Table of Contents – Unit 0 – How to Understand Mathematics and Solve Problems

WHAT IS PROBLEM SOLVING?

Introduction: What is the Difference between Solving a Problem and Performing an Exercise?

- **Performing an Exercise:** This requires you to *follow a procedure* that you have learned.
  Performing mathematical exercises is analogous to executing drills like "suicides" when practicing for a sport or playing scales when practicing a musical instrument. Very little original thinking is required.
- **Solving a Problem:** This requires you to *think and be imaginative*.
  You can consider yourself a problem solver *only when you devise the strategy*. If you are following a strategy devised by someone else, then you are merely performing an exercise NOT solving a problem. This is analogous to playing a game in sports or improvising on a musical instrument. At any point in time, you can never be certain of exactly what will happen next. You must adapt to the circumstances as they change. The "game plan" evolves as the game is played!

Performing an Exercise

- Mechanical ("Auto Pilot")
- Very Predictable
- Follows a Set Procedure or Strategy Devised by Someone Else
- The Path to the Destination is Known and entirely Clear
- Little Thinking or Imagination Needed
- Like Doing Drills in Sports or Playing Scales in Music
- Also like a Police Officer Writing out a Ticket for a Traffic Violation

Solving a Problem

- Not Mechanical
- Somewhat Unpredictable
- Does not Follow a Strategy Devised by Someone Else
- The Problem Solver Devises the Strategy
- The Path to the Destination is not entirely Clear
- Thinking and Imagination are Required
- Like Improvising in Music or Playing a Game in Sports
- Also like a Police Detective trying to Solve a Crime

George Polya's Four Steps of Problem Solving

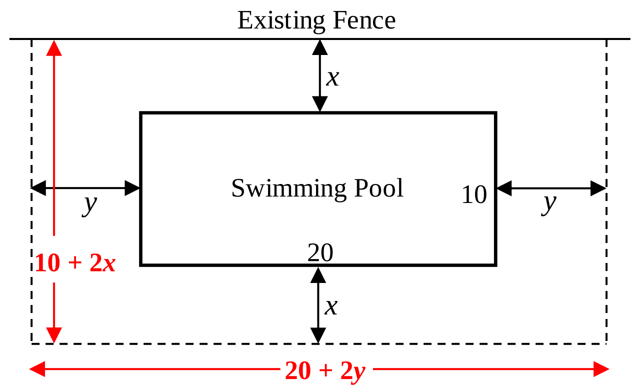
1. **Understand the Problem**
   Do you understand all the terminology used in the question? Do you understand what are you being asked to do? What information is given? Is all the given information required? Is there any missing information? What would a reasonable answer look like? Can you represent the problem in different ways? (**e.g.** diagram, graph, model, table of values, equation, etc.) …
2. **Devise a Strategy**
   What mathematical concepts are relevant and do you understand them? Do you know any strategies that could work? Do you need to invent a new strategy? Can you solve a simplified version of the problem? Can you solve a related problem? Can you work backwards and then reverse the steps? ...
3. **Carry out the Strategy**
   Carefully carry out the strategy that you devised in step 2.
4. **Check the Solution**
   Carefully check your solution. Does your answer make sense? Does it agree with the prediction you made in step 1? Have others arrived at the same answer?

Example

To help prevent drowning accidents, a protective fence is to be erected around a pool whose dimensions are 20 m by 10 m. Since there is an existing fence parallel to one of the 20-m sides of the pool, new fencing is only required around three sides of the pool. In addition, the gap between the fence and the edge of the pool must be uniform on opposite sides of the pool. If 100 m of fencing material is available, what is the maximum area that can be enclosed by the fencing?

Solution

1. Understand the Problem



Given

- pool is 20 m × 10 m
- existing fence ∥ to 20-m side of pool
- gap is uniform on opposite sides of pool
- 100 m of fencing available
  → $10 + 2x + 20 + 2y + 10 + 2x = 100$ [*]

Required to Find

- max area that can be enclosed by fencing
- i.e. max value of $A = (20 + 2y)(10 + 2x)$ [**]

Reasonable Prediction

Distribute the fencing according to the ratio of length to width of the swimming pool. Since the width is twice the length, we might expect the length and width of the rectangular area to be 50 m and 25 m respectively, yielding an area of 1250 m^2.

This turns out to be *flawed* reasoning but it still produces the correct answer in this specific case. (See question 1 on p. 5)

2. Devise a Strategy

- Find a relationship between x and y (see [*] above).
- Use the relationship to rewrite equation [**] in terms of a single unknown.
- Graphically or algebraically find the maximum value of A.

3. Carry Out the Strategy

Solve for One of the Unknowns in Terms of the Other

$10 + 2x + 20 + 2y + 10 + 2x = 100$

$\therefore 4x + 2y + 40 = 100$

$\therefore 4x + 2y = 60$

$\therefore 2x + y = 30$

$\therefore 2x = 30 - y$

Write the Equation for *A* in terms of a Single Unknown

$A = (20 + 2y)(10 + 2x)$

$\therefore A = (20 + 2y)(10 + 30 - y)$

$\therefore A = (20 + 2y)(40 - y)$

$\therefore A = 2(10 + y)(40 - y)$

Zeros of the Parabola

$2(10+y)(40-y)=0$

$\therefore 10+y=0$ or $40-y=0$

$\therefore y=-10$ or $y=40$

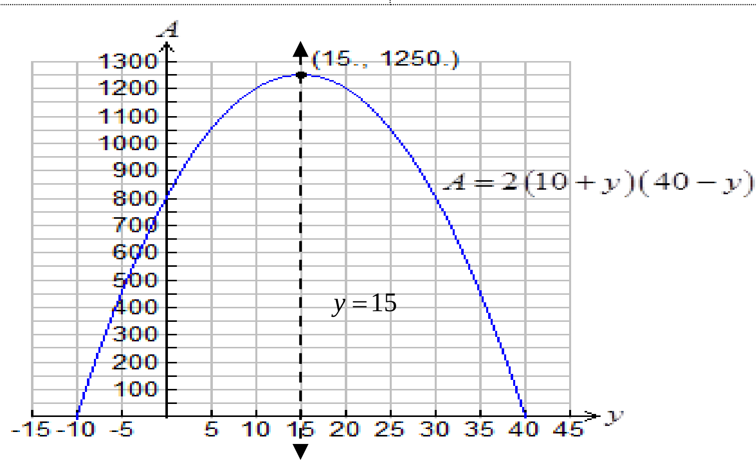
Axis of Symmetry of the Parabola

The axis of symmetry lies half-way between the zeros. Therefore, its equation is $y=\dfrac{-10+40}{2}=15$.

Co-ordinates of the Vertex

Since the vertex lies on the axis of symmetry,

$$A=2(10+y)(40-y)$$
$$=2(10+15)(40-15)$$
$$=2(25)(25)$$
$$=1250$$



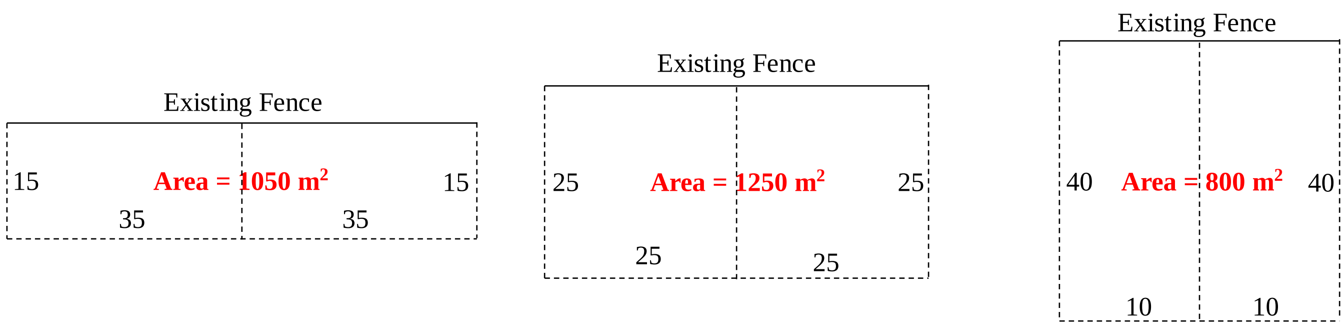
Conclusion

The maximum area that can be enclosed by the fence is 1250 m^2.

4. Check the Solution

It is well known that the area of a rectangle with a given perimeter is maximized when it's a square. However, we must be careful in this situation not to jump to conclusions. Since the pool is not a square and the fence is installed only along three sides of the rectangular area, we ***cannot assume*** that the area of the rectangular region is maximized if it's a square. If we do assume this, we arrive at an area of about 1100 m^2, which is clearly ***not*** the maximum area.

The following diagrams show that the rectangular area can be divided into two smaller congruent rectangles, ***each of which has a perimeter of 100 m***. Since each of these rectangles has a fixed perimeter of 100 m, the maximum area is obtained when each rectangle is a square that is 25 m by 25 m. Therefore, the maximum area of each square is 625 m^2, yielding a maximum total area of 1250 m^2.



Questions

1. To help prevent drowning accidents, a protective fence is to be erected around a pool whose dimensions are 30 m by 10 m. Since there is an existing fence parallel to one of the 30-m sides of the pool, new fencing is only required around three sides of the pool. In addition, the gap between the fence and the edge of the pool must be uniform on opposite sides of the pool. If 100 m of fencing material is available, what is the maximum area that can be enclosed by the fencing? ***Solve this problem using both of the methods described above.***

2. To help prevent drowning accidents, a protective fence is to be erected around a pool whose dimensions are *c* metres by *d* metres. Since there is an existing fence parallel to one of the *c*-metre sides of the pool, new fencing is only required around three sides of the pool. In addition, the gap between the fence and the edge of the pool must be uniform on opposite sides of the pool. If *L* metres of fencing material is available, what is the maximum area that can be enclosed by the fencing? ***Solve this problem using both of the methods described above.***

3. Repeat question 2 but this time, all four sides of the rectangular area need to be fenced. Do you expect a different answer this time?

Answers

1. 1250 m^2

2. $\frac{L^2}{8}$ provided that both c and d are smaller than the dimensions of the fence https://www.desmos.com/calculator/xl1ivv2t6l

3. $\frac{L^2}{16}$ provided that both c and d are smaller than $\frac{L}{4}$ https://www.desmos.com/calculator/an6iya7wdz

Do not repeat after me words that you do not understand. Do not merely put on a mask of my ideas, for it will be an illusion and you will thereby deceive yourself.

-Jiddu Krishnamurti

Introduction – Focus on a Small Number of Important Ideas

The best way to succeed in mathematics is to focus primarily on understanding important ***ideas*** and on applying these ideas in a wide variety of contexts. Unfortunately, many students instead direct most of their attention to memorizing the steps performed in ***examples*** given in class as well as those found in textbooks and various other sources. Invariably, this approach fails miserably because it depends mostly on ***mimicry***. Relying solely or too heavily on mimicry ***prevents*** students from developing the mathematical insight that is required for tackling novel, more challenging problems.

To avoid falling into the trap of devoting too much time to playing "the imitation game," students must constantly remind themselves to pay a great deal more attention to ***fundamental ideas***. Given below are some examples from Cartesian geometry that illustrate the power of understanding and applying such ideas.

The only Equation you need to know to find an Equation of a Line

Slope = Slope

Example

Find an equation of the line with slope $-\frac{2}{3}$ and passing through the point (4, –7).

Solution

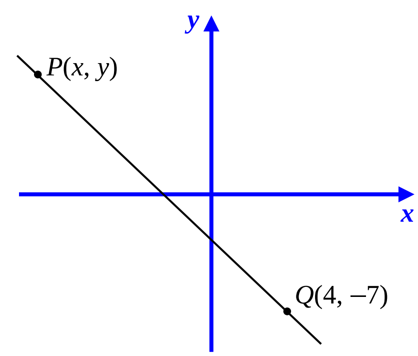
Let $P(x, y)$ be any point on the line other than (4, –7). Since slope = slope,

$$\therefore \frac{\Delta y}{\Delta x} = -\frac{2}{3}$$

$$\therefore \frac{y-(-7)}{x-4} = -\frac{2}{3}$$

$$\therefore y+7 = -\frac{2}{3}(x-4)$$

$$\therefore y = -\frac{2}{3}x - \frac{13}{3}$$



Therefore, $y = -\frac{2}{3}x - \frac{13}{3}$ is ***an*** (not "the") equation of the required line.

The only Equation you need to know to find the Distance between Two Points

The Pythagorean Theorem

Example

Find the distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Solution

Let P, Q and R be as shown in the diagram.

Notice the use of "absolute value."

