MCR 3U9 — **Semester 1, 2016 - 2017**

Grade 11 Pre-AP Mathematics

Unit 2 – Exponential and Logarithmic Functions – Major Test

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***Victim*: Mr. Solutions** Beautiful work Mr. S.!!

KU	APP	TIPS	COM
19/19	18/18	15/15	10/10

Modified True/False (5 KU)

State whether each statement is ***true*** or ***false***. If false, ***change*** the ***underlined part*** to make the statement true.

$a^x a^y = a^{x+y} = f(x+y)$

1. T/F F For $f(x)=a^x$, $\underline{f(x)f(y)} = f(xy)$. **Change:** $f(x+y)$

$a^{nx} = (a^x)^n = f(x)^n$

2. T/F F For $f(x)=a^x$, $f(nx) = \underline{nf(x)}$. **Change:** $f(x)^n$

3. T/F F $y=\log_a x$ means that $\underline{a^x = y}$. ✓ = ½ mark **Change:** $a^y = x$

4. T/F F $\underline{\log_{243} 9} = \frac{5}{2}$. **Change:** $\log_9 243$

5. T/F F $\underline{\log_a(x+y)} = \log_a x + \log_a y$ **Change:** $\log_a xy$

Problems

6. Suppose that $g(x) = 1.5\log_2(0.5(x+6))+2$. **(14 KU)**

(a) State the transformations required to obtain g from the base function $f(x)=\log_2 x$.

Horizontal	*Vertical*
1. Stretch by a factor of two	1. Stretch by a factor of 1.5
2. Translate six units to the left	2. Translate two units up

2

(b) Express the transformation in ***mapping notation***.

$(x, y) \to (2x-6, 1.5y+2)$ 2

(c) Now apply the transformation to a few key points on the graph of the base function $f(x)=\log_2 x$.

Pre-image Point on $y=f(x)$	*Image Point on* $y=g(x)$
$(1/2, -1)$	$(-5, -0.5)$
$(1, 0)$	$(-4, 2)$
$(2, 1)$	$(-2, 3.5)$
$(4, 2)$	$(2, 5)$
$(8, 3)$	$(10, 6.5)$

4

(d) Apply the transformation to the asymptote of $y=f(x)$.

Pre-image Asymptote on $y=f(x)$	*Image Asymptote on* $y=g(x)$
$x=0$	$x=-6$

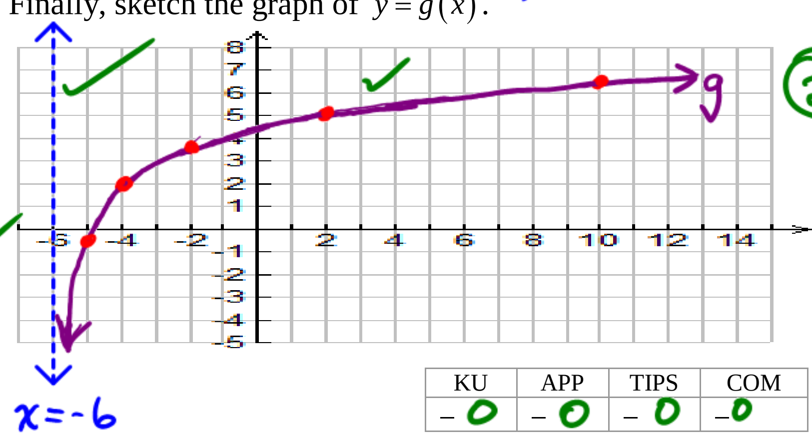
2

(e) State the domain and range of g.

domain = $\{x \in \mathbb{R} \mid x > -6\}$

range = $\mathbb{R}$ 2

(f) Finally, sketch the graph of $y=g(x)$.



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– 0	– 0	– 0	–0

7. Opie lent Brian and Stewie an amount of money at a rate of 3.60% p.a. (per year), compounded ***monthly***. Brian and Stewie finally repaid Opie ten years later in one lump sum of \$10,000. How much money did Brian and Stewie borrow from Opie?

