Table of Contents – Properties and Applications of Exponential and Logarithmic Functions

EXPONENTIAL FUNCTIONS – INTRODUCTORY PROBLEM

An amoeba propagates by simple division; each split takes 3 minutes to complete. When such an amoeba is put into a glass container with a nutrient fluid, the container is full of amoebas in one hour. How long would it take for the container to be filled if we start with not one amoeba, but two?

PROPERTIES AND APPLICATIONS OF EXPONENTIAL FUNCTIONS

Exponential Growth and Exponential Decay

Exponential functions are used to model ***very fast growth*** or ***very fast decay***. Specifically, exponential functions model

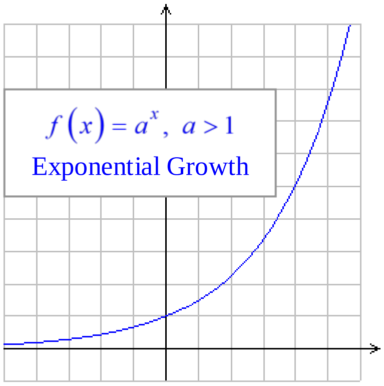
- growth that involves doubling, tripling, etc. at regular intervals

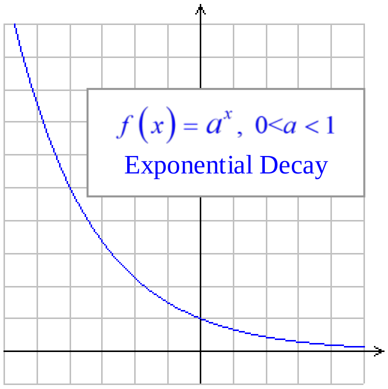
  e.g. Since 1950, the Earth's population has been doubling approximately every 40 years.

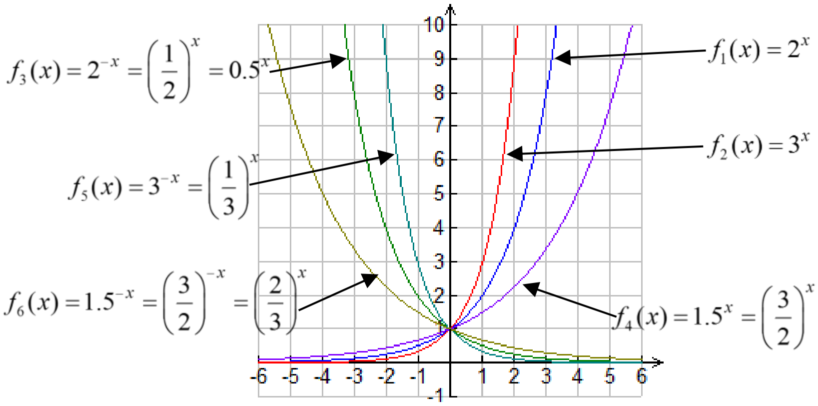
- decay that involves cutting in half, cutting in thirds, etc. at regular intervals

  e.g. Because the nuclei of radioactive isotopes are unstable, they emit particles and energy from time to time as the nuclei tend toward a more stable configuration. Sodium-24 is a radioactive isotope of sodium with a ***half-life*** of 14.9 hours. This means that after 14.9 hours, half the atoms in a sample of sodium-24 will have decayed to magnesium-24. Sodium-24 decays by beta decay to an excited magnesium-24 (^{24}Mg) nucleus. Two gamma rays are rapidly emitted and the excitation energy is carried off, whereby the stable ground state of magnesium-24 is reached.

