MCR3U9 Review for the Final Exam

Critical Thinking: True or False

Derivation: the process of deducing a new formula, theorem, etc., from previously proven or accepted statements; a ***line of reasoning*** that shows how a conclusion follows logically from accepted propositions.

(a) If a statement is true, prove that it is or provide an explanation.

(b) If a series of statements leads to a conclusion that is true, justify each step in the derivation.

(c) If a statement is false, provide a ***counterexample*** or an explanation. In either case, ***correct*** the statement.

(d) If a series of statements leads to a conclusion that is false, find the flaws in the derivation and ***correct*** them.

(e) Also correct any errors in the usage of mathematical notation.

Statement or Series of Statements	*True or False?*	*Proof, Counterexample, Explanation, Correction*
1. $f(x+y)=f(x)+f(y)$		
2. $f^{-1}(x)=\dfrac{1}{f(x)}$		
3. $\left(f(x)\right)^{-1}=\dfrac{1}{f(x)}$		
4. The function $f(u)=\dfrac{u-2}{u^2-5u+6}$ has vertical asymptotes $u=2$ and $u=3$ (Don't take $f(u)$ personally!)		
5. Suppose that r = 5 m and θ = 225° $\because l=r\theta$ $\therefore l=(5)(225°)$ $\therefore l=1125$ Therefore, arc length is 1125 m.		
6. $(x+2)^3=x^3+2^3=x^3+8$		

Statement or Series of Statements	*True or False?*	*Proof, Counterexample, Explanation, Correction*
7. $(x-5)^2 \geq 0$ $\therefore x-5 \geq 0$ $\therefore x \geq 5$		
8. $6x^2 - 5x = 6$ $\therefore x(6x-5) = 6$ $\therefore x = 6$ or $6x-5=6$ $\therefore x = 6$ or $x = \dfrac{11}{6}$		
9. $\sec\theta = 2$ $\therefore \dfrac{1}{\cos}\theta = 2$ $\therefore \dfrac{\theta}{\cos} = 2$ $\therefore \theta = 2\cos$		
10. $\csc\theta = 2$ $\therefore \dfrac{1}{\sin\theta} = 2$ $\therefore \sin\theta = \dfrac{1}{2}$ $\therefore \theta = \sin^{-1}\left(\dfrac{1}{2}\right)$ $\therefore \theta = \dfrac{1}{\sin 2}$		
11. $\tan\left(\dfrac{\pi}{4}+\dfrac{\pi}{3}\right) = \tan\dfrac{\pi}{4} + \tan\dfrac{\pi}{3}$ $= 1+\sqrt{3}$		
12. $\tan\left[\dfrac{\pi}{4}\left(\dfrac{\pi}{3}\right)\right] = \tan\dfrac{\pi}{4}\left(\tan\dfrac{\pi}{3}\right)$ $= 1\left(\sqrt{3}\right) = \sqrt{3}$		

Statement or Series of Statements	***True or False?***	***Proof, Counterexample, Explanation, Correction***
13. $\sin 2x = 2\sin x$		
14. $\sin 3x = 3\sin x$		
15. $\cos 4x = 4\cos x$		
16. $\cos\frac{x}{2} = \frac{\cos x}{2}$		
17. A rational function can have two horizontal asymptotes.		
18. No function can have two horizontal asymptotes.		
19. The solution set to the inequality $-(x-3)^2(x-5)(x+5)\le 0$ is $(-\infty,5]$. (As always, look at equations and inequalities graphically as well as algebraically.)		
20. The solution set to the inequality $12x^6+2x^4+3x^2+1\le 0$ is $\{\ \}=\varnothing$ (i.e. the empty set).		

Statement or Series of Statements	True or False?	Proof, Counterexample, Explanation, Correction
21. $\lvert x+y\rvert = \lvert x\rvert + \lvert y\rvert$		
22. $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$		
23. $\sqrt{xy} = \sqrt{x}\sqrt{y}$		
24. $\sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}}$		
25. $\lvert xy\rvert = \lvert x\rvert \lvert y\rvert$		
26. $\left\lvert \dfrac{x}{y} \right\rvert = \dfrac{\lvert x\rvert}{\lvert y\rvert}$		
27. The rational function $f(x) = \dfrac{-3(x-7)^2(x-3)(x+1)}{2(x-7)(x-1)(x+3)}$ has vertical asymptotes $x=-3$, $x=1$ and $x=7$, as well as a horizontal asymptote $y=-\dfrac{3}{2}$.		
28. The equation $\csc\theta = \dfrac{1}{2}$ has an infinite number of solutions.		