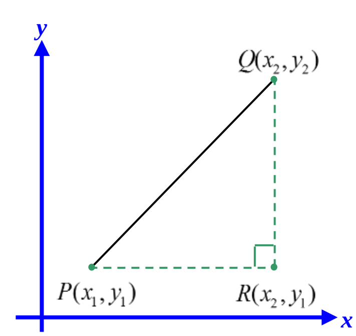
Clearly, $PR = |x_2 - x_1|$ and $QR = |y_2 - y_1|$. Therefore, by the Pythagorean Theorem,

$$PQ^2 = PR^2 + QR^2$$

$$\therefore PQ^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$\therefore PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

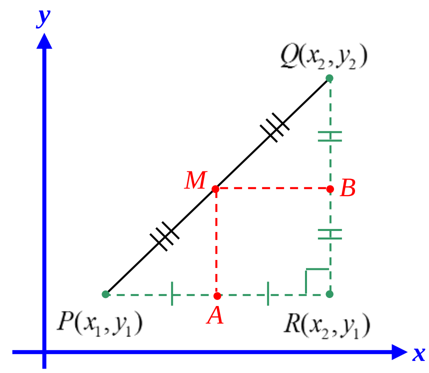
$$\therefore PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



> ***The only Equation you need to know to find the Midpoint of a Line Segment***
>
> **The average of two numbers *a* and *b* is $\dfrac{a+b}{2}$.**

Example

Find the midpoint of the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.



Solution

As shown in the diagram, let *M*, *A* and *B* represent the midpoints of the line segments *PQ*, *PR* and *QR* respectively. Since *A* lies on *PR* and *PR* is parallel to the *x*-axis, the *y*-co-ordinate of *A* must be y_1. Also, since *A* lies exactly half way between *P* and *R*, its *x*-co-ordinate must be the average of the *x*-co-ordinates of *P* and *R*. Therefore, the co-ordinates of *A* must be $\left(\dfrac{x_1+x_2}{2}, y_1\right)$. Using similar reasoning, the co-ordinates of *B* must be $\left(x_2, \dfrac{y_1+y_2}{2}\right)$.

Since *MA* is parallel to the *y*-axis, the *x*-co-ordinate of *M* must equal that of *A*. Since *MB* is parallel to the *x*-axis, the *y*-co-ordinate of *M* must equal that of *B*. Therefore, the co-ordinates of *M* must be

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Important Questions

1. Using a diagram and an argument similar to those given above, explain why parallel lines have equal slope.

2. Using a diagram and an argument similar to those given above, explain why any line perpendicular to a line with slope *m* must have slope $-\dfrac{1}{m}$. (That is, explain why perpendicular lines must have negative reciprocal slopes.)

3. Give a physical interpretation of slope.

4. Using a diagram and an argument similar to those given above, explain why finding intercepts involves setting some quantity equal to zero.

5. Explain the ***logic*** behind the following:

$(x-2)(x+3)=0$

$\therefore x-2=0 \text{ or } x+3=0$

6. Why is "absolute value" used in the derivation of the formula for finding the distance between two points?

7. Given the points $P(-3,5)$ and $Q(11,11)$, find an equation of the line passing through the midpoint of the line segment *PQ* and having slope $-\dfrac{2}{7}$.

8. Given the points $P(-3,5)$ and $Q(11,11)$, find an equation of the line perpendicular to *PQ* and passing through *Q*.

9. Find the distance from the point $P(-3,5)$ to the line $y=2x+3$. (Draw a diagram!)

Answers

1, 2, 3, 4, 5, 6: To be discussed in class

7. $y=-\frac{2}{7}x+\frac{64}{7}$

8. $y=\frac{-7}{3}x+\frac{110}{3}$

9. $\frac{8}{\sqrt{5}}=\frac{8\sqrt{5}}{5}$

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Understanding Math Lesson 2:
View Mathematical Relationships from Different Perspectives

> *The most fatal illusion is the settled point of view. Since life is growth and motion, a fixed point of view kills anybody who has one.*
> -Brooks Atkinson

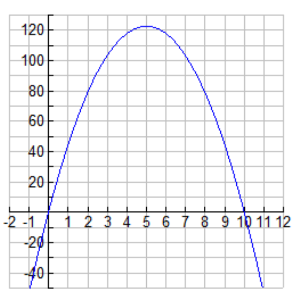
Introduction – There is much more to Algebra than Meets the Eye

The core ideas of "modern" algebra were conceived in the Arabic world, roughly between 800 AD and 1100 AD. These ideas turned out to be so powerful that they gave rise to a period, which continues to this day, of very rapid growth of mathematical and scientific knowledge. In fact, algebra is so important in mathematics that, as stated in a Wikipedia article on the subject, it is considered "… a unifying thread of almost all of mathematics." (See https://en.wikipedia.org/wiki/Algebra)

Nevertheless, algebra has its limitations. Over time, mathematicians progressively made use of algebraic ideas to formalize mathematical concepts, that is, to build such concepts upon a much more rigorous logical foundation. While this greatly helped to make mathematical knowledge more "certain" from a theoretical point of view, it also caused mathematics to seem increasingly far removed from the so-called "real world." Because of this, it is not at all surprising that many people tremble in horror when faced with the prospect of studying algebra. To such people, algebra seems so disconnected from reality that it has little or no meaning to them.

The solution to this problem is quite simple but requires a great deal of personal discipline. Students must strongly resist the temptation to lose themselves entirely in a world of algebraic forms. Instead, they should learn to journey seamlessly through many different worlds, each of which lends itself to viewing mathematical ideas from a different perspective.

This idea is illustrated in the following table. The ***same*** mathematical relationship is represented in each column of the table. However, the ***point of view*** in any given column is ***different*** from that of all the others. This allows us to get a much better picture of the nature of the relationship than would otherwise be possible. Imagine how shallow our understanding would be if we only viewed the relationship from the algebraic perspective!

<table>
<tr><th>Algebraic</th><th>Geometric</th><th>"Real-World"</th><th>Verbal</th><th>Numerical</th></tr>
<tr>
<td>$h = 49t - 4.9t^2$</td>
<td>
</td>
<td>A cannonball is fired vertically into the air with an initial speed of 49 m/s. Its height above the ground at a given time t is quadratically related to t.</td>
<td>The value of h is equal to the product of 49 and t reduced by the product of 4.9 and the square of t.</td>
<td>
<table>
<tr><th>x</th><th>y1 (x)
49x-4.9x^2</th></tr>
<tr><td>-2</td><td>-117.6</td></tr>
<tr><td>-1</td><td>-53.9</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>44.1</td></tr>
<tr><td>2</td><td>78.4</td></tr>
<tr><td>3</td><td>102.9</td></tr>
<tr><td>4</td><td>117.6</td></tr>
<tr><td>5</td><td>122.5</td></tr>
<tr><td>6</td><td>117.6</td></tr>
<tr><td>7</td><td>102.9</td></tr>
<tr><td>8</td><td>78.4</td></tr>
<tr><td>9</td><td>44.1</td></tr>
<tr><td>10</td><td>-5.68E-14</td></tr>
<tr><td>11</td><td>-53.9</td></tr>
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</td>
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To summarize, each of these perspectives plays an important role in the understanding of mathematical ideas.

Exercise

Complete the following table: (Continued on the next page)

Algebraic	*Geometric*	*"Real-World"*	*Verbal*
$y = 0.13x$			

Algebraic	***Geometric***	***"Real-World"***	***Verbal***
$y = 150 + 80x$			
		A taxi company charges $2.50 for the first fifth of a kilometre and $0.45 for each additional fifth of a kilometre.	
		A pizza store charges $8.00 per medium two-topping pizza plus a $1.50 delivery charge per order. On weekends, a special is offered: for six or more medium two-topping pizzas, the price per pizza is discounted by $0.50 and delivery is free.	
		The manufacturing of a new kind of sports bicycle costs $110 per bike plus $700,000 to set up the manufacturing equipment. Marketing studies have determined that the number of bikes sold will roughly be 70,000 decreased by 200 times the selling price.	
		A 3-hour river cruise goes 15 km upstream and then back again. The river has a current of 2 km/h.	
$\lvert x-7 \rvert = 2$			
$\lvert x-7 \rvert \le 2$			

Understanding Math Lesson 3: Don't just Scratch the Surface! Learn in Depth!

Look beneath the surface; let not the several quality of a thing nor its worth escape thee.
-Marcus Aurelius

Note: In this context, the word "several" means separate, distinct or particular. This is an obsolete (archaic) usage.

Example: Understanding Quadratic Relations in Depth

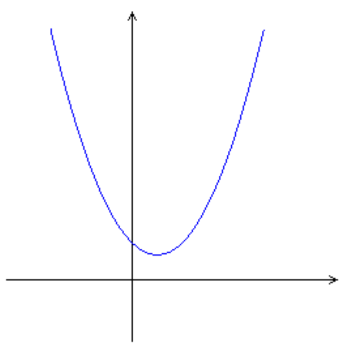
General Facts

Some students believe that to understand mathematics, it is enough to know formulas and how to apply them. This couldn't be further from the truth. By examining quadratic relations in depth, for example, we'll see how much more there is to understand than the quadratic formula!

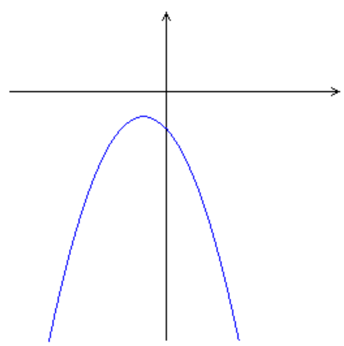
The quadratic formula, that is, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, is the ***general solution*** of the ***quadratic equation*** $ax^2+bx+c=0$. This formula can be derived by ***completing the square***. The ***discriminant***, b^2-4ac, characterizes the ***roots*** of a quadratic equation.	Graphically, the solutions of $ax^2+bx+c=0$ are the ***points of intersection*** of $y=ax^2+bx+c$ and $y=0$, that is, the ***x-intercepts*** of $y=ax^2+bx+c$.	The equation of a quadratic relation can be written in the following forms: **Standard Form:** $y=ax^2+bx+c$ **Vertex Form:** $y=a(x-h)^2+k$ **Factored Form:** $y=a(x-r_1)(x-r_2)$ **Partially Factored Form:** $y=ax(x+b)+c$

Relationship to Graphs

No Zeros: Parabola Lies Entirely Above or Below *x*-axis (i.e. there are ***no*** *x*-intercepts)

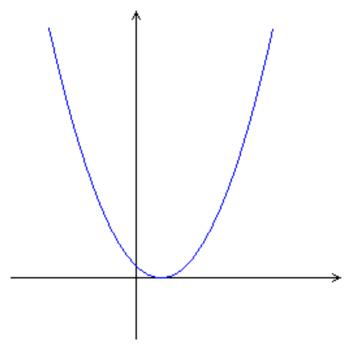


$a>0$, $b^2-4ac<0$

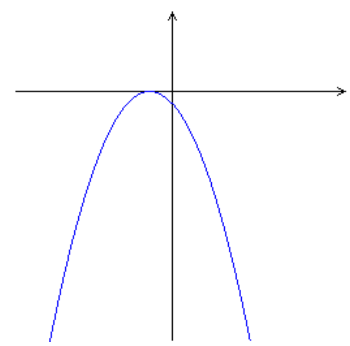


$a<0$, $b^2-4ac<0$

One Zero: Parabola Just Touches the *x*-axis (i.e. there is exactly ***one*** *x*-intercept)

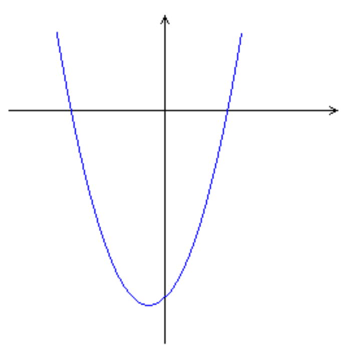


$a>0$, $b^2-4ac=0$

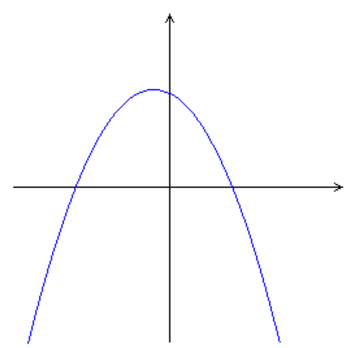


$a>0$, $b^2-4ac=0$

Two Zeros: Parabola Intersects the *x*-axis Exactly Twice (i.e. there are ***two*** *x*-intercepts)



