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SEQUENCES

Intuitive Definition of a Sequence

Intuitively, a sequence is simply any ***finite*** or (countably) ***infinite*** ordered list of numbers (or any other types of objects). When a sequence is written out, the numbers (or other objects) are separated by commas. Here are some examples.

(a) 1, 3, 5, 7, 9, 11, … **(b)** 2, 4, 6, 8, 10, … **(c)** 1, 4, 9, 16, 25, … **(d)** , …

Exercise

For the sequences in (a), (b) and (c), predict the next three ***terms***. Are you certain that your answers are the only possible correct answers?

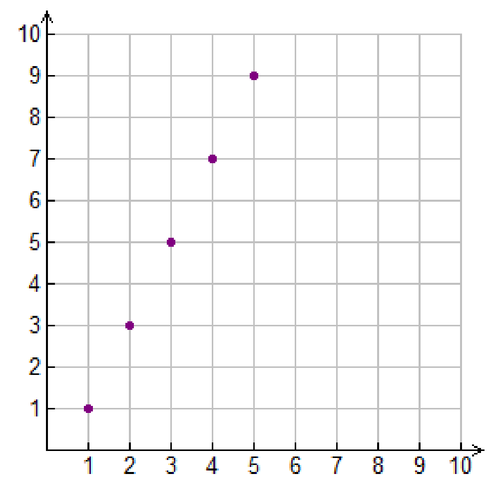
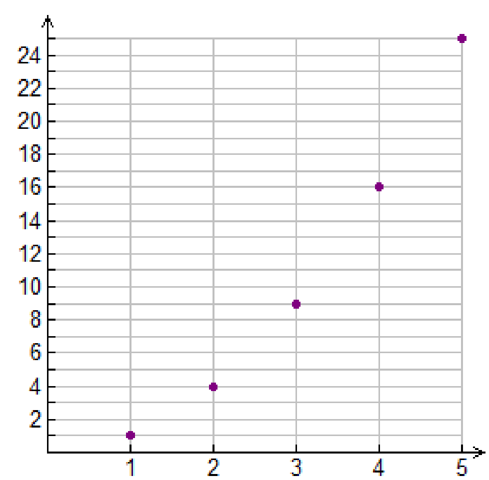
Terminology and Notation of Sequences

- The members of a sequence are called ***elements*** or ***terms***.
- The symbol t_n is used to denote the n^{th} term of a sequence. For example, t_1 represents the first term, t_2 represents the second term and so on. In the example given in (a) above, $t_1 = 1$, $t_2 = 3$, $t_3 = 5$, $t_4 = 7$, $t_5 = 9$ and $t_6 = 9$.

Formal Definition of a Sequence

From a formal perspective, a ***sequence*** is simply a ***discrete function*** whose domain is restricted to the natural numbers. Stated more explicitly, a ***sequence*** is a ***function*** whose ***domain*** is $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ (the set of ***natural numbers***) or some subset of $\mathbb{N}$. (The set of natural numbers is often called the set of ***counting*** numbers.)

Examples

Sequence Written as an a Ordered List	*Formula for nth Term*		*Graph*
	Term Notation	*Function Notation*	
1, 3, 5, 7, 9, 11, …	$t_n = 2n - 1, n \in \mathbb{N}$	$f(n) = 2n - 1, n \in \mathbb{N}$	
1, 4, 9, 16, 25, …	$t_n = n^2, n \in \mathbb{N}$	$f(n) = n^2, n \in \mathbb{N}$	

THE FIBONACCI SEQUENCE

Background – Who was Fibonacci?

