(a) Plot the data.

(b) Determine a linear function $f(x) = ax + b$ that models these data, where x is the year. Plot f and the data on the same coordinate axes.

(c) Find $f^{-1}(x)$. Explain the significance of f^{-1}.

(d) Use f^{-1} to predict the year in which there were 11,987 radio stations. Compare it with the true value, which is 1995.

4.2 Exponential Functions

Previously, we considered functions having terms of the form

$$\text{variable base}^{\text{constant power}},$$

such as x^2, $0.2x^{1.3}$, and $8x^{2/3}$. We now turn our attention to functions having terms of the form

$$\text{constant base}^{\text{variable power}},$$

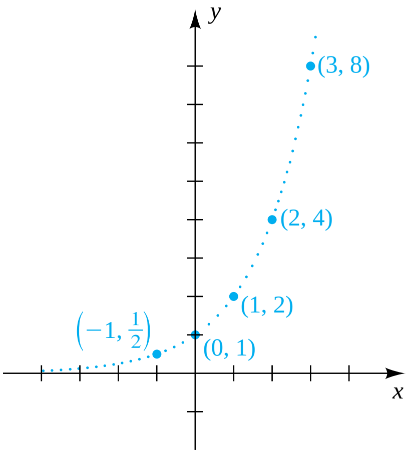
such as 2^x, $(1.04)^{4x}$, and 3^{-x}. Let us begin by considering the function f defined by

$$f(x) = 2^x,$$

where x is restricted to *rational* numbers. $\left(\text{Recall that if } x = m/n \text{ for integers } m \text{ and } n \text{ with } n > 0, \text{ then } 2^x = 2^{m/n} = \left(\sqrt[n]{2}\right)^m.\right)$ Coordinates of several points on the graph of $y = 2^x$ are listed in the following table.

x	−10	−3	−2	−1	0	1	2	3	10
$y = 2x$	$\frac{1}{1024}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	1024

FIGURE 1



Other values of y for x rational, such as $2^{1/3}$, $2^{-9/7}$, and $2^{5.143}$, can be approximated with a calculator. We can show algebraically that if x_1 and x_2 are rational numbers such that $x_1 < x_2$, then $2^{x_1} < 2^{x_2}$. Thus, f is an increasing function, and its graph rises. Plotting points leads to the sketch in Figure 1, where the small dots indicate that only the points with *rational* x-coordinates are on the graph. There is a *hole* in the graph whenever the x-coordinate of a point is irrational.

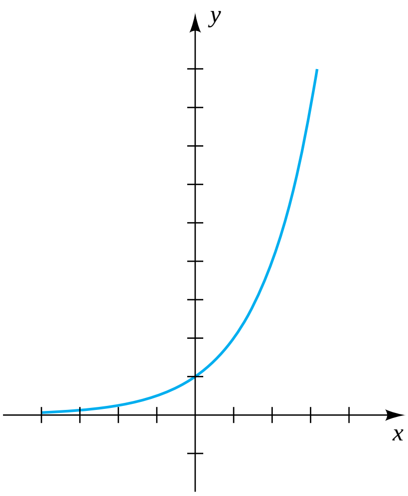
To extend the domain of f to all real numbers, it is necessary to define 2^x for every *irrational* exponent x. To illustrate, if we wish to define 2^π, we could use the nonterminating decimal representing 3.1415926 . . . for π and consider the following rational powers of 2:

$$2^3, \quad 2^{3.1}, \quad 2^{3.14}, \quad 2^{3.141}, \quad 2^{3.1415}, \quad 2^{3.14159}, \quad \ldots$$

It can be shown, using calculus, that each successive power gets closer to a unique real number, denoted by 2^π. Thus,

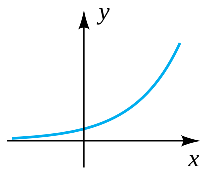
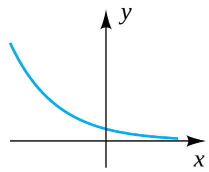
$$2^x \to 2^\pi \quad \text{as} \quad x \to \pi, \quad \text{with } x \text{ rational}.$$

FIGURE 2



The same technique can be used for any other irrational power of 2. To sketch the graph of $y = 2^x$ with x *real*, we replace the holes in the graph in Figure 1 with points, and we obtain the graph in Figure 2. The function f defined by $f(x) = 2^x$ for every real number x is called the **exponential function** *with base* 2.

Let us next consider *any* base a, where a is a positive real number different from 1. As in the preceding discussion, to each real number x there corresponds exactly one positive number a^x such that the laws of exponents are true. Thus, as in the following chart, we may define a function f whose domain is $\mathbb{R}$ and range is the set of positive real numbers.

Terminology	Definition	Graph of f for $a > 1$	Graph of f for $0 < a < 1$
Exponential function f with base a	$f(x) = a^x$ for every x in $\mathbb{R}$, where $a > 0$ and $a \neq 1$		

The graphs in the chart show that if $a > 1$, then f is increasing on $\mathbb{R}$, and if $0 < a < 1$, then f is decreasing on $\mathbb{R}$. (These facts can be proved using calculus.) The graphs merely indicate the *general* appearance—the *exact* shape depends on the value of a. Note, however, that since $a^0 = 1$, the y-intercept is 1 for every a.

Note that if $a > 1$, then $a = 1 + d$ ($d > 0$) and the base a in $y = a^x$ can be thought of as representing multiplication by more than 100% as x increases by 1, so the function is increasing. For example, if $a = 1.15$, then $y = (1.15)^x$ can be considered to be a 15% per year growth function. More details on this concept appear later.

If $a > 1$, then as x *decreases* through negative values, the graph of f approaches the x-axis (see the third column in the chart). Thus, the x-axis is a *horizontal asymptote.* As x increases through positive values, the graph rises rapidly. This type of variation is characteristic of the **exponential law of growth,** and f is sometimes called a **growth function.**

If $0 < a < 1$, then as x *increases,* the graph of f approaches the x-axis asymptotically (see the last column in the chart). This type of variation is known as **exponential decay.**

When considering a^x, we exclude the cases $a \leq 0$ and $a = 1$. Note that if $a < 0$, then a^x is not a real number for many values of x such as $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{11}{6}$. If $a = 0$, then $a^0 = 0^0$ is undefined. Finally, if $a = 1$, then $a^x = 1$ for every x, and the graph of $y = a^x$ is a horizontal line.