(a) If P represents the amount borrowed, complete the following: **(4 APP)**

annual rate = 0.036

monthly rate = 0.003

Time (months)	*Amount (\$)*
0	P
1	$P(1.003)$
2	$P(1.003)(1.003)$
3	$P(1.003)(1.003)(1.003)$
⋮	⋮
t	$P(1.003)^t$

(b) Now solve the problem! **(6 APP)**

Given: future value = \$10000, $t = 10$ years $= 120$ months

Unknown: P

Let $V(t)$ represent the future value of the amount borrowed by Stewie and Brian.

$$\therefore V(t) = P(1.003)^t$$

$$\therefore 10000 = P(1.003)^{120}$$

$$\therefore P = \frac{10000}{(1.003)^{120}}$$

$$\doteq 6980.52$$

Stewie and Brian borrowed \$6980.52.

8. Through detailed studies, scientists have determined that in ***living*** carbonaceous material, the ratio of number of ^{14}C atoms to number of ^{12}C atoms is $1:10^{12}$. In a wooden artifact found in an archaeological excavation, the ratio of number of ^{14}C atoms to number of ^{12}C atoms is measured to be $1:2.7\times10^{12}$. Estimate the age of the wood used to make the artifact. (Recall that the half-life of ^{14}C is 5730 years.) **(8 APP)**

t (years)	# ^{14}C atoms
0	N_0
5730	$N_0\left(\frac{1}{2}\right)$
11460	$N_0\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$
⋮	⋮
t	$N_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

$$\frac{10^{12}}{2.7\times10^{12}} = \frac{1}{2.7} = \text{fraction of } ^{14}\text{C atoms remaining}$$

Let $N(t)$ represent the # of ^{14}C atoms remaining after t years. Then,

$$N(t) = N_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\therefore \frac{N_0}{2.7} = N_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\therefore \left(\frac{1}{2}\right)^{\frac{t}{5730}} = \frac{1}{2.7}$$

$$\therefore \log_{10}\left(\frac{1}{2}\right)^{\frac{t}{5730}} = \log_{10}\frac{1}{2.7}$$

$$\therefore \frac{t}{5730}\log_{10}\frac{1}{2} = \log_{10}\frac{1}{2.7}$$

$$\therefore t = \frac{5730\log_{10}\frac{1}{2.7}}{\log_{10}\frac{1}{2}}$$

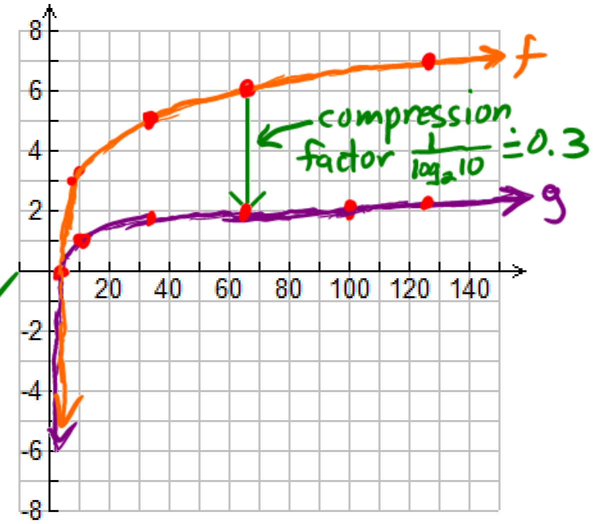
$$\doteq 8210.9$$

The artifact is made of wood that is about 8200 years old.

KU	APP	TIPS	COM
–0	–0	–0	–0

9. Recall the "change of base" formula for logarithms: $\log_b x = \dfrac{\log_a x}{\log_a b}$.

Use this formula to explain how the graph of $g(x) = \log_{10} x$ can be obtained by transforming the graph of $f(x) = \log_2 x$. Illustrate your answer with graphs. **(8 TIPS)**



$$g(x) = \log_{10} x = \frac{\log_2 x}{\log_2 10} = \frac{1}{\log_2 10}\log_2 x = \frac{1}{\log_2 10} f(x)$$

Therefore, the graph of g can be obtained by compressing the graph of f vertically by a factor of $\frac{1}{\log_2 10} \doteq 0.3$

10. The dynamic range of human hearing spans roughly from 10^{-12} W/m^2 to 10 W/m^2. This means that the highest sound intensity that can be heard is about 10,000,000,000,000 times louder than the quietest! Such a wide span of sound intensity is very impractical for most applications.

