x is in quad. I

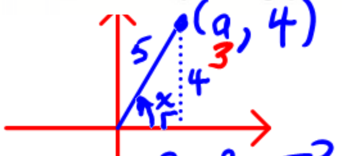
>0 $\therefore$ ccw rotation

9. If $\sin x = \frac{4}{5}$ and $\sin y = -\frac{12}{13}$, $0 < x < \frac{\pi}{2}$, $\frac{3\pi}{2} < y < 2\pi$, evaluate

a) $\cos(x + y)$ c) $\cos(x - y)$ e) $\tan(x + y)$

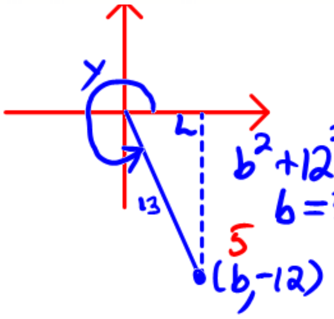
b) $\sin(x + y)$ d) $\sin(x - y)$ f) $\tan(x - y)$

y is in quad 4



$a^2 + 4^2 = 5^2$

$\therefore a = 3$



$b^2 + 12^2 = 13^2$

$b = 5$

I or IV

(a) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\therefore \cos(x+y) > 0$

$= \frac{3}{5}\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{-12}{13}\right)$

$= \frac{15}{65} + \frac{48}{65}$

$= \frac{63}{65}$ positive

SMART

PGRASP

prediction

given

required

analysis

solution

$\sin\left(\frac{C}{2} + \frac{D}{2}\right) = \sin\frac{C}{2}\cos\frac{D}{2} + \cos\frac{C}{2}\sin\frac{D}{2}$

Too messy!

WTF? IDK

Extending

16. Prove $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$.

Can we write C and D in a way that simplifies these expressions?

i.e. let $x = \frac{C+D}{2}$

$y = \frac{C-D}{2}$

Proof: Let $x = \frac{C+D}{2}$ ①

$y = \frac{C-D}{2}$ ②

From ① $C + D = 2x$ ③

" ② $C - D = 2y$ ④

③+④, $2C = 2x + 2y$

$\therefore C = x + y$

③−④, $2D = 2x - 2y$

$\therefore D = x - y$

L.S. $= \sin C + \sin D$

$= \sin(x+y) + \sin(x-y)$

$= \sin x\cos y + \cos x\sin y + \sin x\cos y - \cos x\sin y$

$= 2\sin x\cos y$

$= 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$=$ R.S.

$\because$ L.S. = R.S., the given equation is an identity.

20. Prove that $\dfrac{\tan x \sin x}{\tan x + \sin x} = \dfrac{\tan x - \sin x}{\tan x \sin x}$.

Proof:

$$L.S. = \frac{\cos x\left(\frac{\sin x}{\cos x}\right)\sin x}{\cos x\left(\frac{\sin x}{\cos x} + \sin x\right)}$$

× top & bottom by $\cos x$

write $\tan x = \dfrac{\sin x}{\cos x}$

$$= \frac{\sin^2 x}{\sin x + \sin x \cos x}$$

$$= \frac{\sin x(\sin x)}{\sin x(1+\cos x)}$$

$$= \frac{\sin x}{1+\cos x}$$

Stop for now because this is very simple

$$R.S. = \frac{\tan x - \sin x}{\tan x \sin x}$$

$$= \frac{\cos x\left(\frac{\sin x}{\cos x} - \sin x\right)}{\cos x\left(\frac{\sin x}{\cos x}\right)\sin x}$$

$$= \frac{\sin x - \sin x \cos x}{\sin x(\sin x)}$$

$$= \frac{\sin x(1-\cos x)}{\sin x(\sin x)}$$

$$= \frac{1-\cos x}{\sin x}\left(\frac{1+\cos x}{1+\cos x}\right)$$

$$= \frac{1-\cos^2 x}{\sin x(1+\cos x)}$$

$$= \frac{\sin^2 x}{\sin x(1+\cos x)}$$

$$= \frac{\sin x}{1+\cos x}$$

Logic

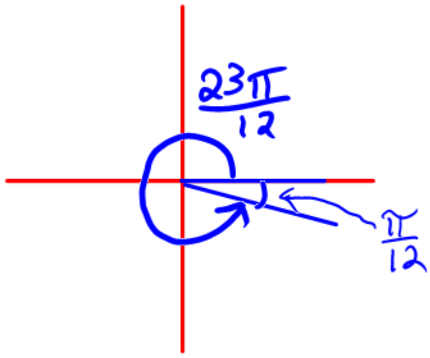
$L.S. = a$

$R.S. = a$

$\therefore L.S. = R.S.$

If $a=b$ and $a=c$ then $b=c$

f) $\tan\left(\frac{23\pi}{12}\right)$ → express in terms of special angles and multiples $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, etc.