$a>0$, $b^2-4ac>0$

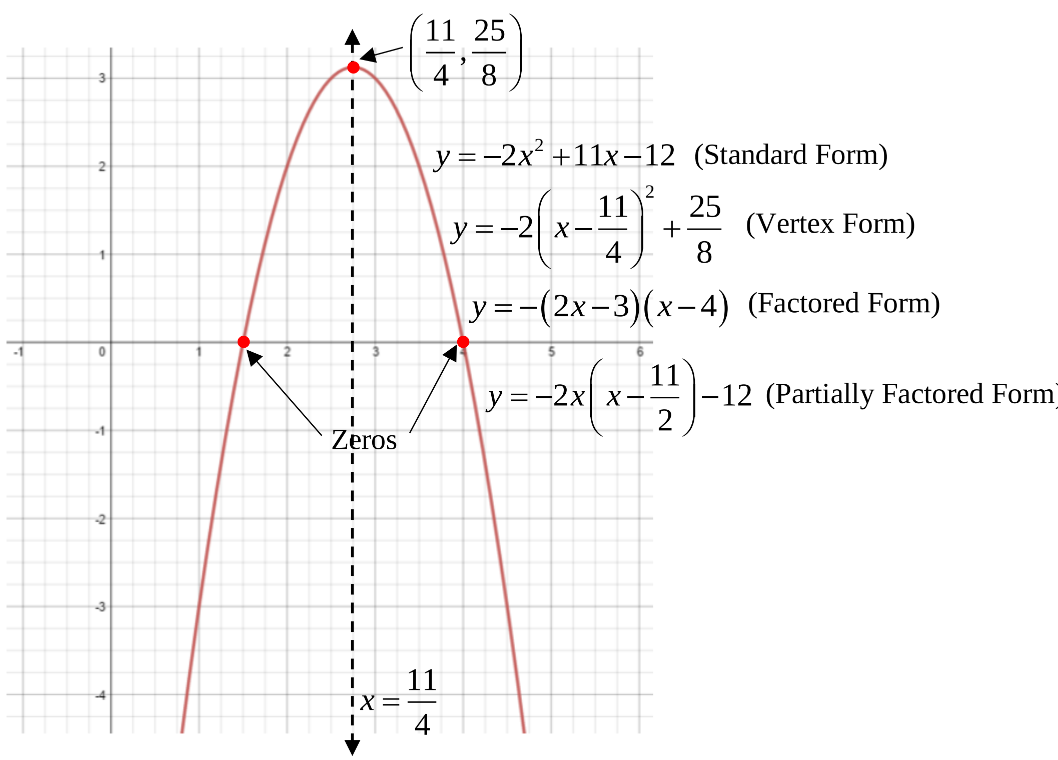


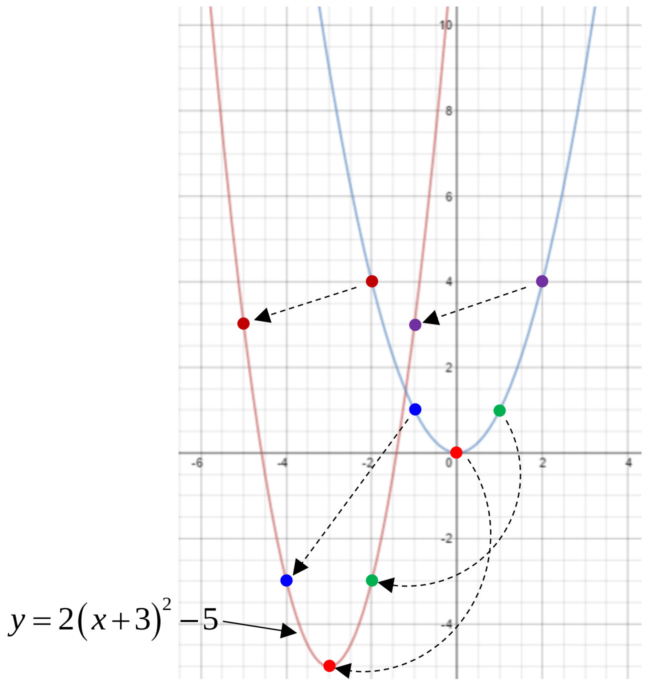
$a<0$, $b^2-4ac>0$

Properties of Quadratic Relations

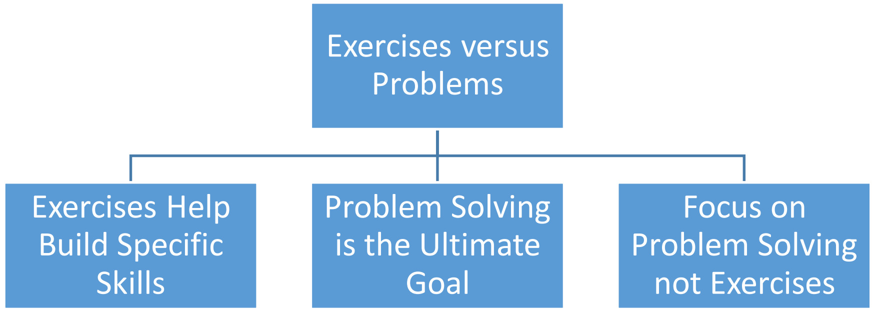
- The graph of a quadratic relation is a ***parabola***.
- In the equations given below, the sign of *a* determines whether the parabola opens upward or downward
  - If $a > 0$, the parabola opens upward
  - If $a < 0$, the parabola opens downward

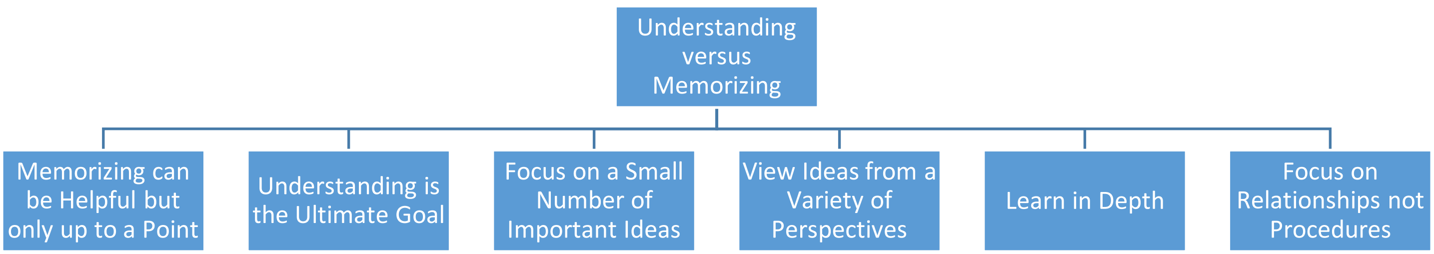
Equation Form	*Characteristics / Important Points*	*Example*
Standard Form $y = ax^2 + bx + c$ **e.g.** $y = -2x^2 + 11x - 12$ **Factored Form** $y = a(x - r_1)(x - r_2)$ **e.g.** $y = -1(2x - 3)(x - 4)$	• The other forms can be converted to standard form by expanding and simplifying. • Standard form is not particularly useful for graphing. • Factored form is useful for graphing. • The zeros (*x*-intercepts) of a parabola can be found by fully factoring a quadratic. • The co-ordinates of the vertex can also easily be found from the factored form (see example). • If the quadratic cannot be fully factored, then the ***quadratic formula*** must be used to find the zeros. • To find the equation of a parabola, ***three points*** on the parabola must be known.	$y = -2x^2 + 11x - 12$ $= -1(2x^2 - 11x + 12)$ $= -1(2x - 3)(x - 4)$ **Zeros:** $\frac{3}{2}$ and 4 **Axis of Symmetry:** $x = \frac{11}{4}$ (average of the zeros) **Vertex:** $\left(\frac{11}{4}, \frac{25}{8}\right)$ (substitute $x = \frac{11}{4}$ to find *y*) **Opens:** downward because the coefficient of x^2 is negative.

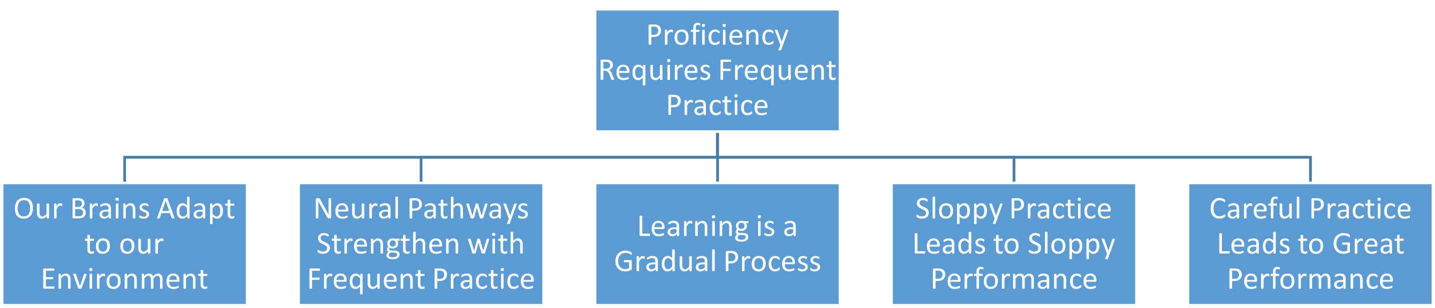


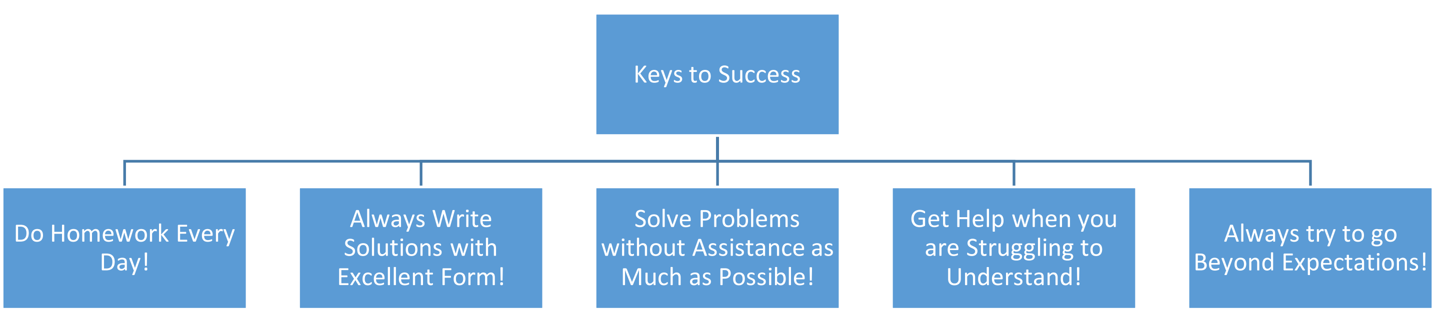
Equation Form	*Characteristics / Important Points*	*Example*
Vertex Form $y=a(x-h)^2+k$ **e.g.** $y=2(x+3)^2-5$	• The other forms can be converted to vertex form using at least two different methods: ○ Completing the square ○ Find the zeros of the parabola. Use the zeros to find the axis of symmetry. The vertex is the point of intersection of the parabola and its axis of symmetry. • Vertex form is very useful for graphing as well as for finding maximum and minimum values. • Vertex form can be used to find the zeros (x-intercepts) of a parabola. • To find the equation of a parabola in vertex form, ***two points*** on the parabola must be known, one of which ***must*** be the vertex.	When the equation of a parabola is specified in vertex form, graphing is a simple matter of ***transforming*** the base function (aka "parent function" or "mother function") $y=x^2$. For the example $y=2(x+3)^2-5$, the mother function $y=x^2$ is stretched vertically by a factor of 2, then translated down 5 units. In addition, the parabola is translated 3 units to the left. Note that the vertical and horizontal transformations are independent of each other.  The transformations can be summarized very succinctly using the following notation: $$(x,y)\to(x-3,2y-5)$$ This means that a point (x,y) on the base graph ("pre-image graph") is transformed into the point $(x-3,2y-5)$ on the image graph. **e.g.** $(0,0)\to(0-3,2(0)-5)=(-3,-5)$
Partially Factored Form $y=ax(x+b)+c$ **e.g.** $y=-6x(x+1)-7$	• Partially factored form is very useful for finding the axis of symmetry when a quadratic in standard form cannot be factored fully.	The quadratic expression $-6x^2-6x-7$ cannot be factored fully. The partially factored form $-6x(x+1)-7$, however, reveals that the axis of symmetry must be the same as the axis of symmetry of $y=-6x(x+1)$. This is because the graph of $y=-6x(x+1)-7$ can be obtained simply by translating the graph of $y=-6x(x+1)$ down 7 units. Since the zeros of $y=-6x(x+1)$ are 0 and −1, the equation of the axis of symmetry must be $x=-\frac{1}{2}$