Sources: http://en.wikipedia.org/wiki/Fibonacci, http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibBio.html

Leonardo of Pisa (Leonardo da Pisa in Italian, circa 1170 – circa 1250) is also known as Leonardo Pisano, Leonardo Bigollo, Leonardo Bonacci, Leonardo Fibonacci or most commonly, simply ***Fibonacci***. He was an Italian mathematician considered by some the most talented European mathematician of the Middle Ages. Fibonacci is best known to the modern world for the following:

- The spreading of the Hindu-Arabic numeral system (also called the Indo-Arabic numeral system) in Europe, primarily through the publication in the early 13th century of his *Book of Calculation*, the *Liber Abaci*.
- A modern number sequence named after him known as the Fibonacci numbers, which he did not discover but used as an example in the *Liber Abaci*.

Leonardo, born in Pisa, Italy (of leaning tower fame) was given the nickname Fibonacci posthumously (i.e. after his death). This sobriquet (synonym of "nickname") was most likely derived from *filius Bonacci*, meaning son of Bonaccio in Latin, because his father Guglielmo (pronounced "gool-yel'-mo") was nicknamed Bonaccio ("good natured" or "simple"). Guglielmo directed a trading post in Bugia, a port east of Algiers in the Almohad dynasty's sultanate in North Africa (now Bejaia, Algeria). As a young boy, Leonardo travelled there to help his father. This is where he learned about the Hindu-Arabic numeral system.

Recognizing that arithmetic with Hindu numerals is much simpler and much more efficient than with Roman numerals, Fibonacci travelled throughout the Mediterranean world to study under the leading Arab mathematicians of the time. Leonardo returned from his travels around 1200. In 1202, he published what he had learned in *Liber Abaci* (*Book of Abacus* or *Book of Calculation*), and thereby introduced Hindu-Arabic numerals to Europe.

The Fibonacci Sequence

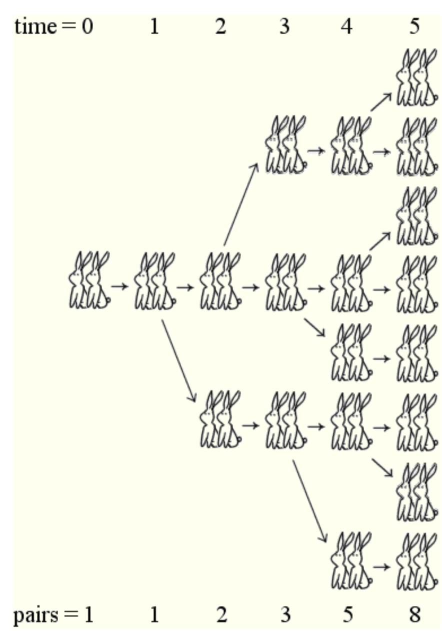
It should be emphasized that Fibonacci ***did not discover*** the sequence that now bears his name. The earliest known reference to what we now call Fibonacci numbers is contained in *Chandas Shastra*, a book on meters (rhythmic patterns) by an Indian mathematician named Pingala (circa 500 BC). As documented by Donald Knuth in *The Art of Computer Programming*, this sequence was also described by the Indian mathematicians Gopala and Hemachandra in 1150, who were investigating the possible ways of exactly packing items of length 1 and 2 in a bin.

The Rabbit Problem

Fibonacci introduced the sequence to Europe in 1202 through the following problem in *Liber Abaci*.