Graphs of Exponential Functions







x	y1(x) 2^x	y2(x) 3^x	y3(x) 0.5^x	y4(x) 1.5^x	y5(x) 3^(-x)	y6(x) 1.5^(-x)
-5	0.03125	0.004115	32	0.131687	243	7.59375
-4	0.0625	0.012346	16	0.197531	81	5.0625
-3	0.125	0.037037	8	0.296296	27	3.375
-2	0.25	0.111111	4	0.444444	9	2.25
-1	0.5	0.333333	2	0.666667	3	1.5
0	1	1	1	1	1	1
1	2	3	0.5	1.5	0.333333	0.666667
2	4	9	0.25	2.25	0.111111	0.444444
3	8	27	0.125	3.375	0.037037	0.296296
4	16	81	0.0625	5.0625	0.012346	0.197531
5	32	243	0.03125	7.59375	0.004115	0.131687
6	64	729	0.015625	11.3906	0.001372	0.087791
7	128	2187	0.007813	17.0859	0.000457	0.058528
8	256	6.56E+03	0.003906	25.6289	1.52E-04	0.039018
9	512	1.97E+04	0.001953	38.4434	5.10E-05	0.026012
10	1024	5.90E+04	0.000977	57.665	1.70E-05	0.017342

Properties of Exponential Functions

An ***exponential function*** is any function $f(x) = a^x$ such that $a > 0$ and $a \neq 1$.

1. All exponential functions have the following domain and range: Domain = $\mathbb{R}$, Range = $\{y \in \mathbb{R} : y > 0\}$

2. All exponential functions are ***one-to-one***.

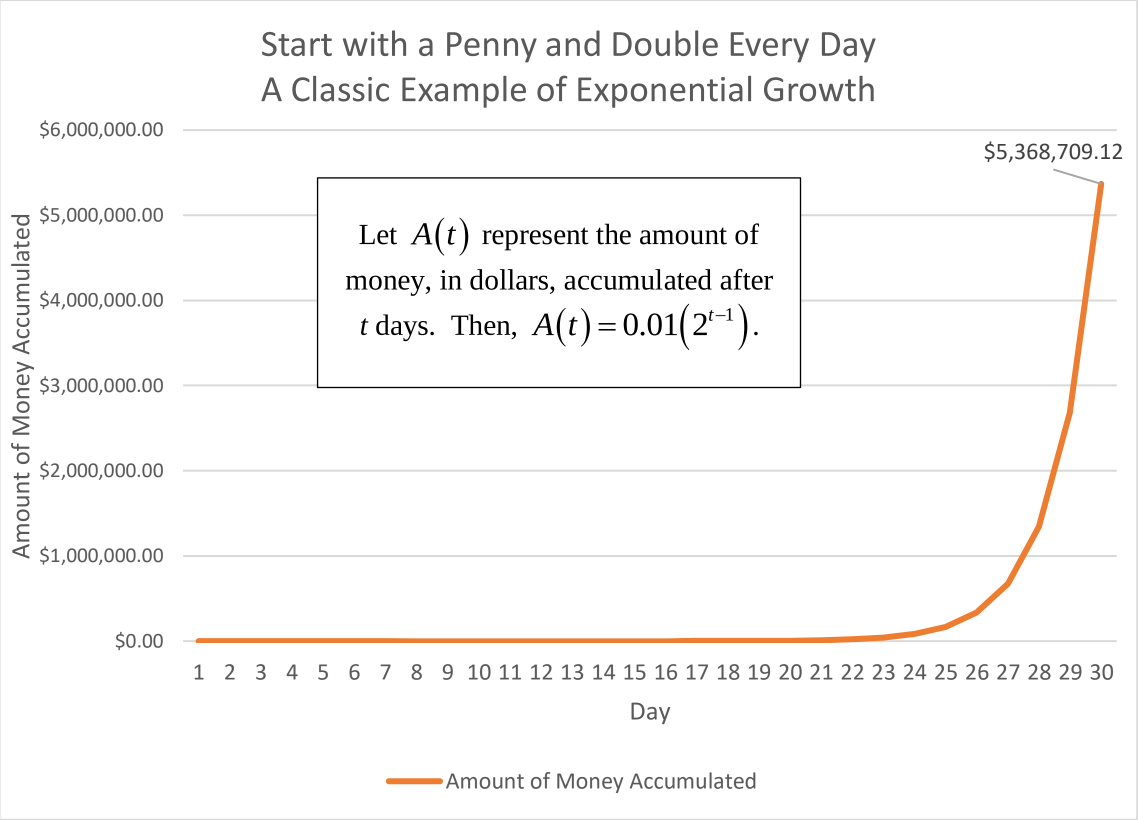
3. All exponential functions with $a > 1$ are ***strictly increasing*** (rise from left to right).
All exponential functions with $0 < a < 1$ are ***strictly decreasing*** (fall from left to right).