Summary of Main Ideas









Review Exercises and Problems

Mechanical Practice: Simplifying, Factoring, Solving Equations, Inequalities

1. Simplifying expressions Expand and simplify.

a) $3(4t-8)+6(2t-1)$
b) $7(3w-4)-5(5w-3)$
c) $6(m+3)+2(m-11)-4(3m-9)$
d) $5(3y-4)-2(y+7)-(3y-8)$
e) $4(3x^2-2x+5)-6(x^2-2x-1)$
f) $6(x-y)-2(2x+7y)-(3x-2y)$
g) $3(x^2-2xy+2y^2)-5(2x^2-2xy-y^2)$

2. Solving linear equations Solve and check.

a) $2(2r-1)+4=5(r+1)$
b) $5(x-3)-2x=-6$
c) $7-2(1-3x)+16=8x+11$
d) $4y-(3y-1)-3+6(y-2)=0$
e) $4(w-5)-2(w+1)=3(1-w)$
f) $0=2(t-6)+8+4(t+7)$
g) $4(y-2)=3(y+1)+1-3y$

3. Solving linear equations Solve and check.

a) $\frac{x}{3}+\frac{1}{2}=0$
b) $\frac{y-1}{3}=6$
c) $\frac{x}{3}-\frac{1}{2}=\frac{1}{4}$
d) $\frac{m+2}{2}=\frac{m-1}{3}$
e) $\frac{w+1}{2}+\frac{w+1}{3}=5$
f) $\frac{2x+1}{3}-\frac{x+1}{4}=3$
g) $0.4(c-8)+3=4$
h) $0.5x-0.1(x-3)=4$
i) $1.5(a-3)-2(a-0.5)=10$
j) $1.2(10x-5)-2(4x+7)=8$

4. Common factors Factor.

a) $7t^2-14t^3$
b) $36x^7+24x^5$
c) $4xy-2xz+10x$
d) $8x^3-16x^2+4x$
e) $9x^2y+6xy-3xy^2$
f) $10a^2b+5ab-15a$

5. Factoring $ax^2+bx+c,\ a=1$ Factor.

a) $x^2+7x+12$ **b)** y^2-2y-8
c) $d^2+3d-10$ **d)** $x^2-8x+15$
e) w^2-81 **f)** t^2-4t
g) $y^2-10y+25$ **h)** $x^2-3x-40$

6. Factoring $ax^2+bx+c,\ a\neq 1$ Factor.

a) $2x^2+7x+3$ **b)** $2x^2-3x+1$
c) $3t^2-11t-20$ **d)** $2y^2-7y+5$
e) $6x^2+x-1$ **f)** $4x^2+12x+9$
g) $9a^2-16$ **h)** $6s^2-7s-3$
i) $2u^2+7u+6$ **j)** $9x^2-6x+1$
k) $3x^2+7x-20$ **l)** $4v^2+10v$

7. Solving quadratic equations by factoring Solve by factoring. Check your solutions.

a) $x^2-x-2=0$
b) $y^2-9=0$
c) $n^2-7n=0$
) $x^2-4x=-4$
) $6x+8=-x^2$
$z^2+12=-z$
g) $2x^2-5x+2=0$
h) $2y^2+7y+3=0$

8. Inequalities Graph the following integers on a number line.

a) $x>-2$ **b)** $x<3$
c) $x\geq 0$ **d)** $x\leq -1$

Solving Quadratic Equations, Graphing Quadratic Relations, "Real-World" Applications of Quadratics

1. Solve:
   a) $(x+7)(x-3)=(x+7)(5-x)$
   b) $(3x-9)(x+2)=(x-3)(2x+1)$
   c) $(x+4)(x-4)=-9(x+1)(x-1)$
   d) $(x+5)(2x-3)=(x+3)(x+4)$
2. Calculate the radius of a circle that has an area of:
   a) 169 cm^2; b) 1772 mm^2; c) $16\pi\text{ km}^2$.
3. Solve graphically:
   a) $2x^2+11x-6=0$ b) $2x^2-5x-12=0$
   c) $4x^2-25=0$ d) $16x^2+8x-143=0$
4. Write a quadratic equation with roots:
   a) $7, -1$; b) $0, \frac{11}{2}$; c) $\frac{4}{3}, -\frac{3}{4}$; d) $1.125, -5.875$
5. Solve.
   a) $x^2-5x-14=0$ b) $m^2+4m-32=0$
   c) $3v^2-2v-1=0$ d) $6t^2-11t-10=0$
6. Solve:
   a) $x^2-3x-22=4(x-1)$ b) $7v(v-1)=5(v^2-1.2)$
   c) $2(x-3)(x+3)+5x=0$
   d) $(z-4)(3z+2)=(z-5)(2z+1)-1$
7. One side of a right triangle is 2 cm shorter than the hypotenuse and 7 cm longer than the third side. Find the lengths of the sides of the triangle.
8. The height, h, in metres, of an infield fly ball t seconds after being hit is given by the formula: $h=30t-5t^2$. How long is the ball in the air?
9. The length of a rectangular picture is 5 cm greater than the width. Find the dimensions of the picture if its area is:
   a) 150 cm^2; b) 300 cm^2.
10. Solve by completing the square:
    a) $x^2-8x-30=0$ b) $x^2+6x-90=0$
    c) $x^2-5x+2=0$ d) $x^2+15x+25=0$
11. Solve by completing the square:
    a) $2x^2+9x+3=0$ b) $6x^2+2x-5=0$
    c) $7x^2-16x+5=0$ d) $10x^2+7x-10=0$
12. Solve:
    a) $5x^2+11x-12=0$ b) $3x^2+10x-32=0$
    c) $5x^2-15x+11=0$ d) $9x^2-6x-143=0$
    e) $12x^2-29x+14=0$ f) $20x^2+x-12=0$
13. The surface area, A, of a closed cylinder of radius r is given by the formula: $A=6.28r^2+92.1r$. Find the radius of the cylinder if the surface area is:
    a) 1138.72 cm^2; b) 1772.98 cm^2.

2. The zeros of a quadratic relation are -3 and 9. The second differences are positive.
   (a) Explain whether the optimal value will be a maximum or a minimum.
   (b) What value of the independent variable will produce the optimal value?
   (c) Explain whether the optimal value is a negative or positive value.

3. **(a)** Points $(-9, 0)$ and $(19, 0)$ lie on the curve of a parabola. What is the axis of symmetry for the parabola?
   (b) What are the zeros of the parabola?
   (c) The optimal value of the parabola is -28. Write the algebraic expression of the parabola in standard form.

4. Sketch each graph, using the x-intercepts and the optimal value as reference points. Clearly identify the x-intercepts and vertex.
   (a) $y=(x-6)(x+2)$ **(b)** $y=(4+x)(6-x)$

5. Expand and simplify each expression.
   (a) $(2x-3)(5x+2)$ **(b)** $(5a+2b)(3a-4b)$ **(c)** $-5(x-6)(2x+7)$

6. Factor each expression.
   (a) $x^2-9x+14$ **(b)** $16x^2-25$
   (c) $6x^2+5x-4$ **(d)** $2x^2+10x+12$

Simultaneous Systems of Linear Equations and their Applications

1. Solve graphically:
   a) $2x - y = 3$, $3x - 5y = 15$
   b) $-2x + y = 3$, $x + y = 6$
   c) $3x - y = -4$, $x - y = -8$
   d) $2y - 5x = 15$, $3y - 4x = 12$
2. Solve graphically:
   a) $x + 3y = 2$, $3x + 9y = 8$
   b) $3x + 7y = 27$, $5x + 2y = 16$
   c) $4x - 6y = -10$, $-6x + 9y = 15$
   d) $4x - 3y = 0$, $5x - 6y = -18$
3. Given: lines L_1: $5x + 2y + 10 = 0$, L_2: $x + 3y - 11 = 0$. On the same grid, graph L_1 and L_2, and the equations represented by:
   a) $L_1 + L_2$; b) $L_1 - L_2$; c) $3L_1 - 2L_2$; d) $L_1 + 3L_2$.
4. Solve algebraically:
   a) $2x + y = 3$, $4x - y = -6$
   b) $y = 3x - 7$, $y = 5 - x$
   c) $5x - 7y = 0$, $7x + 5y = 74$
   d) $3x + y = 6$, $2x - 3y - 9 = 0$
5. Find the solution correct to two decimal places:
   a) $5x - 3y = 9$, $2x + 5y = 8$
   b) $10x - 3y = 1$, $y = -2x + 7$
   c) $8x + 7y = 3$, $2x - 6y - 1 = 0$
   d) $3x + 8y + 2 = 0$, $2x - y + 1 = 0$
6. The lines $2x + y = 10$ and $7x + 8y = 53$ intersect at A, and the lines $2x - y = 12$ and $x + 3y = 27$ intersect at B. Find the equation of the line through A and B.
7. A triangle has vertices $(-1, 3)$, $(4, -2)$, $(8, 6)$. It is intersected by the line $3x - y - 10 = 0$ at P and Q. Find:
   a) the coordinates of P and Q; b) the length of PQ.
8. An office is equipped with two card-sorting machines, A and B. If A is operated for 2 min and B for 5 min, 20 500 cards can be sorted. If A is operated for 5 min and B for 2 min, 25 000 can be sorted. What are the sorting rates of the machines?
9. A motorist travels 400 km partly at 100 km/h and partly at 80 km/h. If he had travelled at 80 km/h instead of 100 km/h and 100 km/h instead of 80 km/h, the journey would have taken 0.5 h longer. Find his time for the trip.
10. Solve graphically:
    a) $x^2 + y^2 = 17$, $y = 4x$
    b) $x^2 + y^2 = 18$, $y = 2x + 3$
    c) $y = x^2$, $x - y + 6 = 0$
    d) $y = x^2 - 3x + 5$, $x + y = 8$

1. **(a)** Determine graphically the point of intersection between the lines defined by $y = -2x + 6$ and $8 = 5x - y$.
(b) Verify that you determined the correct point by solving the system of equations in part **(a)** algebraically.

2. Solve by substitution.

(a) $3x + y = 5$
$x - 2y = 11$

(b) $5x - 2y = -16$
$-2x + y = 7$

(c) $4x - 3y = 10$
$2x + 3y = 4$

3. Solve by elimination.

(a) $a - 15b = 3$
$3b + a = 21$

(b) $2x + 5y = 19$
$3x - y = 3$

(c) $3x - 2y = -8$
$3y - 21 = 9x$

4. Confirm or deny: The ordered pair $(3, -5)$ is the solution to the linear system defined by $2x + 5y = -19$ and $6y - 8x = -54$. Justify your answer.

5. Jeff is a cashier at the grocery store. He has a total of \$580 in bills. He has 76 bills, consisting of \$5 bills and \$10 bills. How many of each type does he have?

6. A traffic helicopter pilot finds that with a tailwind her 120 km trip away from the airport takes 30 min. On her return trip to the airport, into the wind, she finds that her trip is 10 min longer. What is the speed of the helicopter? What is the speed of the wind?

7. Rani is comparing the monthly costs from two Internet service providers. Netaxes charges a flat monthly fee of \$10, plus \$0.75 per hour spent on-line. Webz charges a flat monthly fee of \$5, plus \$1 per hour.
(a) Determine when the monthly costs are the same.
(b) Rani plans to use the Internet for at least 30 h each month. Which provider should she choose? Explain.

8. Premium gasoline sells for 78.9¢/L. Regular gas sells for 71.9¢/L. To boost sales, a middle octane gasoline is formed by mixing premium and regular. If 1000 L of this middle octane gas is produced, and is sold at 73.9¢/L, then how much of each type of gasoline can you assume was used in the mixture?

9. Graph a linear system with no solution. Determine two possible equations that could represent both lines in your graph.

10. Solve.

(a) $12(x - 2) - (2y - 1) = 14$
$5(x - 1) + 2(1 - 2y) = 14$

(b) $\frac{x-2}{3} - \frac{y+5}{2} = -3$
$3x - \frac{2y}{3} = 13$

More Quadratics

1. Find the vertex, axis of symmetry, and direction of opening of the parabola. Use this information to sketch the graph.

(a) $y = x^2 - 7$ **(b)** $y = (x - 3)^2$ **(c)** $y = -(x + 1)^2 + 10$

(d) $y = 3x^2 - 12$ **(e)** $y = -\frac{1}{2}(x + 2)^2 - 3$ **(f)** $y = -2(x - 5)^2 + 6$

2. Describe how the graph of $y = x^2$ can be transformed, step by step, to obtain the graph of each quadratic relation in question 1.