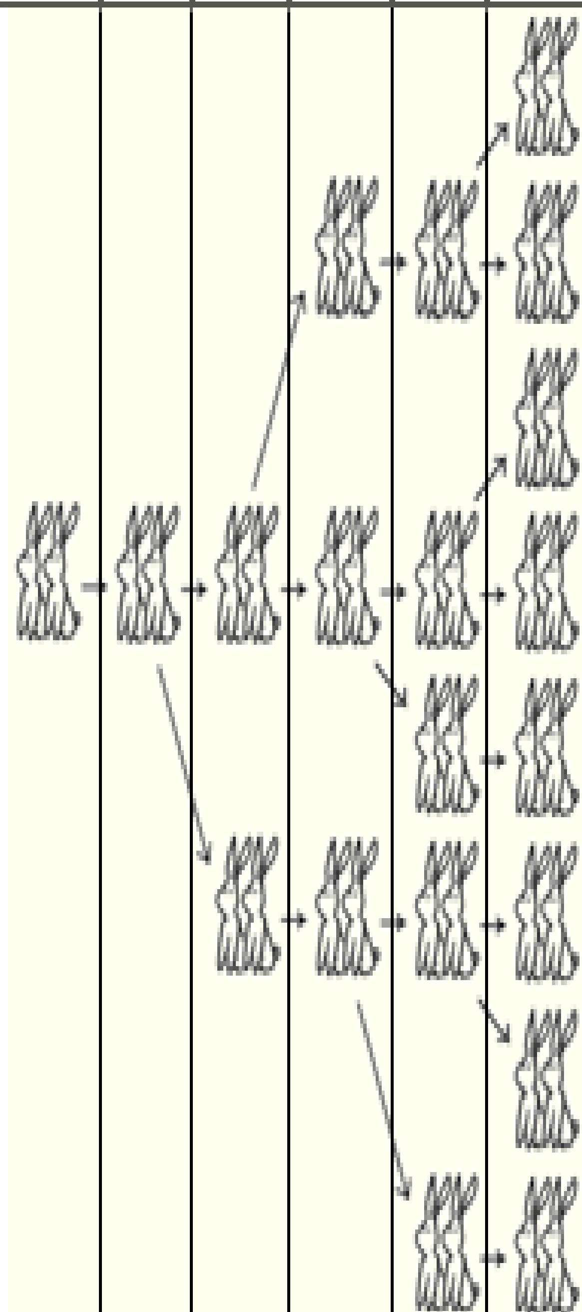
Under the following conditions, ***how many rabbits will there be after one year***?

- in the first month, there is just one female and one male rabbit, both born on January 1
- each month is of equal length
- newborn pairs become fertile from their second month on
- each month, every fertile pair begets a new mixed pair (one male, one female)
- no rabbit dies during the twelve month period



Exercises

1. Complete the diagram on the previous page to determine number of rabbits present at the end of one year.

Time	0	1	2	3	4	5	6	7	8	9	10	11	12
													
Pairs	1	1	2	3	5	8							

2. Now focus on the ***sequence*** of the number of pairs of rabbits present in each month. Describe the pattern that you see. Once you have done your best to spot the pattern, turn to the next page to find out if you are correct.

How to Generate the Fibonacci Sequence

If you have performed the exercise on the previous page correctly, you should have obtained the following sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233$$

$$8 \quad + \quad 13 \quad = \quad 21$$

Any given term of the sequence is obtained by adding the previous two terms. In terms of the generations of rabbits, this means that the number of pairs of rabbits in a given month is equal to the sum of the pairs of rabbits in the previous two months.

Formal Definition of the Fibonacci Sequence

The Fibonacci sequence can be defined formally in the following way:

$$\begin{cases} t_1 = 2 \\ t_2 = 1 \\ t_n = t_{n-2} + t_{n-1}, n \geq 3 \end{cases}$$

Translated into plain English, this means the following:

$$\begin{cases} \text{The first term is 1.} \\ \text{The second term is 1.} \\ \text{The } n^{\text{th}} \text{ term is obtained by} \\ \text{adding terms } n-1 \text{ and } n-2. \end{cases}$$

Explicitly Defined Sequences versus Recursively Defined Sequences

The sequences described on page 2 are defined in terms of an ***explicit formula*** (also called a ***closed formula***). To calculate the value of t_n, it is only necessary to know the value of n. The definition of the Fibonacci sequence, however, is given in a different manner. The value of n in and of itself is not enough to calculate the value of t_n. Besides knowing the value of n, it is also necessary to know the value of one or more previous terms. (In the case of the Fibonacci sequence, one must know the values of the ***two*** previous terms to calculate the value of t_n.) Such sequences are said to be defined ***recursively***.