$$\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y}$$

$$= -\tan\frac{\pi}{12}$$

$$= -\tan\left(\underbrace{\frac{4\pi}{12}}_{\frac{\pi}{3}} - \underbrace{\frac{3\pi}{12}}_{\frac{\pi}{4}}\right)$$

$$= -\tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= -\frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}}$$

$$= -\frac{\sqrt{3}-1}{1+\sqrt{3}(1)}$$

$$= -\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\left(\frac{\sqrt{3}-1}{\sqrt{3}-1}\right)$$

$$= -\frac{3-2\sqrt{3}+1}{3-1}$$

$$= -\frac{4-2\sqrt{3}}{2}$$

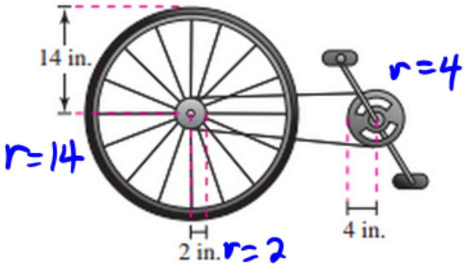
$$= -\frac{2-\sqrt{3}}{1} = \sqrt{3}-2$$

negative ✓

$\frac{23\pi}{12}$ is in quadrant IV where tan is negative

68. Speed of a Bicycle

The radii of the pedal sprocket, the wheel sprocket, and the wheel of the bicycle in the figure are 4 inches, 2 inches, and 14 inches, respectively. A cyclist is pedaling at a rate of 1 revolution per second.



(a) Find the speed of the bicycle in feet per second and miles per hour.

(b) Use your result from part (a) to write a function for the distance d (in miles) a cyclist travels in terms of the number n of revolutions of the pedal sprocket.

(c) Write a function for the distance d (in miles) a cyclist travels in terms of the time t (in seconds). Compare this function with the function from part (b).

Ratio of Radii $\} \; 4:2:14 = 2:1:7$

Ratio of Circumferences $\} \; 8\pi : 4\pi : 28\pi = 2\pi : \pi : 7\pi = 2:1:7$

Same ratio! This happens because $\ell = r\theta$ is a linear function.

(a) one complete rotation of front sprocket
→ two complete rotations of back sprocket
→ two complete rotations of back wheel because sprocket is attached to the wheel

Since cyclist pedals at rate of 1 rev/s, distance travelled in one second

$= 2$ times circumference of wheel

$= 2[2\pi(14 \text{ inches})]$

$= 56\pi$ inches $= \frac{56\pi}{12}$ feet $= \frac{14\pi}{3}$ feet

1 mile = 5280 feet

1 hour = 3600 s

$\therefore$ speed in m.p.h.

$$= \frac{\frac{\cancel{14}^{7}\pi}{3} \text{ feet/s}}{\cancel{5280}_{\cancel{22}\ 11} \text{ ft/mile}} \left(\cancel{3600}^{15} \text{ s/h}\right) = \frac{105\pi}{33} \text{ miles/h} \doteq 10 \text{ miles/h}$$

(b) Distance travell d (in feet) $=$ (# revolutions of front sprocket)(distance travelled per revolution)

$= n\left(\frac{14\pi}{3} \text{ feet/rev}\right)$

$\therefore d(n) = \frac{14\pi n}{3} = \frac{14\pi}{3} n$ feet

$\frac{14\pi}{3}$ feet

$= \frac{14\pi}{3} \div 5280$

$= \frac{\cancel{14}^{7}\pi}{3}\left(\frac{1}{\cancel{5280}_{2640}}\right)$ miles

$\therefore d(n) = \frac{7\pi}{7920} n$ miles

(c) Assumption: cyclist pedals at a rate of 1 rev/s

$d(t) = \frac{7\pi}{7920} t$ miles, t in seconds

Formula-Oriented Method

Angular Velocity

$$\omega = \frac{\theta}{t}$$

angle through which something rotates, divided by the time it takes

Linear Velocity

For rotational motion.

$$v = \frac{d}{t} = \frac{\text{arc length}}{t} = \frac{r\theta}{t} = r\left(\frac{\theta}{t}\right) = r\omega$$

Recall that θ MUST be in radians!