Statement or Series of Statements	***True or False?***	***Proof, Counterexample, Explanation, Correction***
29. An even function is the same as an even-degree polynomial function.		
30. An odd function is the same as an odd-degree polynomial function.		
31. The graph of $f(x) = -3\sin\left(\frac{\pi}{4}(x-5)\right)+6$ can be obtained by performing the following transformations to the graph of $y = \sin x$: **Vertical:** Stretch by a factor of −3, then translate 6 units up. **Horizontal:** Compress by a factor of $\frac{\pi}{4}$, then translate 5 units left.		
32. The meaning of $\frac{\pi}{4}$ in $f(x) = -3\cos\left(\frac{\pi}{4}(x-5)\right)+6$ is that there are $\frac{\pi}{4}$ cycles per 2π radians.		
33. The meaning of $\frac{\pi}{4}$ in $f(x) = -3\tan\left(\frac{\pi}{4}(x-5)\right)+6$ is that there are $\frac{\pi}{4}$ cycles per 2π radians.		
34. The remainder obtained when $f(x) = x^3 - 8x^2 - 3x + 90$ is divided by $x+2$ can be obtained by evaluating $f(2)$.		

Statement or Series of Statements	*True or False?*	*Proof, Counterexample, Explanation, Correction*
35. $h(t)=0.5\cos\left(\frac{200}{\pi}t\right)+2.5$ models how the height (in metres) above the floor of the tip of one of the blades of a motorized fan changes over time (in seconds). In this equation, 0.5 represents the diameter of the fan, 2.5 represents the maximum height and $\frac{200}{\pi}$ represents the rate of rotation in of the fan in rotations per second.		
36. $(x,y)\rightarrow\left(2x-7,\pi y-\frac{\pi}{2}\right)$ models the following transformation: **Vertical:** Stretch by a factor of π, then translate $\frac{\pi}{2}$ units down. **Horizontal:** Compress by a factor of $\frac{1}{2}$, then translate 7 units right.		
37. $(x,y)\rightarrow\left(2(x-7),\pi\left(y-\frac{\pi}{2}\right)\right)$ models the following transformation: **Vertical:** Stretch by a factor of π, then translate $\frac{\pi}{2}$ units down. **Horizontal:** Stretch by a factor of 2, then translate 7 units left.		
38. Success in mathematics ***does not*** require ***any*** of the following: • thought • communication skills • understanding concepts • correct usage of terminology • understanding the meaning of mathematical notation • logical reasoning • creativity • problem-solving skills • number sense • spatial sense • pattern recognition It is sufficient to memorize formulas and unthinkingly mimic examples. It also doesn't hurt to suck up to the teacher! ;)		

Critical Thinking: Classification of Equations

The following is a list of types of equations that we have encountered in this course.

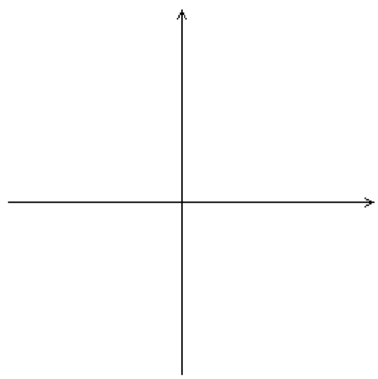
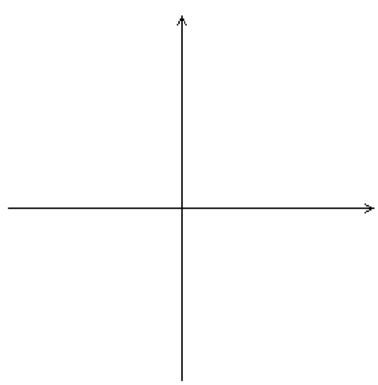
(a) Classify each equation as an ***identity***, an ***equation to be solved*** for the unknown, an ***equation of a function*** or an ***equation of a relation***.