The graph of an exponential function f is either increasing throughout its domain or decreasing throughout its domain. Thus, f is one-to-one by the theorem on page 251. Combining this result with the definition of a one-to-one function (see page 250) gives us parts (1) and (2) of the following theorem.

Theorem: Exponential Functions Are One-to-One

The exponential function f given by

$$f(x) = a^x \quad \text{for} \quad 0 < a < 1 \quad \text{or} \quad a > 1$$

is one-to-one. Thus, the following equivalent conditions are satisfied for real numbers x_1 and x_2.

(1) If $x_1 \neq x_2$, then $a^{x_1} \neq a^{x_2}$.

(2) If $a^{x_1} = a^{x_2}$, then $x_1 = x_2$.

When using this theorem as a reason for a step in the solution to an example, we will state that *exponential functions are one-to-one.*

ILLUSTRATION

Exponential Functions Are One-to-One

- If $7^{3x} = 7^{2x+5}$, then $3x = 2x + 5$, or $x = 5$.

In the following example we solve a simple *exponential equation*—that is, an equation in which the variable appears in an exponent.

EXAMPLE 1 Solving an exponential equation

Solve the equation $3^{5x-8} = 9^{x+2}$.

SOLUTION

$3^{5x-8} = 9^{x+2}$	given
$3^{5x-8} = (3^2)^{x+2}$	express both sides with the same base
$3^{5x-8} = 3^{2x+4}$	law of exponents
$5x - 8 = 2x + 4$	exponential functions are one-to-one
$3x = 12$	subtract $2x$ and add 8
$x = 4$	divide by 3

■

Note that the solution in Example 1 depended on the fact that the base 9 could be written as 3 to some power. We will consider only exponential equations of this type for now, but we will solve more general exponential equations later in the chapter.

In the next two examples we sketch the graphs of several different exponential functions.

EXAMPLE 2 Sketching graphs of exponential functions

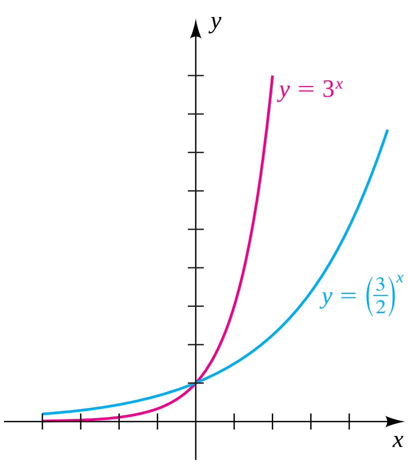
If $f(x) = \left(\frac{3}{2}\right)^x$ and $g(x) = 3^x$, sketch the graphs of f and g on the same coordinate plane.

SOLUTION Since $\frac{3}{2} > 1$ and $3 > 1$, each graph *rises* as x increases. The following table displays coordinates for several points on the graphs.

x	-2	-1	0	1	2	3	4
$y = \left(\frac{3}{2}\right)^x$	$\frac{4}{9} \approx 0.4$	$\frac{2}{3} \approx 0.7$	1	$\frac{3}{2}$	$\frac{9}{4} \approx 2.3$	$\frac{27}{8} \approx 3.4$	$\frac{81}{16} \approx 5.1$
$y = 3x$	$\frac{1}{9} \approx 0.1$	$\frac{1}{3} \approx 0.3$	1	3	9	27	81

Plotting points and being familiar with the general graph of $y = a^x$ leads to the graphs in Figure 3. ■

FIGURE 3



Example 2 illustrates the fact that if $1 < a < b$, then $a^x < b^x$ for positive values of x and $b^x < a^x$ for negative values of x. In particular, since $\frac{3}{2} < 2 < 3$, the graph of $y = 2^x$ in Figure 2 lies between the graphs of f and g in Figure 3.

FIGURE 4

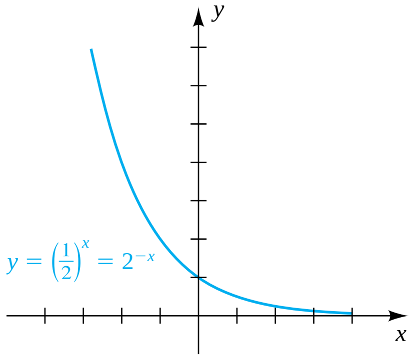


FIGURE 5

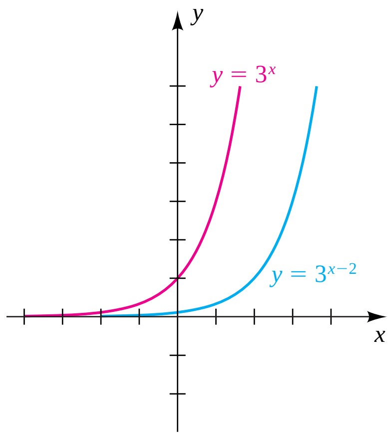
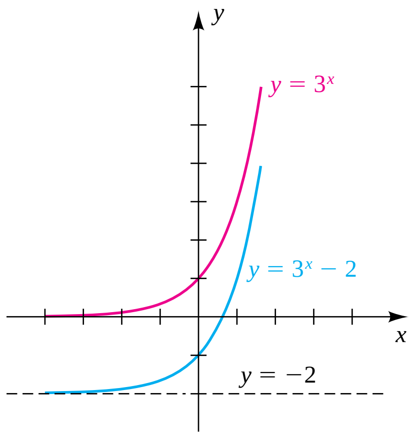


FIGURE 6



EXAMPLE 3 **Sketching the graph of an exponential function**

Sketch the graph of the equation $y = \left(\frac{1}{2}\right)^x$.