Exercise

Use a dictionary of your choice to find the meanings of the following terms:

Explicit	***Recursive***

Example of an Explicitly Defined Sequence

e.g. 1, 2, 4, 8, 16, 32, 64, …

This is an example of a ***geometric sequence***. Let t_n represent the n^{th} term of this sequence. If we can find a formula that expresses t_n in terms of n, then we say that t_n is ***defined explicitly in terms of n***. The following is an ***explicit definition*** of the sequence given above:

$$t_n = 2^{n-1}, n \in \mathbb{N}$$

Notice that you can determine any term in the sequence ***just by knowing the value of n***.

Example of a Recursively Defined Sequence

e.g. 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, …

This is a variation of the Fibonacci sequence known as the ***Lucas Numbers***. Being closely related to the Fibonacci sequence, it can be defined in exactly the same manner.

$$\begin{cases} t_1 = 2 \\ t_2 = 1 \\ t_n = t_{n-2} + t_{n-1}, n \geq 3 \end{cases}$$

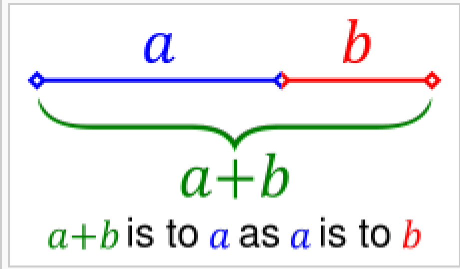
Notice that in this case you need to know the values of the two previous terms of the sequence to calculate t_n. ***Knowing n is not enough!***

(See http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html for a great site on how the Fibonacci sequence arises in nature. In addition, see http://en.wikipedia.org/wiki/Fibonacci_number for a detailed discussion of the Fibonacci numbers.)

Explicit (Closed) Formula for the Fibonacci Sequence

If you have read the *Da Vinci Code* by Dan Brown or if you simply are fascinated by numbers, then you probably have encountered a number called the ***golden ratio***. As shown in the diagram below, the numbers a and b are said to be in the golden ratio φ ("phi") if

$$\frac{a+b}{a}=\frac{a}{b}=\varphi$$



The "golden section" is a line segment sectioned into two according to the **golden ratio**. The total length ***a+b*** is to the longer segment ***a*** as ***a*** is to the shorter segment ***b***.

By the definition of φ, $a = \varphi b$. Therefore, $\dfrac{\varphi b+b}{\varphi b}=\dfrac{\varphi b}{b}$.

$$\therefore \frac{b(\varphi+1)}{\varphi b}=\frac{\varphi b}{b}$$

$$\therefore \frac{\varphi+1}{\varphi}=\varphi$$

$$\therefore \varphi^2=\varphi+1$$

$$\therefore \varphi^2-\varphi-1=0$$

By applying the quadratic formula, we find that $\varphi=\dfrac{1+\sqrt{5}}{2}\doteq 1.6180339887$. Strangely enough, although at first glance the Fibonacci sequence seems to be completely unrelated to the golden ratio, it turns out that an explicit formula for the Fibonacci numbers involves φ! The formula is given below without proof. Let $F(n)$ represent the nth Fibonacci number. Then, $F(n)=\dfrac{\varphi^n-(1-\varphi)^n}{\sqrt{5}}=\dfrac{\varphi^n-(-\varphi)^{-n}}{\sqrt{5}}, n\in\mathbb{N}$.

(For more information on the golden ratio, see http://en.wikipedia.org/wiki/Golden_ratio.)

Exercises

1. Prove that $\varphi-1=\dfrac{1}{\varphi}$. Use this result to show that $\dfrac{\varphi^n-(1-\varphi)^n}{\sqrt{5}}=\dfrac{\varphi^n-(-\varphi)^{-n}}{\sqrt{5}}$.

