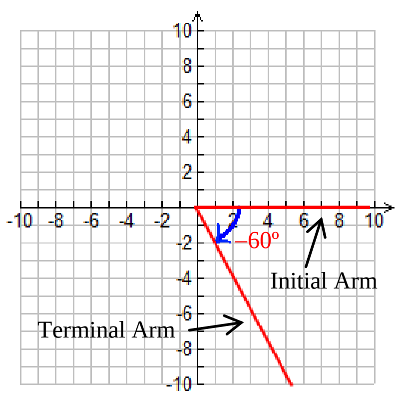
pp. 272 – 275, #3, 4, 9, 10, 11, 13, 15, 19

Trigonometric Ratios of Angles of Revolution (Rotation) – Trigonometric Ratios of Angles of Any Size

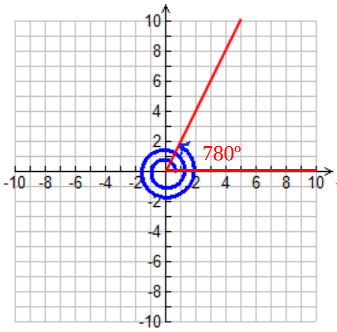
We can extend the idea of trigonometric ratios to angles of any size by introducing the concept of ***angles of revolution*** (also called ***angles of rotation***).

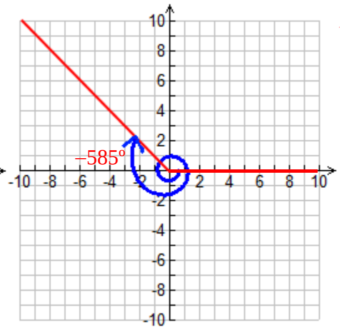


- A ***positive*** angle of revolution (rotation) in ***standard form***.
- "Standard form" means that the ***initial arm*** of the angle lies on the positive *x*-axis and the ***vertex*** of the angle is at the origin.
- A positive angle results from a ***counter-clockwise*** revolution. (The British say "anti-clockwise.")



- A ***negative*** angle of revolution (rotation) in ***standard form***.
- "Standard form" means that the ***initial arm*** of the angle lies on the positive *x*-axis and the ***vertex*** of the angle is at the origin.
- A negative angle results from a ***clockwise*** revolution.

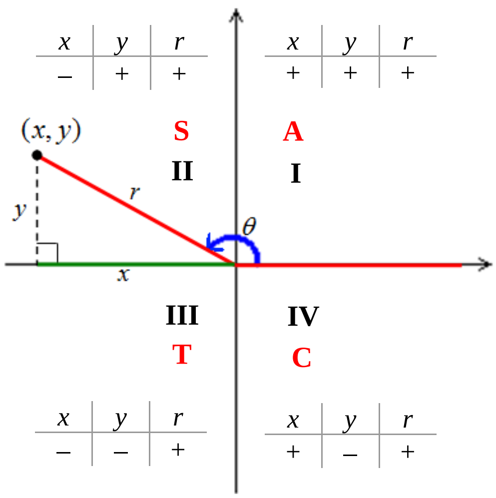




Why Angles of Revolution?

To describe motion that involves moving from one place to another, it makes sense to use units of distance. For instance, it is easy to find your destination if you are told that you need to move 2 km north and 1 km west of your current position.

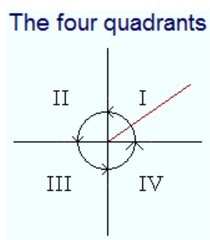
Consider a spinning figure skater. It does not make sense to describe his/her motion using units of distance because he/she is fixed in one spot and rotating. However, it is very easy to describe the motion through angles of rotation.



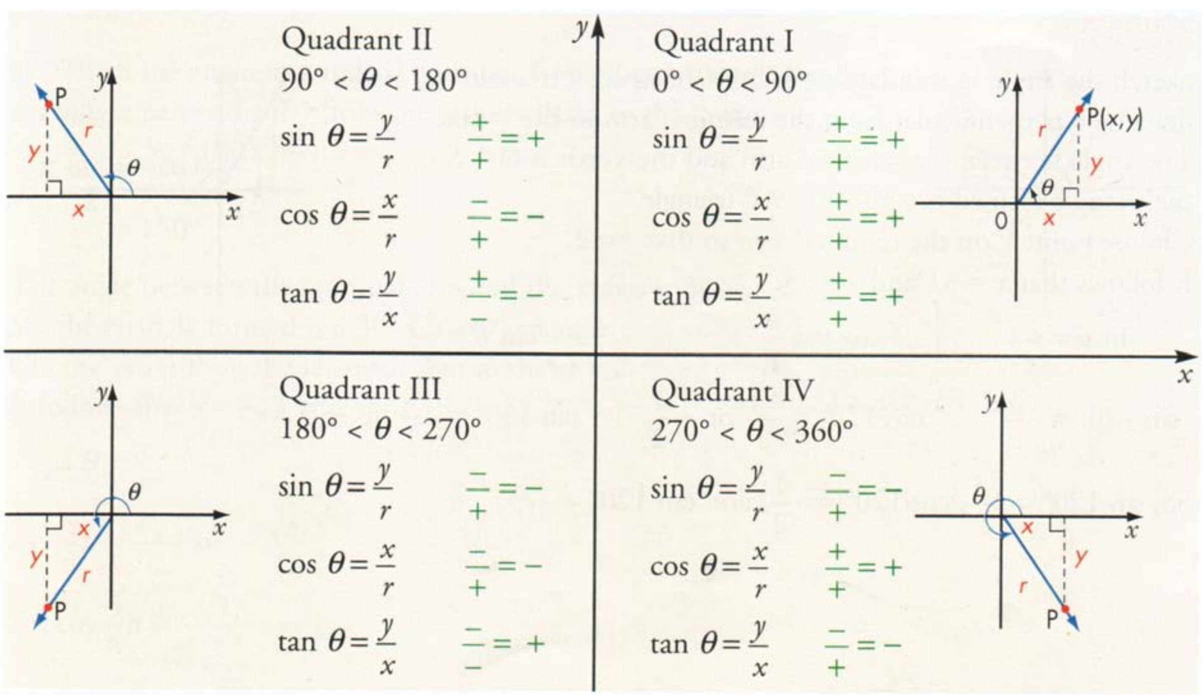
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$



- Since r represents the length of the terminal arm, $r > 0$.
- In quadrant I, $x > 0$ and $y > 0$. Therefore **A**LL the trig ratios are positive.
- In quadrant II, $x < 0$ and $y > 0$. Therefore only **S**INE is positive. The others are negative.
- In quadrant III, $x < 0$ and $y < 0$. Therefore only **T**ANGENT is positive. The others are negative.
- In quadrant IV, $x > 0$ and $y < 0$. Therefore only **C**OSINE is positive. The others are negative.
- Hence the mnemonic, "**A**LL **S**TUDENTS **T**ALK on **C**ELLPHONES"



Coterminal Angles

Angles of revolution are called ***coterminal*** if, when in standard position, they share the same terminal arm. For example, –90º, 270º and 630º are coterminal angles. An angle coterminal to a given angle can be found by adding or subtracting any multiple of 360º.

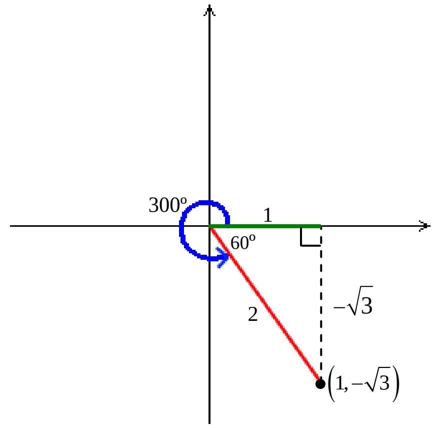
Example

Find the trigonometric ratios of 300º.

> Every angle of rotation has a ***related first quadrant angle***. The related first quadrant angle is found by taking the ***acute between the terminal arm and the x-axis***.
>
> For 300º, the related first quadrant angle is 60º.

Solution

From the diagram at the right, we can see that for an angle of rotation of 300º, we obtain a 30º-60º-90º right triangle in quadrant IV. In addition, by observing the ***acute angle*** between the ***terminal arm and the x-axis***, we find the ***related first quadrant angle***, 60º.



Therefore,

$$\sin 300° = \frac{y}{r} = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$

$$\cos 300° = \frac{x}{r} = \frac{1}{2}$$

$$\tan 300° = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

> Compare these answers to
>
> $$\sin 60° = \frac{\sqrt{3}}{2}$$
>
> $$\cos 60° = \frac{1}{2}$$
>
> $$\tan 60° = \sqrt{3}$$

Question

How are the trigonometric ratios of 300º related to the trigonometric ratios of 60º?

Answer

The ***magnitudes*** of the trig ratios of 300º are equal to the ***magnitudes*** of the trig ratios of the related first quadrant angle 60º. The ratios may differ only in ***sign***. To determine the correct sign, use the ***ASTC*** rule. In case you forget how to apply the ASTC rule, just think about the ***signs*** of *x* and *y* in each quadrant (see previous page). Don't forget that *r* is always positive because it represents the length of the terminal arm. Thus, the above ratios could have been calculated as follows:

Angle of Rotation: 300º (quadrant IV) **Related First Quadrant Angle:** 60º

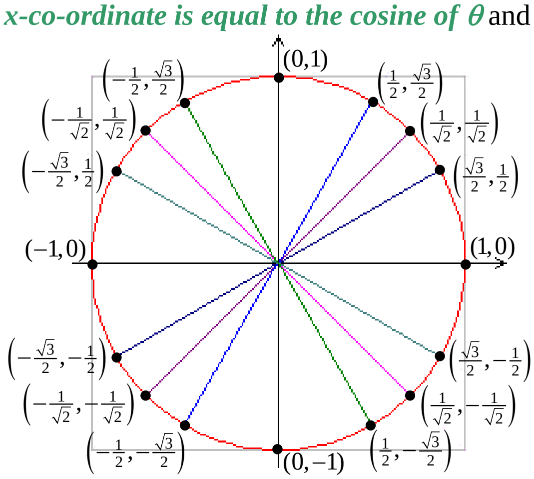
In quadrant IV, $\sin\theta = \frac{y}{r} < 0$ because $\frac{-}{+} = -$, $\cos\theta = \frac{x}{r} > 0$ because $\frac{+}{+} = +$ and $\tan\theta = \frac{y}{x} < 0$ because $\frac{-}{+} = -$.

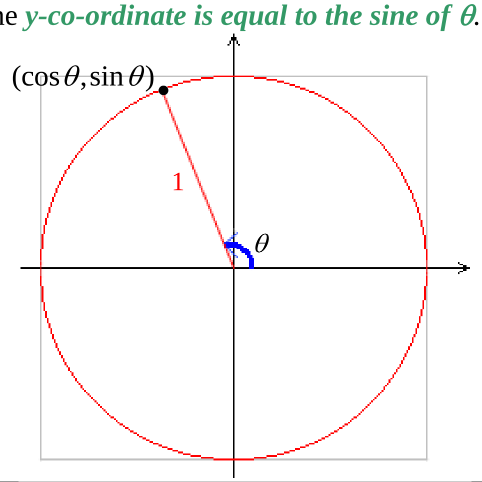
Hence, $\sin 300° = -\sin 60° = -\frac{\sqrt{3}}{2}$, $\cos 300° = \cos 60° = \frac{1}{2}$ and $\tan 300° = -\tan 60° = -\sqrt{3}$.

Additional Tools for Determining Trig Ratios of Special Angles

The Unit Circle

A ***unit circle*** is any circle having a radius of ***one unit***. If a unit circle is centred at the origin, it is described by the equation $x^2 + y^2 = 1$, meaning that for any point (x, y) lying on the circle, the value of $x^2 + y^2$ must equal 1. Furthermore, for any point (x, y) lying on the unit circle and for any angle θ, $r = 1$. Therefore, $\cos\theta = \frac{x}{r} = \frac{x}{1} = x$ and $\sin\theta = \frac{y}{r} = \frac{y}{1} = y$. In other words, for any point (x, y) lying on the unit circle, the ***x-co-ordinate is equal to the cosine of θ*** and the ***y-co-ordinate is equal to the sine of θ***.





The Rule of Quarters (Beware of Blind Memorization!)

The rule of quarters makes it easy to remember the sine of special angles. ***Be aware, however, that this rule invites blind memorization!***

$$\sin(0°) = \sqrt{\frac{0}{4}} = 0$$

$$\sin(30°) = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\sin(45°) = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\sin(60°) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\sin(90°) = \sqrt{\frac{4}{4}} = 1$$

Exercise

1. Evaluate each of the following. Whenever possible, give exact values based on the special triangles. Otherwise, round off your answers to 4 decimal places.

(a) $\cos 135°$ **(b)** $\sin 495°$ **(c)** $\tan 225°$ **(d)** $\cos 585°$

(e) $\cos(-90°)$ **(f)** $\sin 180°$ **(g)** $\tan 0°$ **(h)** $\cos 90°$

(i) $\tan 120°$ **(j)** $\tan(-150°)$ **(k)** $\tan 90°$ **(l)** $\sin 180°$

(m) $\tan 270°$ **(n)** $\sin 270°$ **(o)** $\sin(-210°)$ **(p)** $\cos 570°$

(q) $\cos 247°$ **(r)** $\sin(-321°)$ **(s)** $\tan 222°$ **(t)** $\tan(-359°)$

2. Solve for θ. In each case, $0 \le \theta \le 360°$.

(a) $\cos\theta = -\frac{1}{\sqrt{2}}$ **(b)** $\sin\theta = \frac{\sqrt{3}}{2}$ **(c)** $\tan\theta = -\frac{1}{\sqrt{3}}$ **(d)** $\tan\theta = \frac{1}{\sqrt{3}}$

(e) $\cos\theta = -1$ **(f)** $\sin\theta = 1$ **(g)** $\tan\theta = \sqrt{3}$ **(h)** $\sin\theta = 0$

(i) $\cos\theta = -\frac{1}{2}$ **(j)** $\sin\theta = \frac{1}{2}$ **(k)** $\sin\theta = -\frac{1}{2}$ **(l)** $\cos\theta = \frac{1}{2}$

(m) $\tan\theta = -1$ **(n)** $\tan\theta = 1$ **(o)** $\sin\theta = 0.2344536$ **(p)** $\cos\theta = -0.9342572$

Homework

pp. 281 – 282, #1, 2i, 3, 4, 5, 8, 9, 11

pp. 348 – 350, #1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16

pp. 353 – 354, #2, 3abcd, 4

Law of Sines and Law of Cosines

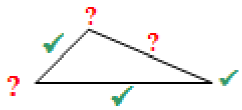
Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

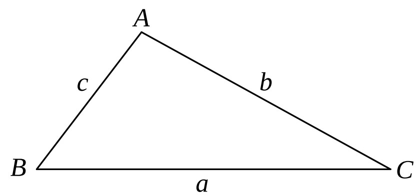
The law of cosines is a *generalization* of the Pythagorean Theorem.

The law of cosines is most useful in the following two cases.

The law of cosines also can be used in the following case. However, a quadratic equation needs to be solved.



While the Pythagorean Theorem ***holds only for right triangles***, the Sine Law and the Cosine Law hold for ***all triangles***!

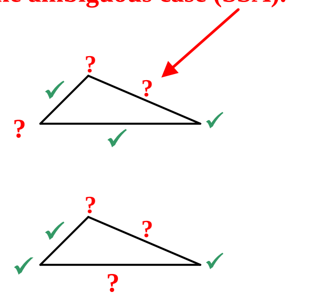


However, you ***should not*** use these laws when working with right triangles. It is much easier to use the basic trigonometric ratios and the Pythagorean Theorem.

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The law of sines is used in the following two cases. **However, you must beware of the ambiguous case (SSA).**

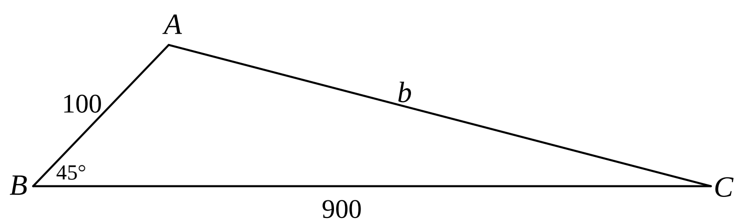


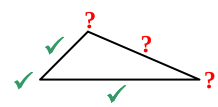
Important Exercise

Rearrange the equation for the law of cosines in such a way that you solve for $\cos C$. In what situation might you want to solve for $\cos C$?

