Prove that

f) $\dfrac{1 + 2\sin x \cos x}{\sin x + \cos x} = \sin x + \cos x$ is an identity

"Cancelling" Logic

$$\frac{2(7)}{2} = \frac{2}{2}\left(\frac{7}{1}\right) = 1\left(\frac{7}{1}\right) = 7$$

This logic can only be applied when the numerator and denominator have a common factor

Proof: "Fractions" → you may × or ÷ num. and den. by the same value

Why? This is equivalent to × or ÷ the fraction by 1

$$\text{L.S.} = \left(\frac{1+2\sin x\cos x}{\sin x+\cos x}\right)\underbrace{\left(\frac{\sin x+\cos x}{\sin x+\cos x}\right)}_{\text{must equal 1}}$$

$$= \frac{(1+2\sin x\cos x)(\sin x+\cos x)}{\sin^2 x+2\sin x\cos x+\cos^2 x}$$

$$\frac{ab}{c} = \frac{a}{c}\left(\frac{b}{1}\right)$$

Fundamental Knowledge (Rule for multiplying fractions)

$$= \frac{(1+2\sin x\cos x)(\sin x+\cos x)}{\sin^2 x+\cos^2 x+2\sin x\cos x}$$

$$= \frac{(1+2\sin x\cos x)(\sin x+\cos x)}{1+2\sin x\cos x}$$

$$= \left(\frac{1+2\sin x\cos x}{1+2\sin x\cos x}\right)\left(\frac{\sin x+\cos x}{1}\right)$$

$$= \sin x+\cos x$$

$\cancel{\dfrac{x+7}{x}}$

"Cancelling" CANNOT be done.

$$\frac{x+7}{x} = \frac{x}{x}+\frac{7}{x} = 1+\frac{7}{x}$$

No common factors.

These expressions are equivalent because of the rule for adding fractions.

Again, fundamental knowledge plays a critical role.

Prove that

j) $\dfrac{\sin x - 1}{\sin x + 1} = \dfrac{-\cos^2 x}{(\sin x + 1)^2}$ is an identity

$$\text{R.S.} = \frac{-(1-\sin^2 x)}{(\sin x+1)^2}$$

$$= \frac{\sin^2 x - 1^2}{(\sin x+1)^2}$$

a^2-b^2
$=(a-b)(a+b)$

$$= \frac{(\sin x-1)(\sin x+1)}{(\sin x+1)(\sin x+1)}$$

$$= \frac{\sin x-1}{\sin x+1} = \text{L.S.}$$

∴ L.S. = R.S.

∴ the given equation is an identity

Prove that

a) $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$ is an identity

$$\begin{aligned}
L.S. &= \sin^4 x - \cos^4 x \\
&= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) \\
&= (\sin^2 x - \cos^2 x)(1) \quad \text{Pyth. Id.} \\
&= \sin^2 x - \cos^2 x \\
&= R.S.
\end{aligned}$$

$$\begin{aligned}
&x^4 - y^4 \\
&= (x^2)^2 - (y^2)^2 \\
&= (x^2+y^2)(x^2-y^2) \\
&= (x^2+y^2)(x+y)(x-y)
\end{aligned}$$

Difference of Squares in Disguise

$\therefore$ L.S. = R.S.

$\therefore$ the given equation is an identity

Prove that

c) $\dfrac{4}{\cos^2 x} - 5 = 4\tan^2 x - 1$ is an identity

Proof:

$$\begin{aligned}
L.S. &= \frac{4}{\cos^2 x} - \frac{5}{1} \\
&= \frac{4}{\cos^2 x} - \frac{5\cos^2 x}{\cos^2 x} \\
&= \frac{4 - 5\cos^2 x}{\cos^2 x}
\end{aligned}$$

$$\begin{aligned}
R.S. &= 4\tan^2 x - 1 \\
&= 4\left(\frac{\sin^2 x}{\cos^2 x}\right) - \frac{1}{1} \\
&= \frac{4\sin^2 x}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x} \\
&= \frac{4\sin^2 x - \cos^2 x}{\cos^2 x} \\
&= \frac{4(1-\cos^2 x) - \cos^2 x}{\cos^2 x} \\
&= \frac{4 - 4\cos^2 x - \cos^2 x}{\cos^2 x} \\
&= \frac{4 - 5\cos^2 x}{\cos^2 x}
\end{aligned}$$

$\therefore$ L.S. = R.S.

$\therefore$ the given equation is an identity

Logic

L.S. R.S.

If $a = c$ and $b = c$

then $a = b$

20. Prove that $\dfrac{\tan x \sin x}{\tan x + \sin x} = \dfrac{\tan x - \sin x}{\tan x \sin x}$. is an identity.

Proof:

$$L.S. = \frac{\tan x \sin x}{\tan x + \sin x}$$

$$= \frac{\left(\frac{\sin x}{\cos x}\right)\left(\frac{\sin x}{1}\right)}{\left(\frac{\sin x}{\cos x} + \frac{\sin x}{1}\right)}$$

$$= \frac{\left(\frac{\sin^2 x}{\cos x}\right)}{\left(\frac{\sin x + \sin x \cos x}{\cos x}\right)}$$

$$= \left(\frac{\sin^2 x}{\cos x}\right)\left(\frac{\cos x}{\sin x + \sin x \cos x}\right)$$

$$= \frac{\sin^2 x}{\sin x + \sin x \cos x}$$

$$= \frac{1 - \cos^2 x}{\sin x(1 + \cos x)}$$

$$= \frac{(1 - \cos x)(1 + \cos x)}{\sin x(1 + \cos x)}$$

$$= \frac{1 - \cos x}{\sin x}$$

$$R.S. = \frac{\tan x - \sin x}{\tan x \sin x}$$

$$= \frac{\left(\frac{\sin x}{\cos x} - \frac{\sin x}{1}\right)}{\left(\frac{\sin x}{\cos x}\right)\left(\frac{\sin x}{1}\right)}$$

$$= \frac{\left(\frac{\sin x - \sin x \cos x}{\cos x}\right)}{\left(\frac{\sin^2 x}{\cos x}\right)}$$

$$= \left(\frac{\sin x - \sin x \cos x}{\cos x}\right)\left(\frac{\cos x}{\sin^2 x}\right)$$

$$= \frac{\sin x - \sin x \cos x}{\sin^2 x}$$

$$= \frac{\sin x(1 - \cos x)}{\sin x(\sin x)}$$

$$= \frac{1 - \cos x}{\sin x}$$

$\therefore$ L.S. = R.S.

$\therefore$ the given equation is an identity

Summary of Strategies and How to Improve Understanding

① Always keep **FUNDAMENTAL** (ie. basic) ideas in mind

e.g. order of operations, how to +, −, ×, ÷ fraction, etc.

This helps keep the focus on "why" instead of "how."

Why does the rule make sense?

rule

② Start with the more complex side

③ Express all ratios in terms of sin and cos

④ Use identities already proved

⑤ Always think ahead more than one step

⑥ Don't expand unless you have a good reason for doing so

⑦ Multiply or divide by 1
(multiply or divide both num. and den. by same expression)

⑧ Factor if possible

⑨ Work on both sides simultaneously (but separately of course)
Then apply logic: "If $a=b$ and $b=c$ then $a=c$."