3. Find the quadratic relation in vertex form that

(a) has its vertex at $(-6, 0)$ and passes through $(-3, 27)$

(b) has its vertex at $(3, 7)$ and passes through $(-1, -17)$

(c) has its vertex at $(-4, -2)$ and has a y-intercept of -8

(d) has zeros -3 and 5 and passes through $(3, 6)$

4. Express each relation you found in (3) in standard form $y = ax^2 + bx + c$.

5. Without graphing, tell how many zeros (x-intercepts) the quadratic relation has. Justify your answers.

(a) $y = x^2 + 3$ **(b)** $y = -3(x + 5)^2$ **(c)** $y = \frac{2}{3}(x - 2)^2 - 7$

6. A concrete bridge over a river has an underside in the shape of a parabolic arch. At the water level, the arch is 30 m wide. It has a maximum height of 10 m above the water. The minimum vertical thickness of the concrete is 1.5 m.

(a) Find an algebraic relation that represents the shape of the arch.

(b) What is the vertical thickness of the concrete 3 m from the centre of the arch?

(c) If the water level rises 2 m, how wide will the arch be at this new level?

7. A baseball is hit into the air by the Blue Jays' batting coach. Its height h, in metres, after t seconds is $h = -4.9(t - 2.8)^2 + 39$.

(a) How high off the ground was the ball when it was hit?

(b) What is the maximum height of the ball?

(c) What is the height of the ball after 2.5 s? Is it on the way up or down? Justify your answer.

(d) Is the ball still in the air after 6 s? Explain how you know.

(e) To one decimal place, when does the ball hit the ground?

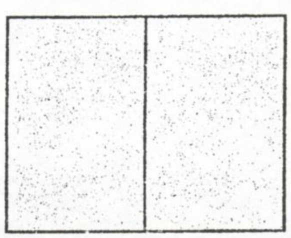
8. Find the coordinates of the vertex of the quadratic relation using the most appropriate method. List two other points that lie on the graph of the relation. Express the relation in vertex form.

(a) $y = 2(x - 3)(x + 7)$ **(b)** $y = x(2x + 6) - 11$

(c) $y = -3x^2 + 12x + 15$ **(d)** $y = x^2 - 6x - 4$

(e) $y = 4x^2 - 11x - 55$ **(f)** $y = 5x^2 + 20x - 11$

9. A farmer has \$2400 to spend to fence two rectangular pastures as shown in the diagram. The local contractor will build the fence at a cost of \$6.25/m. What is the largest total area that the farmer can have fenced for that price?



Mixed Review of Algebra

Write an equation to model each problem. Then solve.

1. The product of 3 and a number is 27.	2. Eight less than ten times a number is 62.
3. One-third of Natalie's age is 16 years less than her age 20 years ago. How old is Natalie?	4. Nine cards and four boxes of candy cost \$24.73. Ten cards and five boxes of candy cost \$29.20. How much do two cards cost?

Solve. In each case, sketch a diagram so that you also have a *geometric viewpoint*.

5. Write an equation of a line that passes through the point (1, 29) and is parallel to the line with equation $19x + 8y = 251$.	6. Write an equation of a line that passes through the point (40, 4) and is perpendicular to the line with the following equation: $\frac{7x+2}{2} = 144$	7. Write an equation of a line that passes through the point (-38, -34) and is perpendicular to the line with the following equation: $y = \frac{-1}{9}x - \frac{344}{9}$

Simplify and write in lowest terms.

8. $\frac{4h}{6} + \frac{7h}{3}$	9. $\frac{8d}{28} + \frac{9d}{7}$	10. $\frac{9}{b} + \frac{2}{p}$

Expand and simplify.

11. $(5x^7 - 10x^6 - 10x)(-11x^2 + 12x + 4)$	12. $(3x^2 - 11x - 9)(11x + 10)$	13. $(-2x^7 + 12x - 2)(-9x^2 + 8x + 9)$

Write an equation for a parabola that passes through the given points. (Sketch a graph of each parabola before using algebra.)

14. (6, –55), (0, –7), (4, –31), (–1, -6)	15. (–2, –7), (6, –87), (2, –31)

Solve each equation by factoring.

16. $15x^2 + 99x = 168$	17. $-120x^2 + 390x - 315 = 0$	18. $\frac{2}{5}x^2 + 3x = 10$

Simplify.

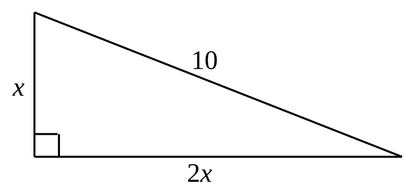
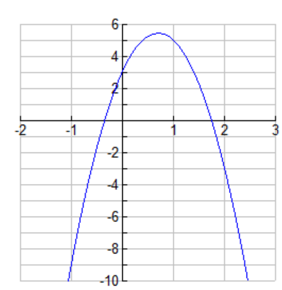
19. $(-4h^{-5})(-4h^4t^{-4})$	20. $(f^{-5})(3f^{-3}v^3)$	21. $(-9q^{-5}y^5)(10q^2)$

What is Mathematics?

From a theoretical point of view, ***mathematics*** is the ***investigation*** of ***axiomatically defined abstract structures*** using ***logic*** and ***mathematical notation***. On a more practical level, mathematics can be seen as an ***extension*** of ***spoken*** and ***written natural languages***, with an extremely precisely defined vocabulary and grammar, for the purpose of ***describing and exploring physical and conceptual RELATIONSHIPS***. (Math is like a ***dating service***. It's all about ***relationships***! The lonely and very simple "Mr. x" is looking for a lovely but perhaps somewhat complex "Miss y.")

Mathematical Relationships

A ***formula*** is an ***equation*** that expresses a ***mathematical relationship*** between ***two or more unknowns***. ***All formulas are equations but not all equations are formulas!***

Example Formulas	***Relationship in Words***	***Example***
$c^2 = a^2 + b^2$ This is the famous Pythagorean Theorem	This formula expresses the relationship among the sides of a right triangle.	**The hypotenuse of a right triangle has a length of 10 m and the lengths of other sides are in the ratio 2:1. Find the lengths of the other sides.** $x^2 + (2x)^2 = 10^2$ $\therefore x^2 + 4x^2 = 100$ $\therefore 5x^2 = 100$ $\therefore x^2 = 20$ $\therefore x = \sqrt{20} = 2\sqrt{5}$  The lengths of the other two sides are $2\sqrt{5}$ m and $4\sqrt{5}$ m.
$Ax + By + C = 0$ This is the so-called ***standard form*** of a linear equation.	This formula expresses the relationship between the x-co-ordinate and the y-co-ordinate of any point lying on a straight line with slope equal to $-\dfrac{A}{B}$ and y-intercept equal to $-\dfrac{C}{B}$.	**Find the slope and y-intercept of the line $Ax + By + C = 0$.** First solve for y. Then compare to the $y = mx + b$ form. $Ax + By + C = 0$ $\therefore By = -Ax - C$ $\therefore y = -\dfrac{A}{B}x - \dfrac{C}{B}$ By comparing to $y = mx + b$, we see that $m = -\dfrac{A}{B}$ and $b = -\dfrac{C}{B}$.
$y = ax^2 + bx + c$ This is the so-called ***standard form*** of a quadratic equation.	This formula expresses the relationship between the x-co-ordinate and the y-co-ordinate of any point lying on a parabola with vertex $\left(-\dfrac{b}{2a}, \dfrac{4ac-b^2}{4a}\right)$ and vertical stretch factor a. If $a > 0$ then the parabola opens ***upward***. Otherwise, if $a < 0$ then the parabola opens ***downward***.	**Find the vertex, vertical stretch factor and direction of opening of a parabola with equation $y = -5x^2 + 7x + 3$.** $y = -5x^2 + 7x + 3$ $\therefore y = -5\left(x^2 - \dfrac{7}{5}x\right) + 3$ $\therefore y = -5\left(x^2 - \dfrac{7}{5}x + \left(\dfrac{7}{10}\right)^2 - \left(\dfrac{7}{10}\right)^2\right) + 3$ $\therefore y = -5\left(x - \dfrac{7}{10}\right)^2 + 5\left(\dfrac{7}{10}\right)^2 + 3$ $\therefore y = -5\left(x - \dfrac{7}{10}\right)^2 + \dfrac{109}{20}$  Therefore, the vertex is $\left(\dfrac{7}{10}, \dfrac{109}{20}\right)$, the direction of opening is downward and the vertical stretch factor is 5.

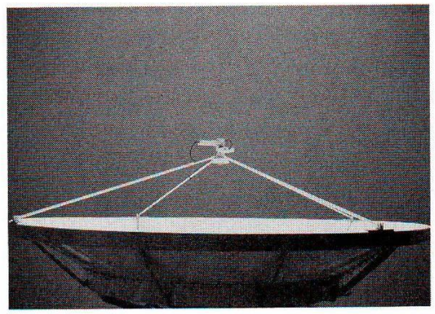
Linear Relationships

- These are the ***simplest*** of all mathematical relationships. They are very easy to analyze mathematically and are completely understood (i.e. there are no unresolved problems regarding linear relationships).
- The graphs of linear relationships are ***straight lines***.
- If y is linearly related to x, then the ***rate of change of y is constant with respect to x***. That is, for a given Δx, Δy is always the same. Another way of expressing this is that the ***first differences are constant***.
- Linear relationships can be used to ***model*** any quantity that ***changes at a constant rate***. For example, for a car that is travelling at a constant speed, distance travelled is linearly related to time elapsed. That is, the graph of distance versus time would be a straight line.
- The general equation of a linear relationship is $Ax + By + C = 0$. This equation can be rewritten in the form $y = -\frac{A}{B}x - \frac{C}{B}$. By comparing this to the slope-intercept form a linear equation, we see that slope $= -\frac{A}{B}$ and y-intercept $= -\frac{C}{B}$. By comparing to the slope-intercept form of a linear relationship we find $m = -\frac{A}{B}$ and $b = -\frac{C}{B}$.
- Any equation involving only linear terms can be solved by using the ***balancing method***. For example, an equation such as $\frac{2}{3}(5x-7) - \frac{1}{2} = \frac{7}{9}x + 10$ can be solved by balancing.
- While using the balancing method, it is helpful to remember the "dressing/undressing" analogy. Remember that as long as you perform the ***same operation to both sides of any equation***, you will obtain an equivalent equation (i.e. having the same solutions).

Quadratic Relationships

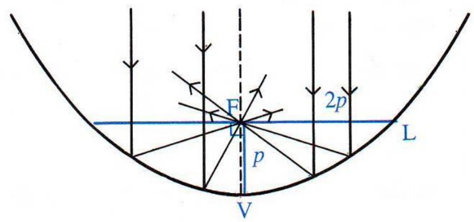
- These are not as simple as linear relationships but are still quite easy to analyze mathematically. Like linear relationships, quadratics are completely understood (i.e. there are no unresolved problems regarding quadratics).
- The graphs of quadratic relationships are ***parabolas***.
- If y is quadratically related to x, then the ***rate of change of y is NOT constant***. That is, for a given Δx, Δy is ***NOT*** always the same. For quadratic relationships, the ***first differences change linearly*** and ***the second differences are constant***.
- Quadratic relationships can be used to ***model*** many different quantities. For example, the position of any object moving solely under the influence of gravity (close to the surface of the Earth) changes quadratically. (See cannonball example in the "What is Mathematics?" section.
- The general equation of a quadratic relationship is $y = ax^2 + bx + c$. By ***completing the square***, this equation can be rewritten in the more convenient form $y = a(x-h)^2 + k$, where (h,k) is the vertex of the parabola.
- An equation involving quadratic terms ***cannot*** be solved entirely by using the ***balancing method***. Quadratic equations can be solved by ***factoring***, ***the quadratic formula***, ***partial factoring*** and ***completing the square***.
- The roots of a quadratic equation are completely characterized by the discriminant, which equals $b^2 - 4ac$. This expression appears under the square root sign in the quadratic formula, which is what allows us to decide the nature of the roots.
- Once we know the nature of the roots of a quadratic, we can deduce whether its associated parabola lies entirely above the x-axis, entirely below the x-axis, crosses the x-axis at two points or just "touches" it at one point.
- Contrary to the claims of many students, quadratics have a wide variety of applications. An interesting one is given on the next page. Satellite dishes are just antennas that are designed to receive signals originating from satellites in geostationary orbits around the Earth. Most satellite dishes have a parabolic cross-section.

Reflector Property of the Parabola



We have all seen dish antennas for receiving TV signals from satellites. These antennas have parabolic cross sections. When the antenna is aimed at a satellite, the signals entering the antenna are reflected to the receiver, which is placed at the focus of the antenna.