Examples

1. Solve the following triangle.





Solution

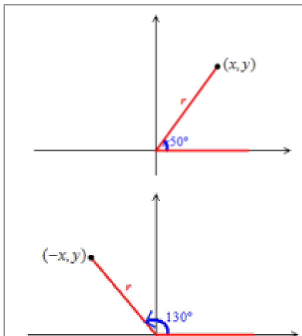
Given: $a = 900$, $c = 100$, $\angle B = 45°$

Required to Find (RTF): $b = ?$, $\angle A = ?$, $\angle C = ?$ (Note that "RTF" should not be confused with "WTF.")

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$= 900^2 + 100^2 - 2(900)(100)\cos 45°$$
$$= 810000 + 10000 - 180000\left(\frac{1}{\sqrt{2}}\right)$$
$$= 820000 - \frac{180000}{\sqrt{2}}$$
$$= \frac{820000\sqrt{2} - 180000}{\sqrt{2}}$$
$$\therefore b = \sqrt{\frac{820000\sqrt{2} - 180000}{\sqrt{2}}} \doteq 832.2985$$

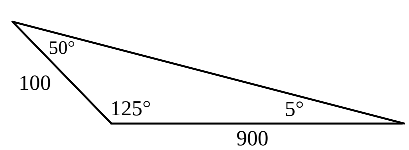
$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\therefore \frac{\sin A}{900} = \frac{\sin 45°}{\sqrt{\frac{820000\sqrt{2} - 180000}{\sqrt{2}}}}$$
$$\therefore \sin A \doteq \frac{900\left(\frac{1}{\sqrt{2}}\right)}{832.2985} \doteq 0.764624834$$
$$\therefore \angle A \doteq \sin^{-1}(0.764624834) \doteq 50° \text{ or } 130°$$

This is an example of how the *ambiguous* case of the sine law can arise.



The problem here is that $\sin 50° = \sin 130°$. Which answer is correct, 50° or 130°?

Because the given angle (45°) is enclosed by the two given sides (SAS), we must conclude that $A \doteq 130°$. If we allow $\angle A$ to have a measure of 50°, then the only triangle that can be constructed is the one shown at the right. Clearly, this triangle contradicts the given information because it lacks a 45° angle.



Final Answer: $b \doteq 832.2985$, $\angle A \doteq 130°$, $\angle C \doteq 5°$

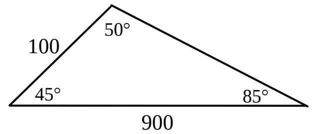
2. Answer the following questions.

(a) Explain the meaning of the word "ambiguous."

(b) Explain why the law of sines ***has*** an ambiguous case.

(c) Explain why the law of cosines ***does not have*** an ambiguous case.

(d) Refer to example 1. Explain why it is not possible for $\angle A$ to have a measure of 50º. That is, explain why it is not possible to construct the triangle shown at the right.



(e) Suppose that a solution of the equation $\sin\theta = k$ is the first quadrant angle α. What would be a second quadrant solution?

(f) In ΔABC, $\angle A = 37°$, a = 3 cm and c = 4 cm. How many different triangles are possible?

Solutions

(a) **Ambiguous:** Open to two or more interpretations; or of uncertain nature or significance; or intended to mislead e.g. "The polling had a complex and ***ambiguous*** message for potential female candidates."

(b) Since $\sin\theta > 0$ in quadrants I and II, solving an equation such as $\sin\theta = 0.5$ will result in two answers for θ, one that is in quadrant I and one that is in quadrant II. Since angles in a triangle can have any measure between 0º and 180º, then ***both*** answers are possible!

(c) Since $\cos\theta > 0$ in quadrant I and $\cos\theta < 0$ in quadrant II, there is no ambiguity. A positive cosine implies a first quadrant angle while a negative cosine implies a second quadrant angle.

(d) In any triangle, the largest angle must be opposite the longest side. In the triangle shown above, the largest angle is opposite the shortest side, which is impossible. To confirm that this is the case, we can calculate $\angle A$ by using the law of cosines. Beginning with the equation $a^2 = b^2 + c^2 - 2bc\cos A$, we solve for $\cos A$:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore \cos A \doteq \frac{832.2985^2 + 100^2 - 900^2}{2(832.2985)(100)}$$

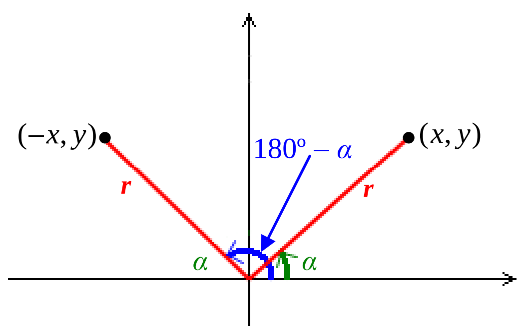
$$\therefore \cos A \doteq -0.64447555$$

$$\therefore \angle A \doteq \cos^{-1}(-0.64447555)$$

$$\therefore \angle A \doteq 130°$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\therefore 2bc\cos A = b^2 + c^2 - a^2$$
$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

(e) Consider the diagram at the right.



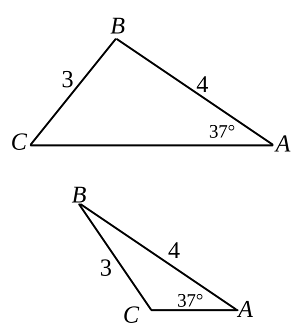
$$\sin\alpha = \frac{y}{r}$$

$$\sin(180° - \alpha) = \frac{y}{r}$$

$$\therefore \sin\alpha = \sin(180° - \alpha)$$

Therefore, if α is a first quadrant solution to the given equation, then 180º – α is a second quadrant solution.

(f) As shown at the right, there are two possible triangles that meet the given criteria. To confirm this algebraically, the law of cosines can be used.



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\therefore 3^2 = b^2 + 4^2 - 2(b)(4)\cos 37°$$

$$\therefore 9 = b^2 + 16 - (8\cos 37°)b$$

$$\therefore b^2 - (8\cos 37°)b + 7 = 0$$

$$\therefore b = \frac{8\cos 37° \pm \sqrt{(8\cos 37°)^2 - 4(1)(7)}}{2(1)}$$

$$\therefore b \doteq 4.99 \text{ or } b \doteq 1.40$$

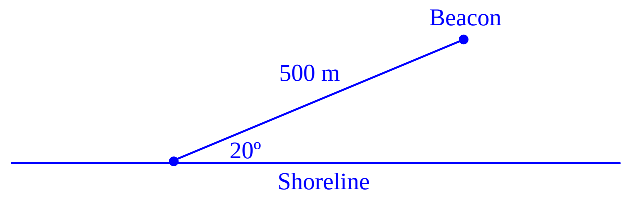
Since we obtain two answers for b, there are two possible triangles.

The number of roots of this quadratic equation can be determined very quickly by calculating the discriminant:

$D = (8\cos 37°)^2 - 4(1)(7) \doteq 12.8$

Since $D > 0$, there are two real roots.

3. The light from a rotating offshore beacon can illuminate effectively up to a distance of 250 m. A point on the shore is 500 m from the beacon. From this point, the sight line to the beacon makes an angle of 20º with the shoreline.

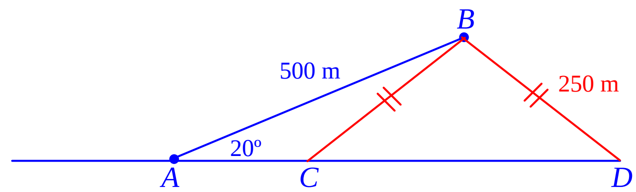


(a) What length of shoreline is illuminated effectively by the beacon?

(b) What area of the shore is illuminated effectively by the beacon?

Solution

(a) Using the diagram at the right, it is clear that *CD* is the portion of the shoreline that is illuminated effectively by the beacon.



Using ΔABD and the law of sines, we obtain

$$\frac{\sin D}{AB}=\frac{\sin A}{BD}$$

$$\therefore \frac{\sin D}{500}=\frac{\sin 20^\circ}{250}$$

$$\therefore \sin D = 500\left(\frac{\sin 20^\circ}{250}\right)$$

$$\therefore \sin D = 2\sin 20^\circ$$

$$\therefore \angle D = \sin^{-1}(2\sin 20^\circ)$$

$$\therefore \angle D \doteq 43.2^\circ$$

Since ΔBCD is isosceles,

$$\therefore \angle BCD \doteq 43.2^\circ$$

$$\therefore \angle CBD \doteq 180^\circ - 43.2^\circ - 43.2^\circ = 93.6^\circ$$

Using ΔBCD and the law of sines, we obtain

$$\frac{CD}{\sin\angle CBD}=\frac{BC}{\sin D}$$

$$\therefore \frac{CD}{\sin 93.6^\circ}=\frac{250}{\sin 43.2^\circ}$$

$$\therefore CD = \sin 93.6^\circ\left(\frac{250}{\sin 43.2^\circ}\right)$$

$$\therefore CD \doteq 364$$

Therefore, the length of the shoreline effectively illuminated by the beacon is approximately 364 m.

(b) Let *S* represent the required area. The portion of the shore illuminated by the beacon is shaded in the diagram below. The area of this portion of the shore can be found by subtracting the area of ΔBCD from the area of sector *BCD* (shape of a slice of pie). That is,

S = (area of sector *BCD*) – (area of ΔBCD)

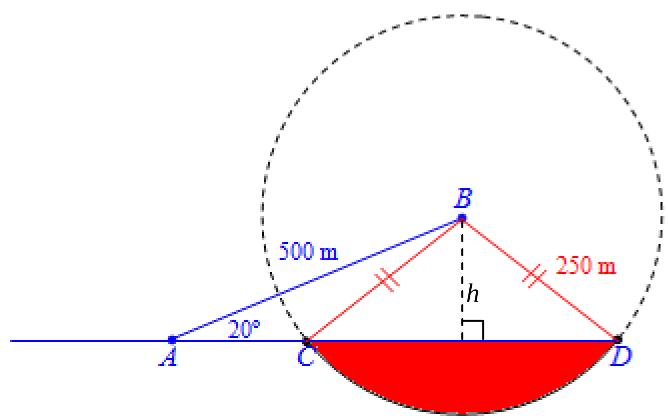
Since $\angle CBD \doteq 93.6^\circ$, the area of sector *BCD* is about $\frac{93.6}{360}$ the area of the entire circle. Thus,

$$S \doteq \frac{\theta}{360}\left(\pi r^2\right)-\frac{1}{2}bh$$

$$\therefore S \doteq \frac{93.6}{360}\left(\pi(250)^2\right)-\frac{1}{2}(364)\sqrt{250^2-182^2}$$

$$\therefore S \doteq 19857$$

The area of the shore effectively illuminated by the beacon is approximately 19587 m^2.



Homework

pp. 291 – 294, #4, 8, 9, 10, 14, 20, Achievement Check (at bottom of p. 294)

pp. 348 – 350, #2bdegh, 5, 6, 8, 9, 10, 14, 15, 17, 18, 19

Proofs of the Pythagorean Theorem, the Law of Cosines and the Law of Sines

Introduction

"A demonstration is an argument that will convince a reasonable man. A proof is an argument that can convince even an unreasonable man." (Mark Kac, 20th century Polish American mathematician)

Until now, you have simply ***accepted*** the "truth" of much of what you have been taught in mathematics. But how do you know that the claims made by your math teachers really are true? To be sure that you are not being duped, you should always seek ***proof***, or at the very least, a very convincing demonstration.

Nolfi's Intuitive Definition of "Proof"

A proof is a ***series*** or ***"chain" of inferences*** (i.e. "if…then" statements, formally known as ***logical implications*** or ***conditional statements***) that allows us to make ***logical deductions*** that lead from an ***initial premise***, which is known or assumed to be true, to a desired ***final conclusion***.

Hopefully, my definition is somewhat easier to understand than that of a former Prime Minister:

Jean Chrétien, a former Prime Minister of Canada, was quoted by CBC News as saying, "A proof is a proof. What kind of a proof? It's a proof. A proof is a proof. And when you have a good proof, it's because it's proven."

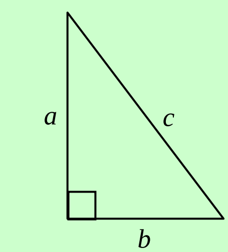
Pythagorean Theorem Example

The ***initial premise***

The ***final conclusion***

In ***any*** right triangle,
the square of the hypotenuse is equal to the sum of the squares of the other two sides.

That is, $c^2 = a^2 + b^2$



Proof:

Begin with ***right*** ΔABC and construct the altitude AD.

Since $\Delta ABC \sim \Delta DBA$ (AA similarity theorem),

$$\therefore \frac{AB}{BC} = \frac{BD}{AB}.$$

Since $\Delta ABC \sim \Delta DAC$ (AA similarity theorem),

$$\therefore \frac{AC}{BC} = \frac{DC}{AC}.$$

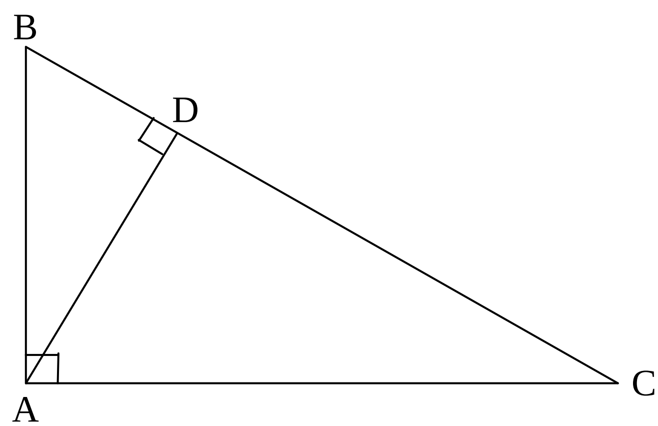
Therefore,

$$(AB)(AB) = (BD)(BC) \text{ and } (AC)(AC) = (DC)(BC)$$

Summing up, we obtain

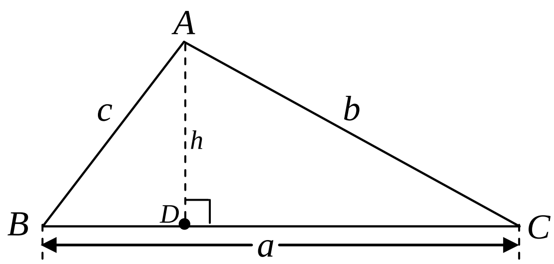
$$\begin{aligned}(AB)(AB) + (AC)(AC) &= (BD)(BC) + (DC)(BC)\\ &= BC(BD + DC)\\ &= (BC)(BC)\\ &= BC^2\end{aligned}$$

$$\therefore AB^2 + AC^2 = BC^2 \text{ //}$$



See http://www.cut-the-knot.org/pythagoras/index.shtml for a multitude of proofs of the Pythagorean Theorem.

Now that we have proved the Pythagorean Theorem we are allowed to use it to derive more relationships.



For the following proofs, refer to the diagram at the left.

Proof of the Law of Cosines

By the Pythagorean Theorem,

$$c^2 = h^2 + BD^2 \text{ and } b^2 = h^2 + DC^2.$$

Therefore,

$$h^2 = c^2 - BD^2 \text{ and } h^2 = b^2 - DC^2 \text{, from which we conclude that}$$

$$c^2 - BD^2 = b^2 - DC^2 \text{ or equivalently,}$$

$$\mathbf{c^2 = b^2 + BD^2 - DC^2 \quad (1).}$$

Since $a = BD + DC$,

$$a^2 = (BD + DC)^2 = BD^2 + 2(BD)(DC) + DC^2.$$

Subtracting $2(BD)(DC)$ and $2DC^2$ from both sides we obtain

$$a^2 - 2(BD)(DC) - 2DC^2 = BD^2 - DC^2.$$

Substituting in (1),

$$\begin{aligned} c^2 &= b^2 + BD^2 - DC^2 \\ &= b^2 + a^2 - 2(BD)(DC) - 2DC^2 \\ &= a^2 + b^2 - 2DC(BD + DC) \\ &= a^2 + b^2 - 2DC(a) \end{aligned}$$

But $\cos C = \dfrac{DC}{b}$, which means that $DC = b\cos C$. Hence,

$$\begin{aligned} c^2 &= a^2 + b^2 - 2DC(a) \\ &= a^2 + b^2 - 2(b\cos C)(a) \\ &= a^2 + b^2 - 2ab\cos C \ // \end{aligned}$$

Proof of the Law of Sines

$$\frac{h}{c} = \sin B \text{ and } \frac{h}{b} = \sin C$$

$$\therefore h = c\sin B \text{ and } h = b\sin C$$

$$\therefore c\sin B = b\sin C$$

$$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$$

Similarly, it can be shown that $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

Therefore, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ //

Important Questions – Test your Understanding

1. Would it be correct to use the law of cosines to prove the Pythagorean Theorem?

2. Complete the proof of the law of sines. (The word “similarly” was used to indicate that the remaining part of the proof would proceed in exactly the same manner.)

