MCR 3U9 — **Semester 2, 2015 - 2016**

Grade 11 Pre-AP Mathematics
Trigonometry Unit Final Test

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Victim: Mr. Solutions

Inspirational once again Mr. S.!!

KU	APP	TIPS	COM
21/21	23/23	21/21	10/10

1. Use the three methods indicated below to demonstrate that the equation $\sin(\theta+\frac{\pi}{2})=\cos\theta$ ***is*** an identity.

(a) Compound angle identity **(3 KU)**

$L.S. = \sin(\theta+\frac{\pi}{2})$

$= \sin\theta\cos\frac{\pi}{2}+\cos\theta\sin\frac{\pi}{2}$

$= \sin\theta(0)+\cos\theta(1)$

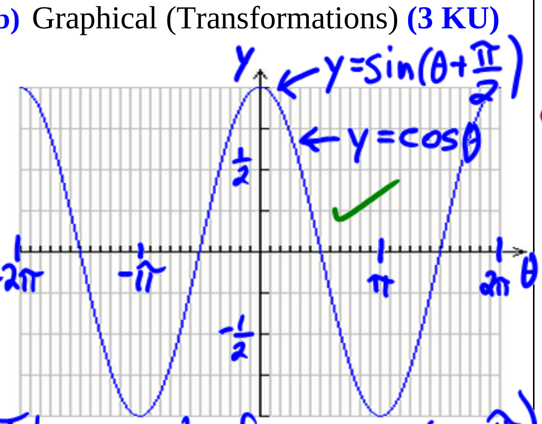
$= 0+\cos\theta$

$= \cos\theta$

$= R.S.$

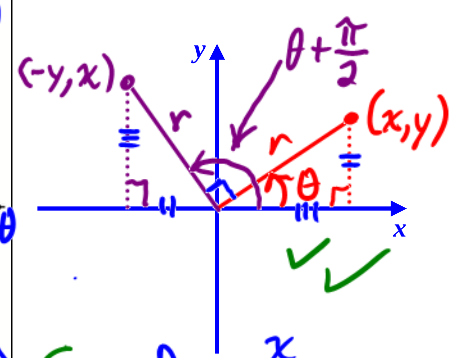
$\therefore \sin(\theta+\frac{\pi}{2})=\cos\theta$ is an identity

(b) Graphical (Transformations) **(3 KU)**



The graph of $y=\sin(\theta+\frac{\pi}{2})$ can be obtained by translating the graph of $y=\sin\theta$ $\frac{\pi}{2}$ to the left. This produces the graph of $y=\cos\theta$. $\therefore \sin(\theta+\frac{\pi}{2})=\cos\theta$

(c) Angles of Rotation **(3 KU)**



$\cos\theta=\frac{x}{r}$

$\sin(\theta+\frac{\pi}{2})=\frac{x}{r}$

$\therefore \sin(\theta+\frac{\pi}{2})=\cos\theta$

2. Using the methods listed below, demonstrate that the equation $\cos\left(x+\frac{\pi}{4}\right)=\cos x+\cos\frac{\pi}{4}$ ***is not*** an identity.

(a) ***Counterexample*** **(3 KU)**

Let $x=0$.

Then,

$L.S. = \cos(0+\frac{\pi}{4})=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