4. For all exponential functions $f(x) = a^x$, $f(0) = 1$. This happens because $a^0 = 1$ for all $a \in \mathbb{R}$.

Other ways of stating this: The *y*-intercept of *f* is 1, the graph of *f* passes through $(0,1)$, $(0,1)$ lies on the graph of *f*.

A Classic Example of Exponential Growth – Start with a Penny and Double Every Day!

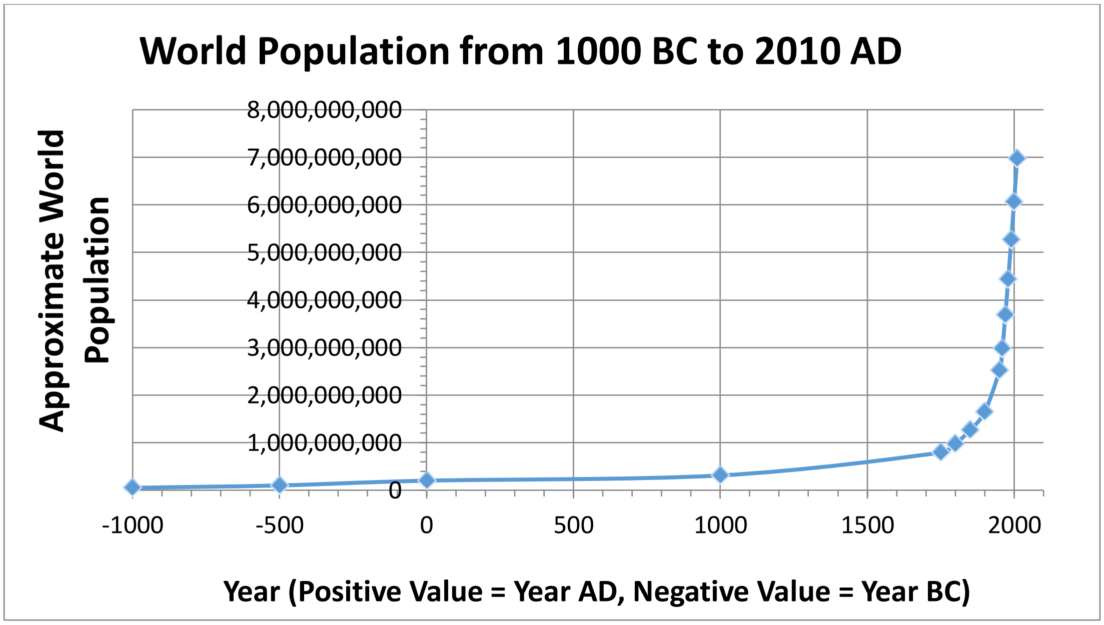
Day	Amount of Money
1	$0.01
2	$0.02
3	$0.04
4	$0.08
5	$0.16
6	$0.32
7	$0.64
8	$1.28
9	$2.56
10	$5.12
11	$10.24
12	$20.48
13	$40.96
14	$81.92
15	$163.84
16	$327.68
17	$655.36
18	$1,310.72
19	$2,621.44
20	$5,242.88
21	$10,485.76
22	$20,971.52
23	$41,943.04
24	$83,886.08
25	$167,772.16
26	$335,544.32
27	$671,088.64
28	$1,342,177.28
29	$2,684,354.56
30	$5,368,709.12



Let $A(t)$ represent the amount of money, in dollars, accumulated after t days. Then, $A(t)=0.01\left(2^{t-1}\right)$.