3. In the given proof of the Pythagorean Theorem, it is stated that $\Delta ABC \sim \Delta BDA$ and that $\Delta ABC \sim \Delta ADC$. Explain why each of these statements must be true.

4. Later in this unit we shall be learning about trigonometric identities. An identity is an equation that is true for all possible values of the variable(s) that appear(s) in the equation. For example, ***for all possible values of θ***, $\sin^2\theta + \cos^2\theta = 1$. It is worth repeating that this equation is true for all possible values of θ. It is not an equation that is to be solved for θ because every choice of θ will work. Use angles of rotation and the Pythagorean Theorem to prove this identity.

5. Explain why

(a) $|\sin\theta| \le 1$ for all values of θ

(b) $|\cos\theta| \le 1$ for all values of θ

(c) $\tan(90° + 180°n)$ is undefined for all $n \in \mathbb{Z}$

(d) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for all values of θ such that $\cos\theta \ne 0$

(e) $\sin(90° - \theta) = \cos\theta$ for all values of θ

(f) $\cos(90° - \theta) = \sin\theta$ for all values of θ

(g) $\sin(180° - \theta) = \sin\theta$ for all values of θ

(h) $\cos(180° - \theta) = -\cos\theta$ for all values of θ

(i) $\tan(180° - \theta) = -\tan\theta$ for all values of θ

(j) $\sin(-\theta) = -\sin\theta$ for all values of θ

(k) $\cos(-\theta) = \cos\theta$ for all values of θ

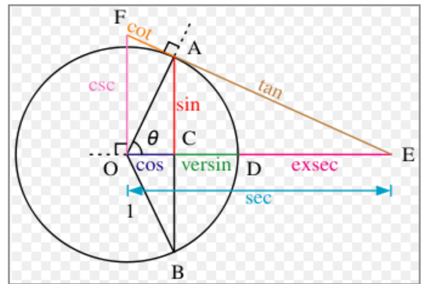
(l) $\tan(-\theta) = -\tan\theta$ for all values of θ

(m) $\tan(180° + \theta) = \tan\theta$ for all values of θ

THE RECIPROCAL TRIGONOMETRIC RATIOS

Review

- ☺ Trigonometric ratios of ***acute angle*** → **right triangles, SOH CAH TOA**
- ☺ Trigonometric ratios of ***any angles*** → **angles of rotation, related first quadrant (acute) angle, ASTC**
- ☺ Trigonometric ratios of ***special angle*** → **45º-45º-90º triangle (1:1: $\sqrt{2}$), 30º-60º-90º right triangle (1: $\sqrt{3}$:2), unit circle, rule of quarters**
- ☺ Solve ***right triangle*** → **trigonometric ratios and Pythagorean Theorem**
- ☺ Solve ***any triangle*** → **law of cosines, law of sines (beware of ambiguous case)**

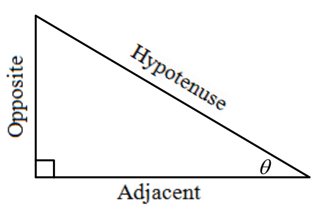


The trigonometric functions of an angle θ can be constructed geometrically using a unit circle (circle of radius 1). (The functions "versin" and "exsec" are no longer used due to the availability of electronic calculators.)

Why on Earth do we need more Ratios?

It is easy to show that sin and cos alone are sufficient for expressing all trigonometric relationships. However, using sin and cos alone can cause certain situations to be needlessly complicated. This is where the other trig ratios come into play. In addition to tan, the reciprocal trigonometric ratios ***cosecant*** (csc), ***secant*** (sec) and ***cotangent*** (cot) can help to simplify matters in certain cases.

The following table lists all the trigonometric ratios in modern usage.

	$\sin\theta=\frac{\text{opposite}}{\text{hypotenuse}}$ $\quad\csc\theta=\frac{\text{hypotenuse}}{\text{opposite}}=\frac{1}{\sin\theta}$ $\cos\theta=\frac{\text{adjacent}}{\text{hypotenuse}}$ $\quad\sec\theta=\frac{\text{hypotenuse}}{\text{adjacent}}=\frac{1}{\cos\theta}$ $\tan\theta=\frac{\text{opposite}}{\text{adjacent}}$ $\quad\cot\theta=\frac{\text{adjacent}}{\text{opposite}}=\frac{1}{\tan\theta}$	SOH CAH TOA CHO SHA COTAO

Examples

1. Evaluate all six trigonometric ratios of 210º.

Solution

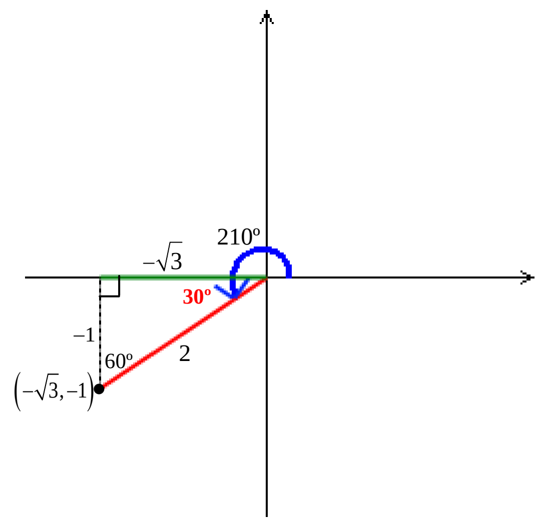
From the diagram at the right, we can see that for an angle of rotation of 210º, the ***related first quadrant angle*** is 30º. Using the ***ASTC*** rule and the 30º-60º-90º special triangle, we obtain

Therefore,

$$\sin 210^\circ=-\sin 30^\circ=-\frac{1}{2} \qquad \csc 210^\circ=\frac{1}{\sin 210^\circ}=-2$$

$$\cos 210^\circ=-\cos 30^\circ=-\frac{\sqrt{3}}{2} \qquad \sec 210^\circ=\frac{1}{\cos 210^\circ}=-\frac{2}{\sqrt{3}}$$

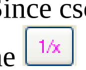
$$\tan 210^\circ=\tan 30^\circ=\frac{1}{\sqrt{3}} \qquad \cot 210^\circ=\frac{1}{\tan 210^\circ}=\sqrt{3}$$



2. Use your calculator to

(a) evaluate $\csc 337^\circ$

Solution

Using a calculator, we find that $\sin 337^\circ \doteq -0.390731128$. Since csc is the reciprocal of sin, press the  button to obtain the required value:

$$\csc 337^\circ=\frac{1}{\sin 337^\circ}\doteq -2.559304665$$

(b) solve $\cot\theta=-2.24636$ for $-180^\circ\le\theta\le 180^\circ$

Solution

Solving $\cot\theta=-2.24636$ is equivalent to solving $\tan\theta=-\frac{1}{2.24636}$.

$$\therefore \theta=\tan^{-1}\left(-\frac{1}{2.24636}\right)$$

$$\therefore \theta\doteq -24^\circ \text{ or } \theta\doteq 180^\circ-24^\circ=156^\circ$$

Quadrant II Solution

Quadrant IV Solution found using Calculator

Exercises

1. Evaluate each of the following. Whenever possible, give exact values based on the special triangles. Otherwise, round off your answers to 4 decimal places.

(a) $\sec 135°$ **(b)** $\csc 510°$ **(c)** $\cot 205°$ **(d)** $\sec 750°$

(e) $\sec(-90°)$ **(f)** $\csc 180°$ **(g)** $\cot 0°$ **(h)** $\sec 90°$

(i) $\cot 690°$ **(j)** $\cot(-420°)$ **(k)** $\cot 90°$ **(l)** $\csc 180°$

(m) $\cot 270°$ **(n)** $\csc 270°$ **(o)** $\csc(-210°)$ **(p)** $\sec 1080°$

(q) $\sec 247°$ **(r)** $\csc(-321°)$ **(s)** $\cot 222°$ **(t)** $\cot(-359°)$

2. Solve for θ. In each case, $-180 \le \theta \le 180°$.

(a) $\sec\theta = -\sqrt{2}$ **(b)** $\csc\theta = \frac{2}{\sqrt{3}}$ **(c)** $\cot\theta = -\frac{1}{\sqrt{3}}$ **(d)** $\cot\theta = \frac{1}{\sqrt{3}}$

(e) $\sec\theta = -1$ **(f)** $\csc\theta = \frac{1}{2}$ **(Think!!!)** **(g)** $\cot\theta = \sqrt{3}$ **(h)** $\csc\theta = 2$

(i) $\sec\theta = -2$ **(j)** $\csc\theta = -2$ **(k)** $\csc\theta = -\sqrt{2}$ **(l)** $\sec\theta = \frac{1}{2}$ **(Think!!!)**

(m) $\cot\theta = -1$ **(n)** $\cot\theta = 5.7865$ **(o)** $\csc\theta = 10.2344536$ **(p)** $\sec\theta = -7.9342572$

Applications of Simple Geometry and Trigonometric Ratios

1. As shown very crudely in the diagram at the right, the data stored on a compact disc are arranged along a continuous spiral that begins near the centre of the disc. Because of this, a CD must spin at different rates to read data from different parts of the disc. For instance, audio CDs spin at a rate of about 28000º per minute when data are read near the centre, decreasing gradually to about 12000º per minute when data are read near the edge.

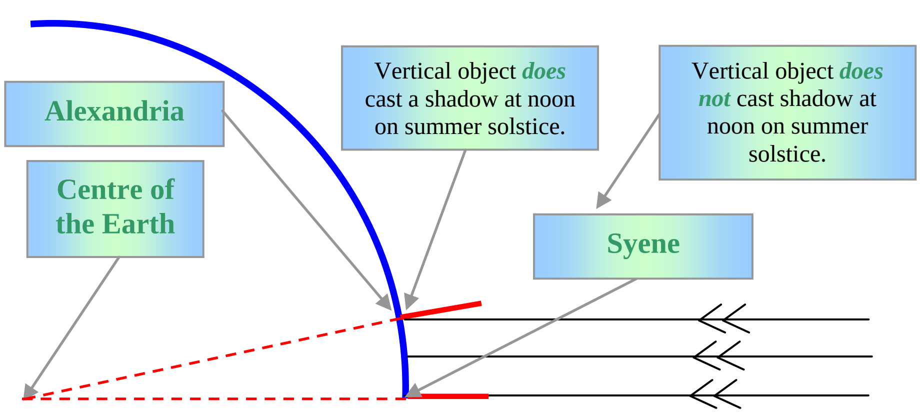
(a) Through how many degrees per second does an audio CD spin when data are read near the centre of the disc? Through how many degrees per second does an audio CD spin when data are read near the edge of the disc?

(b) Computer CD-ROM drives rotate CDs at multiples of the values used in audio CD players. A 1× drive uses the same angular velocity as a music player, a 2× drive is twice as fast, a 4× drive is four times as fast and so on. Through how many degrees per second does a CD rotate in a 52× drive when data are read near the centre of the disc?

2. It is a popular misconception that until the time of Christopher Columbus, people believed that the Earth was a flat plate. In reality, the spherical shape of the Earth was known to the ancient Greeks and quite likely, even to earlier civilizations. In the fourth century B.C., Aristotle put forth two strong arguments for supporting the theory that the Earth was a sphere. First, he observed that during lunar eclipses, the Earth's shadow on the moon was always circular. Second, he remarked that the North Star appeared at different ***angles of elevation*** in the sky, depending on whether the observer viewed the star from northerly or southerly locations. These two observations, being entirely inconsistent with the "flat Earth" hypothesis, led Aristotle to conclude that the Earth's surface must be curved. Even sailors in ancient Greece, despite being less scholarly than Aristotle, also realized that the Earth's surface was curved. They noticed that the sails of a ship were always visible before the hull as the ship emerged over the horizon and that the sails appeared to "dip" into the ocean as the ship would retreat beyond the horizon.

A Greek named Eratosthenes (born: 276 BC in Cyrene, North Africa, which is now Shahhat, Libya, died: 194 BC, Alexandria, Egypt) took these observations one step further. He was the chief librarian in the great library of Alexandria in Egypt and a leading all-round scholar. At his disposal was the latest scientific knowledge of his day. One day, he read that a deep vertical well near Syene, in southern Egypt, was entirely lit up by the sun at noon once a year (on the summer solstice). This seemingly mundane fact probably would not have captured the attention of someone of ordinary intellectual abilities, but it instantly piqued Eratosthenes' curiosity. He reasoned that at this time, the sun must be directly overhead, with its rays shining directly into the well. Upon further investigation, he learned that in Alexandria, almost due north of Syene, the sun was not directly overhead at noon on the summer solstice because a vertical object would cast a shadow. He deduced, therefore, that the Earth's surface must be curved or the sun would be directly overhead in both places at the same time of day. By adding two simple assumptions to his deductions, Eratosthenes was able to calculate the Earth's circumference to a high degree of accuracy! First, he knew that the Earth's surface was curved so he assumed that it was a sphere. This assumption was strongly supported by Aristotle's observations in the fourth century B.C. Second, he assumed that the sun's rays are parallel to each other. This was also a very reasonable assumption because he knew that since the sun was so distant from the Earth, its rays would be virtually parallel as they approached the Earth.

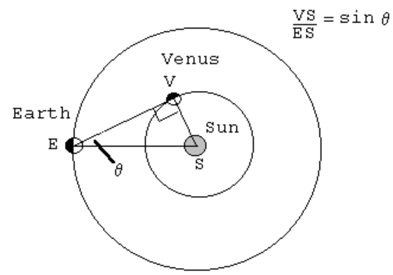
Now it's your turn to reproduce Eratosthenes' calculation. The diagram below summarizes all the required information.



Eratosthenes hired a member of a camel-powered trade caravan to "pace out" the distance between Syene and Alexandria. This distance was found to be about 5000 "stadia." The length of one "stadion" varied from ancient city to ancient city so there is some debate concerning how to convert Eratosthenes' measurement in stadia to a modern value in kilometres. However, it is usually assumed that Eratossthenes' stadion measured 184.98 m. Eratosthenes measured the "shadow angle" at Alexandria and found that it was approximately 7.2°.

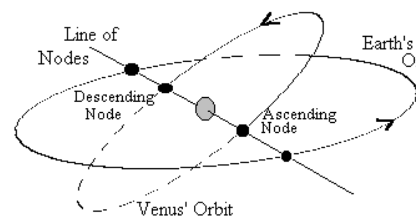
3. Have you ever wondered how astronomers calculate distances from the Earth to celestial bodies? In this problem, you will learn about how the distance from the Earth to the Sun was first calculated in 1882.

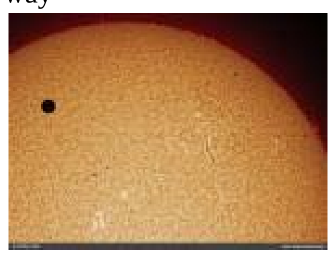
(a) Using the ***angle of separation*** (as measured from the Earth) between a body in our solar system and the Sun, its distance from the sun can be determined in terms of the distance from the Earth to the Sun. To facilitate this process, the "AU" (astronomical unit) was created, the distance from the Earth to the sun being defined as 1 AU. As a result of astronomical measurements made prior to 1882, the distances from the Sun of all the planets known at the time had been calculated in terms of the AU. Unfortunately, however, it was not possible to convert these distances into kilometres because the distance from the Earth to the Sun was not known. The diagram at the right shows how astronomers calculated the distance from Venus to the sun in terms of the AU. When Venus and the sun were conveniently positioned in the sky to make the angle of separation as large as possible, the angle θ was measured to be approximately 46.054°.

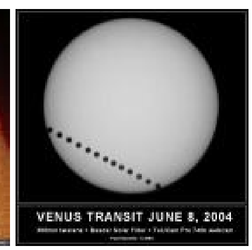


Use this information to calculate the distance from Venus to the Sun in terms of the astronomical unit. In addition, explain why ΔSVE must be a right triangle.

