MCR 3U9 **Semester 2, 2016 - 2017**

Grade 11 Pre-AP Functions

Unit 1 – Major Test 1 – Functions, Relations and Transformations

Mr. N. Nolfi

***Victim*:** Mr. Solutions

Another brilliant display of mathematical insight Mr. N.!

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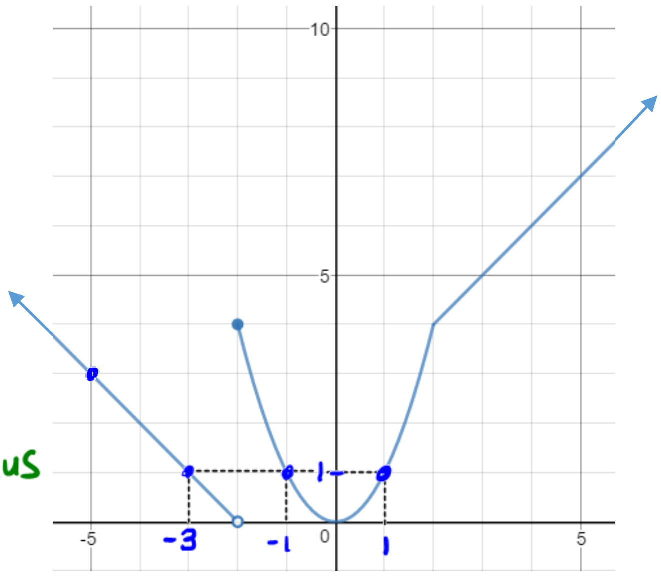
23/23	26/26	20/20	10/10

1. Carefully study the graph of the function *f* at the right and then answer the questions given below. **(10 KU)**



(a) State the domain and range of *f*.

②

Domain = $\mathbb{R}$ ✓

Range = $\{y \in \mathbb{R} \mid y \geq 0\}$ ✓

(b) Evaluate each of the following.

⑤

$f(2) =$ 4 ✓ $f(-2) =$ 4 ✓

$f(-3-2) =$ $f(-5) = 3$ ✓

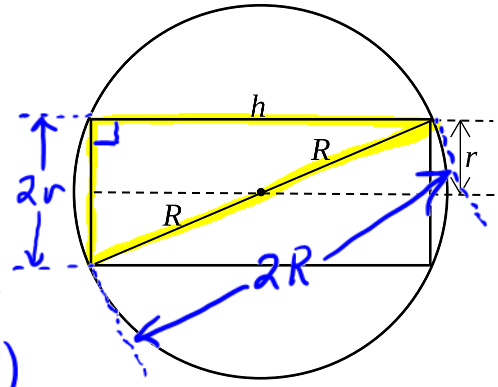
$f(a) = 1 \quad \therefore a =$ 1 ✓ or −1 ✓ or −3 ✓ +1 Bonus

(c) Suggest a possible equation for *f*.

③

$$f(x) = \begin{cases} -x-2, & x < -2 \ ✓ \\ x^2, & -2 \leq x \leq 2 \ ✓ \\ x+2, & x > 2 \ ✓ \end{cases}$$

2. A cylinder with radius *r* and height *h* is inscribed in a sphere of constant radius *R*. Express the volume of the cylinder as a function of *h*. (Recall that $V_{\text{cylinder}} = \pi r^2 h$.) **(6 APP)**



By the Pythagorean Theorem,

$h^2 + (2r)^2 = (2R)^2$ ✓

$\therefore h^2 + 4r^2 = 4R^2$ ✓

$\therefore 4r^2 = 4R^2 - h^2$ ✓

$\therefore r^2 = \dfrac{4R^2 - h^2}{4}$ ✓

$\therefore V = \pi\left(\dfrac{4R^2 - h^2}{4}\right)h$

$\therefore V = \frac{1}{4}\pi h(4R^2 - h^2)$ ✓ ✓

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3. Given below is the graph of $f(x)=|x^3-1|$. The transformation expressed in mapping notation below is applied to the function *f* to produce the function *g*. Complete the table for this transformation. **(12 APP)**