Another Example of Exponential Growth – Human Population Growth from 1000 BC to 2010 AD

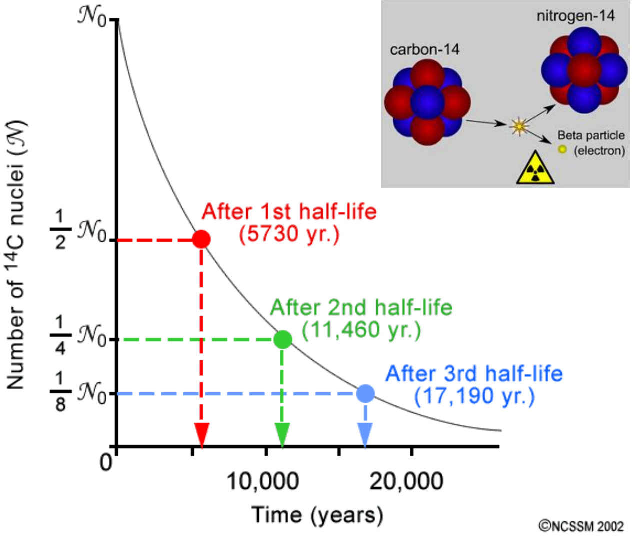
Year	*Population*
−1000	50000000
−500	100000000
1	200000000
1000	310000000
1750	791000000
1800	978000000
1850	1262000000
1900	1650000000
1950	2519000000
1960	2982000000
1970	3692000000
1980	4435000000
1990	5263000000
2000	6070000000
2010	6972000000



Note

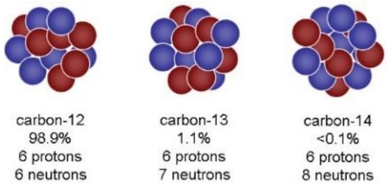
- For graphing convenience, the dividing line between the BC and AD eras is shown as year 0. However, there was no "year 0"in reality. The BC era ended with year 1, which was immediately followed by year 1 in the AD era.
- Some authors refer to the BC ("before Christ") era as BCE ("before the current/common era") and to the AD (*anno domini* or "In the year of the Lord") era as CE ("current/common era").

Example: Exponential Decay of Carbon-14 (The Basis of Radiocarbon Dating)



The nucleus of an atom consists of protons, which are positively charged and neutrons, which are not electrically charged. Due to the nature of the ***electromagnetic force***, like charges normally repel one another. However, on the scale of the nucleus of an atom, the ***strong nuclear force*** overcomes the electromagnetic force, allowing the protons and neutrons (both made of quarks) to be tightly bound together. In spite of the strong nuclear force, not all configurations of protons and neutrons are stable. When the protons and neutrons in the nucleus of an atom are arranged in an unstable configuration, the atom is said to be ***radioactive***. From time to time, radioactive nuclei emit particles and energy as they tend toward more stable configurations.

For example, carbon-14 (^{14}C) is a radioactive isotope of carbon. It is the least common isotope of carbon, accounting for less than 0.1% of all carbon found on the earth (see https://en.wikipedia.org/wiki/Carbon-14 for more detailed information).



Half-Life of a Radioactive Isotope

Carbon-14 has a ***half-life*** of about 5730 years. This means that in a sample of ^{14}C, after 5730 years, about half the atoms in the sample would decay into nitrogen-14 (^{14}N), the stable and most abundant isotope of nitrogen found on earth.