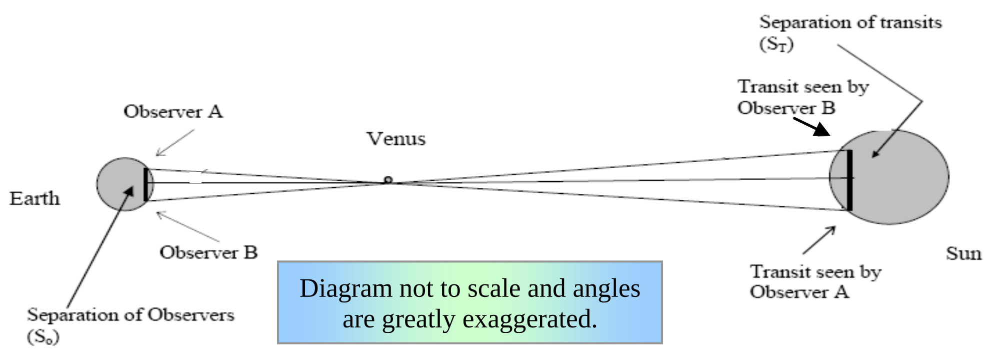
(b) As shown in the diagram at the right, the plane of Venus' orbit is inclined to that of the Earth. (The actual orbital inclination of Venus is only 3.39° but it is highly exaggerated in the illustration for the sake of clarity.) Notice that Venus passes through the Earth's orbital plane twice per orbit. When Venus intersects both the orbital plane of the Earth and the line connecting the Earth to the Sun, ***a transit of Venus*** occurs. From the Earth, Venus is observed as a small black disk that slowly makes its way across the face of the Sun. Transits of Venus are exceedingly rare, usually occurring in pairs spaced apart by 121.5±8 years (4 transits per 243-year cycle).



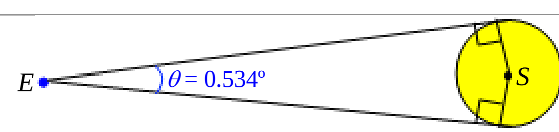




In 1882, a transit of Venus occurred. This opportunity was seized by astronomers to calculate, once and for all, the Earth-Sun distance. Observers on Earth separated by a North-South distance S_O (separation of observers) would observe transits separated by a North-South distance S_T (separation of transits). Observer *A* in the northern hemisphere would see Venus lower on the face of the Sun than observer *B* in the southern hemisphere. This is due to an effect called ***parallax***, the same phenomenon that causes an apparent shift in the position of objects when viewed successively with one eye and then with the other.



Use the information in the following table and your answer from 3(a) to express 1 AU in kilometres.

Symbols	Measured Values	RTF	Strategies / Hints
S_O = separation of observers S_T = separation of transits	S_O = 2000 km	S_T = ?	Use similar triangles to calculate S_T. Your answer from 3(a) is important!
S_{16} = separation of transits on circle of radius 16 cm r_S = radius of the sun	S_{16} = 0.059198 cm	r_S = ?	$\frac{r_S}{S_T} = \frac{0.5(16)}{S_{16}}$
θ = angle of separation (see diagram at the right)	θ = 0.534° This is the average of many measurements made by astronomers.	ES = ?	

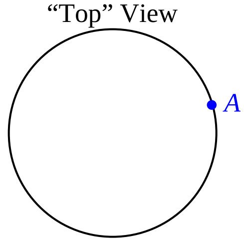
4. The circumference of the Earth at the equator is approximately 40074 km. A ***sidereal day*** is defined as the time required for the Earth to make one complete rotation relative to its axis of rotation. Careful scientific measurements have shown that the length of a sidereal day is about 23.9344696 hours (23 hours, 56 minutes, 4.09056 s) and that the circumference of the Earth at the equator is approximately 40074 km.

Therefore, a point on the equator moves in a very large circle at a speed of $\frac{40074 \text{ km}}{23.9344696 \text{ h}} \doteq 1674.3 \text{ km/h}$.

(a) Use the given information to calculate the rotational speed of a point on any line of latitude, relative to the Earth's axis of rotation.
Hint: Take any point A located at θ degrees north or θ degrees south. Keeping in mind that a line of latitude is nothing more than a large circle, you should be able to calculate its circumference in terms of the Earth's radius and the angle θ.

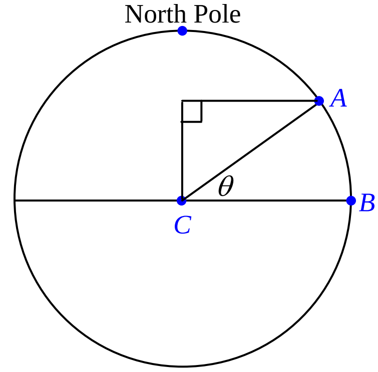
(b) Central Peel is located at 43.6964 degrees north. How fast is CPSS moving in a large circle about the Earth's axis of rotation?

“Top” View



The great circle at latitude θ degrees north as seen when looking “down” from a point directly above the North Pole

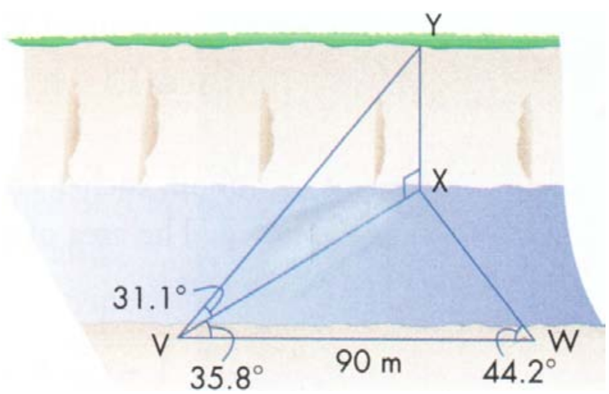
North Pole



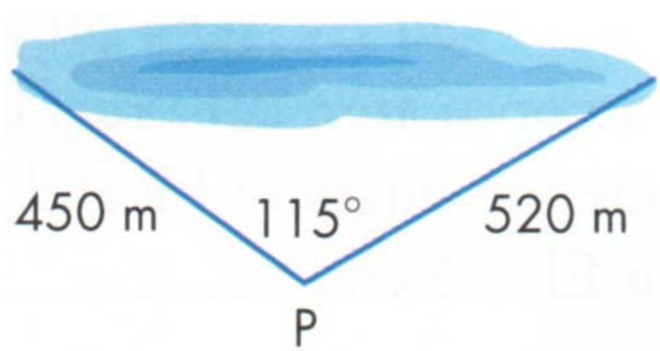
“Side” View

(c) What is the speed of the North Pole relative to the Earth's axis of rotation? What is the speed of the South Pole relative to the Earth's axis of rotation?

5. To measure the height XY of an inaccessible cliff, a surveyor recorded the data shown in the diagram at the right. If the height of the theodolite used was 1.7 m, find the height of the cliff.

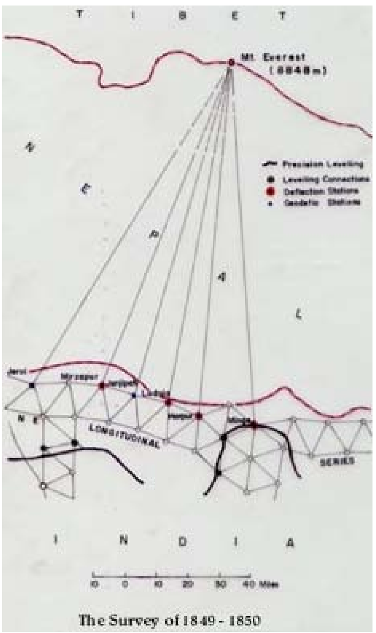


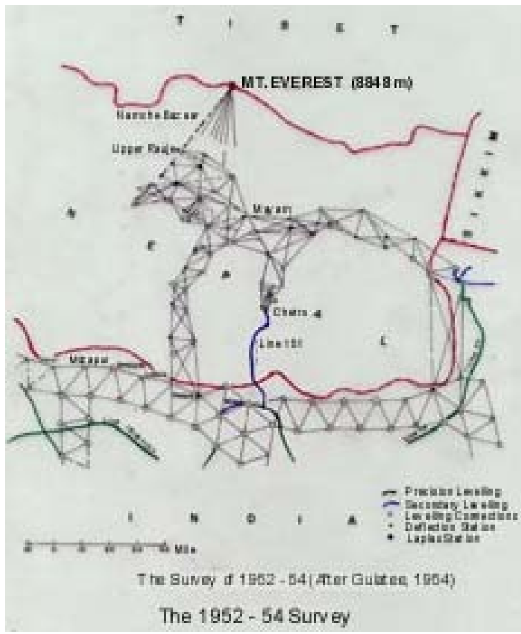
6. From point P, the distance to one end of a pond is 450 m and the distance to the other end is 520 m. The angle formed by the line of sight is 115º. Find the length of the pond



7. In 1852, as part of the Great Trigonometric Survey of India, Radhanath Sikhdar, an Indian mathematician and surveyor from Bengal, was the first to identify Mount Everest (called ***Sagarmatha*** by the Nepalese people) as the world's highest peak. Sikhdar used ***theodolites*** to make measurements of "Peak XV," as it was then known, from a distance of about 240 km. He then used trigonometry to calculate the height of Sagarmatha. Based on the average of measurements made from six different observation stations, Peak XV was found to be exactly 29,000 feet (8,839 m) high but was publicly declared to have a height of 29,002 feet (8,840 m). The arbitrary addition of 2 feet (0.6 m) was to avoid the impression that an exact height of 29,000 feet was nothing more than a rounded estimate.

An aerial view of Mount Everest from the south, with Lhotse in front and Nuptse on the left

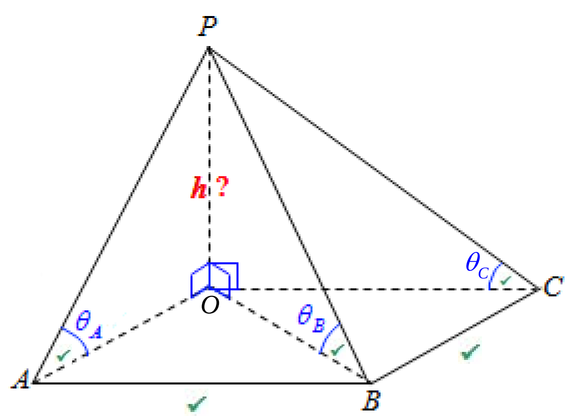




Radhanath Sikhdar

A ***theodolite*** is an instrument for measuring both horizontal and vertical angles,

(a) In the diagram at the right, the point *P* represents the peak of a mountain. Points *A*, *B* and *C* represent points on the ground that are at the same elevation above sea level. From these points, the ***angles of elevation*** θ_A, θ_B and θ_C are measured. As indicated in the diagram, the lengths of *AB* and *BC* are known, as are the measures of the angles θ_A, θ_B and θ_C. Does this give us enough information to calculate *h*, the height of the mountain above the surface of the plane through quadrilateral *ABCO*? If so, use an example to outline a method for calculating the height *h* in terms of the given information. If not, describe what other information would be needed.



(b) Is it necessary for the points *A*, *B* and *C* to be at the same elevation above sea level? Explain.

(c) Explain why it would be impractical to measure directly the lengths of *AO*, *BO* and *CO*. (Obviously, the length of *PO* = *h* cannot be measured directly.)

(d) Describe the orientation of quadrilateral *ABCO* relative to the surface of the Earth.

(e) Once *h* is calculated, how would the height of the mountain above sea level be determined?

(f) What is the currently accepted height above sea level of Mount Everest? Are you surprised that Sikhdar's measurement from 1852 is so close to the modern value?

Trigonometric Identities

Review – Types of Equations Studied thus Far

☺ ***Equation that is Solved for the Unknown.***

e.g. Solve $x^2 - 5x + 6 = 0$

- This means that we need to ***find*** the value(s) of x that make the left-hand-side equal to the right-hand-side.
- Geometrically, this equation describes the intersection between the graphs of $y = x^2 - 5x + 6$ and $y = 0$.

☺ ***Equation that Expresses a Function or a Relation***

e.g. $f(x) = x^2 - 5x + 6$, $\quad x^2 + y^2 = 16$

- Such equations express a ***relationship*** between an independent variable and a dependent variable.
- They cannot be solved in the same sense as equations such as $x^2 - 5x + 6 = 0$ are solved because such equations ***do not*** represent the intersection of two (or more) graphs.
- However, it does make sense to use algebraic manipulations to rewrite them in a different form.
- If x is allowed to vary continuously, such equations usually describe (piecewise) continuous curves.
- If x is restricted to integral or rational values (i.e. whole numbers or fractions), the graphs of such functions will be a discrete collection of points in the Cartesian plane.

What is an Identity?

☺ An ***identity*** is yet another type of equation. ***Identities express the equivalence of two expressions***.

e.g. $\cos^2\theta + \sin^2\theta = 1$, $\quad \tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (We have already seen and proved these identities. See page 13.)

- The given equations are identities. ***For all values of θ that make sense***, the left-hand-side equal equals the right-hand-side. That is, the expression on the left side ***is equivalent to*** the expression on the right side.
- For the identity $\cos^2\theta + \sin^2\theta = 1$, there are no restrictions on the value of θ.
- For the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, we cannot allow θ to take on values that make $\cos\theta = 0$ because this would lead to the undefined operation of dividing by zero.

List of Identities that we Already Know

- To discourage the erroneous notion that θ is the only symbol that is allowed to express trigonometric identities or trigonometric functions, x will often be used in place of θ. It should be clear that any symbol whatsoever can be used as long as meaning is not compromised.
- Since the identities in the following list have already been proved to be true, they can be used to construct proofs of other identities. Examples are given below.

Pythagorean Identities	*Quotient Identity*	*Reciprocal Identities*		
For all $x \in \mathbb{R}$, $\cos^2 x + \sin^2 x = 1$, $\sin^2 x = 1 - \cos^2 x$, $\cos^2 x = 1 - \sin^2 x$	For all $x \in \mathbb{R}$ such that $\cos x \neq 0$, $\tan x = \dfrac{\sin x}{\cos x}$	For all $x \in \mathbb{R}$ such that $\sin x \neq 0$, $\csc x = \dfrac{1}{\sin x}$	For all $x \in \mathbb{R}$ such that $\cos x \neq 0$, $\sec x = \dfrac{1}{\cos x}$	For all $x \in \mathbb{R}$ such that $\tan x \neq 0$, $\cot x = \dfrac{1}{\tan x}$

Important Note about Notation

- $\sin^2 x$ is a shorthand notation for $(\sin x)^2$, which means that ***first*** $\sin x$ is evaluated, ***then*** the result is squared

e.g. $\sin^2 45° = (\sin 45°)^2 = \left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1}{2}$ $\qquad \cos^2 60° = (\cos 60°)^2 = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

- This notation is used to avoid the excessive use of parentheses
- $\sin x^2 \neq \sin^2 x$ $\qquad$ The expression $\sin x^2$ means that ***first*** x is squared and ***then*** "sin" is applied

Examples

Prove each of the following identities. (Here L.S. means "left side" and R.S. means "right side.")

1. $\cot x = \dfrac{\cos x}{\sin x}$

Proof

$\text{L.S.} = \cot x$ ← Assertion

$= \dfrac{1}{\tan x}$ (definition of cot)

$= \dfrac{1}{\left(\dfrac{\sin x}{\cos x}\right)}$ (quotient identity)

$= \dfrac{1}{1} \times \left(\dfrac{\cos x}{\sin x}\right)$

$= \dfrac{\cos x}{\sin x}$

Justification

$\text{R.S.} = \dfrac{\cos x}{\sin x}$

$\therefore$ L.S. = R.S. so $\cot x = \dfrac{\cos x}{\sin x}$ //

2. $1 + \tan^2 x = \sec^2 x$

Proof

$\text{L.S.} = 1 + \tan^2 x$

$= 1 + \left(\dfrac{\sin x}{\cos x}\right)^2$ (quotient identity)

$= 1 + \dfrac{\sin^2 x}{\cos^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \dfrac{1}{\cos^2 x}$ (Pythagorean identity)

$= \left(\dfrac{1}{\cos x}\right)^2$

$= \sec^2 x$ (definition of sec)

$\text{R.S.} = \sec^2 x$

$\therefore$ L.S. = R.S. so $1 + \tan^2 x = \sec^2 x$ //

3. $\dfrac{\sin^2 x}{1 - \cos x} = 1 + \cos x$

Proof

$\text{L.S.} = \dfrac{\sin^2 x}{1 - \cos x}$

$= \dfrac{1 - \cos^2 x}{1 - \cos x}$ (Pythagorean identity)

$= \dfrac{(1 - \cos x)(1 + \cos x)}{1 - \cos x}$ (factoring)

$= 1 + \cos x$

$\text{R.S.} = 1 + \cos x$

$\therefore$ L.S. = R.S. so $\dfrac{\sin^2 x}{1 - \cos x} = 1 + \cos x$ //

Logical and Notational Pitfalls – Please Avoid Absurdities!