SOLUTION Since $0 < \frac{1}{2} < 1$, the graph *falls* as x increases. Coordinates of some points on the graph are listed in the following table.

x	-3	-2	-1	0	1	2	3
$y = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

The graph is sketched in Figure 4. Since $\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$, the graph is the same as the graph of the equation $y = 2^{-x}$. Note that the graph is a reflection through the y-axis of the graph of $y = 2^x$ in Figure 2. ■

Equations of the form $y = a^u$, where u is some expression in x, occur in applications. The next two examples illustrate equations of this form.

EXAMPLE 4 **Shifting graphs of exponential functions**

Sketch the graph of the equation:

(a) $y = 3^{x-2}$ **(b)** $y = 3^x - 2$

SOLUTION

(a) The graph of $y = 3^x$, sketched in Figure 3, is resketched in Figure 5. From the discussion of horizontal shifts in Section 2.5, we can obtain the graph of $y = 3^{x-2}$ by shifting the graph of $y = 3^x$ two units to the right, as shown in Figure 5.

The graph of $y = 3^{x-2}$ can also be obtained by plotting several points and using them as a guide to sketch an exponential-type curve.

(b) From the discussion of vertical shifts in Section 2.5, we can obtain the graph of $y = 3^x - 2$ by shifting the graph of $y = 3^x$ two units downward, as shown in Figure 6. Note that the y-intercept is -1 and the line $y = -2$ is a horizontal asymptote for the graph. ■

EXAMPLE 5 **Finding an equation of an exponential function satisfying prescribed conditions**

Find an exponential function of the form $f(x) = ba^{-x} + c$ that has horizontal asymptote $y = -2$, y-intercept 16, and x-intercept 2.

SOLUTION The horizontal asymptote of the graph of an exponential function of the form $f(x) = ba^{-x}$ is the x-axis—that is, $y = 0$. Since the desired horizontal asymptote is $y = -2$, we must have $c = -2$, so $f(x) = ba^{-x} - 2$.

Because the y-intercept is 16, $f(0)$ must equal 16. But $f(0) = ba^{-0} - 2 = b - 2$, so $b - 2 = 16$ and $b = 18$. Thus, $f(x) = 18a^{-x} - 2$.

Lastly, we find the value of a:

$$f(x) = 18a^{-x} - 2 \quad \text{given form of } f$$
$$0 = 18(a)^{-2} - 2 \quad f(2) = 0 \text{ since 2 is the } x\text{-intercept}$$

$$2 = 18 \cdot \frac{1}{a^2} \qquad \text{add 2; definition of negative exponent}$$

$$a^2 = 9 \qquad \text{multiply by } a^2/2$$

$$a = \pm 3 \qquad \text{take square root}$$

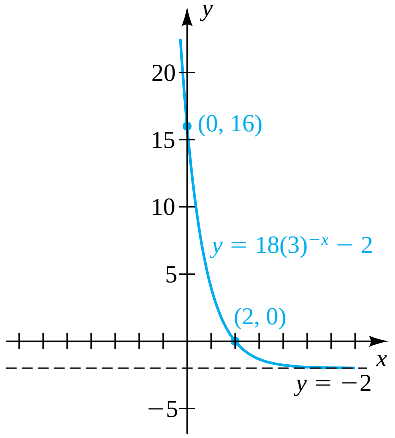
Since a must be positive, we have

$$f(x) = 18(3)^{-x} - 2.$$

Figure 7 shows a graph of f that satisfies all of the conditions in the problem statement. Note that $f(x)$ could be written in the equivalent form

$$f(x) = 18\left(\tfrac{1}{3}\right)^x - 2.$$

FIGURE 7



The bell-shaped graph of the function in the next example is similar to a *normal probability curve* used in statistical studies.

EXAMPLE 6 Sketching a bell-shaped graph

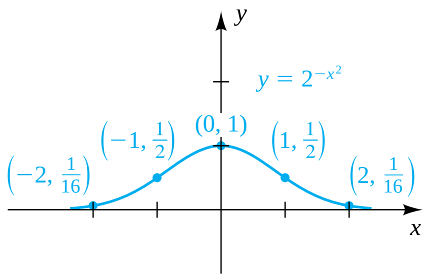
If $f(x) = 2^{-x^2}$, sketch the graph of f.

SOLUTION If we rewrite $f(x)$ as

$$f(x) = \frac{1}{2^{(x^2)}},$$

we see that as x increases through positive values, $f(x)$ decreases rapidly; hence the graph approaches the x-axis asymptotically. Since x^2 is smallest when $x = 0$, the maximum value of f is $f(0) = 1$. Since f is an even function, the graph is symmetric with respect to the y-axis. Some points on the graph are $(0, 1)$, $\left(1, \frac{1}{2}\right)$, and $\left(2, \frac{1}{16}\right)$. Plotting and using symmetry gives us the sketch in Figure 8.

FIGURE 8



APPLICATION Bacterial Growth

Exponential functions may be used to describe the growth of certain populations. As an illustration, suppose it is observed experimentally that the number of bacteria in a culture doubles every day. If 1000 bacteria are present at the start, then we obtain the following table, where t is the time in days and $f(t)$ is the bacteria count at time t.

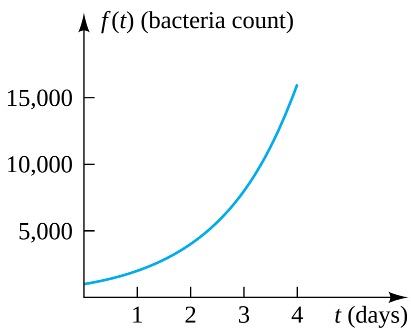
***t* (time in days)**	0	1	2	3	4
***f*(*t*) (bacteria count)**	1000	2000	4000	8000	16,000

It appears that $f(t) = (1000)2^t$. With this formula we can predict the number of bacteria present at any time t. For example, at $t = 1.5 = \frac{3}{2}$,

$$f(t) = (1000)2^{3/2} \approx 2828.$$

The graph of f is sketched in Figure 9.