Given: $\omega_{FS} = 1$ rev/s, $r_{FS} = 4$ inches, $r_{BS} = 2$ inches, $r_{wheel} = 14$ inches

FS = front sprocket, BS = back sprocket

Solution

(a) Since the chain is attached to both sprockets,

velocity of a given point on chain = velocity of a given point on FS
= velocity " " " " BS

Let v represent this velocity.

$$\therefore\ v = r_{FS}\,\omega_{FS} = r_{BS}\,\omega_{BS}$$

$$\therefore\ (4 \text{ inches})(1 \text{ rev/s}) = (2 \text{ inches})\,\omega_{BS}$$

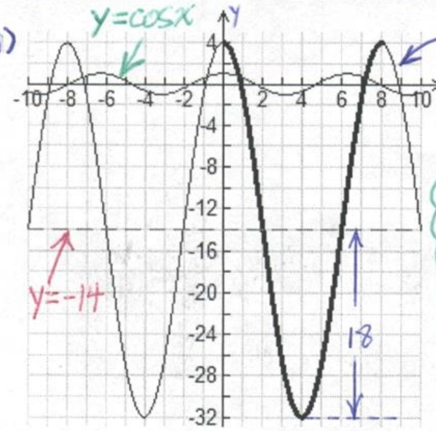
$$\therefore\ \omega_{BS} = \frac{(4 \text{ inches})(1 \text{ rev/s})}{2 \text{ inches}} = 2 \text{ rev/s}$$

Since the back sprocket is connected to the back wheel,

$$\omega_{wheel} = \omega_{BS} = 2 \text{ rev/s}$$

$$\begin{aligned}
\therefore\ v_{Bicycle} &= r_{wheel}\,\omega_{wheel} \\
&= (14 \text{ inches})(2 \text{ rev/s}) \\
&= (14 \text{ inches})[2(2\pi) \text{ rad/s}] \\
&= 56\pi \text{ inches/s} \\
&= \frac{105\pi}{33} \text{ miles/h} \quad \text{(as shown on first page)} \\
&\doteq 10 \text{ miles/h}
\end{aligned}$$

(a)



$y=18\cos\left(\frac{\pi x}{4}\right)-14=18\cos\left(\frac{\pi}{4}x\right)-14$

$(x,y)\rightarrow\left(\frac{4}{\pi}x,\ 18y-14\right)$

Period of $y=\cos x$ is stretched by a factor of $\frac{4}{\pi}$

$\therefore\ T=2\pi\left(\frac{4}{\pi}\right)=8$

$A=18,\ d=-14,\ p=0$

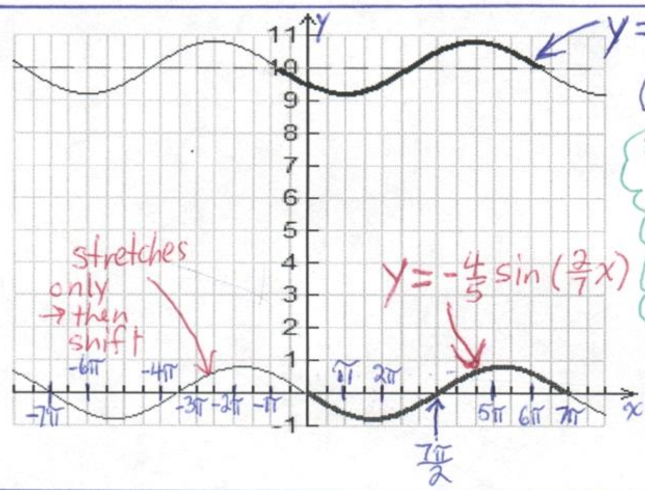
Horizontal
1. Stretch by factor of $\frac{4}{\pi}$
2. No shift

Vertical
1. Stretch by a factor of 18
2. Translate down 14 units

Suitable Scale for x-axis

$T=8$, which is divisible by 4 $\therefore$ 1 square = 1 unit is a good choice

(b)