Transformation

Mapping Notation

$(x,y)\to\left(2x+4,-\frac{3}{2}y+12\right)$

Function Notation

③ $g(x)=-\frac{3}{2}f\left(\frac{1}{2}(x-4)\right)+12$ ✓ ✓ ✓

Verbal ④

Horizontal

1. Stretch by a factor of 2 ✓
2. Translate 4 to the right ✓

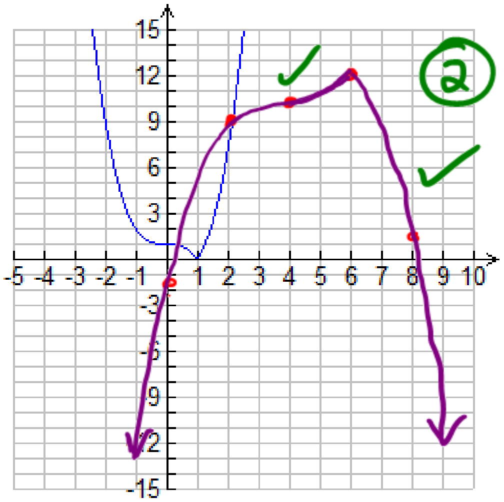
Vertical

1. stretch by a factor of $\frac{3}{2}$ and reflect in x-axis. ✓
2. Translate 12 units upward ✓

Images of Five Key Points ①

Pre-Image	Image
$(0,1)$	$(4,\frac{21}{2})$
$(1,0)$	$(6,12)$
$(2,7)$	$(8,\frac{3}{2})$ ✓
$(-1,2)$	$(2,9)$
$(-2,9)$	$(0,-\frac{3}{2})$

Sketch the Graph of y = g(x) ②



Equation of the Image Function y = g(x) ②

$g(x)=-\frac{3}{2}f\left(\frac{1}{2}(x-4)\right)+12$

$\therefore g(x)=-\frac{3}{2}\left|\left[\frac{1}{2}(x-4)\right]^3-1\right|+12$ ✓

$\therefore g(x)=-\frac{3}{2}\left|\frac{1}{8}(x-4)^3-1\right|+12$ ✓

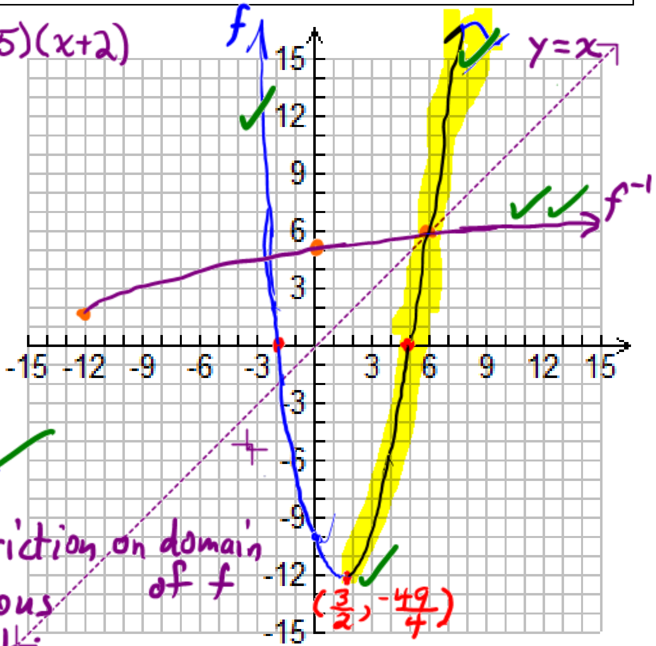
4. Consider the function $f(x)=x^2-3x-10$. **(13 KU)** $=(x-5)(x+2)$

③ **(a)** Use the zeros of *f* to sketch its graph on the provided grid.

(b) Restrict the domain of *f* in such a way that f^{-1} is defined.

① $\{x\in\mathbb{R}\mid x\geq\frac{3}{2}\}$ ✓

② **(c)** Sketch the graph of f^{-1} for the restricted domain in (b).



(d) Find the equation of f^{-1} for the restricted domain in (b).

⑤ $f(x)=\left(x-\frac{3}{2}\right)^2-\frac{49}{4}$ ✓ (Obvious because of (a))

To form f^{-1}, perform transf. $(x,y)\to(y,x)$ ✓

$x=\left(y-\frac{3}{2}\right)^2-\frac{49}{4}$

$\therefore y=\frac{3}{2}\pm\sqrt{x+\frac{49}{4}}$ ✓ ✓

because of the restriction on domain of f

$\therefore y=\frac{3}{2}+\sqrt{x+\frac{49}{4}}=\underbrace{\sqrt{x+\frac{49}{4}}+\frac{3}{2}}$ ✓

also obvious from graph: $\sqrt{x}$ translated $\frac{49}{4}$ left and $\frac{3}{2}$ up.

(e) State the domain and range of f^{-1}.