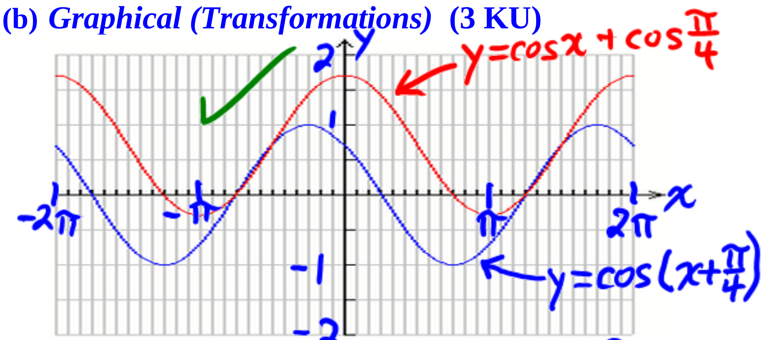
$R.S. = \cos 0+\cos\frac{\pi}{4}$

$= 1+\frac{1}{\sqrt{2}} > \frac{1}{\sqrt{2}}$

$\therefore L.S. \neq R.S.$

$\therefore$ the equation is not an identity

(b) ***Graphical (Transformations)*** **(3 KU)**



$y=\cos(x+\frac{\pi}{4})$: $y=\cos x$ shifted $\frac{\pi}{4}$ left

$y=\cos x+\cos\frac{\pi}{4}=\cos x+\frac{1}{\sqrt{2}}$: $y=\cos x$ shifted $\frac{1}{\sqrt{2}}$ units upward

3. Evaluate the following trig ratios without using a calculator. Exact values are required!

(a) $\tan\frac{7\pi}{12}$ **(4 APP)**

$=\tan(\frac{3\pi}{12}+\frac{4\pi}{12})$

$=\tan(\frac{\pi}{4}+\frac{\pi}{3})$

$=\dfrac{\tan\frac{\pi}{4}+\tan\frac{\pi}{3}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{3}}$

$=\dfrac{1+\sqrt{3}}{1-1(\sqrt{3})}$

$=\dfrac{1+\sqrt{3}}{1-\sqrt{3}}$

(b) $\sin 112.5°$ **(4 APP)**

$112.5° = \frac{225°}{2} = x$

$\therefore 2x = 225°$

$\because \cos 2x = 1-2\sin^2 x$

$\therefore \cos 225° = 1-2\sin^2 112.5°$

$\therefore -\cos 45° = 1-2\sin^2 112.5°$

$\therefore -\frac{1}{\sqrt{2}} = 1-2\sin^2 112.5°$

$\therefore 2\sin^2 112.5° = 1+\frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}}$

$\therefore \sin^2 112.5° = \frac{\sqrt{2}+1}{2\sqrt{2}}$

$\therefore \sin 112.5° = \pm\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$

$\because 112.5°$ is in quad II, $\sin 112.5° = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$

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– 0	– 0	–	–

4. Write $\sin 3\theta$ entirely in terms of $\sin\theta$. **(10 APP)**

$\sin 3\theta$
$= \sin(2\theta+\theta)$
$= \sin 2\theta\cos\theta + \cos 2\theta \sin\theta$
$= (2\sin\theta\cos\theta)(\cos\theta) + (1-2\sin^2\theta)(\sin\theta)$
$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$
$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$
$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta$

Note: The original question wasn't wrong after all. Here is a solution.

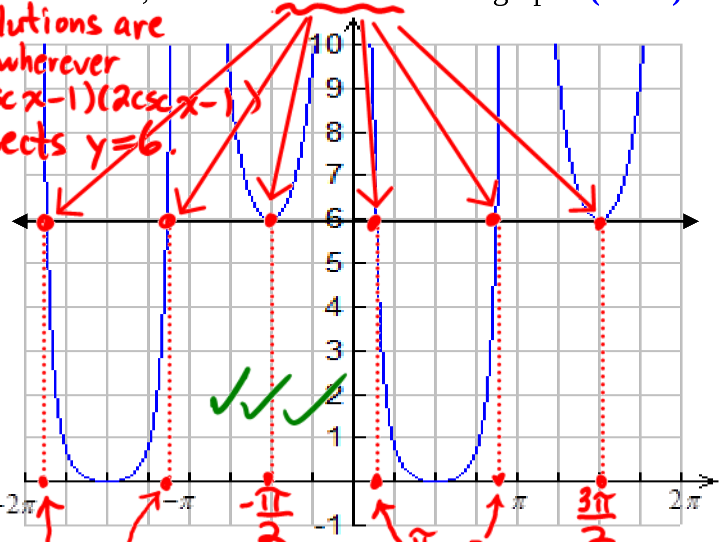
$\cos 3\theta$
$= \cos(2\theta+\theta)$
$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$
$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$
$= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta$
$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta)$
$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$
$= 4\cos^3\theta - 3\cos\theta$

5. The following question deals with solving trigonometric equations both graphically and algebraically.

(a) Shown below are the graphs of $y=(\csc x-1)(2\csc x-1)$ and $y=6$. State ***approximate*** solutions to the equation $(\csc x-1)(2\csc x-1)=6$ for $x\in[-2\pi,2\pi]$. In addition, ***mark*** the solutions on the graph. **(6 KU)**

Not zero!