2. Show numerically that for n large enough, $\dfrac{F(n+1)}{F(n)}\doteq\varphi$ and that $\dfrac{F(n)}{F(n+1)}\doteq\varphi-1$.

Homework

pp. 433 – 435, #1f, 2afhik, 3bfjl, 4ag, 5cd, 7, 8, 9, 12, 14

pp. 461 – 464, #1efgh, 2ef, 3d, 4d, 9

Appendix – Ontario Ministry of Education Guidelines – Unit 3

OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of recursive sequences, represent recursive sequences in a variety of ways, and make connections to Pascal's triangle;
2. demonstrate an understanding of the relationships involved in arithmetic and geometric sequences and series, and solve related problems;
3. make connections between sequences, series, and financial applications, and solve problems involving compound interest and ordinary annuities.

SPECIFIC EXPECTATIONS

1. Representing Sequences

By the end of this course, students will:

1.1 make connections between sequences and discrete functions, represent sequences using function notation, and distinguish between a discrete function and a continuous function [e.g., $f(x) = 2x$, where the domain is the set of natural numbers, is a discrete linear function and its graph is a set of equally spaced points; $f(x) = 2x$, where the domain is the set of real numbers, is a continuous linear function and its graph is a straight line]

1.2 determine and describe (e.g., in words; using flow charts) a recursive procedure for generating a sequence, given the initial terms (e.g., 1, 3, 6, 10, 15, 21, …), and represent sequences as discrete functions in a variety of ways (e.g., tables of values, graphs)

1.3 connect the formula for the nth term of a sequence to the representation in function notation, and write terms of a sequence given one of these representations or a recursion formula

1.4 represent a sequence algebraically using a recursion formula, function notation, or the formula for the nth term [e.g., represent 2, 4, 8, 16, 32, 64, … as $t_1 = 2$; $t_n = 2t_{n-1}$, as $f(n) = 2^n$, or as $t_n = 2^n$, or represent $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$ as $t_1 = \frac{1}{2}$; $t_n = t_{n-1} + \frac{1}{n(n+1)}$, as $f(n) = \frac{n}{n+1}$, or as $t_n = \frac{n}{n+1}$, where n is a natural number], and describe the information that can be obtained by inspecting each representation (e.g., function notation or the formula for the nth term may show the type of function; a recursion formula shows the relationship between terms)

Sample problem: Represent the sequence 0, 3, 8, 15, 24, 35, … using a recursion formula, function notation, and the formula for the nth term. Explain why this sequence can be described as a discrete quadratic function. Explore how to identify a sequence as a discrete quadratic function by inspecting the recursion formula.

1.5 determine, through investigation, recursive patterns in the Fibonacci sequence, in related sequences, and in Pascal's triangle, and represent the patterns in a variety of ways (e.g., tables of values, algebraic notation)

1.6 determine, through investigation, and describe the relationship between Pascal's triangle and the expansion of binomials, and apply the relationship to expand binomials raised to whole-number exponents [e.g., $(1 + x)^4$, $(2x - 1)^5$, $(2x - y)^6$, $(x^2 + 1)^5$]

2. Investigating Arithmetic and Geometric Sequences and Series

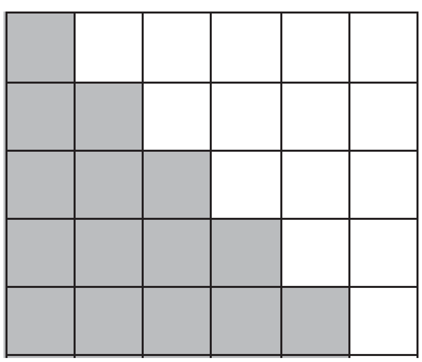
By the end of this course, students will:

2.1 identify sequences as arithmetic, geometric, or neither, given a numeric or algebraic representation