② domain $=\{x\in\mathbb{R}\mid x\geq-\frac{49}{4}\}$ ✓ range $=\{y\in\mathbb{R}\mid y\geq\frac{3}{2}\}$ ✓

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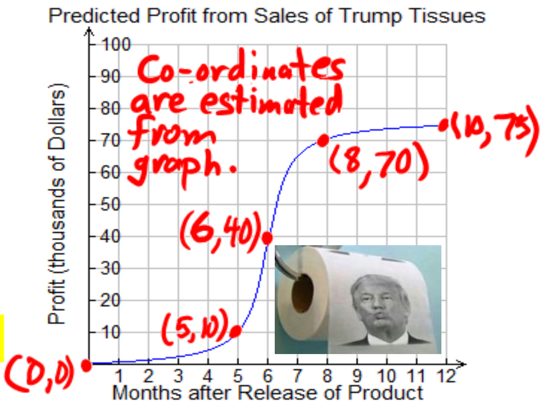
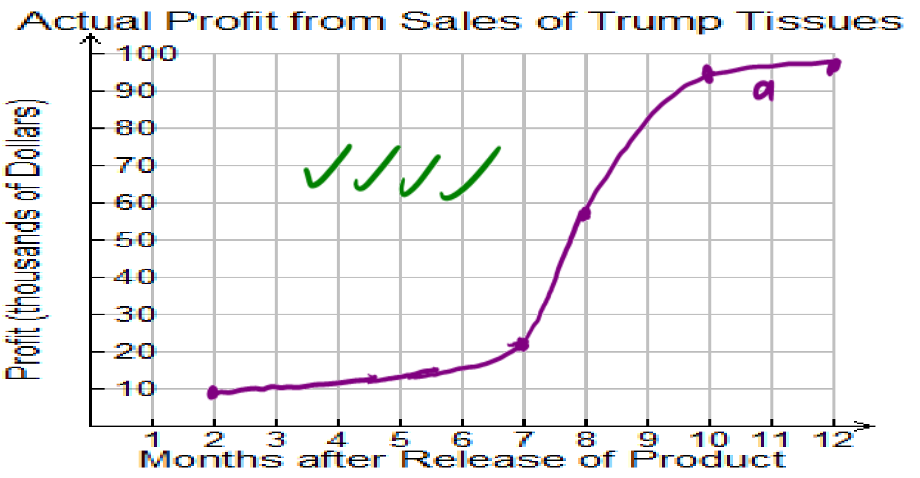
5. The graph of the function p at the right shows the ***predicted profit*** from sales of Donald Trump Bathroom Tissues for the first 12 months after introduction of the product.



Due to manufacturing problems, the introduction of the product was ***delayed by two months***. In spite of the delay, the ***actual profit*** turned out to be $10,000 more than 1.2 times the predicted profit.

Express this using mapping notation, function notation and graphically. For the purposes of function notation, let p represent the predicted-profit function and let a represent the actual-profit function. **(8 APP)**

Mapping Notation	***Graph***
$(x,y) \to (x+2,\ 1.2y+10)$	
Function Notation	
$a(t) = 1.2p(t-2)+10$ $t \to$ time in months	

6. Smartphone "finger" stands normally sell for $15 apiece. At this price, 200,000 are expected to be sold. For every 2-dollar increase in price, 10,000 fewer are sold. **(8 TIPS)**

(a) What selling price will produce the maximum revenue and what will the maximum revenue be?

Let x represent the number of \$2 increases in price and let $R(x)$ represent the revenue obtained (in dollars)

Then,

⑥ $R(x) = (200000 - 10000x)(15+2x)$

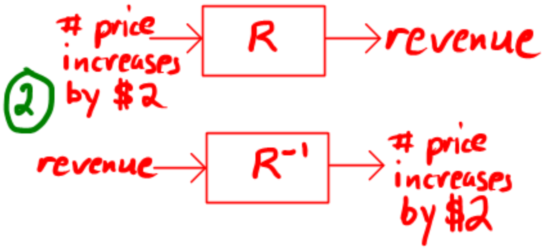
Zeros: $-\frac{15}{2}$, 20

axis of symmetry: $x = \frac{20+(-\frac{15}{2})}{2} = \frac{25}{4}$

vertex: $\left(\frac{25}{4}, 3781250\right)$

∴ the selling price should be $15+2\left(\frac{25}{4}\right) = \$27.50$, which produces a maximum revenue of $R\left(\frac{25}{4}\right) = \$3,781,250$

(b) Explain the ***meaning*** of the ***inverse*** of the function that you obtained in part (a).



As shown in the diagram, the function R^{-1} takes the revenue as input and outputs the # of \$2 price increases needed to produce that revenue.