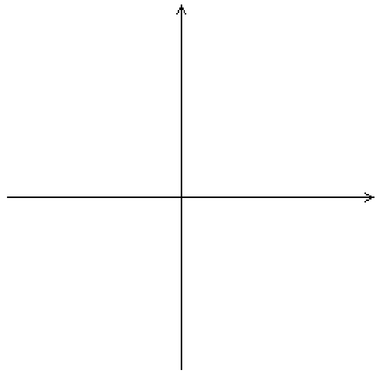
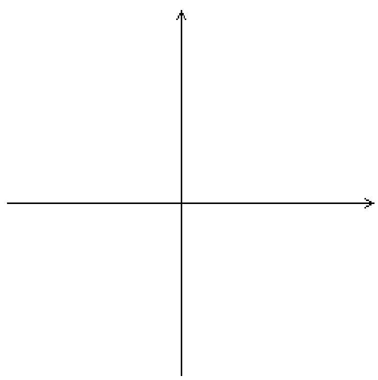
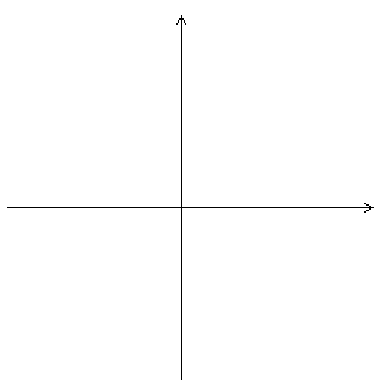
(b) Give a geometric (graphical) representation of each equation.

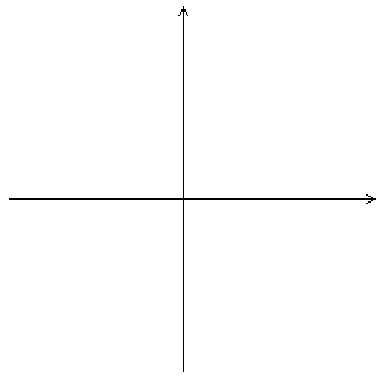
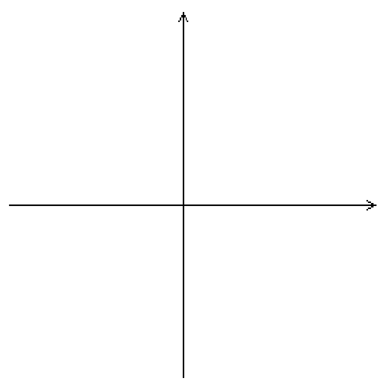
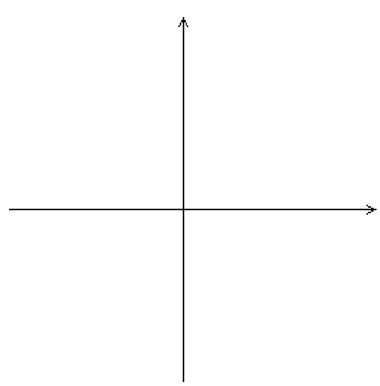
(c) For the equations that are identities, prove that the expression on the L.S. is ***equivalent*** to that on the R.S.

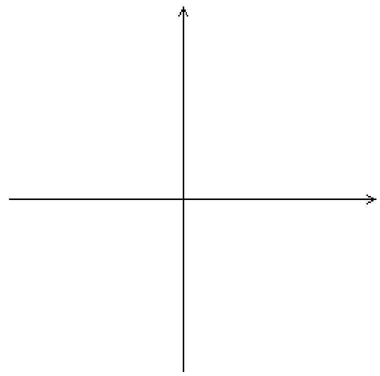
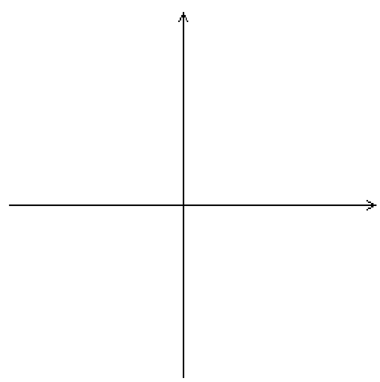
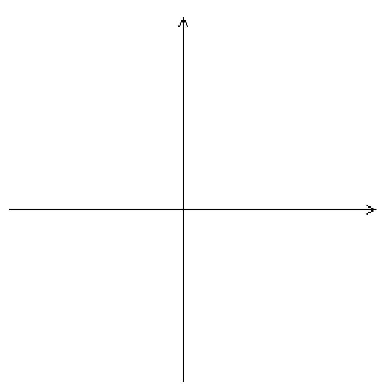
(d) For the equations of functions/relations, use the equation to find a point that lies on the graph of the function/relation. (Mark that point on the graph.)