FIGURE 9

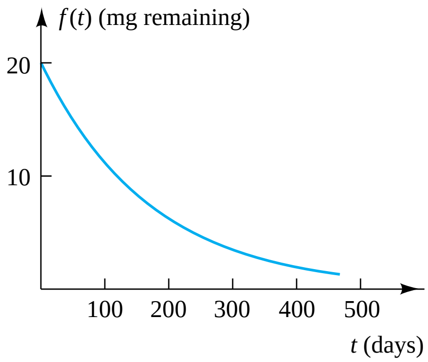


APPLICATION Radioactive Decay

Certain physical quantities *decrease* exponentially. In such cases, if a is the base of the exponential function, then $0 < a < 1$. One of the most common examples of exponential decrease is the decay of a radioactive substance, or isotope. The **half-life** of an isotope is the time it takes for one-half the original amount in a given sample to decay. The half-life is the principal characteristic used to distinguish one radioactive substance from another. The polonium isotope ^{210}Po has a half-life of approximately 140 days; that is, given any amount, one-half of it will disintegrate in 140 days. If 20 milligrams of ^{210}Po is present initially, then the following table indicates the amount remaining after various intervals of time.

***t* (time in days)**	0	140	280	420	560
***f*(*t*) (mg remaining)**	20	10	5	2.5	1.25

FIGURE 10



The sketch in Figure 10 illustrates the exponential nature of the disintegration.

Other radioactive substances have much longer half-lives. In particular, a by-product of nuclear reactors is the radioactive plutonium isotope ^{239}Pu, which has a half-life of approximately 24,000 years. It is for this reason that the disposal of radioactive waste is a major problem in modern society.

APPLICATION Compound Interest

Compound interest provides a good illustration of exponential growth. If a sum of money P, the *principal*, is invested at a *simple* interest rate r, then the interest at the end of one interest period is the product Pr when r is expressed as a decimal. For example, if $P = \$1000$ and the interest rate is 9% per year, then $r = 0.09$, and the interest at the end of one year is \$1000(0.09), or \$90.

If the interest is reinvested with the principal at the end of the interest period, then the new principal is

$$P + Pr \quad \text{or, equivalently,} \quad P(1 + r).$$

Note that to find the new principal we may multiply the original principal by $(1 + r)$. In the preceding example, the new principal is \$1000(1.09), or \$1090.

After another interest period has elapsed, the new principal may be found by multiplying $P(1 + r)$ by $(1 + r)$. Thus, the principal after two interest periods is $P(1 + r)^2$. If we continue to reinvest, the principal after three periods is $P(1 + r)^3$; after four it is $P(1 + r)^4$; and, in general, the amount A accumulated after k interest periods is

$$A = P(1 + r)^k.$$

Interest accumulated by means of this formula is **compound interest.** Note that A is expressed in terms of an exponential function with base $1 + r$. The interest period may be measured in years, months, weeks, days, or any other suitable unit of time. When applying the formula for A, remember that *r is the interest rate per interest period expressed as a decimal.* For example, if the rate is stated as *6% per year compounded monthly,* then the rate per month is $\frac{6}{12}$% or, equivalently, 0.5%. Thus, $r = 0.005$ and k is the number of months. If \$100 is invested at this rate, then the formula for A is

$$A = 100(1 + 0.005)^k = 100(1.005)^k.$$

In general, we have the following formula.

Compound Interest Formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

where P = principal
r = annual interest rate expressed as a decimal
n = number of interest periods per year
t = number of years P is invested
A = amount after t years.

The next example illustrates the use of the compound interest formula.

EXAMPLE 7 Using the compound interest formula

Suppose that \$1000 is invested at an interest rate of 9% compounded monthly. Find the new amount of principal after 5 years, after 10 years, and after 15 years. Illustrate graphically the growth of the investment.

SOLUTION Applying the compound interest formula with $r = 9\% = 0.09$, $n = 12$, and $P = \$1000$, we find that the amount after t years is

$$A = 1000\left(1 + \frac{0.09}{12}\right)^{12t} = 1000(1.0075)^{12t}.$$

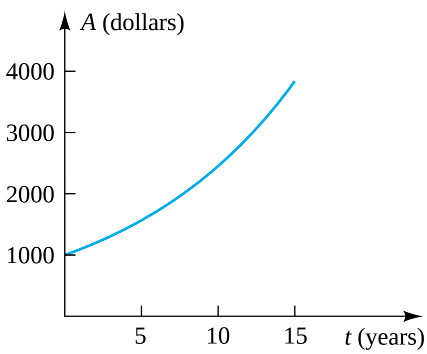
Substituting $t = 5$, 10, and 15 and using a calculator, we obtain the following table.

Note that when working with monetary values, we use = instead of ≈ and round to two decimal places.

Number of years	Amount
5	$A = \$1000(1.0075)^{60} = \$1565.68$
10	$A = \$1000(1.0075)^{120} = \$2451.36$
15	$A = \$1000(1.0075)^{180} = \$3838.04$

FIGURE 11

Compound interest: $A = 1000(1.0075)^{12t}$



The exponential nature of the increase is indicated by the fact that during the first five years, the growth in the investment is \$565.68; during the second five-year period, the growth is \$885.68; and during the last five-year period, it is \$1386.68.

The sketch in Figure 11 illustrates the growth of \$1000 invested over a period of 15 years. ■

EXAMPLE 8 Finding an exponential model

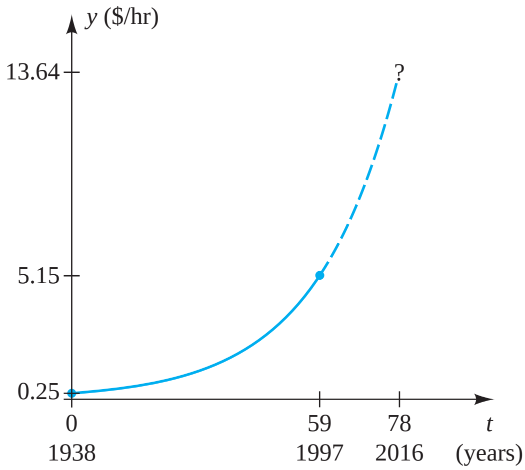
In 1938, a federal law establishing a minimum wage was enacted, and the wage was set at \$0.25 per hour; the wage had risen to \$5.15 per hour by 1997. Find a simple exponential function of the form $y = ab^t$ that models the federal minimum wage for 1938–1997.