The solutions are found wherever $y=(\csc x-1)(2\csc x-1)$ intersects $y=6$.



$-\frac{11\pi}{6}$, $-\frac{7\pi}{6}$, $-\frac{\pi}{2}$, $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{3\pi}{2}$

3 solutions in $[-\pi,\pi]$

Approximate Solutions:
$x \doteq -\frac{11\pi}{6}, -\frac{7\pi}{6}, -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

(b) Use an algebraic method to solve the equation $(\csc x-1)(2\csc x-1)=6$, where $x\in[-\pi,\pi]$. Verify that your solutions agree with those that you obtained in (a). **(5 APP)**

$2\csc^2 x - 3\csc x + 1 = 6$
$\therefore 2\csc^2 x - 3\csc x - 5 = 0$
$\therefore (2\csc x - 5)(\csc x + 1) = 0$
$\therefore \csc x = \frac{5}{2}$ or $\csc x = -1$
$\therefore \sin x = \frac{2}{5}$ or $\sin x = -1$
$\therefore x = \sin^{-1}(\frac{2}{5})$ or $x = \sin^{-1}(-1)$
In the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$,
$x \doteq 0.4115$ or $x = -\frac{\pi}{2}$
In the interval $[-\pi,\pi]$ there is one additional solution,
$x \doteq \pi - 0.4115 \doteq 2.7301$
$\therefore x \doteq 0.4115$ or $x \doteq 2.7301$ or $x = -\frac{\pi}{2}$

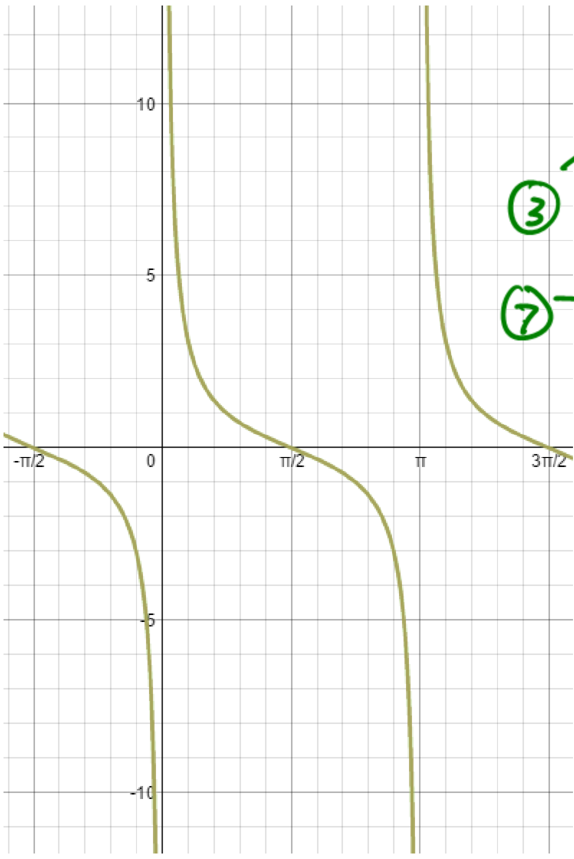
6. Solve the equation $-3\cos 2x - 7\sin x = 17$ for the interval $0 \le x \le 2\pi$. If necessary, and only if necessary, round your answers to 4 decimal places. **(7 TIPS)**

$-3\cos 2x - 7\sin x - 17 = 0$
$\therefore -3(1-2\sin^2 x) - 7\sin x - 17 = 0$
$\therefore -3 + 6\sin^2 x - 7\sin x - 17 = 0$
$\therefore 6\sin^2 x - 7\sin x - 20 = 0$
$\therefore 6\sin^2 x - 15\sin x + 8\sin x - 20 = 0$
$\therefore 3\sin x(2\sin x - 5) + 4(2\sin x - 5) = 0$
$\therefore (2\sin x - 5)(3\sin x + 4) = 0$
$\therefore \sin x = \frac{5}{2}$ or $\sin x = -\frac{4}{3}$
But for all $x\in\mathbb{R}$, $-1 \le \sin x \le 1$.
Since $\frac{5}{2} > 1$ and $-\frac{4}{3} < -1$, the given equation has no solutions.