1. The purpose of a proof is to ***establish*** the "truth" of a mathematical statement. ***Therefore, you must never assume what you are trying to prove!*** A common error is shown at the right. ***The series of steps shown is wrong and would be assigned a mark of zero!*** To write a correct proof, the left and right sides of the equation must be treated separately. Only once you have demonstrated that the left side is equal to the right side are you allowed to declare their equality.

2. Keep in mind that words like "sin," "cos" and "tan" ***are function names, not numerical values!*** Therefore, you must not treat them as numbers. For example, it makes sense to write $\dfrac{\sin 2x}{\sin x}$ but it makes no sense whatsoever to "cancel" the sines. Many students will write statements such as ~~$\dfrac{\sin 2x}{\sin x} = \dfrac{2x}{x}$~~ $= 2$, which are completely nonsensical. First of all, dividing the numerator and the denominator by "sin" is invalid because "sin" is not a number. Furthermore, a simple test reveals that $\dfrac{\sin 2x}{\sin x} \neq 2$: $\dfrac{\sin 2(45°)}{\sin 45°} = \dfrac{\sin 90°}{\sin 45°} = \dfrac{1}{1/\sqrt{2}} = \sqrt{2}$.

   Clearly, $\sqrt{2} \neq 2$. Therefore, the assertion that was made is entirely false!

~~***"Proof"***~~

~~$\cot x = \dfrac{\cos x}{\sin x}$~~

~~$\therefore \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$~~

~~$\therefore \dfrac{1}{\left(\dfrac{\sin x}{\cos x}\right)} = \dfrac{\cos x}{\sin x}$~~

~~$\therefore \dfrac{1}{1} \times \left(\dfrac{\cos x}{\sin x}\right) = \dfrac{\cos x}{\sin x}$~~

~~$\therefore \dfrac{\cos x}{\sin x} = \dfrac{\cos x}{\sin x}$~~

Suggestions for Proving Trig Identities

1. Write the given expressions in terms of sin and cos.
2. Begin with the more complicated side and try to simplify it.
3. Keep a list of important identities in plain view while working.
4. Expect to make mistakes! If one approach seems to lead to a dead end, try another. Don't give up!

Homework

pp. 398 – 401, #1, 2acegik, 4dhij, 5, 7, 10, 11, 12, 13, 20

Prove (a) $\dfrac{\sec x}{\csc x} = \tan x$ (b) $\dfrac{\csc x}{\sec x} = \cot x$ (c) $\cot x \csc x(\sec x - 1) = \dfrac{1}{1 + \cos x}$ (d) $1 + \cot^2 x = \csc^2 x$

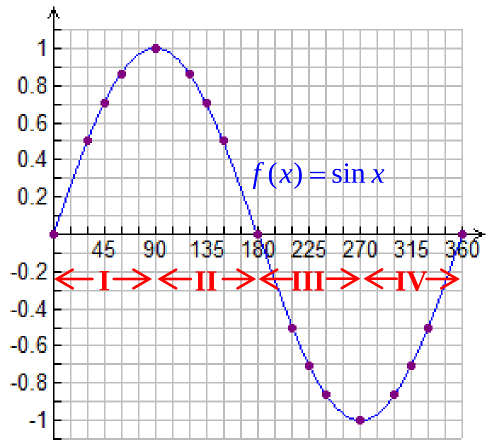
Introduction to Trigonometric Functions

Overview

Now that we have developed a thorough understanding of trigonometric ratios, we can proceed to our investigation of trigonometric functions. The old adage "a picture says a thousand words" is very fitting in the case of the graphs of trig functions. Each curve summarizes everything that we have learned about trig ratios. Answer the questions below to discover the details.

Graphs

x	$y=\sin x$
0º	0
30º	$1/2=0.5$
45º	$1/\sqrt{2} \doteq 0.70711$
60º	$\sqrt{3}/2 \doteq 0.86603$
90º	1
120º	$\sqrt{3}/2 \doteq 0.86603$
135º	$1/\sqrt{2} \doteq 0.70711$
150º	$1/2=0.5$
180º	0
210º	$-1/2=-0.5$
225º	$-1/\sqrt{2} \doteq -0.70711$
240º	$-\sqrt{3}/2 \doteq -0.86603$
270º	-1
300º	$-\sqrt{3}/2 \doteq -0.86603$
315º	$-1/\sqrt{2} \doteq -0.70711$
330º	$-1/2=-0.5$
360º	0



Questions about the Sine Function

1. State the domain and range of the sine function.

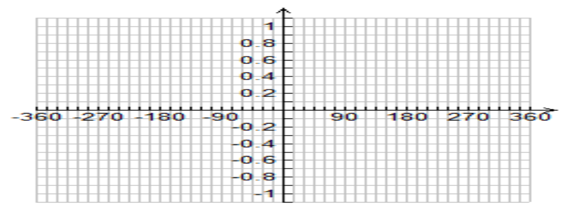
2. Is the sine function one-to-one or many-to-one?

3. State a suitable domain over which the inverse of sine ($\sin^{-1}$) is defined.

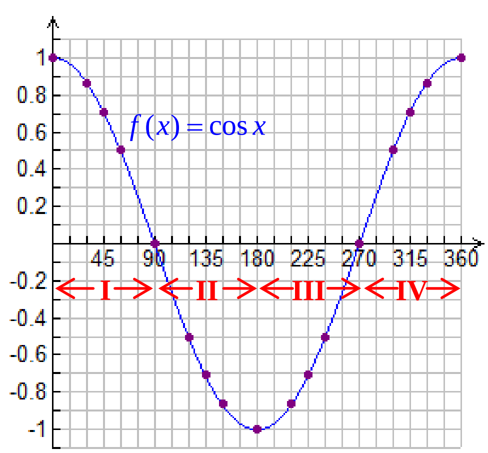
4. How can the graph of the sine function help you to remember the ***sign*** (i.e. + or –) of a sine ratio for any angle?

5. How can the graph of the sine function help you to remember the sine ratios of the special angles?

6. Sketch the graph of $f(x)=\sin x$ for $-360° \le x \le 360°$.



x	$y=\cos x$
0º	1
30º	$\sqrt{3}/2 \doteq 0.86603$
45º	$1/\sqrt{2} \doteq 0.70711$
60º	$1/2=0.5$
90º	0
120º	$-1/2=-0.5$
135º	$-1/\sqrt{2} \doteq -0.70711$
150º	$-\sqrt{3}/2 \doteq -0.86603$
180º	-1
210º	$-\sqrt{3}/2 \doteq -0.86603$
225º	$-1/\sqrt{2} \doteq -0.70711$
240º	$-1/2=-0.5$
270º	0
300º	$1/2=0.5$
315º	$1/\sqrt{2} \doteq 0.70711$
330º	$\sqrt{3}/2 \doteq 0.86603$
360º	1



Questions about the Cosine Function

1. State the domain and range of the cosine function.

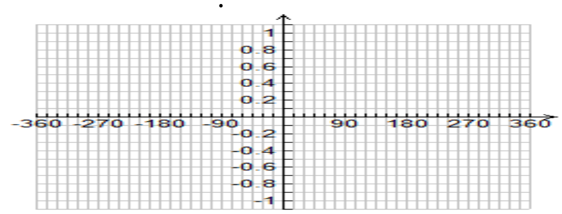
2. Is the cosine function one-to-one or many-to-one?

3. State a suitable domain over which the inverse of cosine ($\cos^{-1}$) is defined.

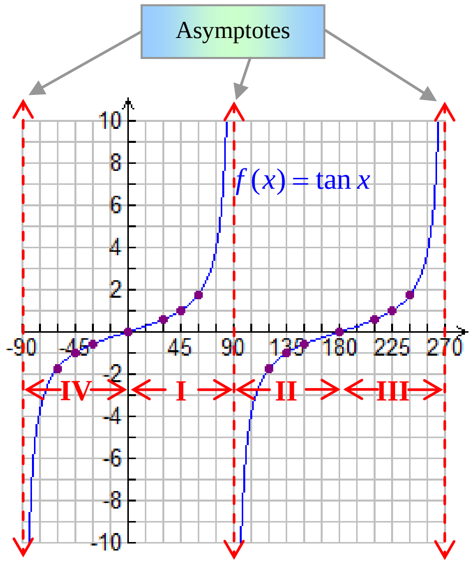
4. How can the graph of the cosine function help you to remember the ***sign*** (i.e. + or –) of a cosine ratio for any angle?

5. How can the graph of the cosine function help you to remember the cosine ratios of the special angles?

6. Sketch the graph of $f(x)=\cos x$ for $-360° \le x \le 360°$.



x	$y = \tan x$
–90º	undefined
–60º	$-\sqrt{3} \doteq -1.73205$
–45º	–1
–30º	$-1/\sqrt{3} \doteq -0.57735$
0º	0
30º	$1/\sqrt{3} \doteq 0.57735$
45º	1
60º	$\sqrt{3} \doteq 1.73205$
90º	undefined
120º	$-\sqrt{3} \doteq -1.73205$
135º	–1
150º	$-1/\sqrt{3} \doteq -0.57735$
180º	0
210º	$1/\sqrt{3} \doteq 0.57735$
225º	1
240º	$\sqrt{3} \doteq 1.73205$
270º	undefined



Questions about the Tangent Function

1. State the domain and range of the tangent function.

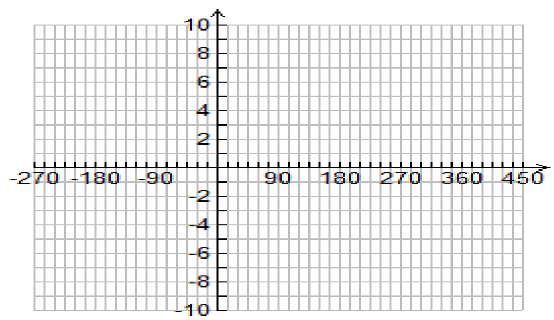
2. Is the tangent function one-to-one or many-to-one?

3. State a suitable domain over which the inverse of tangent ($\tan^{-1}$) is defined.

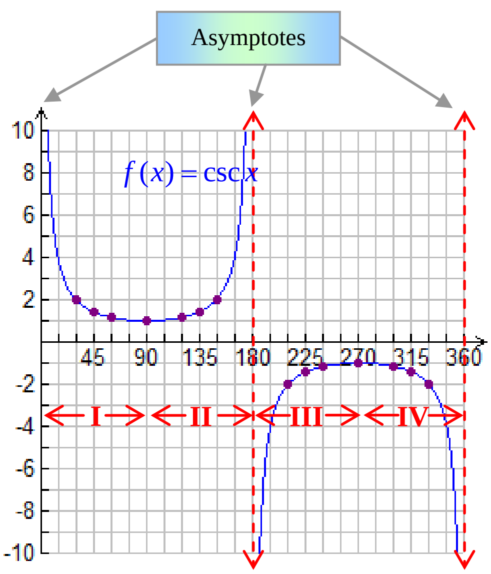
4. How can the graph of the tangent function help you to remember the sign (i.e. + or –) of a tangent ratio for any angle?

5. How can the graph of the tangent function help you to remember the tangent ratios of the special angles?

6. Sketch the graph of $f(x) = \tan x$ for $-270° \le x \le 450°$.



x	$y = \csc x$
0º	undefined
30º	2
45º	$\sqrt{2} \doteq 1.41421$
60º	$2/\sqrt{3} \doteq 1.15470$
90º	1
120º	$2/\sqrt{3} \doteq 1.15470$
135º	$\sqrt{2} \doteq 1.41421$
150º	2
180º	undefined
210º	–2
225º	$-\sqrt{2} \doteq -1.41421$
240º	$-2/\sqrt{3} \doteq -1.15470$
270º	–1
300º	$-2/\sqrt{3} \doteq -1.15470$
315º	$-\sqrt{2} \doteq -1.41421$
330º	–2
360º	undefined



Questions about the Cosecant Function

1. State the domain and range of the cosecant function.

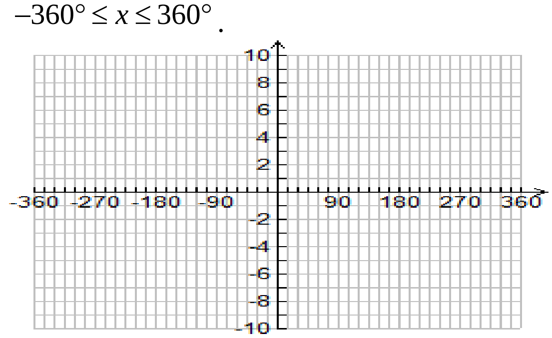
2. Is the cosecant function one-to-one or many-to-one?

3. State a suitable domain over which the inverse of cosecant ($\csc^{-1}$) is defined.

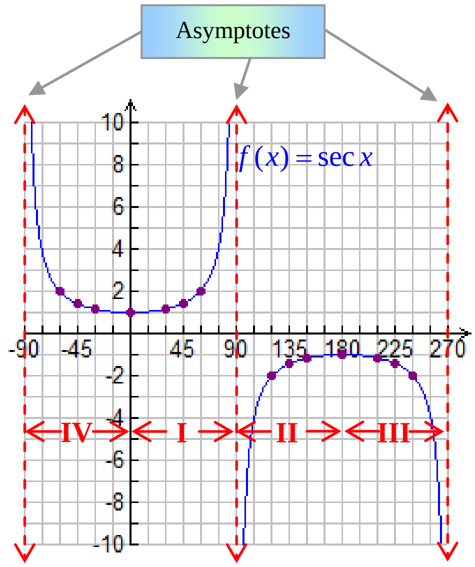
4. How can the graph of the cosecant function help you to remember the sign (i.e. + or –) of a cosecant ratio for any angle?

5. How can the graph of the cosecant function help you to remember the cosecant ratios of the special angles?

6. Sketch the graph of $f(x) = \csc x$ for $-360° \le x \le 360°$.



x	$y=\sec x$
–90º	undefined
–60º	2
–45º	$\sqrt{2} \doteq 1.41421$
–30º	$2/\sqrt{3} \doteq 1.15470$
0º	1
30º	$2/\sqrt{3} \doteq 1.15470$
45º	$\sqrt{2} \doteq 1.41421$
60º	2
90º	undefined
120º	–2
135º	$-\sqrt{2} \doteq -1.41421$
150º	$-2/\sqrt{3} \doteq -1.15470$
180º	–1
210º	$-2/\sqrt{3} \doteq -1.15470$
225º	$-\sqrt{2} \doteq -1.41421$
240º	–2
270º	undefined



Questions about the Secant Function

1. State the domain and range of the secant function.

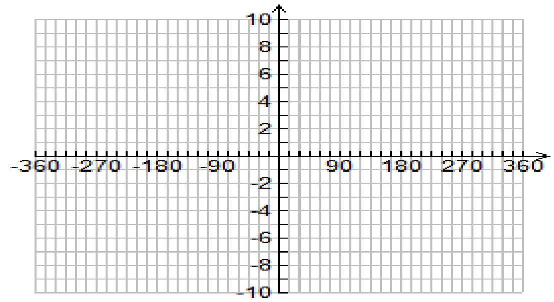
2. Is the secant function one-to-one or many-to-one?

3. State a suitable domain over which the inverse of secant ($\sec^{-1}$) is defined.

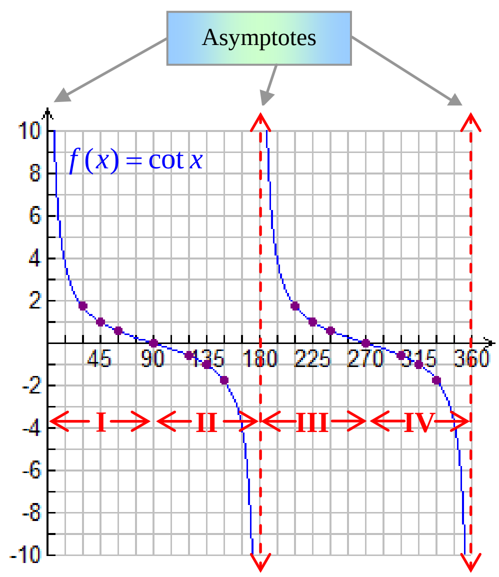
4. How can the graph of the secant function help you to remember the sign (i.e. + or –) of a secant ratio for any angle?