SOLUTION

$y = ab^t$	given
$0.25 = ab^0$	let $t = 0$ for 1938
$0.25 = a$	$b^0 = 1$
$y = 0.25b^t$	replace a with 0.25
$5.15 = 0.25b^{59}$	$t = 1997 - 1938 = 59$
$b^{59} = \frac{5.15}{0.25} = 20.6$	divide by 0.25
$b = \sqrt[59]{20.6}$	take 59th root
$b \approx 1.0526$	approximate

We obtain the model $y = 0.25(1.0526)^t$, which indicates that the federal minimum wage rose about 5.26% per year from 1938 to 1997. A graph of the model is shown in Figure 12. Do you think this model will hold true through the year 2016?

FIGURE 12



We conclude this section with an example involving a graphing utility.

EXAMPLE 9 **Estimating amounts of a drug in the bloodstream**

If an adult takes a 100-milligram tablet of a certain prescription drug orally, the rate R at which the drug enters the bloodstream t minutes later is predicted to be

$$R = 5(0.95)^t \text{ mg/min.}$$

It can be shown using calculus that the amount A of the drug in the bloodstream at time t can be approximated by

$$A = 97.4786[1 - (0.95)^t] \text{ mg.}$$

(a) Estimate how long it takes for 50 milligrams of the drug to enter the bloodstream.

(b) Estimate the number of milligrams of the drug in the bloodstream when the drug is entering at a rate of 3 mg/min.

FIGURE 13
[0, 100, 10] by [0, 100, 10]

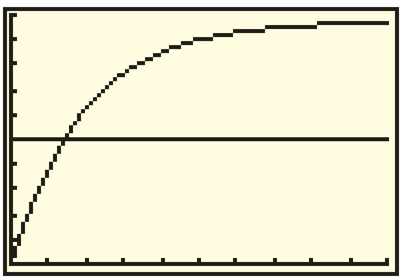
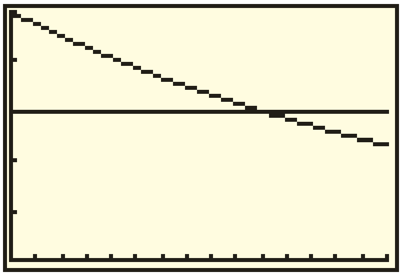


FIGURE 14
[0, 15] by [0, 5]



SOLUTION

(a) We wish to determine t when A is equal to 50. Since the value of A cannot exceed 97.4786, we choose the viewing rectangle to be [0, 100, 10] by [0, 100, 10].

We next assign $97.4786[1 - (0.95)^x]$ to Y_1, assign 50 to Y_2, and graph Y_1 and Y_2, obtaining a display similar to that in Figure 13 (note that $x = t$). Using the intersect feature, we estimate that $A = 50$ mg when $x \approx 14$ min.

(b) We wish to determine t when R is equal to 3. Let us first assign $5(0.95)^x$ to Y_3 and 3 to Y_4. Since the maximum value of Y_3 is 5 (at $t = 0$), we use a viewing rectangle of dimensions [0, 15] by [0, 5] and obtain a display similar to that in Figure 14. Using the intersect feature again, we find that $y = 3$ when $x \approx 9.96$. Thus, after almost 10 minutes, the drug will be entering the bloodstream at a rate of 3 mg/min. (Note that the initial rate, at $t = 0$, is 5 mg/min.) Finding the value of Y_1 at $x = 10$, we see that there is almost 39 milligrams of the drug in the bloodstream after 10 minutes. ■

4.2 Exercises

Exer. 1–10: Solve the equation.

1 $7^{x+6} = 7^{3x-4}$

2 $6^{7-x} = 6^{2x+1}$

3 $3^{2x+3} = 3^{(x^2)}$

4 $9^{(x^2)} = 3^{3x+2}$

5 $2^{-100x} = (0.5)^{x-4}$

6 $\left(\frac{1}{2}\right)^{6-x} = 2$

7 $25^{x-3} = 125^{4-x}$

8 $27^{x-1} = 9^{2x-3}$

9 $4^x \cdot \left(\frac{1}{2}\right)^{3-2x} = 8 \cdot (2^x)^2$

10 $9^{2x} \cdot \left(\frac{1}{3}\right)^{x+2} = 27 \cdot (3^x)^{-2}$

11 Complete the statements for $f(x) = a^x + c$ with $a > 1$.

(a) As $x \to \infty$, $f(x) \to$ ____.

(b) As $x \to -\infty$, $f(x) \to$ ____.

12 Complete the statements for $f(x) = a^{-x} + c$ with $a > 1$.

(a) As $x \to \infty$, $f(x) \to$ ____.

(b) As $x \to -\infty$, $f(x) \to$ ____.

13 Sketch the graph of f if $a = 2$.

(a) $f(x) = a^x$

(b) $f(x) = -a^x$

(c) $f(x) = 3a^x$

(d) $f(x) = a^{x+3}$

(e) $f(x) = a^x + 3$

(f) $f(x) = a^{x-3}$

(g) $f(x) = a^x - 3$

(h) $f(x) = a^{-x}$

(i) $f(x) = \left(\frac{1}{a}\right)^x$

(j) $f(x) = a^{3-x}$

14 Work Exercise 13 if $a = \frac{1}{2}$.

Exer. 15–28: Sketch the graph of f.

15 $f(x) = \left(\frac{2}{5}\right)^{-x}$

16 $f(x) = \left(\frac{2}{5}\right)^x$

17 $f(x) = 5\left(\frac{1}{2}\right)^x + 3$

18 $f(x) = 8(4)^{-x} - 2$

19 $f(x) = -\left(\frac{1}{2}\right)^x + 4$

20 $f(x) = -3^{-x} + 9$

21 $f(x) = -\left(\frac{1}{2}\right)^{-x} + 8$

22 $f(x) = -3^x + 9$

23 $f(x) = 2^{|x|}$

24 $f(x) = 2^{-|x|}$

25 $f(x) = 3^{1-x^2}$

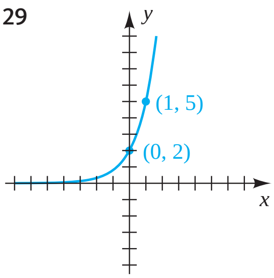
26 $f(x) = 2^{-(x+1)^2}$

27 $f(x) = 3^x + 3^{-x}$

28 $f(x) = 3^x - 3^{-x}$

Exer. 29–32: Find an exponential function of the form $f(x) = ba^x$ or $f(x) = ba^x + c$ that has the given graph.