Sound	Loudness (dB)
soft whisper	30
normal conversation	60
shouting	80
subway	90
screaming	100
rock concert	120
jet engine	140
space-shuttle launch	180

A much more convenient way to measure the "loudness" of a sound is to use a *relative logarithmic scale*. The most commonly used such scale defines the "loudness" L of a sound, in units called *decibels* (dB), in terms of the intensity I of the sound and the intensity I_0 (10^{-12} W/m^2) of a sound at the threshold of human hearing. The following equation relates L, I and I_0:

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

I → intensity (W/m^2)

L → loudness (dB)

A car has an interior loudness of $\boxed{70 \text{ dB}}$ at 50 km/h. A second car, travelling at the same speed, has an interior sound level 3 times more $\boxed{\text{intense}}$ than the first car. Determine the interior loudness of the second car. **(7 TIPS)**

Car 1 Sound Intensity → I
" 2 " " → 3I

Car 1

$$70 = 10\log_{10}\left(\frac{I}{I_0}\right)$$

Car 2

$$L = 10\log_{10}\left(\frac{3I}{I_0}\right)$$

$$\therefore L = 10\log_{10}\left[3\left(\frac{I}{I_0}\right)\right] = 10\left[\log_{10}3 + \log_{10}\left(\frac{I}{I_0}\right)\right] = 10\log_{10}3 + 10\log_{10}\left(\frac{I}{I_0}\right) = 10\log_{10}3 + 70 \doteq 74.8$$

The interior loudness of the second car is about 74.8 dB.

KU	APP	TIPS	COM
– 0	– 0	– 0	– 0

Sudanshu's Alternative Approach to Question 8

The decay of $\frac{1}{2}$ the ^{14}C atoms every 5730 years is equivalent to the ratio of ^{12}C to ^{14}C doubling every 5730 years.

t	$^{12}C : {}^{14}C$
0	$10^{12} : 1$
5730	$2\times10^{12} : 1$
11460	$4\times10^{12} : 1$
⋮	
t	$2^{\frac{t}{5730}}\times10^{12}$

$$\therefore 2.7\times10^{12} = 2^{\frac{t}{5730}}\times10^{12}$$

$$\therefore 2.7 = 2^{\frac{t}{5730}}$$

Since $2^1=2$, $2^2=4$ and $2<2.7<4$,

$$1<\frac{t}{5730}<2$$

By trial and error we find that $2^{1.435} \doteq 2.70$

How equation can still be solved without knowing laws of logs

$$\therefore \frac{t}{5730} \doteq 1.435$$

$$\therefore t \doteq 8200$$

Yet another approach is to observe that the ratio of $^{14}C : {}^{12}C$ is cut in half every 5730 years

$$\therefore \frac{1}{2.7\times10^{12}} = \left(\frac{1}{2}\right)^{\frac{t}{5730}}\left(\frac{1}{10^{12}}\right)$$

$$\therefore \frac{10^{12}}{2.7\times10^{12}} = \left(\frac{1}{2}\right)^{\frac{t}{5730}} \longrightarrow \frac{1}{2.7} = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

Sudanshu's Alternative Approach to Question 10

Car 1: $70 = 10\log_{10}\left(\frac{I}{I_0}\right)$

$$\therefore 7 = \log_{10}\left(\frac{I}{I_0}\right)$$

$$\therefore \frac{I}{I_0} = 10^7$$

$$\therefore I = 10^7 I_0$$

$$\therefore 3I = 3\times10^7 I_0$$

Car 2: $L = 10\log_{10}\left(\frac{3I}{I_0}\right)$

$$= 10\log_{10}\left(\frac{3\times10^7 I_0}{I_0}\right)$$

$$= 10\log_{10}(3\times10^7)$$

calculator can be used at this step to avoid laws of logs

$$= 10(\log_{10}3 + \log_{10}10^7)$$

$$= 10(\log_{10}3 + 7) \doteq 74.8$$