Exponential Model

Let N_0 represent the number of ^{14}C nuclei present at time 0 years and let $N(t)$ represent the number of ^{14}C nuclei remaining after *t* years. Then as shown in the table below, $N(t) = N_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$.

t (years)	***$N(t)$ (# of ^{14}C nuclei remaining after t years)***	***Exponent on $\frac{1}{2}$***
0	N_0	$0 = \frac{0}{5730}$
5730	$N_0\left(\frac{1}{2}\right)$	$1 = \frac{5730}{5730}$
11460	$N_0\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$	$2 = \frac{11460}{5730}$
17190	$N_0\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$	$3 = \frac{17190}{5730}$
⋮	⋮	⋮
t	$N_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$	$\frac{t}{5730}$

$N(t)$ as a Transformation of $f(t) = \left(\frac{1}{2}\right)^t$

Let $f(t) = \left(\frac{1}{2}\right)^t$.

Then $N(t) = N_0 f\left(\frac{t}{5730}\right) = N_0 f\left(\frac{1}{5730}t\right)$.

Therefore, the graph of the function *N* can be obtained by:

Stretching *f* horizontally by a factor of 5730

Stretching *f* vertically by a factor of N_0

That is, *f* can be transformed to *N* by applying the following transformation:

$$(x, y) \to (5730x, N_0 y)$$

In general, let $t_{\frac{1}{2}}$ represent the half-life of a radioactive isotope. Then $N(t) = N_0\left(\frac{1}{2}\right)^{\frac{t}{t_{\frac{1}{2}}}}$.

Example: Modelling Population Growth with Exponential Functions

In 1950, the Earth's population was about 2.5 billion people. Since then, the world's population has been doubling roughly every 40 years. If the current trend continues, ***predict*** the world's population in 2100. Do you think that the current trend will continue?

Solution

Use a Table to help us Understand the Situation

Year	t	Population
1950	0	2.5×10^9
1990	40	$(2.5\times10^9)(2)$
2030	80	$(2.5\times10^9)(2)(2)$
2070	120	$(2.5\times10^9)(2)(2)(2)$
2110	160	$(2.5\times10^9)(2)(2)(2)(2)$
1950 + t	t	$\left(2.5\times10^9\right)\left(2^{\frac{t}{40}}\right)$

Writing an Equation

To ***double*** means to ***multiply by 2***. Furthermore, we must multiply by 2 every 40 years. Thus, if we set $t = 0$ years to correspond to 1950, it is clear that the following exponential function models the given situation:

$$\boxed{P(t)=2.5\times10^9\left(2^{\frac{t}{40}}\right)}$$

The population t years after 1950 is equal to…

…the initial population (2.5 billion)…

…multiplied by two, $\dfrac{t}{40}$ times.

Solving the Problem

The year 2100 is 150 years after 1950. Therefore, $t = 150$.

$$P(150)=2.5\times10^9\left(2^{\frac{150}{40}}\right)$$

$$\doteq 3.4\times10^{10}$$

If the current trend continues, in 2100 the Earth's population will be about 34 billion. $N(t)$

It is difficult to imagine that the current rate of population growth will continue indefinitely. First of all, it is very unlikely that the Earth's food supply can grow at a pace that matches or exceeds the growth of population. To compound matters, as the population size increases, the amount of arable land tends to decrease because extra space is required for residential, commercial and industrial purposes.

Example: Modelling Radioactive Decay with Exponential Functions

Palladium-100 has a half-life of 3.6 days. If one had 6.02×10^{23} atoms at the start, how many atoms would be present after 20 days?

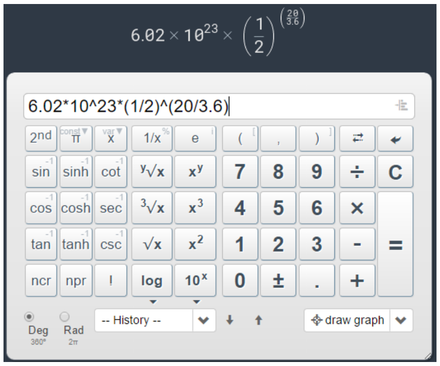
Solution

Time (Days)	Atoms Remaining
0	6.02×10^{23}
3.6	$\left(6.02\times10^{23}\right)\left(\frac{1}{2}\right)$
7.2	$\left(6.02\times10^{23}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$
10.8	$\left(6.02\times10^{23}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$
14.4	$\left(6.02\times10^{23}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$
t	$\left(6.02\times10^{23}\right)\left(\frac{1}{2}\right)^{\frac{t}{3.6}}$

Let $N(t)$ represent the number of palladium atoms remaining after t days. Then as shown in the table,

$$N(t)=\left(6.02\times10^{23}\right)\left(\frac{1}{2}\right)^{\frac{t}{3.6}}$$

After 20 days, $N(20)$ atoms of palladium will remain.