29



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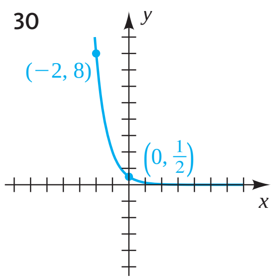

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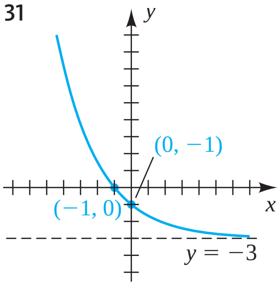

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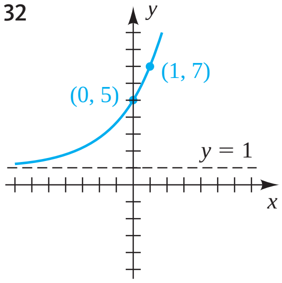


Exer. 33–34: Find an exponential function of the form $f(x) = ba^x$ that has the given y-intercept and passes through the point P.

33 y-intercept 8; $P(3, 1)$

34 y-intercept 5; $P\left(2, \frac{5}{16}\right)$

Exer. 35–36: Find an exponential function of the form $f(x) = ba^{-x} + c$ that has the given horizontal asymptote and y-intercept and passes through point P.

35 $y = 32$; y-intercept 212; $P(2, 112)$

36 $y = 72$; y-intercept 425; $P(1, 248.5)$

37 **Elk population** One hundred elk, each 1 year old, are introduced into a game preserve. The number $N(t)$ alive after t years is predicted to be $N(t) = 100(0.9)^t$.

(a) Estimate the number alive after 5 years.

(b) What percentage of the herd dies each year?

38 **Drug dosage** A drug is eliminated from the body through urine. Suppose that for an initial dose of 10 milligrams, the amount $A(t)$ in the body t hours later is given by $A(t) = 10(0.8)^t$.

(a) Estimate the amount of the drug in the body 8 hours after the initial dose.

(b) What percentage of the drug still in the body is eliminated each hour?

39 **Bacterial growth** The number of bacteria in a certain culture increased from 600 to 1800 between 7:00 A.M. and 9:00 A.M. Assuming growth is exponential, the number $f(t)$ of bacteria t hours after 7:00 A.M. is given by $f(t) = 600(3)^{t/2}$.

(a) Estimate the number of bacteria in the culture at 8:00 A.M., 10:00 A.M., and 11:00 A.M.

(b) Sketch the graph of f for $0 \le t \le 4$.

40 **Newton's law of cooling** According to Newton's law of cooling, the rate at which an object cools is directly proportional to the difference in temperature between the object and the surrounding medium. The face of a household iron cools from 125° to 100° in 30 minutes in a room that remains at a constant temperature of 75°. From calculus, the temperature $f(t)$ of the face after t hours of cooling is given by $f(t) = 50(2)^{-2t} + 75$.

(a) Assuming $t = 0$ corresponds to 1:00 P.M., approximate to the nearest tenth of a degree the temperature of the face at 2:00 P.M., 3:30 P.M., and 4:00 P.M.

(b) Sketch the graph of f for $0 \le t \le 4$.

41 **Radioactive decay** The radioactive bismuth isotope ^{210}Bi has a half-life of 5 days. If there is 100 milligrams of ^{210}Bi present at $t = 0$, then the amount $f(t)$ remaining after t days is given by $f(t) = 100(2)^{-t/5}$.

(a) How much ^{210}Bi remains after 5 days? 10 days? 12.5 days?

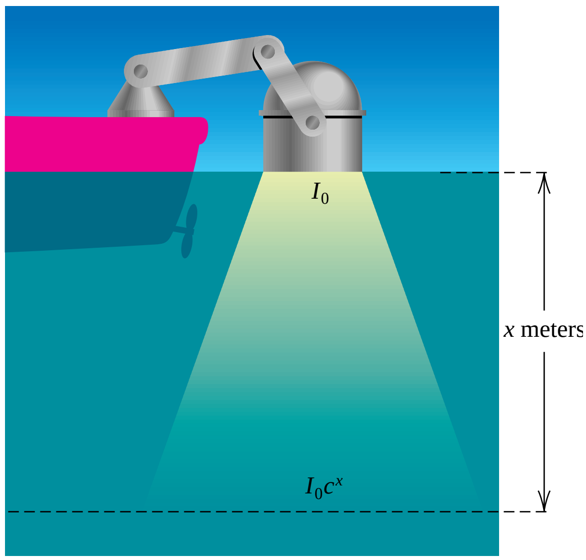
(b) Sketch the graph of f for $0 \le t \le 30$.

42 **Light penetration in an ocean** An important problem in oceanography is to determine the amount of light that can penetrate to various ocean depths. The Beer-Lambert law asserts that the exponential function given by $I(x) = I_0c^x$ is a model for this phenomenon (see the figure). For a certain location, $I(x) = 10(0.4)^x$ is the amount of light (in calories/cm^2/sec) reaching a depth of x meters.

(a) Find the amount of light at a depth of 2 meters.

(b) Sketch the graph of I for $0 \le x \le 5$.

EXERCISE 42



43 **Decay of radium** The half-life of radium is 1600 years. If the initial amount is q_0 milligrams, then the quantity $q(t)$ remaining after t years is given by $q(t) = q_0 2^{kt}$. Find k.

44 **Dissolving salt in water** If 10 grams of salt is added to a quantity of water, then the amount $q(t)$ that is undissolved after t minutes is given by $q(t) = 10\left(\frac{4}{5}\right)^t$. Sketch a graph that shows the value $q(t)$ at any time from $t = 0$ to $t = 10$.