$y=-\frac{4}{5}\sin\left(\frac{2}{7}\left(x+\frac{3\pi}{4}\right)\right)+10$

$(x,y)\rightarrow\left(\frac{7}{2}x-\frac{3\pi}{4},\ -\frac{4}{5}y+10\right)$

Period of $y=\sin x$ is stretched by a factor of $\frac{7}{2}$

$\therefore\ T=2\pi\left(\frac{7}{2}\right)=7\pi$

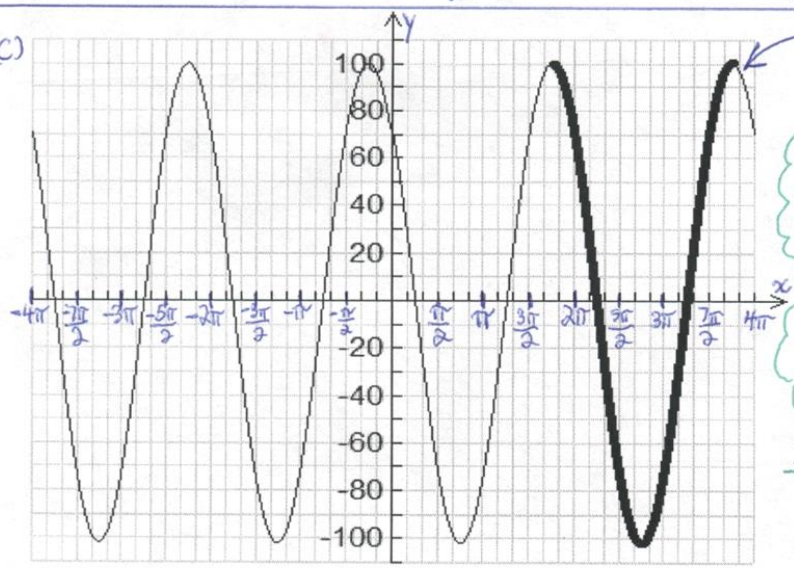
$A=\frac{4}{5},\ d=10,\ p=-\frac{3\pi}{4}$

This one is done in two steps. First the stretches are applied, then the translations are done

Suitable Scale for x-axis

Divide 7π by a number divisible by 4 (e.g. by 12). Made the grid too small → so I used 1 square = $\frac{\pi}{2}$

(c)



$y=101\cos\left(x-\frac{7\pi}{4}\right)-\frac{9}{10}$

$(x,y)\rightarrow\left(x+\frac{7\pi}{4},\ 101y-\frac{9}{10}\right)$

Since there is no horizontal stretch, the period of this function is the same as that of $y=\cos x$

$\therefore\ T=2\pi$

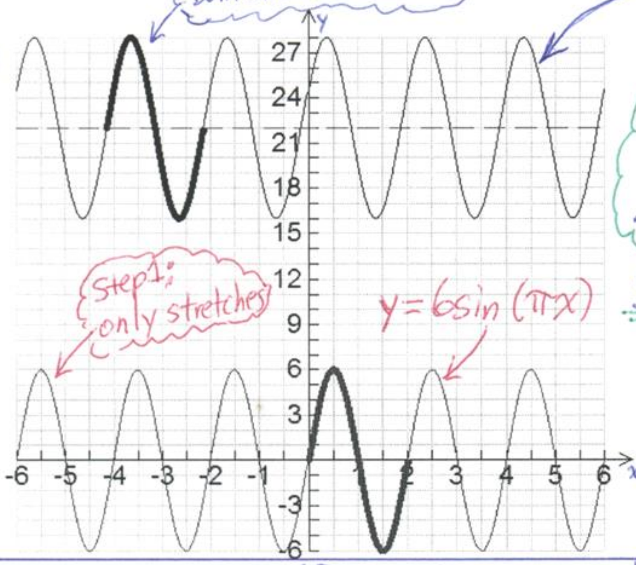
$A=101,\ d=-\frac{9}{10},\ p=\frac{7\pi}{4}$

Suitable Scale for x-axis

Divide 2π by 12

$\frac{2\pi}{12}=\frac{\pi}{6}\rightarrow$ 1 square = $\frac{\pi}{6}$

(d)