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Note: Functions that are their own inverses are called **INVOLUTIONS.**

7. Let $f(x)=\frac{x}{x-1}$. **(12 TIPS)**

The graph of any involution f is symmetric in the line $y=x$. In addition, $f(f(x))=x$.

$$x \rightarrow \boxed{f} \rightarrow f(x) \rightarrow \boxed{f} \rightarrow x$$

(a) Prove that $f(x)=f^{-1}(x)$.

i.e. prove that f is its own inverse (i.e. symmetric in line $y=x$)

⑤

Let $y=\frac{x}{x-1}$

Perform transformation $(x,y)\rightarrow(y,x)$

$\therefore\ x=\frac{y}{y-1}$ ✓

$\therefore\ x(y-1)=y$

$\therefore\ xy-x=y$ ✓

$\therefore\ xy-y=x$

$\therefore\ y(x-1)=x$

$\therefore\ y=\frac{x}{x-1}$

$\therefore\ f^{-1}(x)=\frac{x}{x-1}$

$\because\ f(x)=\frac{x}{x-1}$

$\therefore\ f(x)=f^{-1}(x)$

✓✓✓

(b) Let $g(x)=af\left(\frac{1}{a}x\right)$, where f is as defined above and a represents any ***non-zero*** real number.

Prove that $g(x)=g^{-1}(x)$. i.e. prove that if $f(x)=\frac{x}{x-1}$ is stretched/compressed both horizontally and vertically by the same factor, the resulting function will be its own inverse (i.e. symmetric in $y=x$)

$$\frac{1}{a}x \rightarrow \boxed{f} \rightarrow \frac{\frac{1}{a}x}{\frac{1}{a}x-1}$$

④

$g(x)=af\left(\frac{1}{a}x\right)$

$=a\boxed{\left(\frac{\frac{1}{a}x}{\frac{1}{a}x-1}\right)}$

$=a\left(\frac{\frac{x}{a}}{\frac{x-a}{a}}\right)$

$=a\left(\frac{x}{a}\right)\left(\frac{a}{x-a}\right)$

$=\frac{ax}{x-a}$

✓✓

Let $y=\frac{ax}{x-a}$

To form g^{-1}, perform transformation $(x,y)\rightarrow(y,x)$

$x=\frac{ay}{y-a}$

$\therefore\ x(y-a)=ay$

$\therefore\ xy-ax=ay$

$\therefore\ xy-ay=ax$

$\therefore\ y(x-a)=ax$

$\therefore\ y=\frac{ax}{x-a}$

$\therefore\ g^{-1}(x)=\frac{ax}{x-a}$

$\because\ g(x)=\frac{ax}{x-a}$

$\therefore\ g(x)=g^{-1}(x)$.

✓✓

(c) Let $h(x)=af\left(\frac{1}{a}(x-c)\right)+d$, where f is as defined above and a is any non-zero real number. How must the values of c and d be related for $h(x)=h^{-1}(x)$? Notice that $h(x)=g(x-c)+d$.

The function h can be obtained from g by performing the transformation $(x,y)\rightarrow(x+c,\ y+d)$.

③

This means that g must be translated by c units horizontally ✓ and d units vertically.

The symmetry of g in the line $y=x$ is preserved only if ✓ it is translated horizontally and vertically by the same amount. Therefore, $c=d$ ✓

(See next page for algebraic proof.) (Algebraic proof not required)

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$$h(x) = af\left(\tfrac{1}{a}(x-c)\right)+d$$

$$= a\left[\frac{\frac{1}{a}(x-c)}{\frac{1}{a}(x-c)-1}\right]+d$$

$$= a\left[\frac{\left(\frac{x-c}{a}\right)}{\left(\frac{x-c}{a}-\frac{a}{a}\right)}\right]+d$$

$$= a\left(\frac{x-c}{a}\right)\left(\frac{a}{x-c-a}\right)+d$$

$$= \frac{a(x-c)}{x-c-a}+d$$

Let $y = \dfrac{a(x-c)}{x-c-a}+d$

To find h^{-1}, perform transf. $(x,y)\to(y,x)$

$$x = \frac{a(y-c)}{y-c-a}+d$$

$$\therefore x-d = \frac{a(y-c)}{y-c-a}$$

$$\therefore (x-d)(y-c-a) = a(y-c)$$

$$\therefore xy-xc-xa-dy+cd+ad = ay-ac$$

$$\therefore xy-dy-ay = xa+xc-cd-ad-ac$$

$$\therefore y(x-d-a) = a(x-c)+c(x-d-a)$$

$$\therefore y = \frac{a(x-c)}{x-d-a}+\frac{c(x-d-a)}{x-d-a}$$

$$\therefore y = \frac{a(x-c)}{x-d-a}+c$$

$$\therefore h^{-1}(x) = \frac{a(x-c)}{x-d-a}+c$$

Comparing this to $h(x)$, we see that $h(x)=h^{-1}(x)$ if and only if

$$\frac{a(x-c)}{x-c-a}+d = \frac{a(x-c)}{x-d-a}+c.$$

If $c=d$, this is clearly the case.

If $c\neq d$, this is NOT the case but more algebra is required to show this.

This is left as an exercise for interested students.