Every parabola has a *focus*, which is a particular point on the axis of symmetry. The position of the focus can be defined as follows.



For any parabola, the *focus* is the point on the axis of symmetry which is half as far from the vertex as it is from the parabola, measured along a line perpendicular to the axis of symmetry. For example, in the diagram, FV $= p$, and FL $= 2p$. That is, F is half as far from V as from L. Hence, F is the focus of the parabola. Every parabola has one and only one focus.

You can illustrate the reflector property of the parabola by completing the questions below.

QUESTIONS

1. a) Use a table of values to construct an accurate graph of the parabola defined by $y = \frac{1}{8}x^2$ for values of x between -8 and 8.
   b) Mark the point F(0,2) on the graph. Verify that F satisfies the above definition of the focus.
2. a) Mark any point P on the parabola you constructed in *Question 1*. Join PF, and draw a line PM parallel to the axis of symmetry. By estimation, draw a tangent to the parabola at P. Verify that PF and PM form equal angles with the tangent.
   b) Repeat part a) for other points P on the parabola.
3. Use the above definition of the focus to prove that the coordinates of the focus of the parabola defined by $y = ax^2$ are $\left(0, \frac{1}{4a}\right)$.

Terminology

- By this point in your mathematics education, you must understand and use the following terminology correctly: ***equation***, ***expression***, ***term***, ***factor***, ***polynomial***, ***monomial***, ***binomial***, ***trinomial***, ***factor***, ***expand***, ***solve***, ***simplify***, ***evaluate***, ***roots***, ***discriminant***, ***intercept***, ***intersect***, ***vertex***, ***axis of symmetry***, ***(simultaneous) system of equations***, ***relation***, ***relationship***, ***rate of change***, ***distance***, ***length***, ***area***, ***volume***, ***speed***, ***surface area***

Measurement

- Pythagorean Theorem and distance between two points (length of a line segment)
- Midpoint of a line segment
- Area of rectangle, parallelogram, triangle, trapezoid, circle
- Surface area of cylinder, cone, sphere, prism
- Volume of cylinder, cone, sphere, prism

Systems of Linear Equations (Two Linear Equations in Two Unknowns)

- Solve by using the method of substitution
- Solve by using the method of elimination
- Solve graphically

Operator Precedence (Order of Operations)

Standard Order (no parentheses)	***Notes***	***Example***	***Example with Parentheses***
1. Exponents		$12 \div 4 \times 3 - 2 \times 4^3 \div 32$ $= 3 \times 3 - 2 \times 64 \div 32$ $= 9 - 128 \div 32$ $= 9 - 4$ $= 5$	$12 \div 4 \times (3 - 2) \times 64 \div 32$ $= 3 \times 1 \times 64 \div 32$ $= 3 \times 64 \div 32$ $= 192 \div 32$ $= 6$
2. Multiplication and Division	Performed in order of occurrence from left to right when both of these operations occur in a given term.		
3. Addition and Subtraction	Performed in order of occurrence from left to right when both of these operations occur in a given expression.		

Parentheses are used to ***override*** the standard order of precedence. When a departure from the standard order is required (for example when subtraction needs to be performed before multiplication) parentheses must be used.

Operating with Integers

Adding and Subtracting Integers	***Multiplying and Dividing Integers***
• Movements on a number line • Moving from one floor to another using an elevator • Loss/gain of yards in football • Loss/gain of money in bank account or stock market ***Add a Positive Value or Subtract a Negative Value*** +(+) ***or*** –(–) → ***GAIN*** (move ***up*** or ***right***) ***Add a Negative Value or Subtract a Positive Value*** +(–) ***or*** –(+) → ***LOSS*** (move ***down*** or ***left***)	• Multiplication is ***repeated addition*** **e.g.** 5(–2) = 5 groups of –2 = (–2)+(–2)+(–2)+(–2)+(–2) = –10 • Division is the ***opposite of*** multiplication **e.g.** –10÷(–2) = How many groups of –2 in –10? = 5 ***Multiply or Divide Two Numbers of Like Sign*** (+)(+) ***or*** (–)(–) → ***POSITIVE RESULT*** ***Multiply or Divide Two Numbers of Unlike Sign*** (+)(–) ***or*** (–)(+) → ***NEGATIVE RESULT***

Operating with Fractions

Adding and Subtracting Rational Numbers (Fractions)	Multiplying and Dividing Rational Numbers (Fractions)
• Express each fraction with a ***common denominator*** • Use rules for operating with integers • $\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$ ***Example*** $\frac{-3}{6}+\left(\frac{-5}{9}\right)=\frac{-9}{18}+\left(\frac{-10}{18}\right)=\frac{-9+(-10)}{18}=-\frac{19}{18}$	• ***NO*** common denominator • $\frac{a}{b}\left(\frac{c}{d}\right)=\frac{ac}{bd}$ and $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$ ***Example*** $\frac{-3}{6}\div\left(\frac{-5}{9}\right)=\frac{-3}{6}\times\left(\frac{9}{-5}\right)=\frac{-3}{2}\times\left(\frac{3}{-5}\right)=\frac{9}{10}$

Simplifying Algebraic Expressions

Adding and Subtracting TERMS	Multiplying FACTORS
• Collect ***like terms*** • Use rules for adding/subtracting integers ***Example*** $-3x^2y+5xy-6x^2y-13xy=-9x^2y-8xy$	• Use rules for multiplying integers, laws of exponents and the distributive law ***Examples*** **1.** $-3a^2(5a^6)(-6b^7)=(-3)(5)(-6)a^2a^6b^7=90a^8b^7$ **2.** $(2x-7)(3x-8)=2x(3x)-2x(8)-7(3x)-7(-8)=6x^2-37x+56$

Dividing Algebraic Expressions	
• Use rules for dividing integers • Use laws of exponents	***Examples*** **1.** $\frac{-6a^5b^3}{-18a^3b^5}=\frac{a^2}{3b^2}$, **2.** $\frac{7m^3n-14m^6n^3}{-2m^2n^7}=\frac{7m^3n}{-2m^2n^7}-\frac{14m^6n^3}{-2m^2n^7}=-\frac{7m}{2n^6}+\frac{7m^4}{n^4}$

Factoring

An ***expression*** is ***factored*** if it is written as a ***product***.

Common Factoring	Factor "Simple" Trinomial	Factor "Complex" Trinomial	Difference of Squares
Example $-42m^3n^2+13mn^2p-39m^4n^3q$ $=-13mn^2(4m^2-p+3m^3nq)$	***Example*** $n^2-20n+91$ $=(n-7)(n-13)$ ***Rough Work*** (–7)(–13) = 91 –7 + (–13) = –20	***Example*** $10x^2-x-21$ $=10x^2-15x+14x-21$ $=5x(2x-3)+7(2x-3)$ $=(2x-3)(5x+7)$ ***Rough Work*** (10)(–21)= –210, (–15)(14)= –210 –15 + 14 = –1	***Example*** $98x^2-50y^2$ $=2(49x^2-25y^2)$ $=2((7x)^2-(5y)^2)$ $=2(7x-5y)(7x+5y)$

Solving Equations

- An ***equation*** has an ***expression*** on the left-hand side, an ***expression*** on the right-hand side and an ***equals sign*** between the left and right sides.
- If an equation needs to be ***solved***, then we must find a value of the unknown that ***satisfies*** the equation. That is, when a solution is substituted into the equation, the left side ***must*** equal the right side.
- It is also very important to understand ***graphical solutions*** of equations.
- While using the balancing method, it is helpful to remember the "dressing/undressing" analogy. Remember that if you perform the ***same operation to both sides of any equation***, you will obtain an ***equivalent equation*** (i.e. an equation that has the same solutions as the original).

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Solving Linear Equations

- Use the "balancing" method.
- Eliminate fractions by multiplying both sides of the equation by the ***least common multiple of all denominators***.

Example

$\frac{2}{3}(5x-7)-\frac{1}{2}=\frac{7}{9}x+10$

$\therefore 18\left[\frac{2}{3}(5x-7)-\frac{1}{2}\right]=18\left[\frac{7}{9}x+10\right]$

$\therefore 12(5x-7)-9=14x+180$

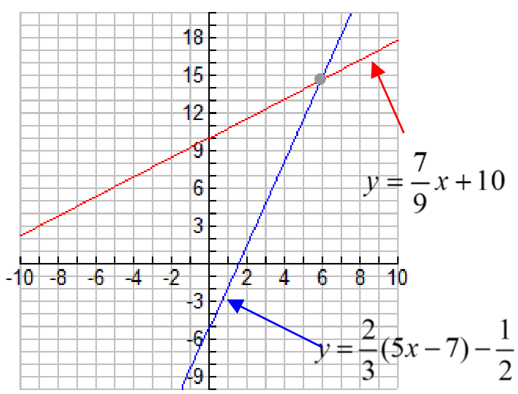
$\therefore 60x-84-9=14x+180$

$\therefore 60x-93=14x+180$

$\therefore 60x-93-14x+93=14x+180-14x+93$

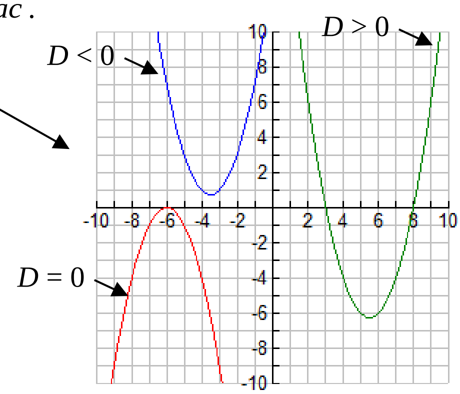
$\therefore 46x=273$

$\therefore x=\frac{273}{46}$



Solving Quadratic Equations

- First write the quadratic equation in the form $ax^2+bx+c=0$.
- Try ***factoring first***.
- If the quadratic does not factor, use the ***quadratic formula***.
- Use the method of "completing the square" only if you are asked to!
- The ***nature of the roots*** can be determined by calculating the ***discriminant*** $D=b^2-4ac$.



Examples

$3n^2-2n-5=0$

$\therefore 3n^2-5n+3n-5=0$

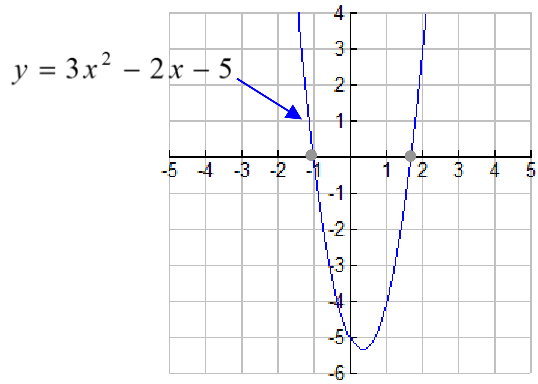
$\therefore (3n^2-5n)+(3n-5)=0$

$\therefore n(3n-5)+1(3n-5)=0$

$\therefore (3n-5)(n+1)=0$

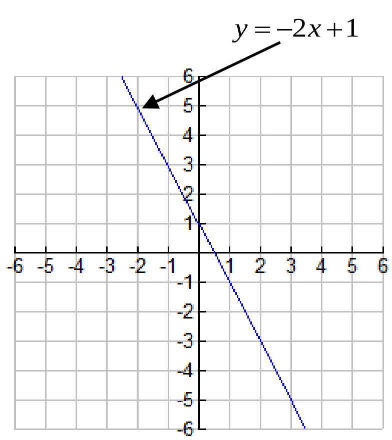
$\therefore 3n-5=0$ or $n+1=0$

$\therefore n=\frac{5}{3}$ or $n=-1$



Rate of Change (How fast a Quantity Changes relative to another Quantity)

- If y is ***linearly related*** to x, then the ***rate of change of y is constant relative to x***. That is, for a given Δx, Δy is always the same. Another way of expressing this is that the ***first differences are constant***.
- If y is quadratically related to x, then the ***rate of change of y is NOT constant relative to x***. That is, for a given Δx, Δy is ***NOT*** always the same. For quadratic relationships, the ***first differences change linearly*** and ***the second differences are constant***.



x	Δx	$y=-2x+1$	First Differences Δy	Second Differences $\Delta(\Delta y)$
−1		3		
0	1	1	−2	
1	1	−1	−2	0
2	1	−3	−2	0
3	1	−5	−2	0
4	1	−7	−2	0
5	1	−9	−2	0
6	1	−11	−2	0