$y=6\sin(\pi x+13)+22=6\sin\left(\pi\left(x+\frac{13}{\pi}\right)\right)+22$

$(x,y)\rightarrow\left(\frac{1}{\pi}x-\frac{13}{\pi},\ 6y+22\right)$

Period of $y=\sin x$ is compressed by a factor of $\frac{1}{\pi}$

$\therefore T=2\pi\left(\frac{1}{\pi}\right)=2,\ A=6,\ d=22,\ p=-\frac{13}{\pi}$

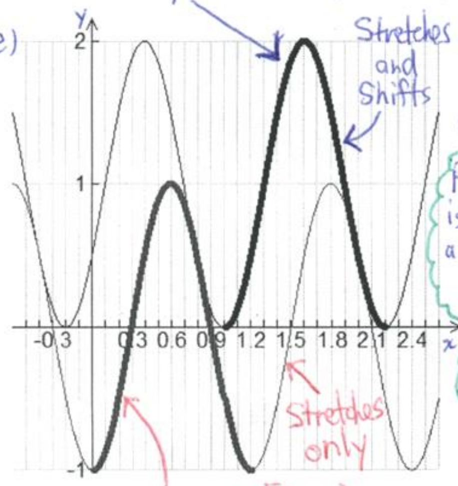
Suitable Scale for x-axis

$T=2\rightarrow$ Divide 2 units into equal parts by a number divisible by 4. I used $\frac{2}{4}=\frac{1}{2}$

$\therefore$ 1 square $=\frac{1}{2}$

(e)

$y=-\cos\left(\frac{5\pi}{3}(x-1)\right)+1$



$(x,y)\rightarrow\left(\frac{3}{5\pi}x+1,\ -y+1\right)$

Period of $y=\cos x$ is compressed by a factor of $\frac{3}{5\pi}$

$\therefore T=2\pi\left(\frac{3}{5\pi}\right)=\frac{6}{5}$

$A=1,\ d=1$
$p=1$

$y=-\cos\left(\frac{5\pi}{3}x\right)$

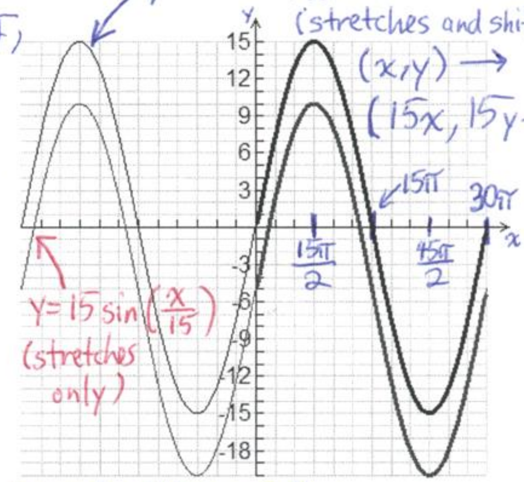
Suitable Scale for x-axis: $\frac{\left(\frac{6}{5}\right)}{12}=\frac{6}{5}\times\frac{1}{12}=\frac{1}{10}=0.1$

(f)

$y=15\sin\left(\frac{x}{15}\right)-5=15\sin\left(\frac{1}{15}x\right)$

(stretches and shifts)

$A=15$
$d=-5$

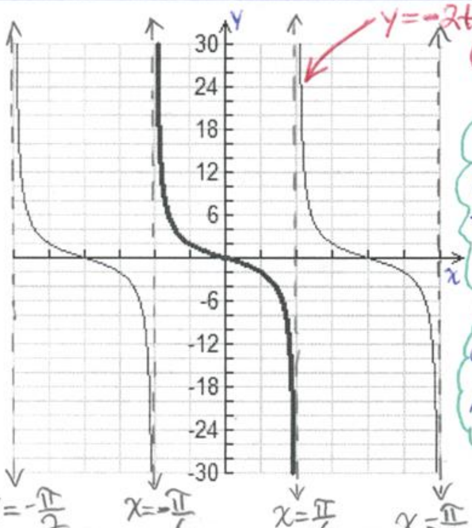


$(x,y)\rightarrow(15x,\ 15y-5)$

$T=2\pi(15)=30\pi$

Suitable Scale for x-axis: $\frac{30\pi}{12}=\frac{5\pi}{2}=2.5\pi$

(g)