5. How can the graph of the secant function help you to remember the secant ratios of the special angles?

6. Sketch the graph of $f(x)=\sec x$ for $-360° \le x \le 360°$.



x	$y=\cot x$
0º	undefined
30º	$\sqrt{3} \doteq 1.73205$
45º	1
60º	$1/\sqrt{3} \doteq 0.57735$
90º	0
120º	$-1/\sqrt{3} \doteq -0.57735$
135º	–1
150º	$-\sqrt{3} \doteq -1.73205$
180º	undefined
210º	$\sqrt{3} \doteq 1.73205$
225º	1
240º	$1/\sqrt{3} \doteq 0.57735$
270º	0
300º	$-1/\sqrt{3} \doteq -0.57735$
315º	–1
330º	$-\sqrt{3} \doteq -1.73205$
360º	undefined



Questions about the Cotangent Function

1. State the domain and range of the cotangent function.

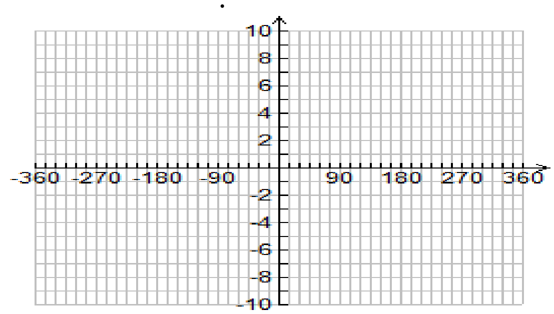
2. Is the cotangent function one-to-one or many-to-one?

3. State a suitable domain over which the inverse of cotangent ($\cot^{-1}$) is defined.

4. How can the graph of the cotangent function help you to remember the sign (i.e. + or –) of a cotangent ratio for any angle?

5. How can the graph of the cotangent function help you to remember the cotangent ratios of the special angles?

6. Sketch the graph of $f(x)=\cot x$ for $-360° \le x \le 360°$.



Sinusoidal Functions

What on Earth is a Sinusoidal Function?

A ***sinusoidal function*** is simply any function that can be obtained by ***stretching*** (compressing) and/or ***translating*** the function $f(x)=\sin x$. That is, a sinusoidal function is any function of the form $g(x)=A\sin\left(k\left(x-p\right)\right)+d = Af\left(k\left(x-p\right)\right)+d$. By applying the general concepts from the unit called "Characteristics of Functions," we can immediately state the following:

Transformation of $f(x)=\sin x$ expressed in Function Notation	***Transformation of $f(x)=\sin x$ expressed in Mapping Notation***
$g(x)=A\sin\left(k\left(x-p\right)\right)+d$	$(x,y)\to\left(k^{-1}x+p, Ay+d\right)$

Vertical Transformations (Apply Operations in Order of Operations)	***Horizontal Transformations (Apply Inverse Operations opposite the Order of Operations)***
1. Stretch or compress vertically by a factor of A. **2.** Translate vertically by d units $(x,y)\to(x,Ay+d)$	**1.** Stretch or compress horizontally by a factor of $k^{-1}=1/k$. **2.** Translate horizontally by p units $(x,y)\to\left(k^{-1}x+p,y\right)$

Since sinusoidal functions look just like ***waves*** and are perfectly suited to modelling wave or wave-like phenomena, special names are given to the quantities *A*, *d*, *p* and *k*.

- *A* is called the ***amplitude***
- *d* is called the ***vertical displacement***
- *p* is called the ***phase shift***
- *k* is called the ***angular frequency***

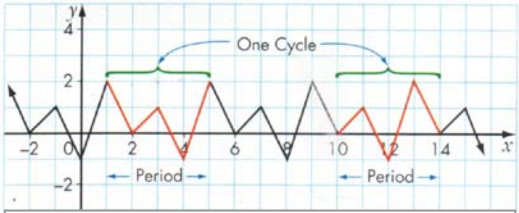
These quantities are described in detail on the next page.

Periodic Functions

There are many naturally occurring and artificially produced phenomena that undergo repetitive cycles. We call such phenomena ***periodic***. Examples of such processes include the following:

- orbits of planets, moons, asteroids, comets, etc
- rotation of planets, moons, asteroids, comets, etc
- phases of the moon
- the tides
- changing of the seasons
- hours of daylight on a given day
- light waves, radio waves, etc
- alternating current (**e.g.** household alternating current has a frequency of 60 Hz, which means that it changes direction 60 times per second)

Intuitively, a function is said to be ***periodic*** if the graph consists of a "basic pattern" that is repeated over and over at ***regular intervals***. One complete pattern is called a ***cycle***.



An example of a ***periodic*** function.

Formally, if there is a number *T* such that $f\left(x+T\right)=f\left(x\right)$ for all values of *x*, then we say that *f* is ***periodic***. The smallest possible positive value of *T* is called the ***period*** of the function. The ***period*** of a periodic function is equal to the ***length of one cycle***.

Exercise

Suppose that the periodic function shown above is called *f*. Evaluate each of the following.

(a) $f(2)$ **(b)** $f(4)$ **(c)** $f(1)$ **(d)** $f(0)$ **(e)** $f(16)$ **(f)** $f(18)$ **(g)** $f(33)$ **(h)** $f(-16)$

(i) $f(-31)$ **(j)** $f(-28)$ **(k)** $f(-27)$ **(l)** $f(-11)$ **(m)** $f(-6)$ **(n)** $f(-9)$ **(o)** $f(-5)$ **(p)** $f(-101)$

Characteristics of Sinusoidal Functions

1. Sinusoidal functions have the general form $f(x) = A\sin\left(k\left(x - p\right)\right) + d$, where *A*, *d*, *p* and *k* are as described above.

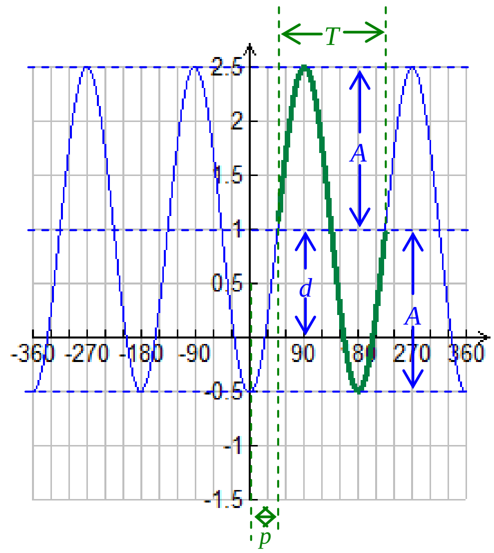
2. Sinusoidal functions are ***periodic***. The letter *T* is used to denote the ***period*** (also called ***primitive period*** or ***wavelength***) of a sinusoidal function. As we shall see very shortly, $T = \frac{360°}{k}$. This makes sinusoidal functions ideal for modelling ***periodic processes*** such as those described on page 25.

3. Sinusoidal functions ***oscillate*** (vary continuously, back and forth) between a maximum and a minimum value. This makes sinusoidal functions ideal for modelling ***oscillatory*** or ***vibratory*** motions. (**e.g.** a pendulum swinging back and forth, a playground swing, a vibrating string, a tuning fork, alternating current, quartz crystal vibrating in a watch, light waves, radio waves, etc)

4. There is a horizontal line that exactly "cuts" a sinusoidal function "in half." The vertical distance from this horizontal line to the peak of the curve is called the ***amplitude***.

Example

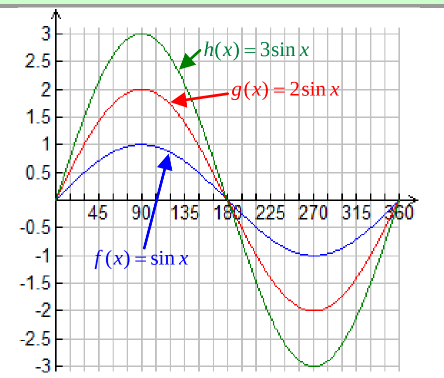
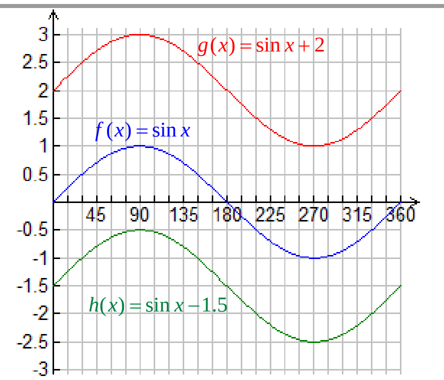
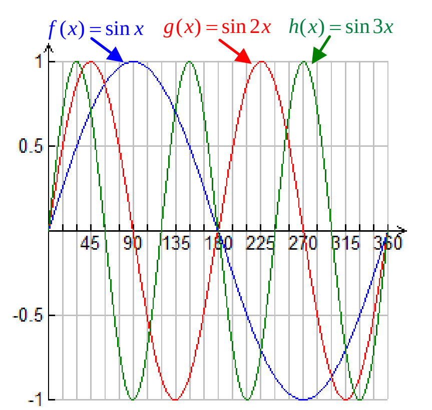
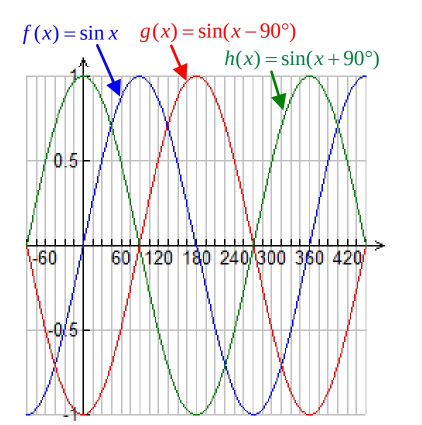
The graph at the right shows a few ***cycles*** of the function $f(x) = 1.5\sin\left(2\left(x - 45°\right)\right) + 1$. One of the cycles is shown as a thick green curve to make it stand out among the others. Notice the following:



- The ***maximum*** value of *f* is 2.5.
- The ***minimum*** value of *f* is –0.5.
- The function *f* oscillates between –0.5 and 2.5.
- The horizontal line with equation $y = 1$ exactly "cuts" the function "in half."
- The ***amplitude*** of this function is $A = 1.5$. This can be seen in a number of ways. Clearly, the vertical distance from the line $y = 1$ to the peak of the curve is 1.5. Also, the amplitude can be calculated by finding half the distance between the maximum and minimum values: $\frac{2.5 - (-0.5)}{2} = \frac{3}{2} = 1.5$
- The ***period***, that is the length of one cycle, is $T = 180°$. This can be seen from the graph (225º – 45º = 180º) or it can be determined by applying your knowledge of transformations. The period of $y = \sin x$ is 360º. Since *f* has undergone a horizontal compression by a factor of $1/2$, its period should be half of 360º, which is 180º. In general, the period $T = \frac{360°}{k}$, where *k* is the angular frequency.
- The function $g(x) = 1.5\sin\left(2\left(x - 45°\right)\right)$ would be "cut in half" by the *x*-axis (the line $y = 0$). The function *f* has exactly the same shape as *g* except that it is ***shifted up by 1 unit***. This is the significance of the ***vertical displacement***. In this example, the vertical displacement $d = 1$.
- The function $g(x) = 1.5\sin\left(2\left(x - 45°\right)\right)$ has exactly the same shape as $h(x) = 1.5\sin 2x$ but is shifted 45º to the right. This horizontal shift is called the ***phase shift***.

Important Exercises

Complete the following table. The first one is done for you.

Function	***A***	***d***	***p***	***k***	***T***	***Description of Transformation***	
f	1	0	0	1	360º	None	
g	2	0	0	1	360º	The graph of $f(x)=\sin x$ is stretched vertically by a factor of 2. The amplitude of *g* is 2.	
h	3	0	0	1	360º	The graph of $f(x)=\sin x$ is stretched vertically by a factor of 3. The amplitude of *g* is 3.	
f							
g							
h							
f							
g							
h							
f							
g							
h							

Example

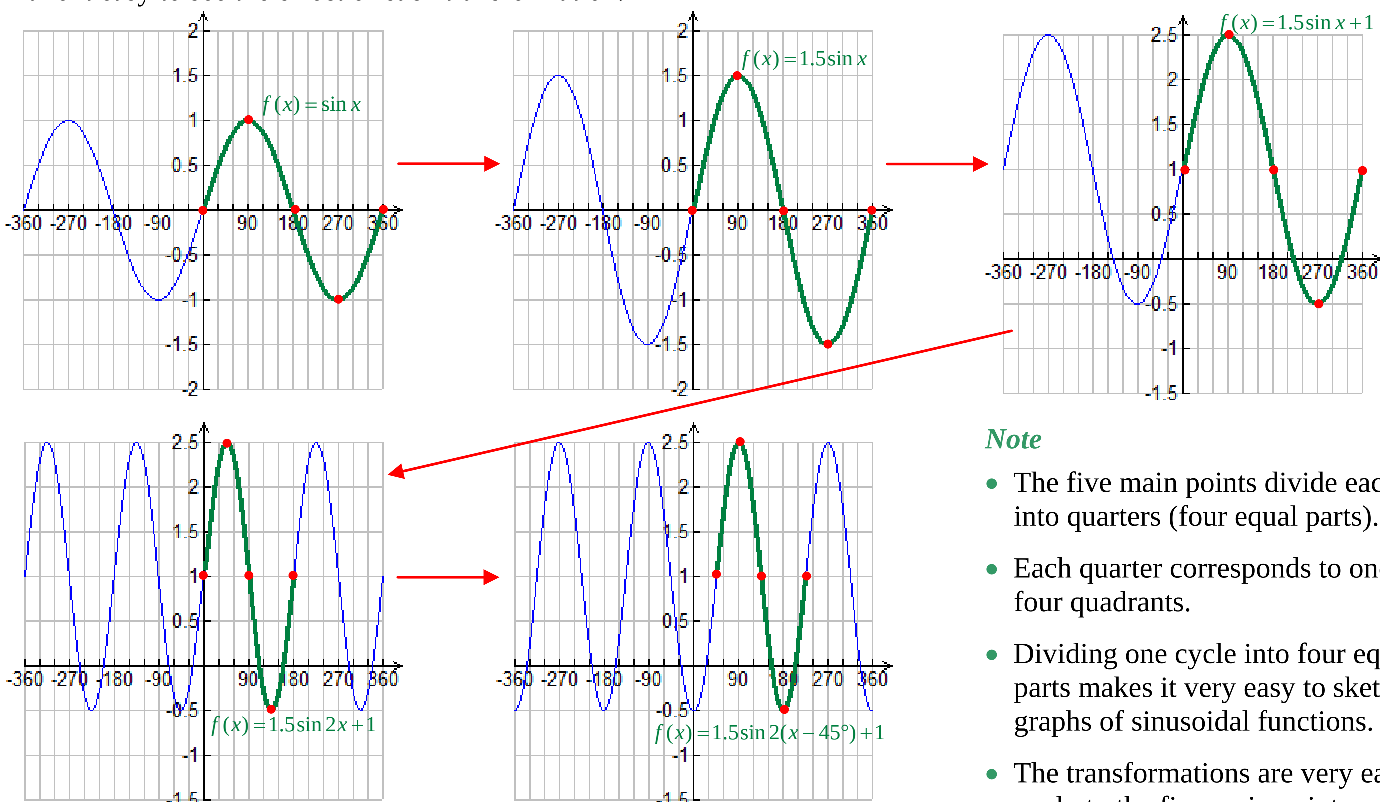
Sketch the graph of $f(x) = 1.5\sin\left(2\left(x - 45°\right)\right) + 1$

Solution

Amplitude	*Vertical Displacement*	*Phase Shift*	*Period*	*Transformations*	
				Vertical	*Horizontal*
$A = 1.5$	$d = 1$	$p = 45°$	$T = \frac{360°}{k} = \frac{360°}{2} = 180°$	Stretch by a factor of 1.5. Translate 1 unit up.	Compress by a factor of $2^{-1} = \frac{1}{2}$ Translate 45° to the right.