(e) Solve the equations that are neither identities nor equations of functions/relations.

Equation	***Type of Equation***	***Geometric(Graphical) Representation***	***Proof/Solution/Evaluation to find Point on Graph***
1. $\sin 2t = 2\sin t \cos t$			
2. $\tan 2\theta = 1$			

Equation	***Type of Equation***	***Geometric(Graphical) Representation***	***Proof/Solution/Evaluation to find Point on Graph***
3. $x^2 + y^2 = 169$			
4. $g(x) = \cot x$			
5. $(x+y)^3$ $= x^3 + 3x^2y + 3xy^2 + y^3$			

Equation	***Type of Equation***	***Geometric(Graphical) Representation***	***Proof/Solution/Evaluation to find Point on Graph***
6. $x^3 - 8x^2 - 3x + 90 = 0$			
7. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$			
8. $\tan x = \dfrac{2\tan 2x - \sec^2 x \tan 2x}{2}$			

Equation	***Type of Equation***	***Geometric(Graphical) Representation***	***Proof/Solution/Evaluation to find Point on Graph***
9. $f(x) = -3\sin\left(\frac{\pi}{4}(x-5)\right)+6$			
10. $f(x) = -3\tan\left(\frac{\pi}{4}(x-5)\right)+6$			
11. $f(x)=\begin{cases} 2x, & x\ge 0 \\ x^2, & x<0 \end{cases}$ $g(x)=\begin{cases} -3x+13, & x\ge 5 \\ -2(x-6)^2, & x<5 \end{cases}$ $f(x)=g(x)$			

Mechanical Practice: Factoring

Fully factor each of the expressions given below. You can check your answers by expanding but you can save a great deal of time by using ***graphing software*** such as Desmos.

Note

- The ***majority*** of the expressions given below can be factored ***without*** first finding one of the zeros of the polynomial expression.
- For the expressions that do require that you first find a zero, you can save time by using software such as Desmos. This is especially helpful whenever trial and error takes too long.

1. x^3+5x^2+6x

2. $6x^4y-13x^3y-5x^2y$

3. a^2-169

4. $25a^2b^4-169c^4d^6$

5. a^3-81

6. a^3+125

7. $64a^3b^6-343c^6d^3$

8. $12x^4+36x^2+15$

9. $6x^5+36x^4+54x^3$

10. $2x^3+5x^2-24x-63$

11. $2\sin^2 x-\sin x-21$

12. $16\sin^2 x\sec^2 x-9\cos^4 x\cot^4 x$

13. $x^4-8x^3-26x^2+168x+441$

14. $8\tan^3 x-27\cot^6 x$

Mechanical Practice: Solving Equations and Inequalities

Solve each equation or inequality given below. You can check your answers by substitution but you can save a great deal of time by using ***graphing software*** such as Desmos.

Note

- Recall that the general approach is first to rewrite the equation or inequality in such a way that one side is some algebraic expression and the other side is ***zero***!
- Using function notation, we can express the previous point more precisely. First, the equation or inequality must be expressed in one of the following forms: $f(x)=0$, $f(x)<0$, $f(x)>0$, $f(x)\le 0$, $f(x)\ge 0$.
- Some of the given equations/inequalities cannot be solved ***analytically*** (i.e. only by exploiting known rules, without using numerical or graphical methods of approximation). In such cases, use graphing software to find approximate solutions.