Rough Work
$6(-20) = -120$
$-15(8) = -120$
$-15 + 8 = -7$

KU	APP	TIPS	COM
-0	-0	-0	-0

7. The graph of $f(x)=\dfrac{2\cot 2x\left(\csc^2 x+1\right)-4\cot 2x}{\cot^2 x-1}$ is shown below. Write an equation for the identity suggested by this graph then prove that the equation is indeed an identity. **(10 TIPS)**



This appears to be the graph of $y=\cot x$

Conjecture The following equation is an identity:

3

$$\cot x=\frac{2\cot 2x(\csc^2x+1)-4\cot 2x}{\cot^2 x-1}$$

7

Proof: $$R.S.=\frac{2\cot 2x(\csc^2x+1-2)}{\cot^2x-1}$$

$$=\frac{2}{1}\left(\frac{\cot^2x-1}{2\cot x}\right)\left(\frac{\csc^2x-1}{1}\right)\left(\frac{1}{\cot^2x-1}\right)$$

$$=\frac{2}{2}\left(\frac{\cot^2x-1}{\cot^2x-1}\right)\left(\frac{\csc^2x-1}{\cot x}\right)$$

$$=1(1)\left(\frac{1+\cot^2x-1}{\cot x}\right)$$ Pythagorean Identity $1+\cot^2x=\csc^2x$

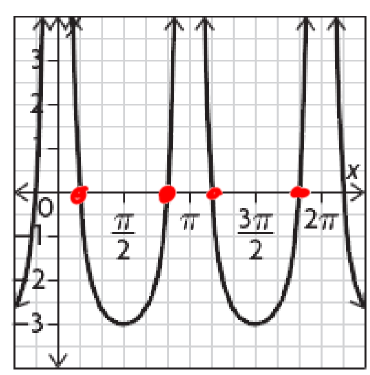
$$=\frac{\cot^2x}{\cot x}$$

$$=\cot x$$

$$=L.S.$$

Since L.S. = R.S., the given equation is an identity.

8. Write a quadratic trigonometric equation involving $\cot x$ whose solutions in the interval $[0,2\pi]$ are the same as the x-intercepts of the graph shown at the right. Show that your equation yields the correct solutions.



Note: The graph shown at the right is NOT the graph of $y=\cot x$. **(4 TIPS)**

x-intercepts of given graph in $[0,2\pi]$:
$\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$

Note that $\cot\frac{\pi}{6}=\sqrt{3}$ and $\cot\frac{5\pi}{6}=-\sqrt{3}$.

Since the period of $y=\cot x$ is π, $\cot(x+\pi)=\cot x$.

$\therefore \cot\left(\frac{7\pi}{6}\right)=\cot\left(\frac{\pi}{6}+\pi\right)=\cot\frac{\pi}{6}=\sqrt{3}$ and

$\cot\left(\frac{11\pi}{6}\right)=\cot\left(\frac{5\pi}{6}+\pi\right)=\cot\frac{5\pi}{6}=-\sqrt{3}$

Therefore, for $x=\frac{\pi}{6},\frac{7\pi}{6}$, $\cot x=\sqrt{3}$ or $\cot x-\sqrt{3}=0$
and for $x=\frac{5\pi}{6},\frac{11\pi}{6}$, $\cot x=-\sqrt{3}$ or $\cot x+\sqrt{3}=0$

$\therefore$ the equation $(\cot x-\sqrt{3})(\cot x+\sqrt{3})=0$ has the required solutions in $[0,2\pi]$.

(Other forms of the equation: $\cot^2x-3=0$, $\cot^2x=3$.)

KU	APP	TIPS	COM
−0	−0	−0	−0