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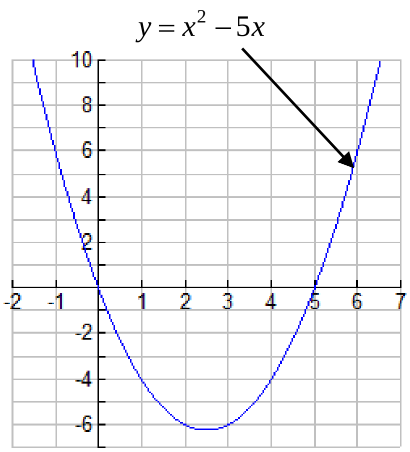
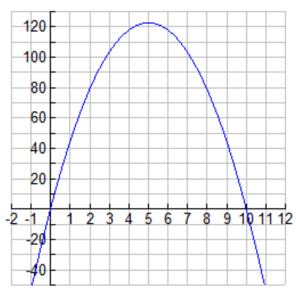
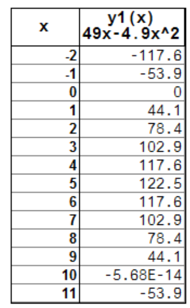


x	Δx	$y = x^2 - 5x$	First Differences Δy	Second Differences $\Delta(\Delta y)$
–1		6		
0	1	0	–6	
1	1	–4	–4	2
2	1	–6	–2	2
3	1	–6	0	2
4	1	–4	2	2
5	1	0	4	2
6	1	6	6	2

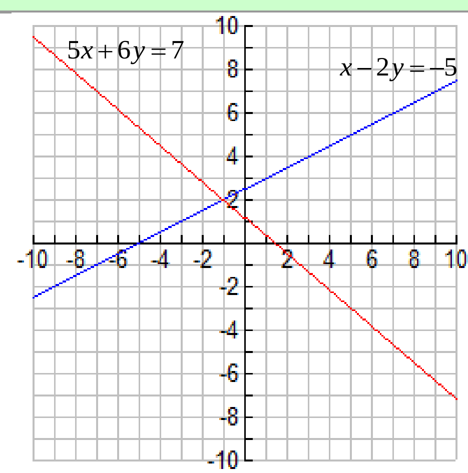
What is Mathematics?

- In a nutshell, ***mathematics*** is the ***investigation*** of ***axiomatically-defined abstract structures*** using ***logic*** and ***mathematical notation***. Accordingly, mathematics can be viewed as an ***extension*** of ***spoken*** and ***written natural languages***, with an extremely precisely defined vocabulary and grammar, for the purpose of ***describing and exploring physical and conceptual RELATIONSHIPS***. (Math is like a ***dating service***. It's all about ***relationships***! The lonely and very simple "Mr. x" is looking for a lovely but perhaps somewhat complex "Miss y.")
- Mathematical relationships can be viewed from a variety of different perspectives as shown below:

Algebraic	***Geometric***	***"Real-World"***	***Verbal***	***Numerical***
$h = 49t - 4.9t^2$		A cannonball is fired vertically into the air with an initial speed of 49 m/s. Its height above the ground at a given time t is quadratically related to t.	The value of h is equal to the product of 49 and t reduced by the product of 4.9 and the square of t.	 x | y1(x) 49x-4.9x^2 -2 | -117.6 -1 | -53.9 0 | 0 1 | 44.1 2 | 78.4 3 | 102.9 4 | 117.6 5 | 122.5 6 | 117.6 7 | 102.9 8 | 78.4 9 | 44.1 10 | -5.68E-14 11 | -53.9

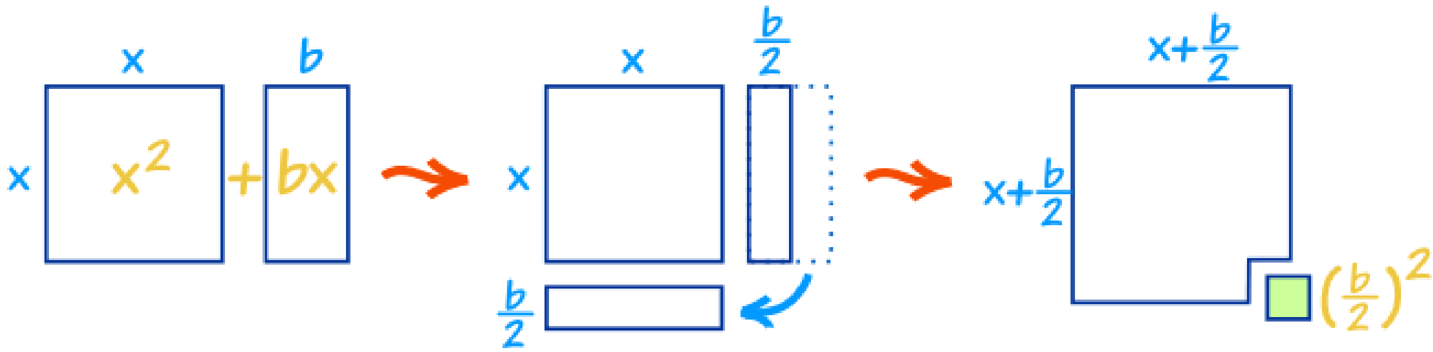
Each of these perspectives has an important role to play in the understanding of mathematical relationships.

Simultaneous Systems of Linear Equations

Solving Simultaneous Systems of Linear Equations	***Graphical Solution***
• Use ***substitution*** or ***elimination***. • The solution ***must satisfy both equations***. ***Example*** $x - 2y = -5$ (1) $5x + 6y = 7$ (2) $(1)\times 3,\quad 3x - 6y = -15$ (3) $(2)+(3),\ 8x = -8$ $\therefore x = -1$ $\therefore y = 2$	 $5x + 6y = 7$; $x - 2y = -5$

Understanding "Completing the Square"

The Geometric Picture



$$x^2+bx=x^2+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2=\left(x+\frac{b}{2}\right)^2-\frac{b^2}{4}$$

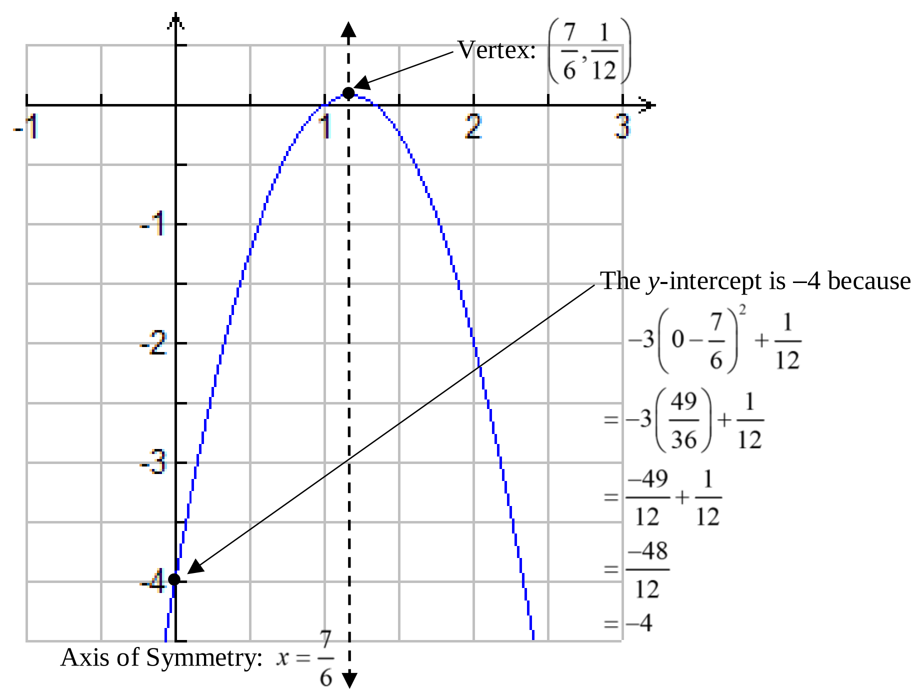
> This quadratic trinomial is in the form $x^2+2kx+k^2=(x+k)^2$, that is, it is a ***perfect square***. We can see this easily if we let $k=\frac{b}{2}$. Then $2k=\left(\frac{2}{1}\right)\frac{b}{2}=2$.

Example

Graph $y=-3x^2+7x-4$ by completing the square.

Solution

$$\begin{aligned}
y&=-3x^2+7x-4\\
&=-3\left(x^2-\frac{7}{3}x\right)-4\\
&=-3\left[x^2-\frac{7}{3}x+\left(\frac{7}{6}\right)^2-\left(\frac{7}{6}\right)^2\right]-4\\
&=-3\left[\left(x-\frac{7}{6}\right)^2-\frac{49}{36}\right]-4\\
&=-3\left(x-\frac{7}{6}\right)^2+\left(\frac{3}{1}\right)\left(\frac{49}{36}\right)-4\\
&=-3\left(x-\frac{7}{6}\right)^2+\frac{49}{12}-\frac{48}{12}\\
&=-3\left(x-\frac{7}{6}\right)^2+\frac{1}{12}
\end{aligned}$$



Extension of the Example

Find the zeros (x-intercepts) of the above graph both by factoring and by using the vertex form of the equation.

Why is "Completing the Square" Important?

1. Makes graphing of quadratic functions much easier because the vertex form of the equation contains explicit information about certain important features of the graph
2. Provides a straightforward approach for deriving the quadratic formula.

Derivation of the Quadratic Formula

Start with: $ax^2+bx+c=0$, where a, b and c are real numbers, $a \neq 0$.

Divide both sides of the equation by a: $x^2+\frac{b}{a}x+\frac{c}{a}=0$

Subtract $\frac{c}{a}$ from both sides: $x^2+\frac{b}{a}x=-\frac{c}{a}$

Add $\left(\frac{b}{2a}\right)^2$ to both sides (complete the square): $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$

The ***left-hand side*** *is now in the form* $x^2+2kx+k^2=(x+k)^2$. Thus, the equation can be rewritten as follows:

$$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

Take the square root of both sides: $x+\frac{b}{2a}=\pm\sqrt{-\frac{c}{a}+\left(\frac{b}{2a}\right)^2}$

Subtract $\frac{b}{2a}$ from both sides: $x=-\frac{b}{2a}\pm\sqrt{-\frac{c}{a}+\left(\frac{b}{2a}\right)^2}$

We have already solved for x but a little simplification is called for:

Evaluate $\left(\frac{b}{2a}\right)^2$: $x=-\frac{b}{2a}\pm\sqrt{\frac{-c}{a}+\frac{b^2}{4a^2}}$

Write "fractions" with a common denominator: $x=-\frac{b}{2a}\pm\sqrt{\frac{-4ac}{4a^2}+\frac{b^2}{4a^2}}$

Write as a single "fraction": $x=-\frac{b}{2a}\pm\sqrt{\frac{-4ac+b^2}{4a^2}}$

Apply the identity $\sqrt{\frac{x}{y}}=\frac{\sqrt{x}}{\sqrt{y}}$: $x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}$

Simplify: $x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$

Simplify fully (common denominator): $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Alternative Derivation of the Quadratic Formula using Partial Factoring

Consider first the graph of

$$y = ax^2 + bx = ax\left(x + \frac{b}{a}\right).$$

Since its zeros are 0 and $-\frac{b}{a}$, the equation of its axis of symmetry must be

$$x = \frac{1}{2}\left(-\frac{b}{a} + 0\right) = -\frac{b}{2a}.$$

Now consider the graph of

$$y = ax^2 + bx + c = ax\left(x + \frac{b}{a}\right) + c.$$

This parabola must have the same axis of symmetry as that of $y = ax\left(x + \frac{b}{a}\right)$ because it is just a vertical translation of $y = ax\left(x + \frac{b}{a}\right)$. Therefore, the co-ordinates of the vertex of