2.2 determine the formula for the general term of an arithmetic sequence [i.e., $t_n = a + (n-1)d$] or geometric sequence (i.e., $t_n = ar^{n-1}$), through investigation using a variety of tools (e.g., linking cubes, algebra tiles, diagrams, calculators) and strategies (e.g., patterning; connecting the steps in a numerical example to the steps in the algebraic development), and apply the formula to calculate any term in a sequence

2.3 determine the formula for the sum of an arithmetic or geometric series, through investigation using a variety of tools (e.g., linking cubes, algebra tiles, diagrams, calculators) and strategies (e.g., patterning; connecting the steps in a numerical example to the steps in the algebraic development), and apply the formula to calculate the sum of a given number of consecutive terms

Sample problem: Given the following array built with grey and white connecting cubes, investigate how different ways of determining the total number of grey cubes can be used to evaluate the sum of the arithmetic series 1 + 2 + 3 + 4 + 5. Extend the series, use patterning to make generalizations for finding the sum, and test the generalizations for other arithmetic series.



2.4 solve problems involving arithmetic and geometric sequences and series, including those arising from real-world applications

3. Solving Problems Involving Financial Applications

By the end of this course, students will:

3.1 make and describe connections between simple interest, arithmetic sequences, and linear growth, through investigation with technology (e.g., use a spreadsheet or graphing calculator to make simple interest calculations, determine first differences in the amounts over time, and graph amount versus time)

Sample problem: Describe an investment that could be represented by the function $f(x) = 500(1 + 0.05x)$.

3.2 make and describe connections between compound interest, geometric sequences, and exponential growth, through investigation with technology (e.g., use a spreadsheet to make compound interest calculations, determine finite differences in the amounts over time, and graph amount versus time)

Sample problem: Describe an investment that could be represented by the function $f(x) = 500(1.05)^x$.

3.3 solve problems, using a scientific calculator, that involve the calculation of the amount, A (also referred to as future value, FV), the principal, P (also referred to as present value, PV), or the interest rate per compounding period, i, using the compound interest formula in the form $A = P(1 + i)^n$ [or $FV = PV(1 + i)^n$]

Sample problem: Two investments are available, one at 6% compounded annually and the other at 6% compounded monthly. Investigate graphically the growth of each investment, and determine the interest earned from depositing $1000 in each investment for 10 years.

3.4 determine, through investigation using technology (e.g., scientific calculator, the TVM Solver on a graphing calculator, online tools), the number of compounding periods, n, using the compound interest formula in the form $A = P(1 + i)^n$ [or $FV = PV(1 + i)^n$]; describe strategies (e.g., guessing and checking; using the power of a power rule for exponents; using graphs) for calculating this number; and solve related problems

3.5 explain the meaning of the term *annuity*, and determine the relationships between ordinary simple annuities (i.e., annuities in which payments are made at the *end* of each period, and compounding and payment periods are the same), geometric series, and exponential growth, through investigation with technology (e.g., use a spreadsheet to determine and graph the future value of an ordinary simple annuity for varying numbers of compounding periods; investigate how the contributions of each payment to the future value of an ordinary simple annuity are related to the terms of a geometric series)

3.6 determine, through investigation using technology (e.g., the TVM Solver on a graphing calculator, online tools), the effects of changing the conditions (i.e., the payments, the frequency of the payments, the interest rate, the compounding period) of ordinary simple annuities (e.g., long-term savings plans, loans)

Sample problem: Compare the amounts at age 65 that would result from making an annual deposit of $1000 starting at age 20, or from making an annual deposit of $3000 starting at age 50, to an RRSP that earns 6% interest per annum, compounded annually. What is the total of the deposits in each situation?

3.7 solve problems, using technology (e.g., scientific calculator, spreadsheet, graphing calculator), that involve the amount, the present value, and the regular payment of an ordinary simple annuity (e.g., calculate the total interest paid over the life of a loan, using a spreadsheet, and compare the total interest with the original principal of the loan)