$(x,y)\rightarrow\left(\frac{1}{3}x-\frac{\pi}{4},\ -2y-3\right)$

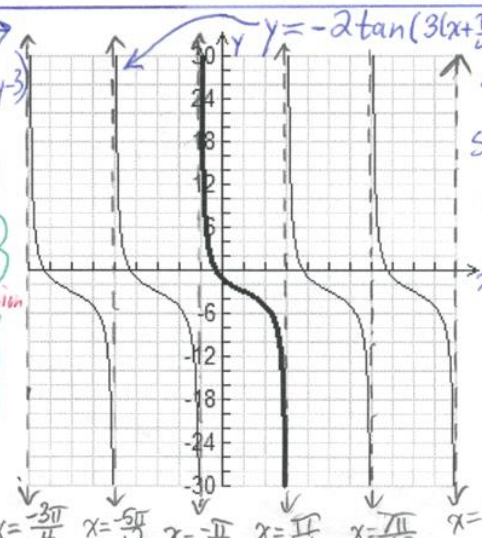
Period of $y=\tan x$ is π. Therefore, $T=\pi\left(\frac{1}{3}\right)=\frac{\pi}{3}$

horizontal compression

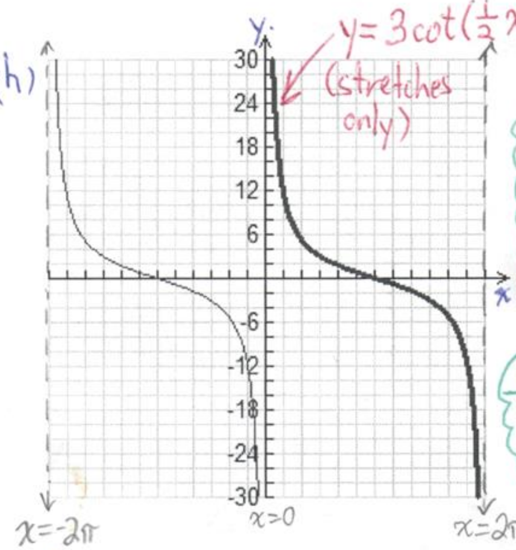
$d=-3,\ p=-\frac{\pi}{4}$
Amplitude is not defined

Scale for x-axis
$\frac{\pi}{3}\div 4=\frac{\pi}{12}$

$y=-2\tan\left(3\left(x+\frac{\pi}{4}\right)\right)-3$



(h) $y = 3\cot\left(\frac{1}{2}x\right)$ (stretches only)



$(x, y) \rightarrow \left(2x + \frac{\pi}{3},\ 3y + 2\right)$

Period of $y = \cot x$ is stretched by a factor of 2

$\therefore T = \pi(2) = 2\pi$

A is undefined

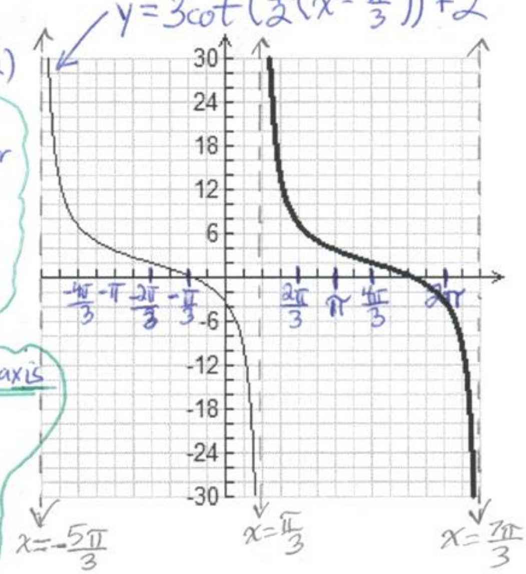
$d = 2,\ p = \frac{\pi}{3}$

Suitable Scale for x-axis

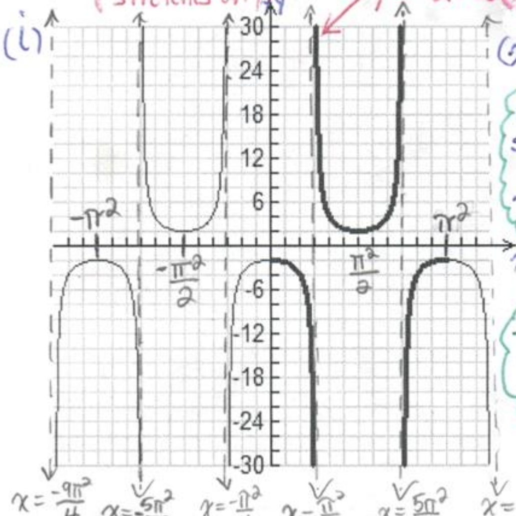
$T = 2\pi,\ \frac{2\pi}{12} = \frac{\pi}{6}$