45 **Compound interest** If \$1000 is invested at a rate of 7% per year compounded monthly, find the principal after

(a) 1 month **(b)** 6 months

(c) 1 year **(d)** 20 years

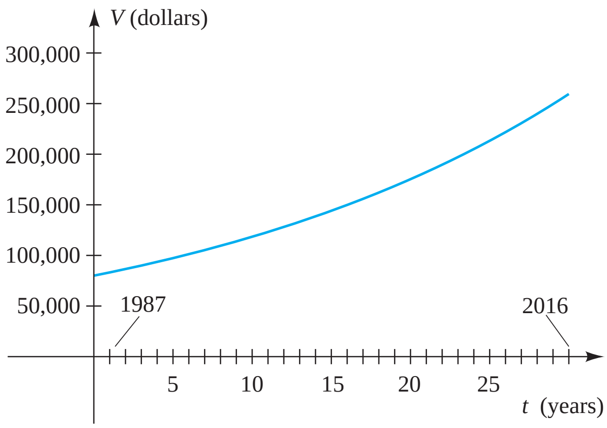
46 **Compound interest** If a savings fund pays interest at a rate of 3% per year compounded semiannually, how much money invested now will amount to \$5000 after 1 year?

47 **Automobile trade-in value** If a certain make of automobile is purchased for C dollars, its trade-in value $V(t)$ at the end of t years is given by $V(t) = 0.78C(0.85)^{t-1}$. If the original cost is \$25,000, calculate, to the nearest dollar, the value after

(a) 1 year **(b)** 4 years **(c)** 7 years

48 **Real estate appreciation** If the value of real estate increases at a rate of 4% per year, after t years the value V of a house purchased for P dollars is $V = P(1.04)^t$. A graph for the value of a house purchased for \$80,000 in 1986 is shown in the figure. Approximate the value of the house, to the nearest \$1000, in the year 2016.

EXERCISE 48



49 **Manhattan Island** The Island of Manhattan was sold for \$24 in 1626. How much would this amount have grown to by 2012 if it had been invested at 6% per year compounded quarterly?

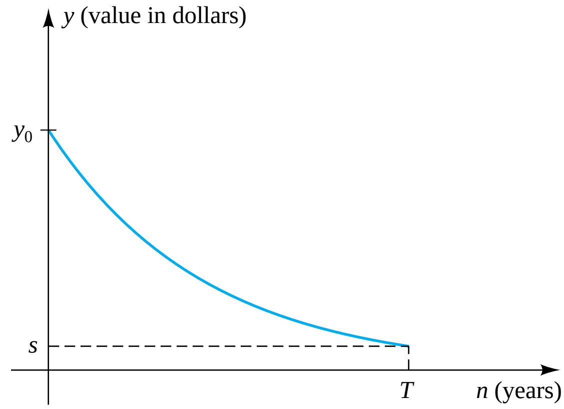
50 **Credit-card interest** A certain department store requires its credit-card customers to pay interest on unpaid bills at the rate of 24% per year compounded monthly. If a customer buys a television set for \$500 on credit and makes no payments for one year, how much is owed at the end of the year?

51 **Depreciation** The declining balance method is an accounting method in which the amount of depreciation taken each year is a fixed percentage of the present value of the item. If y is the value of the item in a given year, the depreciation taken is ay for some depreciation rate a with $0 < a < 1$, and the new value is $(1 - a)y$.

(a) If the initial value of the item is y_0, show that the value after n years of depreciation is $(1 - a)^n y_0$.

(b) At the end of T years, the item has a salvage value of s dollars. The taxpayer wishes to choose a depreciation rate such that the value of the item after T years will equal the salvage value (see the figure). Show that $a = 1 - \sqrt[T]{s/y_0}$.

EXERCISE 51



52 **Language dating** Glottochronology is a method of dating a language at a particular stage, based on the theory that over a long period of time linguistic changes take place at a fairly constant rate. Suppose that a language originally had N_0 basic words and that at time t, measured in millennia (1 millennium = 1000 years), the number $N(t)$ of basic words that remain in common use is given by $N(t) = N_0(0.805)^t$.

(a) Approximate the percentage of basic words lost every 100 years.

(b) If $N_0 = 200$, sketch the graph of N for $0 \le t \le 5$.

Exer. 53–56: Some lending institutions calculate the monthly payment M on a loan of L dollars at an interest rate r (expressed as a decimal) by using the formula

$$M = \frac{Lrk}{12(k - 1)},$$

where $k = [1 + (r/12)]^{12t}$ and t is the number of years that the loan is in effect.

53 **Home mortgage**

(a) Find the monthly payment on a 30-year \$250,000 home mortgage if the interest rate is 8%.

(b) Find the total interest paid on the loan in part (a).

54 **Home mortgage** Find the largest 25-year home mortgage that can be obtained at an interest rate of 7% if the monthly payment is to be \$1500.

55 **Car loan** An automobile dealer offers customers no-down-payment 3-year loans at an interest rate of 10%. If a customer can afford to pay \$500 per month, find the price of the most expensive car that can be purchased.

56 **Business loan** The owner of a small business decides to finance a new computer by borrowing \$3000 for 2 years at an interest rate of 7.5%.

(a) Find the monthly payment.

(b) Find the total interest paid on the loan.

Exer. 57–58: Approximate the function at the value of x to four decimal places.

57 **(a)** $f(x) = 13^{\sqrt{x+1.1}}, \quad x = 3$

(b) $h(x) = (2^x + 2^{-x})^{2x}, \quad x = 1.06$

58 **(a)** $f(x) = 2^{\sqrt{1-x}}, \quad x = 0.5$

(b) $h(x) = \dfrac{3^{-x} + 5}{3^x - 16}, \quad x = 1.4$

Exer. 59–60: Sketch the graph of the equation. (a) Estimate y if $x = 40$. (b) Estimate x if $y = 2$.

59 $y = (1.085)^x$ 60 $y = (1.0525)^x$

Exer. 61–62: Use a graph to estimate the roots of the equation.

61 $1.4x^2 - 2.2^x = 1$

62 $1.21^{3x} + 1.4^{-1.1x} - 2x = 0.5$

Exer. 63–64: Graph f on the given interval. (a) Determine whether f is one-to-one. (b) Estimate the zeros of f.