Method 1 – The Long Way

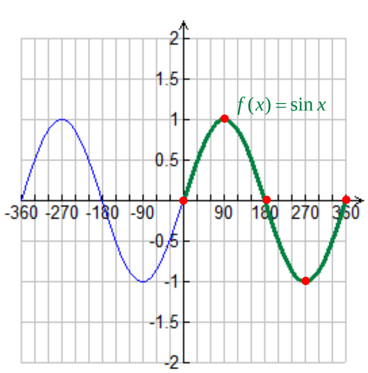
The following shows how the graph of $f(x) = 1.5\sin\left(2\left(x - 45°\right)\right) + 1$ is obtained by beginning with the base function $f(x) = \sin x$ and applying the transformations one-by-one in the correct order. One cycle of $f(x) = \sin x$ is highlighted in green to make it easy to see the effect of each transformation. In addition, ***five main points*** are displayed in red to make it easy to see the effect of each transformation.



Note

- The five main points divide each cycle into quarters (four equal parts).
- Each quarter corresponds to one of the four quadrants.
- Dividing one cycle into four equal parts makes it very easy to sketch the graphs of sinusoidal functions.
- The transformations are very easy to apply to the five main points.

Method 2 – A Much Faster Approach



$f(x) = 1.5\sin\left(2\left(x - 45°\right)\right) + 1$

$A = 1.5,\ d = 1,\ k = 2,\ p = 45°$

$(x, y) \to \left(k^{-1}x + p, Ay + d\right)$

$\therefore (x, y) \to \left(\frac{1}{2}x + 45°, 1.5y + 1\right)$

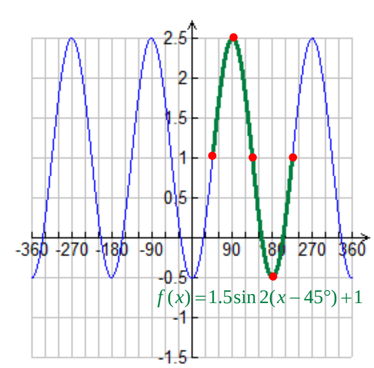
$\therefore (0°, 0) \to \left(\frac{1}{2}(0) + 45°, 1.5(0) + 1\right) = (45°, 1)$

$\therefore (90°, 1) \to \left(\frac{1}{2}(90°) + 45°, 1.5(1) + 1\right) = (90°, 2.5)$

$\therefore (180°, 0) \to \left(\frac{1}{2}(180°) + 45°, 1.5(0) + 1\right) = (135°, 1)$

$\therefore (270°, -1) \to \left(\frac{1}{2}(270°) + 45°, 1.5(-1) + 1\right) = (180°, -0.5)$

$\therefore (360°, 0) \to \left(\frac{1}{2}(360°) + 45°, 1.5(0) + 1\right) = (225°, 1)$



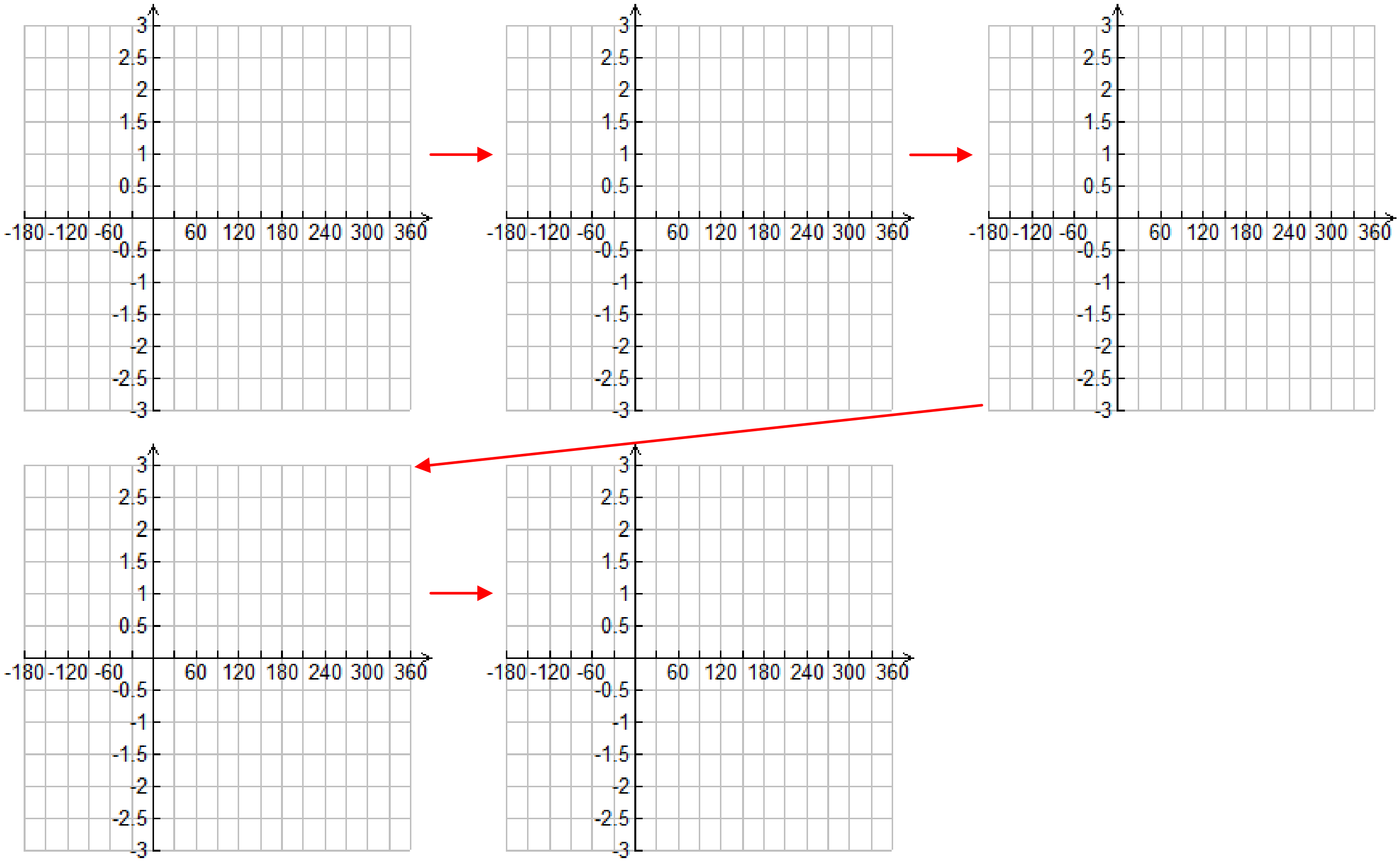
Exercise 1

Using ***both*** of the approaches shown in the previous example, sketch a few cycles of the graph of $f(x) = -2\cos\left(-3\left(x+60°\right)\right)-1$.

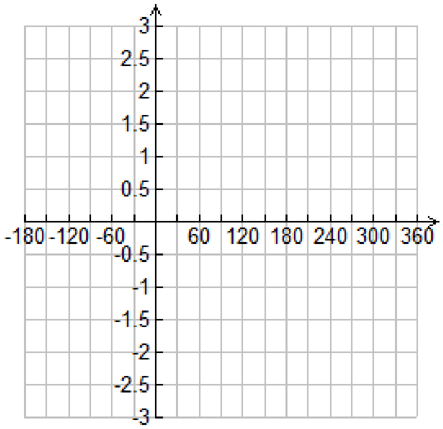
Solution

Amplitude	*Vertical Displacement*	*Phase Shift*	*Period*	*Transformations*	
				Vertical	*Horizontal*

Method 1 – The Long Way



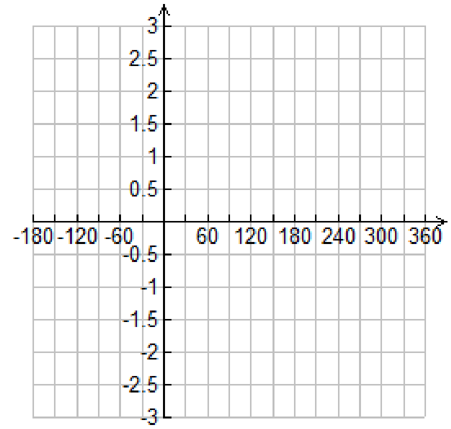
Method 2 – A Much Faster Approach



$f(x) = -2\cos\left(-3\left(x+60°\right)\right)-1$

$A =$ ___, $d =$ ___, $k =$ ___, $p =$ ___

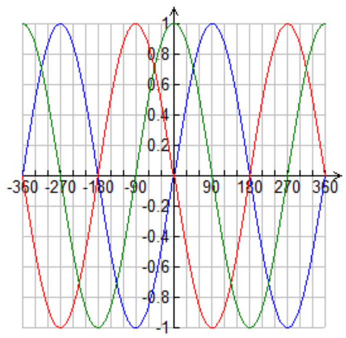
$(x, y) \rightarrow$

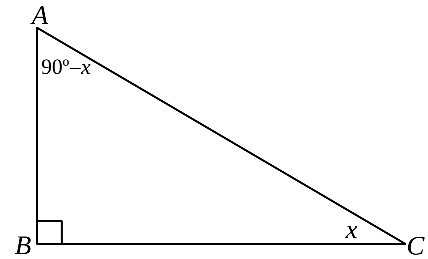


Exercise 2

Complete the following table. The first row is done for you.

<table>
<tr><th>Identity</th><th>Graphical Justification</th><th>Justification using Right Triangles</th></tr>
<tr>
<td>$\sin(90° - x) = \cos x$</td>
<td>Since $\sin(90° - x) = \sin(-1(x - 90°))$, the graph of $y = \sin(90° - x)$ can be obtained by first reflecting $y = \sin x$ in the y-axis, followed by a shift to the right by 90º. Once these transformations are applied, lo and behold, the graph of $y = \cos x$ is obtained!



</td>
<td>


$\cos x = \dfrac{BC}{AC}$

$\sin(90° - x) = \dfrac{BC}{AC}$

$\therefore \cos x = \sin(90° - x)$
</td>
</tr>
<tr><td>$\cos(90° - x) = \sin x$</td><td></td><td></td></tr>
<tr><td>$\cos(90° + x) = -\sin x$</td><td></td><td></td></tr>
</table>

Homework Exercises

There are very few appropriate exercises in our textbook for this section. Do not despair, however. There are plenty of suitable exercises and problems found on this page!

1. Sketch at least three cycles of each of the following functions. In addition, state the domain and range of each, as well as the amplitude, the vertical displacement, the phase shift and the period.

(a) $f(x) = -\cos 3x - 2$

(b) $g(x) = 3\cos(x - 30°)$

(c) $h(x) = 4\sin(2(x + 45°)) - 1$

(d) $p(x) = -2\sin(2x + 60°) + 1$

(e) $q(t) = -5\cos\left(\frac{1}{3}t + 15°\right) + 2$

(f) $r(\theta) = -\frac{2}{3}\sin(-2(\theta - 135°)) - 3$

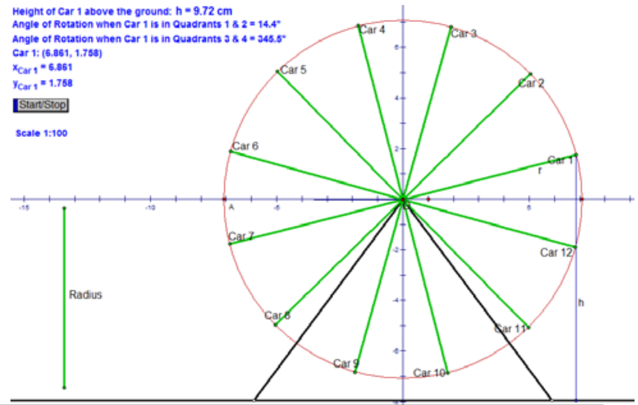
2. Complete the following table.

Identity	*Graphical Justification*	*Justification using Angles of Rotation*
$\sin(180° - \theta) = \sin\theta$		
$\cos(180° - \theta) = -\cos\theta$		
$\sin(-\theta) = -\sin\theta$		
$\cos(-\theta) = \cos\theta$		

Applications of Sinusoidal Functions

Activity 1 – Ferris Wheel Simulation

Either from www.misternolfi.com or from the "I:" drive, load the Geometer's Sketchpad Ferris wheel simulation. Once you understand how to start and stop the animation and how to interpret the given information, answer the questions found below.



Questions

This simulation involves finding out how the height of "Car 1" above the ground ***is related to*** the angle of rotation of the line segment joining "Car 1" to the axis of rotation of the Ferris wheel.

1. Complete the table at the right. Stop the animation each time that a car reaches the *x*-axis (the car ***does not*** need to be exactly on the *x*-axis). Each time that you stop the animation, record the angle of rotation of "Car 1" and its height above the ground. (***Note***: Because of limitations of Geometer's Sketchpad, it was not possible to display the angle of rotation using a single formula. A different formula was used for quadrants III and IV so be careful when recording the data!)

Angle of Rotation of Car 1	*Height of Car 1*

2. Now use the given grid to plot the data that you recorded in question 1. Once you have plotted all the points, join them by sketching a smooth curve that passes through all the points. Does your curve look familiar? Try to write an equation that describes the curve.

3. For this question, you may use either a graphing calculator or TI-Interactive. First, take the data from the above table and create two lists (e.g. L1 and L2). Then perform a ***sinusoidal regression***. (Performing a regression means that the data are "fit" to a mathematical function. A ***sinusoidal regression*** finds the sinusoidal function that ***best fits*** the data.) How does the equation produced by the regression compare to the equation that you wrote in question 2?

4. Now use a graphing calculator or TI-Interactive to graph the function produced by the regression. How does it compare to the graph that you sketched in question 2?

5. Use the equation obtained in question 4 to predict the height of "Car 1" when its angle of rotation is 110º.

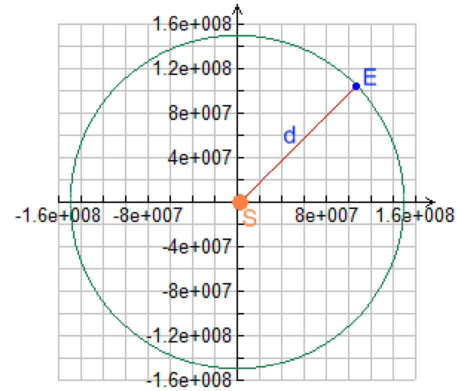
Activity 2 – Earth's Orbit

The table below gives the approximate distance from the Earth to the Sun on certain days of a particular year.

Date	*Day of the Year*	*Earth's Distance (d) from Sun (km)*
January 3	4	1.47098×10^8
February 2	34	1.47433×10^8
March 5	65	1.48349×10^8
April 4	95	1.49599×10^8
May 5	126	1.50848×10^8
June 4	156	1.51763×10^8
July 5	187	1.52098×10^8
August 4	217	1.51763×10^8
September 4	248	1.50848×10^8
October 4	278	1.49599×10^8
November 4	309	1.48349×10^8
December 4	339	1.47433×10^8

Perihelion is the point in the Earth's orbit at which it is closest to the sun. Perihelion occurs in early January.

Aphelion is the point in the Earth's orbit at which it is farthest from the sun. Aphelion occurs in early July.