Works well because phase shift is $\frac{\pi}{3} = 2\left(\frac{\pi}{6}\right)$

$y = 3\cot\left(\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right) + 2$



(i) (stretches only) $y = -2\sec\left(\frac{2}{\pi}x\right)$



$(x, y) \rightarrow \left(\frac{\pi}{2}x - \frac{\pi}{6},\ -2y + 5\right)$

Period of $y = \sec x$ is stretched by a factor of $\frac{\pi}{2}$

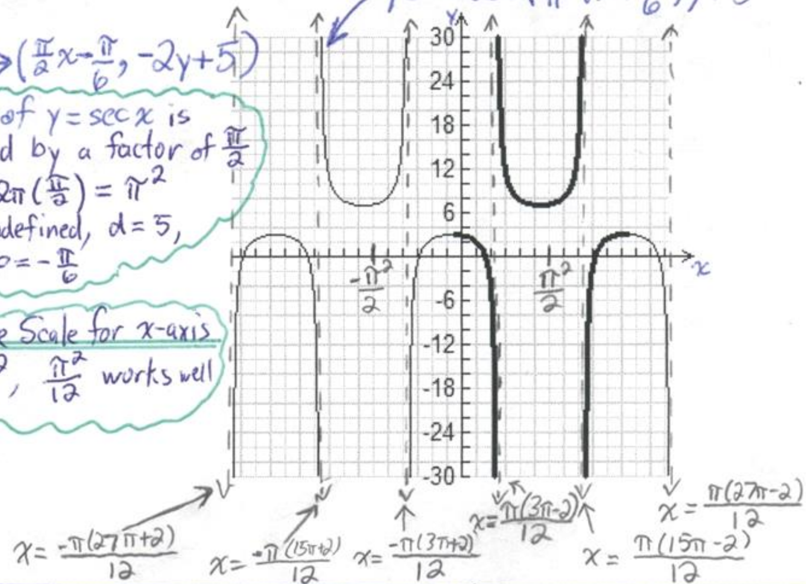
$\therefore T = 2\pi\left(\frac{\pi}{2}\right) = \pi^2$

A is undefined, $d = 5$, $p = -\frac{\pi}{6}$

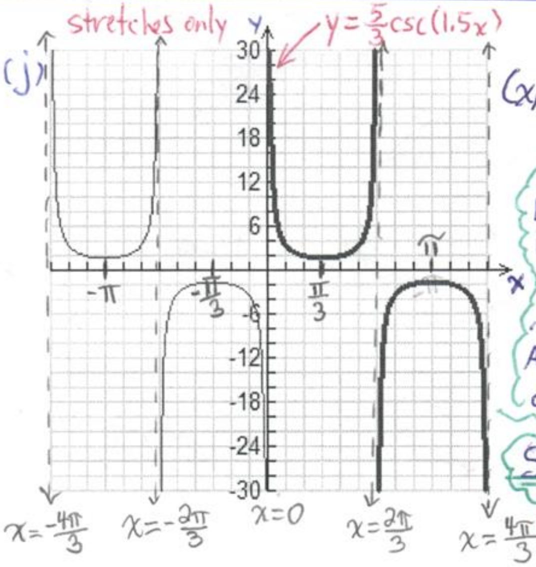
Suitable Scale for x-axis

$T = \pi^2$, $\frac{\pi^2}{12}$ works well

$y = -2\sec\left(\frac{2}{\pi}\left(x + \frac{\pi}{6}\right)\right) + 5$



(j) stretches only $y = \frac{5}{3}\csc(1.5x)$



$y = \frac{5}{3}\csc\left(1.5x + \frac{\pi}{4}\right) + \frac{3}{2} = \frac{5}{3}\csc\left(\frac{3}{2}\left(x + \frac{\pi}{6}\right)\right) + \frac{3}{2}$

$(x, y) \rightarrow \left(\frac{2}{3}x - \frac{\pi}{6},\ \frac{5}{3}y + \frac{3}{2}\right)$