$y = ax^2 + bx + c = ax\left(x + \frac{b}{a}\right) + c$ must be

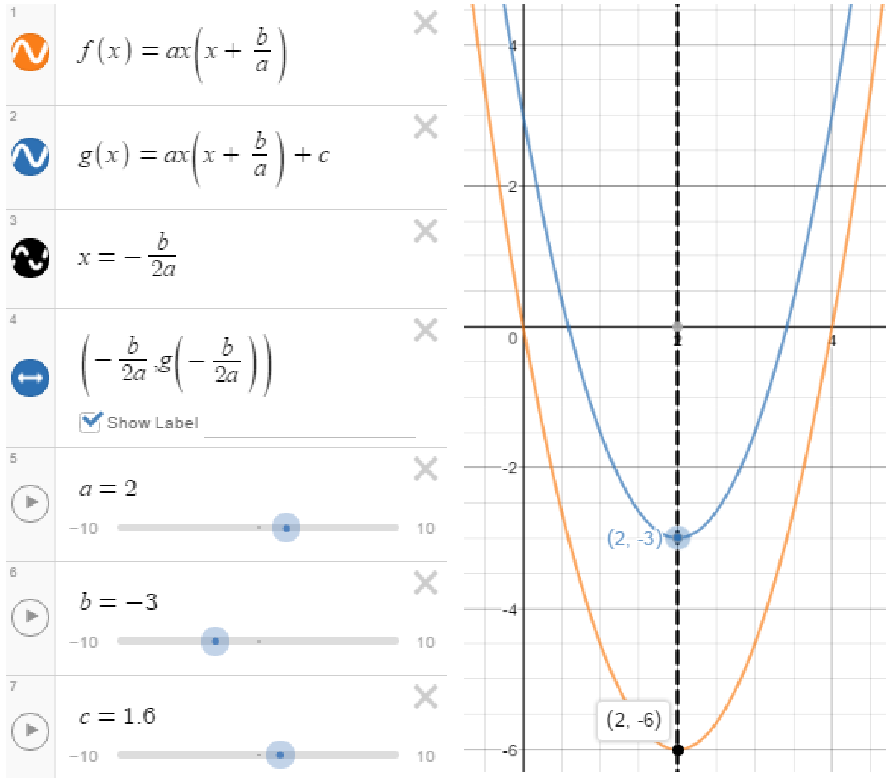
$$\left(-\frac{b}{2a}, a\left(-\frac{b}{2a}\right)\left(-\frac{b}{2a} + \frac{b}{a}\right) + c\right)$$

$$= \left(-\frac{b}{2a}, \left(-\frac{b}{2}\right)\left(-\frac{b}{2a} + \frac{2b}{2a}\right) + c\right)$$

$$= \left(-\frac{b}{2a}, \left(-\frac{b}{2}\right)\left(\frac{b}{2a}\right) + c\right)$$

$$= \left(-\frac{b}{2a}, -\frac{b^2}{4a} + \frac{4ac}{4a}\right)$$

$$= \left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$



See https://www.desmos.com/calculator/t7wgwt9whb to experiment with different values of *a*, *b* and *c*.

Therefore, the vertex form of the equation $y = ax^2 + bx + c$ is

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Now, if $ax^2 + bx + c = 0$ then

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0$$

$$\therefore a\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac - b^2}{4a}$$

$$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

Logic

The study of the ***principles of reasoning***, especially of the ***structure*** of propositions as distinguished from their ***content*** and of ***method***, and ***validity*** in deductive reasoning.

Premise, Conclusion, Logical Implication

- A ***premise*** is a statement that is known or assumed to be true and from which a ***conclusion*** can be drawn.
- A ***conclusion*** is a position, opinion or judgment reached after consideration.
- A ***logical implication*** or ***conditional statement*** is a statement that takes the form "If ***premise*** then ***conclusion***." In such a statement, the truth of the premise ***guarantees*** the truth of the conclusion.

Example

Premise $\rightarrow$ "If he is injured"

Conclusion $\rightarrow$ "then he will not be able to play in tonight's hockey game"

Logical Implication $\rightarrow$ "If he is injured, then he will not be able to play in tonight's hockey game."

Extremely Important Logical Implications in Mathematics

If $x = y$ then $x + a = y + a$, $x - a = y - a$, $ax = ay$, $\frac{x}{a} = \frac{y}{a}$ (if $a \neq 0$), $x^2 = y^2$, $\sqrt{x} = \sqrt{y}$, $\sin x = \sin y$, etc If the same operation is performed to both sides of an equation, then equality is preserved. ***Example*** $2x - 7 = 9$ $\therefore 2x - 7 + 7 = 9 + 7$ $\therefore 2x = 16$ $\therefore \frac{2x}{2} = \frac{16}{2}$ $\therefore x = 8$	If $a = b$ ***and*** $a = c$, then $b = c$. If two quantities are each equal to the same quantity, then they are equal to each other. ***Example*** $h(t) = -4.9t^2 + 49t$ and $h(t) = 0$ $\therefore -4.9t^2 + 49t = 0$	If $xy = 0$, then $x = 0$ or $y = 0$. If the product of two numbers is equal to zero, then at least one of the numbers must be zero. ***Example*** $(x-3)(x+7) = 0$ $\therefore x - 3 = 0$ or $x + 7 = 0$

Logical Fallacy: Ad Hominem Argument – An Example of Studying the Structure of an Argument

Basic Structure of an Ad Hominem Argument	***What is wrong with an ad hominem argument?***
A (fallacious) ad hominem argument has the basic form: Person A makes claim X There is something objectionable about Person A Therefore claim X is false	

Deductive Reasoning

Deductive arguments take the form "If ***Cause*** Then ***Effect***" or "If ***Premise*** Then ***Conclusion***"

In a deductive argument, we know that a ***cause*** (premise) produces a certain ***effect*** (conclusion). If we observe the cause, we can ***deduce*** (conclude by reasoning) that the effect ***must*** occur. These arguments always produce ***definitive*** conclusions; ***general principles*** are applied to reach specific conclusions. We must keep in mind, however, that a false premise can lead to a false result and an inconclusive premise can yield an inconclusive conclusion.

Examples of Deductive Reasoning from Everyday Life

1. If I spill my drink on the floor, the floor will get wet.
2. When students "forget" to do homework, Mr. Nolfi gets angry!
3. If a student is caught cheating, Mr. Nolfi will assign a mark of zero to him/her, ridicule the student publicly, turn red in the face and yell like a raving madman whose underwear are on fire!
4. Drinking too much alcohol causes drunkenness.

Exercise

Rephrase examples 2 and 4 in "If … then" form.

The Meaning of π – An Example of Deductive Reasoning

The following is an example of a typical conversation between Mr. Nolfi and a student who blindly memorizes formulas:

Student: Sir, I can't remember whether the area of a circle is πr^2 or $2\pi r$. Which one is it?

Mr. Nolfi: If you remember the meaning of π, you should be able to figure it out.

Student: How can 3.14 help me make this decision? It's only a number!

Mr. Nolfi: How dare you say something so disrespectful about one of the most revered numbers in the mathematical lexicon! (Just kidding. I wouldn't really say that.) It's true that the number 3.14 is an approximate value of π. But I asked you for its ***meaning***, not its value.

Student: I didn't know that π has a meaning. I thought that it was just a "magic" number.

Mr. Nolfi: Leave magic to the magicians. In mathematics, every term (except for primitive terms) has a very precise definition. Read the following carefully and you'll never need to ask your original question ever again!

In ***any*** circle, the ***ratio*** of the ***circumference*** to the ***diameter*** is equal to a ***constant*** value that we call π. That is,

$$C : d = \pi .$$

Alternatively, this may be written as

$$\frac{C}{d} = \pi$$

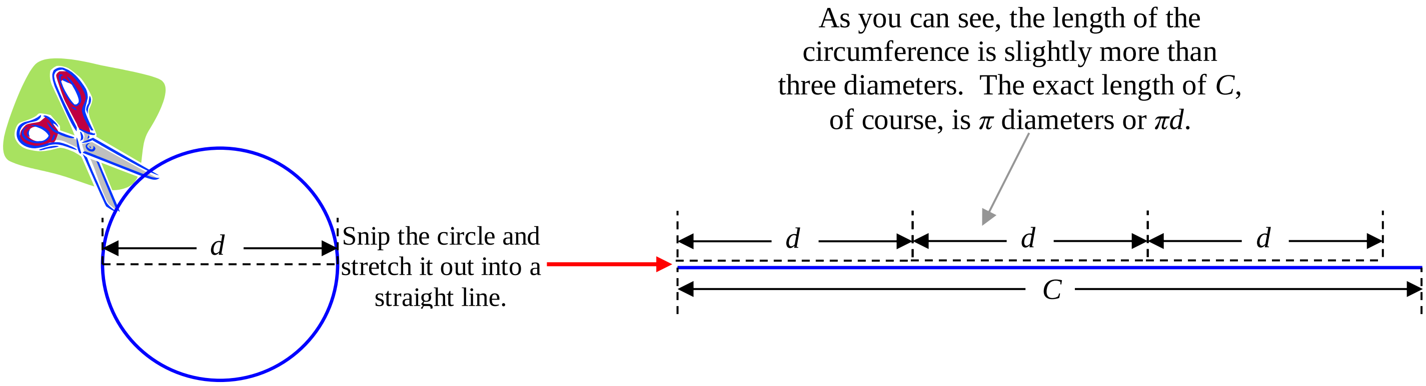
or in the more familiar form

$$C = \pi d .$$

If we recall that $d = 2r$, then we finally arrive at the most common form of this ***relationship***,

$$C = 2\pi r .$$

> This is an example of a ***deductive argument***. Each statement ***follows logically*** from the previous statement.
>
> That is, the argument takes the form "If *P* is true then *Q* must also be true" or more concisely, "*P* implies *Q*."



Mr. Nolfi: So you see, by understanding the meaning of π, you can ***deduce*** that $C = 2\pi r$. Therefore, the formula for the area must be $A = \pi r^2$. Furthermore, it is not possible for the expression $2\pi r$ to yield units of area. The number 2π is dimensionless and r is measured in units of distance such as metres. Therefore, the expression $2\pi r$ must result in a value measured in units of distance. On the other hand, the expression πr^2 must give a value measured in units of area because $r^2 = r(r)$, which involves multiplying a value measured in units of distance by itself. Therefore, by considering units alone, we are drawn to the inescapable conclusion that the area of a circle must be πr^2 and ***not*** $2\pi r$!

Examples

$2\pi r \doteq 2(3.14)(3.6\text{ cm}) = 22.608\text{ cm} \rightarrow$ This answer cannot possibly measure area because cm is a unit of distance.

Therefore, πr^2 must be the correct expression for calculating the area of a circle.

$\pi r^2 \doteq 3.14(3.6\text{ cm})^2 = 3.14(3.6\text{ cm})(3.6\text{ cm}) = 40.6944\text{ cm}^2 \rightarrow$ Notice that the unit cm^2 is appropriate for area.

Extremely Important Terminology

Term	***Meaning***	***Example(s)***
Expression	A combination of constants, operators, and variables representing numbers or quantities	**1.** $3(5)^2 - 4(5)(-1) + (-1)^2$ **2.** $3x^2 - 4xy + y^2$
Equation	A mathematical statement asserting that two expressions have the same value	$2(6z-1)(z-1) = -6(z-1)(2z-5) + 3z + 25$
Evaluate	Ascertain the numerical value of an expression	If $x = 5$ and $y = -1$, ***evaluate*** $3x^2 - 4xy + y^2$. $3x^2 - 4xy + y^2$ $= 3(5)^2 - 4(5)(-1) + (-1)^2$ $= 3(25) + 20 + 1$ $= 96$
Simplify	Convert a mathematical expression to a simpler form	***Simplify*** $5(x-1)(x-2) - (5x-7)(x+10)$ $5(x-1)(x-2) - (5x-7)(x+10)$ $= 5(x^2 - 3x + 2) - (5x^2 + 43x - 70)$ $= 5x^2 - 15x + 10 - 5x^2 - 43x + 70$ $= 5x^2 - 5x^2 - 15x - 43x + 10 + 70$ $= -48x + 80$

Term	A mathematical expression that is associated to another through the operation of addition.	The algebraic expression $3x^2 - 4xy + y^2$ can be rewritten as $3x^2 + (-4xy) + y^2$. Therefore, its ***terms*** are $3x^2$, $-4xy$ and y^2.
Factor	***Noun:*** One of two or more numbers or quantities that can be multiplied together to give a particular number or quantity. **e.g.** The factors of 15 are 1, 3, 5 and 15. ***Verb:*** To determine the factors of a number or expression.	***Factor*** $3n^2 - 2n - 5$ $3n^2 - 2n - 5$ $= 3n^2 - 5n + 3n - 5$ $= (3n^2 - 5n) + (3n - 5)$ $= n(3n - 5) + 1(3n - 5)$ $= (3n - 5)(n + 1)$
Solve	Work out the solution to an equation	***Solve*** $3n^2 - 2n - 5 = 0$ $3n^2 - 2n - 5 = 0$ $\therefore 3n^2 - 5n + 3n - 5 = 0$ $\therefore (3n^2 - 5n) + (3n - 5) = 0$ $\therefore n(3n - 5) + 1(3n - 5) = 0$ $\therefore (3n - 5)(n + 1) = 0$ $\therefore 3n - 5 = 0$ or $n + 1 = 0$ $\therefore n = \frac{5}{3}$ or $n = -1$
Relationship or Relation	A property of association shared by ordered pairs of terms or objects	The equation $y = x^2$ expresses a ***relationship*** between the *x*-co-ordinate and the *y*-co-ordinate of any point that lies on the upward opening parabola with vertex (0, 0) and vertical stretch factor 1.