$6.02\times10^{23}\times\left(\frac{1}{2}\right)^{\left(\frac{20}{3.6}\right)}$

6.02*10^23*(1/2)^(20/3.6)

$$N(20)=\left(6.02\times10^{23}\right)\left(\frac{1}{2}\right)^{\frac{20}{3.6}}\doteq 1.3\times10^{22}$$

About 1.3×10^{22} atoms of palladium remain after 20 days.

General Form of an Exponential Function

<table>
<tr><th>Algebraic Form</th><th>Transformations expressed in Words</th><th>Transf. in Mapping Notation</th></tr>
<tr><td>

$f(x)=a^x$

$g(x)=Af(b(x-h))+k$

$=Aa^{b(x-h)}+k$

Note

Since *a* is being used to denote the base of the exponential function, *A* is used to denote the vertical stretch factor.

</td><td>

Horizontal

1. Stretch/compress by a factor of $1/b=b^{-1}$ depending on whether $0<b<1$ or $b>1$. If *b* is negative, there is also a reflection in the *y*-axis.
2. Shift *h* units right if $h>0$ or *h* units left if $h<0$.

Vertical

1. Stretch/compress by a factor of *A* depending on whether $A>1$ or $0<A<1$. If *A* is negative, there is also a reflection in the *x*-axis.
2. Shift *k* units up/down depending on whether *k* is positive or negative.

</td><td>

$(x,y)\to\left(\frac{1}{b}x+h, Ay+k\right)$

</td></tr>
<tr><td>Example</td><td></td><td></td></tr>
<tr><td>

$f(x)=2^x$

$g(x)=-5f(-1.5(x+1))+6$

$=-5\left(2^{-1.5(x+1)}\right)+6$

To obtain the graph of *g* from the graph of *f*, do the following:

Horizontal

1. Compress horizontally by a factor of 1/1.5=2/3, reflect in the *y*-axis.
2. Translate 1 unit to the left.

Vertical

1. Stretch vertically by a factor of 5, reflect in the *x*-axis.
2. Translate 6 units up.

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<table>
<tr><th>Pre-image</th><th>Image</th></tr>
<tr><td>$(0,1)$</td><td>$(-1,1)$</td></tr>
<tr><td>$(1,2)$</td><td>$\left(-\frac{5}{3},-4\right)$</td></tr>
<tr><td>$\left(-1,\frac{1}{2}\right)$</td><td>$\left(-\frac{1}{3},\frac{7}{2}\right)$</td></tr>
<tr><td>$\left(-2,\frac{1}{4}\right)$</td><td>$\left(\frac{1}{3},\frac{19}{4}\right)$</td></tr>
<tr><td>$\left(-3,\frac{1}{8}\right)$</td><td>$\left(1,\frac{43}{8}\right)$</td></tr>
<tr><td>$\left(-4,\frac{1}{16}\right)$</td><td>$\left(\frac{5}{3},\frac{91}{16}\right)$</td></tr>
</table>

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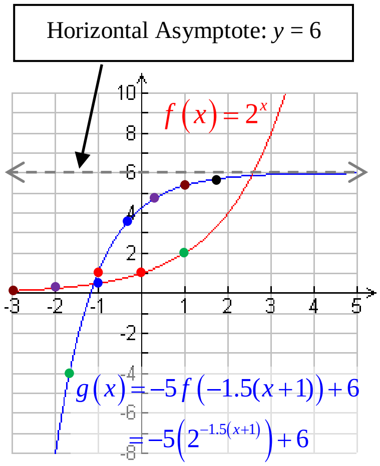
$(x,y)\to\left(-\frac{2}{3}x-1,-5y+6\right)$