63 $f(x) = \dfrac{3.1^x - 2.5^{-x}}{2.7^x + 4.5^{-x}}; \quad [-3, 3]$

64 $f(x) = \pi^{0.6x} - 1.3^{(x^{1.8})}; \quad [-4, 4]$

(*Hint:* Change $x^{1.8}$ to an equivalent form that is defined for $x < 0$.)

Exer. 65–66: Graph f on the given interval. (a) Estimate where f is increasing or is decreasing. (b) Estimate the range of f.

65 $f(x) = 0.7x^3 + 1.7^{(-1.8x)}; \quad [-4, 1]$

66 $f(x) = \dfrac{3.1^{-x} - 4.1^x}{4.4^{-x} + 5.3^x}; \quad [-3, 3]$

67 **Trout population** One thousand trout, each 1 year old, are introduced into a large pond. It is predicted that the number $N(t)$ still alive after t years will be given by the equation $N(t) = 1000(0.9)^t$. Use the graph of N to approximate when 500 trout will be alive.

68 **Buying power** An economist predicts that the buying power $B(t)$ of a dollar t years from now will be given by $B(t) = (0.95)^t$. Use the graph of B to approximate when the buying power will be half of what it is today.

69 **Gompertz function** The **Gompertz function,**

$$y = ka^{(b^x)} \quad \text{with } k > 0,\ 0 < a < 1, \text{ and } 0 < b < 1,$$

is sometimes used to describe the sales of a new product whose sales are initially large but then level off toward a maximum saturation level. Graph, on the same coordinate plane, the line $y = k$ and the Gompertz function with $k = 4$, $a = \frac{1}{8}$, and $b = \frac{1}{4}$. What is the significance of the constant k?

70 **Logistic function** The **logistic function,**

$$y = \frac{1}{k + ab^x} \quad \text{with } k > 0,\ a > 0, \text{ and } 0 < b < 1,$$

is sometimes used to describe the sales of a new product that experiences slower sales initially, followed by growth toward a maximum saturation level. Graph, on the same coordinate plane, the line $y = 1/k$ and the logistic function with $k = \frac{1}{4}$, $a = \frac{1}{8}$, and $b = \frac{5}{8}$. What is the significance of the value $1/k$?

Exer. 71–72: If monthly payments p are deposited in a savings account paying an annual interest rate r, then the amount A in the account after n years is given by

$$A = \frac{p\left(1 + \frac{r}{12}\right)\left[\left(1 + \frac{r}{12}\right)^{12n} - 1\right]}{\frac{r}{12}}.$$

Graph A for each value of p and r, and estimate n for $A = \$100{,}000$.

71 $p = 100, \quad r = 0.05$ 72 $p = 250, \quad r = 0.09$

73 **Government receipts** Federal government receipts (in billions of dollars) for selected years are listed in the table.

Year	1910	1930	1950	1970
Receipts	0.7	4.1	39.4	192.8

Year	1980	1990	2000
Receipts	517.1	1032.0	2025.2

(a) Let $x = 0$ correspond to the year 1910. Plot the data, together with the functions f and g:

(1) $f(x) = 0.786(1.094)^x$

(2) $g(x) = 0.503x^2 - 27.3x + 149.2$

(b) Determine whether the exponential or quadratic function better models the data.

(c) Use your choice in part (b) to graphically estimate the year in which the federal government first collected $1 trillion.

74 **Epidemics** In 1840, Britain experienced a bovine (cattle and oxen) epidemic called epizooty. The estimated number of new cases every 28 days is listed in the table. At the time, the *London Daily* made a dire prediction that the number of new cases would continue to increase indefinitely. William Farr correctly predicted when the number of new cases would peak. Of the two functions

$$f(t) = 653(1.028)^t$$

and

$$g(t) = 54{,}700e^{-(t-200)^2/7500}$$

one models the newspaper's prediction and the other models Farr's prediction, where t is in days with $t = 0$ corresponding to August 12, 1840.

Date	New cases
Aug. 12	506
Sept. 9	1289
Oct. 7	3487
Nov. 4	9597
Dec. 2	18,817
Dec. 30	33,835
Jan. 27	47,191

(a) Graph each function, together with the data, in the viewing rectangle [0, 400, 100] by [0, 60,000, 10,000].

(b) Determine which function better models Farr's prediction.

(c) Determine the date on which the number of new cases peaked.

75 **Cost of a stamp** The price of a first-class stamp was 4¢ for the first time in 1958 and 44¢ in 2009 (it was 2¢ in 1919). Find a simple exponential function of the form $y = ab^t$ that models the cost of a first-class stamp for 1958–2009, and predict its value for 2020.

76 **Super Bowl TV costs** The following table gives the cost (in thousands of dollars) for a 30-second television advertisement during the Super Bowl for various years.

Year	Cost
1967	42
1977	125
1987	600
1997	1200
2007	2600

(a) Plot the data on the xy-plane.

(b) Determine a curve in the form $y = ab^x$, where $x = 0$ is the first year and y is the cost that models the data. Graph this curve together with the data on the same coordinate axes. Answers may vary.

(c) Use this curve to predict the cost of a 30-second commercial in 2002. Compare your answer to the actual value of $1,900,000.

77 **Inflation comparisons** In 1974, Johnny Miller won 8 tournaments on the PGA tour and accumulated $353,022 in official season earnings. In 1999, Tiger Woods accumulated $6,616,585 with a similar record.

(a) Suppose the monthly inflation rate from 1974 to 1999 was 0.0025 (3%/yr). Use the compound interest formula to estimate the equivalent value of Miller's winnings in the year 1999. Compare your answer with that from an inflation calculation on the web (e.g., bls.gov/cpi/home.htm).

(b) Find the annual interest rate needed for Miller's winnings to be equivalent in value to Woods's winnings.

(c) What type of function did you use in part (a)? part (b)?

78 **Consumer Price Index** The CPI is the most widely used measure of inflation. In 1970, the CPI was 37.8, and in 2000, the CPI was 168.8. This means that an urban consumer who paid $37.80 for a market basket of consumer goods and services in 1970 would have needed $168.80 for similar goods and services in 2000. Find a simple exponential function of the form $y = ab^t$ that models the CPI for 1970–2000, and predict its value for 2020.