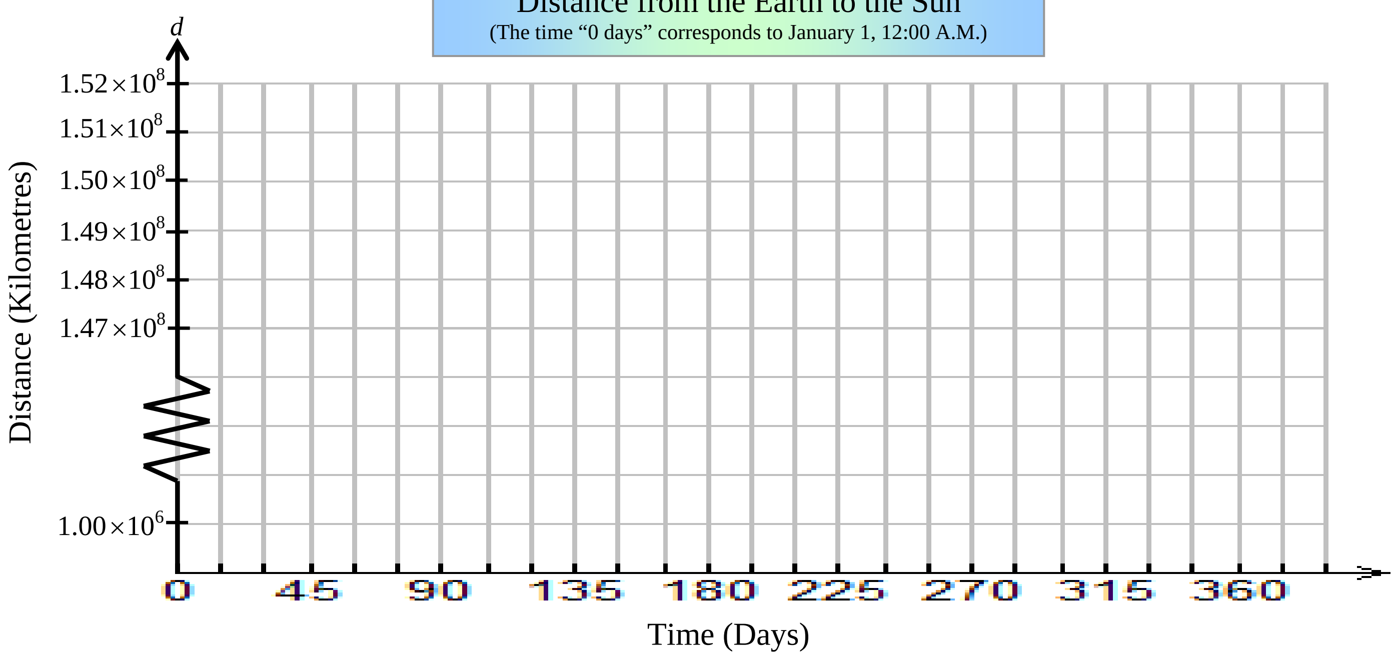


The Earth's orbit around the Sun is an ellipse that is very close to a perfect circle. The Sun is located at one of the two ***foci*** (singular ***focus***) of the ellipse.

Questions

1. Use the grid below to plot the data in the above table. Once you have done so, join the points with a smooth curve. Use your knowledge of trigonometric functions to write an equation of the curve.

Distance from the Earth to the Sun
(The time "0 days" corresponds to January 1, 12:00 A.M.)



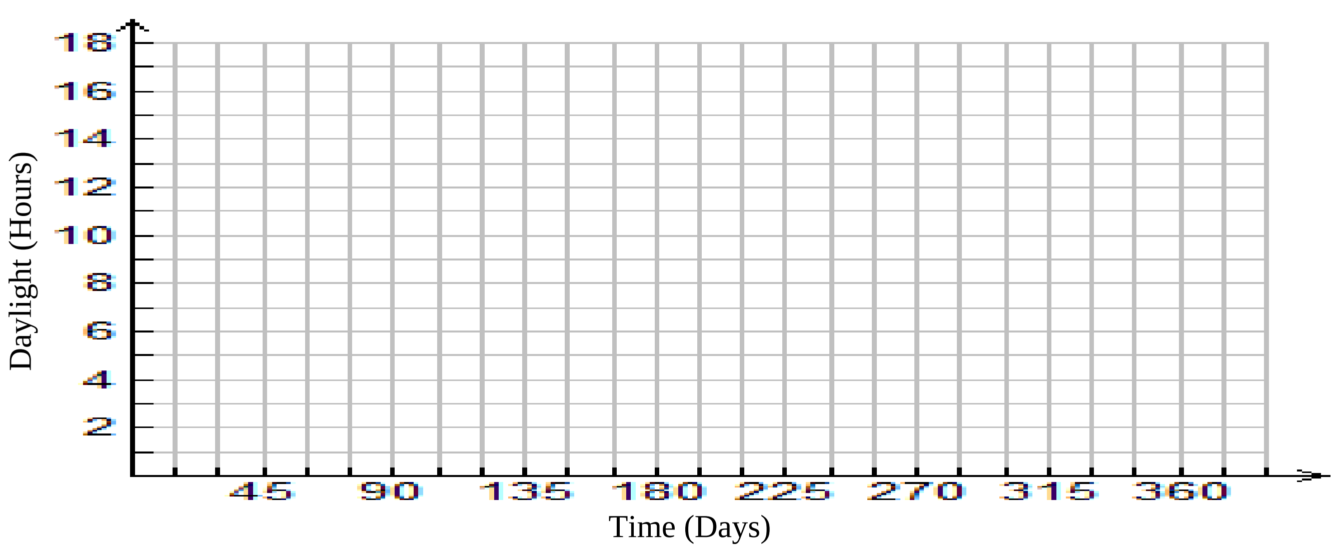
2. Now use TI-Interactive or a graphing calculator to perform a sinusoidal regression on the data in the above table. Compare the equation obtained by regression to the one that you wrote in question 1.
3. Are you surprised that perihelion occurs in early January and that aphelion occurs in early July? Explain.
4. Use the equation obtained in question 2 to predict the distance from the Earth to the Sun on Valentine's Day.
5. Suppose that the Earth's orbit were highly elliptical instead of being nearly a perfect circle. Do you think that life as we know it would still exist? Explain.

Activity 3 – Sunrise/Sunset

The table at the right contains sunrise and sunset data for Toronto, Ontario for the year 2007. (Data obtained from www.sunrisesunset.com .)

1. Complete the table.
2. Use the provided grid to plot a graph of ***number of daylight hours versus the day of the year***. First plot the points and then draw a smooth curve through the points.
3. Write an equation that describes the curve that you obtained in question 2.
4. Use TI-Interactive or a graphing calculator to perform a sinusoidal regression. Enter the values for "day of the year" in L1 and "number of daylight hours" in L2. Compare the equation obtained by the regression to the one that you wrote in question 3.
5. Use your equation to predict the number of daylight hours on December 25, 2007.
6. Suppose that you lived in a town situated exactly on the equator. How would the graph of number of hours of daylight versus day of the year differ from the one for Toronto?

Date	*Day of the Year*	*Sunrise (hh:mm)*	*Sunset (hh:mm)*	*Daylight (hh:mm)*	*Number of daylight hours to the nearest 100th of an hour*
January 1		7:51am	4:51pm		
January 15		7:48am	4:58pm		
January 29		7:38am	5:23pm		
February 12		7:21am	5:42pm		
February 26		7:00am	6:01pm		
March 12		7:36am	7:19pm		
March 26		7:11am	7:36pm		
April 9		6:46am	7:52pm		
April 23		6:23am	8:09pm		
May 7		6:03am	8:25pm		
May 21		5:48am	8:40pm		
June 4		5:38am	8:53pm		
June 18		5:36am	9:01pm		
July 2		5:40am	9:02pm		
July 16		5:51am	8:56pm		
July 30		6:04am	8:44pm		
August 13		6:19am	8:26pm		
August 27		6:35am	8:04pm		
September 10		6:50am	7:39pm		
September 24		7:06am	7:14pm		
October 8		7:22am	6:48pm		
October 22		7:39am	6:25pm		
November 5		6:57am	5:05pm		
November 19		7:15am	4:50pm		
December 3		7:32am	4:42pm		
December 17		7:44am	4:42pm		
December 31		7:50am	4:50pm		



Example

On a particular Ferris wheel, the maximum height of a passenger above the ground is 35 m. The wheel takes 2 minutes to complete one revolution and the passengers board the Ferris wheel 2 m above the ground at the bottom of its rotation.

(a) Sketch two cycles of the graph of height of passenger (in metres) versus time (in seconds). Assume that the passenger is 2 m above the ground at time $t = 0$.

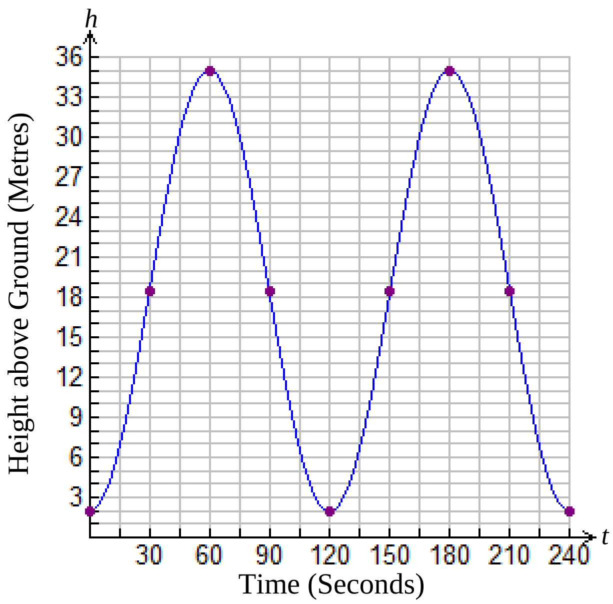
(b) Write an equation of the graph that you obtained in part (a).

(c) How high is the passenger after 25 s?

(d) If the ride lasts six minutes, at what times will the passenger be at the maximum height?

Solution

(a)



(b) Maximum Height = 35 m, Minimum Height = 2 m

$\therefore A = (35-2) \div 2 = 16.5$

From the graph, the following is obvious.

$d = 18.5$, $p = -30°$

$\because T = \dfrac{360°}{k}$ and $T = 120°$ $\quad \therefore k = 3$

$\therefore h(t) = 16.5\sin(3(t-30°)) + 18.5$

(c) $h(25) = 16.5\sin(3(25-30°)) + 18.5 \doteq 14.2$

At 25 seconds, the passenger was about 14.2 m above the ground.

(d) The passenger is at the maximum height whenever $h(t) = 35$. From the graph we can see that this occurs at $t = 60$ s and $t = 180$ s. Since h is periodic, $h(180° + 120°) = h(180°) = 35$ (120º is the period).

Therefore, the passenger is at the maximum height at 60 s, 180 s and 300 s.

Appendix – Ontario Ministry of Education Guidelines – Unit 2

OVERALL EXPECTATIONS

By the end of this course, students will:

1. determine the values of the trigonometric ratios for angles less than 360º; prove simple trigonometric identities; and solve problems using the primary trigonometric ratios, the sine law, and the cosine law;
2. demonstrate an understanding of periodic relationships and sinusoidal functions, and make connections between the numeric, graphical, and algebraic representations of sinusoidal functions;
3. identify and represent sinusoidal functions, and solve problems involving sinusoidal functions, including problems arising from real-world applications.

SPECIFIC EXPECTATIONS

1. Determining and Applying Trigonometric Ratios

By the end of this course, students will:

1.1 determine the exact values of the sine, cosine, and tangent of the special angles: 0º, 30º, 45º, 60º, and 90º

1.2 determine the values of the sine, cosine, and tangent of angles from 0º to 360º, through investigation using a variety of tools (e.g., dynamic geometry software, graphing tools) and strategies (e.g., applying the unit circle; examining angles related to special angles)

1.3 determine the measures of two angles from 0º to 360º for which the value of a given trigonometric ratio is the same

1.4 define the secant, cosecant, and cotangent ratios for angles in a right triangle in terms of the sides of the triangle (e.g., $\sec A = \frac{hypotenuse}{adjacent}$), and relate these ratios to the cosine, sine, and tangent ratios (e.g., $\sec A = \frac{1}{\cos A}$)

1.5 prove simple trigonometric identities, using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$; the quotient identity $\tan x = \frac{\sin x}{\cos x}$; and the reciprocal identities $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$, and $\cot x = \frac{1}{\tan x}$

Sample problem: Prove that $1 - \cos^2 x = \sin x \cos x \tan x$.

1.6 pose problems involving right triangles and oblique triangles in two-dimensional settings, and solve these and other such problems using the primary trigonometric ratios, the cosine law, and the sine law (including the ambiguous case)

1.7 pose problems involving right triangles and oblique triangles in three-dimensional settings, and solve these and other such problems using the primary trigonometric ratios, the cosine law, and the sine law

Sample problem: Explain how a surveyor could find the height of a vertical cliff that is on the other side of a raging river, using a measuring tape, a theodolite, and some trigonometry. Determine what the surveyor might measure, and use hypothetical values for these data to calculate the height of the cliff.

2. Connecting Graphs and Equations of Sinusoidal Functions

By the end of this course, students will:

2.1 describe key properties (e.g., cycle, amplitude, period) of periodic functions arising from real-world applications (e.g., natural gas consumption in Ontario, tides in the Bay of Fundy), given a numeric or graphical representation

2.2 predict, by extrapolating, the future behaviour of a relationship modelled using a numeric or graphical representation of a periodic function (e.g., predicting hours of daylight on a particular date from previous measurements; predicting natural gas consumption in Ontario from previous consumption)

2.3 make connections between the sine ratio and the sine function and between the cosine ratio and the cosine function by graphing the relationship between angles from 0° to 360° and the corresponding sine ratios or cosine ratios, with or without technology (e.g., by generating a table of values using a calculator; by unwrapping the unit circle), defining this relationship as the function $f(x) = \sin x$ or $f(x) = \cos x$, and explaining why the relationship is a function

2.4 sketch the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ for angle measures expressed in degrees, and determine and describe their key properties (i.e., cycle, domain, range, intercepts, amplitude, period, maximum and minimum values, increasing/decreasing intervals)

2.5 determine, through investigation using technology, the roles of the parameters a, k, d, and c in functions of the form $y = af(k(x - d)) + c$, where $f(x) = \sin x$ or $f(x) = \cos x$ with angles expressed in degrees, and describe these roles in terms of transformations on the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ (i.e., translations; reflections in the axes; vertical and horizontal stretches and compressions to and from the x- and y-axes)

Sample problem: Investigate the graph $f(x) = 2\sin(x - d) + 10$ for various values of d, using technology, and describe the effects of changing d in terms of a transformation.

2.6 determine the amplitude, period, phase shift, domain, and range of sinusoidal functions whose equations are given in the form $f(x) = a\sin(k(x - d)) + c$ or $f(x) = a\cos(k(x - d)) + c$

2.7 sketch graphs of $y = af(k(x - d)) + c$ by applying one or more transformations to the graphs of $f(x) = \sin x$ and $f(x) = \cos x$, and state the domain and range of the transformed functions

Sample problem: Transform the graph of $f(x) = \cos x$ to sketch $g(x) = 3\cos 2x - 1$, and state the domain and range of each function.

2.8 represent a sinusoidal function with an equation, given its graph or its properties

Sample problem: A sinusoidal function has an amplitude of 2 units, a period of 180°, and a maximum at (0, 3). Represent the function with an equation in two different ways.

3. Solving Problems Involving Sinusoidal Functions

By the end of this course, students will:

3.1 collect data that can be modelled as a sinusoidal function (e.g., voltage in an AC circuit, sound waves), through investigation with and without technology, from primary sources, using a variety of tools (e.g., concrete materials, measurement tools such as motion sensors), or from secondary sources (e.g., websites such as Statistics Canada, E-STAT), and graph the data

Sample problem: Measure and record distance–time data for a swinging pendulum, using a motion sensor or other measurement tools, and graph the data.

3.2 identify periodic and sinusoidal functions, including those that arise from real-world applications involving periodic phenomena, given various representations (i.e., tables of values, graphs, equations), and explain any restrictions that the context places on the domain and range

Sample problem: Using data from Statistics Canada, investigate to determine if there was a period of time over which changes in the population of Canadians aged 20–24 could be modelled using a sinusoidal function.

3.3 determine, through investigation, how sinusoidal functions can be used to model periodic phenomena that do not involve angles

Sample problem: Investigate, using graphing technology in degree mode, and explain how the function $h(t) = 5\sin(30(t + 3))$ approximately models the relationship between the height and the time of day for a tide with an amplitude of 5 m, if high tide is at midnight.

3.4 predict the effects on a mathematical model (i.e., graph, equation) of an application involving periodic phenomena when the conditions in the application are varied (e.g., varying the conditions, such as speed and direction, when walking in a circle in front of a motion sensor)

Sample problem: The relationship between the height above the ground of a person riding a Ferris wheel and time can be modelled using a sinusoidal function. Describe the effect on this function if the platform from which the person enters the ride is raised by 1 m and if the Ferris wheel turns twice as fast.

3.5 pose problems based on applications involving a sinusoidal function, and solve these and other such problems by using a given graph or a graph generated with technology from a table of values or from its equation

Sample problem: The height above the ground of a rider on a Ferris wheel can be modelled by the sinusoidal function $h(t) = 25\sin(3(t - 30)) + 27$, where $h(t)$ is the height, in metres, and t is the time, in seconds. Graph the function, using graphing technology in degree mode, and determine the maximum and minimum heights of the rider, the height after 30 s, and the time required to complete one revolution.