Period of $y = \csc x$ is compressed by a factor of $\frac{2}{3}$

$\therefore T = 2\pi\left(\frac{2}{3}\right) = \frac{4\pi}{3}$

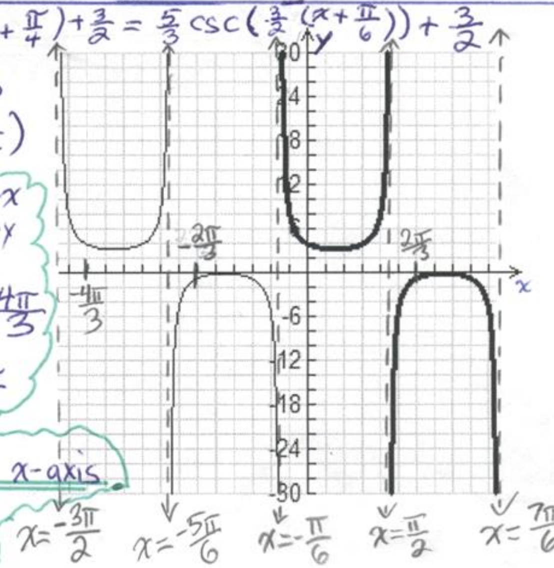
A is undefined,

$d = \frac{3}{2},\ p = -\frac{\pi}{6}$

Suitable Scale for x-axis

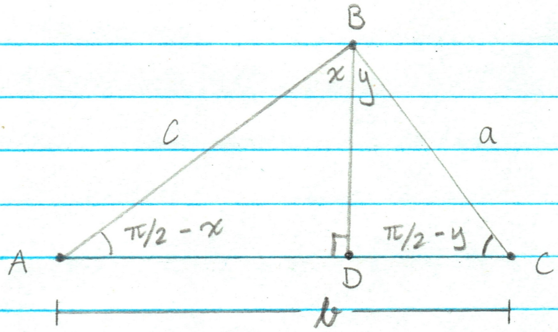
$\frac{4\pi}{3} \div 12$

$= \frac{\pi}{9}$



Challenge Problem

Derive the identity $\cos(x+y) = \cos x \cos y - \sin x \sin y$ form **FIRST PRINCIPLES**.

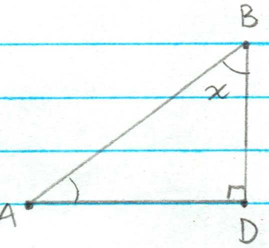


Cosine Law

$$b^2 = a^2 + c^2 - 2ac \cos B$$

rearrange to right in terms of $\cos B$

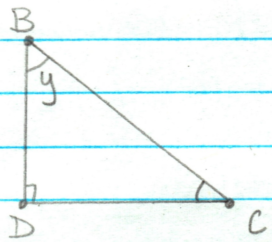
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$



$$\cos x = \frac{BD}{AB}$$

$$\sin x = \frac{AD}{AB}$$

$$(AB)^2 = (BD)^2 + (AD)^2$$



$$\cos y = \frac{BD}{BC}$$

$$\sin y = \frac{CD}{BC}$$

$$(BC)^2 = (BD)^2 + (CD)^2$$

$$\angle B = x + y \qquad \therefore \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos(x+y) = \frac{(AB)^2 + (BC)^2 - (AD + DC)^2}{2(AB)(BC)}$$

pythagorean theorem

$(a+b)^2 = a^2 + 2ab + b^2$

$$\cos(x+y) = \frac{[(BD)^2 + (AD)^2] + [(BD)^2 + (CD)^2] - [(AD)^2 + 2(AD)(DC) + (DC)^2]}{2(AB)(BC)}$$

$$\cos(x+y) = \frac{(BD)^2 + \cancel{(AD)^2} + (BD)^2 + \cancel{(CD)^2} - \cancel{(AD)^2} - [2(AD)(DC)] - \cancel{(DC)^2}}{2(AB)(BC)}$$

$$\cos(x+y) = \frac{2(BD)^2 - 2(AD)(DC)}{2(AB)(BC)}$$

$$\cos(x+y) = \frac{\cancel{2}\,[(BD)^2 - (AD)(DC)]}{\cancel{2}\,(AB)(BC)}$$

$$\cos(x+y) = \frac{BD \cdot BD}{AB \cdot BC} - \frac{(AD)\cdot(DC)}{(AB)\cdot(BC)}$$

$$\cos(x+y) = \frac{BD}{AB} \cdot \frac{BD}{BC} - \frac{AD}{AB} \cdot \frac{DC}{BC}$$] refer back to diagrams.

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\boxed{\therefore \cos(x+y) = \cos x \cos y - \sin x \sin y}$$