1. $x^3+5x^2=-6x$

2. $12x^4=-36x^2-15$

3. $2\sin^2 x-3\sin x-1=0$, $x\in[0,2\pi]$

4. $2x^3+5x^2\ge 24x+63$

5. $a^2-16<0$

6. $x^5+6x^4\ge -9x^3$

7. $\sec x\sin x=3\sin x$, $x\in[0,2\pi]$

8. $4\sin^2 x-1\le 0$

9. $\dfrac{7}{x+2}+\dfrac{5}{x-2}=\dfrac{10x-2}{x^2-4}$

10. $x^4-8x^3-26x^2+168x+441=0$

11. $\tan x=x$

12. $16\sin^2 x-9\cos^2 x=0$, $x\in[-\pi,\pi]$

13. $-4\sin^2 x\cos^2 x\ge 3\sin^4 x-3$

14. $\cos x\le x^2-6x-16$

15. $\dfrac{1}{x-6}+\dfrac{x}{x-2}\le\dfrac{4}{x^2-8x+12}$

16. $\dfrac{1}{\sin x}+\dfrac{2}{\cos x}\le\dfrac{4}{\sin 2x}$

17. $\dfrac{1}{a+1}+\dfrac{1}{a-1}=\dfrac{2}{a^2-1}$

18. $\dfrac{1}{x+1}+\dfrac{x}{x-2}\ge\dfrac{13}{x^2-x-2}$

Theory: Proving that an Equation is an Identity

Prove that each of the following equations is an identity.

1. $(a+b)^2 = a^2 + 2ab + b^2$

2. $\sin^4 x - \cos^4 x = -\cos 2x$

3. $8\csc^2\theta - 3\cot^2\theta = 3 + 5\csc^2\theta$

4. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

5. $\sqrt{xy} = \sqrt{x}\sqrt{y}$, $x \ge 0$, $y \ge 0$

6. $\dfrac{1-\cos 2\theta}{2} = \sin^2\theta$

7. $\dfrac{\sin^2 x + 4\sin x + 3}{\cos^2 x} = \dfrac{3+\sin x}{1-\sin x}$

8. $\sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}}$, $x \ge 0$, $y > 0$

9. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y$

10. $\dfrac{\sin^2\theta + \cos^2\theta}{\sin^2\theta - \cos^2\theta} = -\sec 2\theta$

11. $\cos(x+y)\cos(x-y)$
$= \cos^2 x + \cos^2 y - 1$

12. $\dfrac{(\sin\theta + \cos\theta)^2}{\sin 2\theta} = \csc 2\theta + 1$

13. $\tan x + \tan y = \dfrac{\sin(x+y)}{\cos x \cos y}$

14. $\cos 3x = 4\cos^3 x - 3\cos x$

15. Derive an identity for $\sin 5x$ entirely in terms of $\sin x$ and $\cos x$.

16. $\sin(x+y)\sin(x-y)$
$= \cos^2 y - \cos^2 x$

17. $\sin(x+y) + \cos(x-y)$
$= (\sin x + \cos x)(\sin y + \cos y)$

Theory: Using Compound and Double-Angle Identities to find Exact Values of Trigonometric Ratios

Find exact values for each of the following. Use Wolfram Alpha to check your answers.

1. $\cos\dfrac{\pi}{12}$

2. $\sin\dfrac{5\pi}{12}$

3. $\cos\dfrac{11\pi}{12}$

4. $\sin\dfrac{\pi}{8}$

5. $\cos\dfrac{3\pi}{8}$

6. $\tan\dfrac{13\pi}{8}$

7. $\cos\dfrac{19\pi}{12}$

8. $\cot\dfrac{19\pi}{12}$

Graphing: Using Transformations of Base/Parent/Mother Functions to Sketch Graphs of Functions

Sketch the graphs of each of the following functions. Use Desmos to check your answers.

1. $f(x) = -3\cos\left(\dfrac{1}{3}(x-2\pi)\right) + 1$

2. $f(x) = \dfrac{1}{2}\cot\left(2\left(x+\dfrac{\pi}{4}\right)\right) - 3$

3. $f(x) = \dfrac{5}{2}\left\lfloor \dfrac{1}{4}(x+3) \right\rfloor - 1$

4. $f(x) = -2\csc\left(\dfrac{\pi}{2}(x-3)\right) - 2$

5. $f(x) = \dfrac{1}{2}\left|3(x+7)\right| - 5$

6. $f(x) = -\dfrac{9}{2}\left[2(x+1)\right]^3 + 1$

Graphing: Graph Polynomial and Rational Functions

Sketch the graphs of each of the following functions. Use Desmos to check your answers.

1. $f(x) = x^3 + 5x^2 + 6x$

2. $f(x) = 2x^3 + 5x^2 - 24x - 63$

3. $f(x) = x^4 - 8x^3 - 26x^2 + 168x + 441$

4. $f(x) = \dfrac{x^2 - 6x + 8}{x^2 - 8x + 16}$

5. $f(x) = \dfrac{x^3 - x^2 - 6x}{-3x^2 - 3x + 18}$

6. $f(x) = \dfrac{x^4 - x^3 - 6x^2}{-3x^2 - 3x + 18}$