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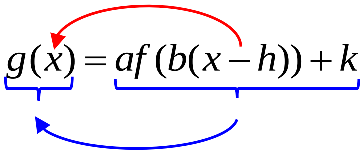
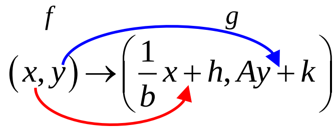
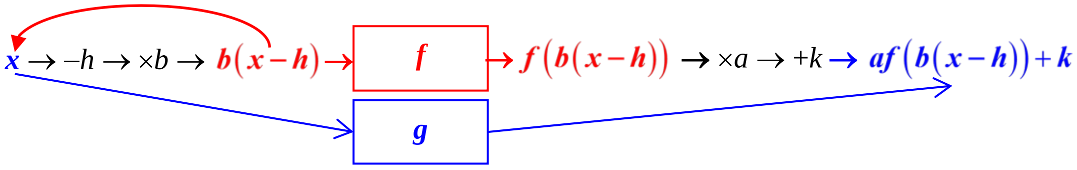
</table>

Why are the Horizontal Transformations the Opposite of the Operations found in the Input to f?







- The input to *f* is $b(x-h)$. Recall that $f(b(x-h))$ means "the *y*-value obtained when $b(x-h)$ is the input given to *f*."
- The input to the function *g*, however, is *x*, not $b(x-h)$.
- Therefore, in ***transforming from f to g***, $\boldsymbol{b(x-h)}$ ***must be transformed to x***.

  This is accomplished by first multiplying $b(x-h)$ by $\frac{1}{b}$ (or equivalently dividing by *b*), then adding *h*.

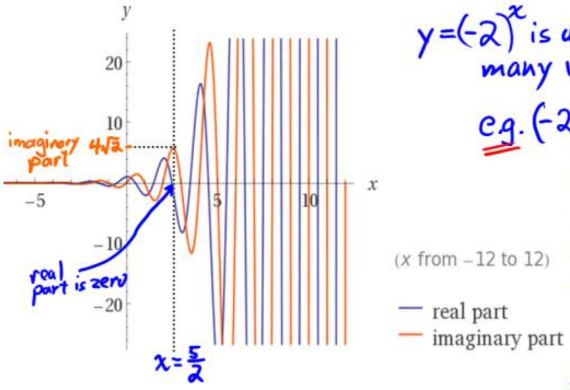
Very Important Restriction on Bases of Exponential Functions

1. ***Negative bases are not allowed!*** When the bases are allowed to be negative, the resulting functions are “badly behaved” in the sense that they are not continuous and smooth. For instance, the table of values below (for the function $f(x)=(-2)^x$) illustrates some of the problems associated with allowing negative bases.

Recall that $x^{\frac{m}{n}}=\left(x^m\right)^{\frac{1}{n}}=\sqrt[n]{x^m}$ and that $x^{\frac{m}{n}}=\left(x^{\frac{1}{n}}\right)^m=\left(\sqrt[n]{x}\right)^m$. Therefore, $\boxed{x^{\frac{m}{n}}=\sqrt[n]{x^m}=\left(\sqrt[n]{x}\right)^m}$.

x	-2	-1	$-1/2$	0	$1/4$	$1/2$	1	$3/2$	2	$5/2$
$f(x)=(-2)^x$	$1/4$	$-1/2$	$1/\sqrt{-2}$ (undefined)	1	$\sqrt[4]{-2}$ (undefined)	$\sqrt{-2}$ (undefined)	-2	$\left(\sqrt{-2}\right)^3$ (undefined)	4	$\left(\sqrt{-2}\right)^5$ (undefined)

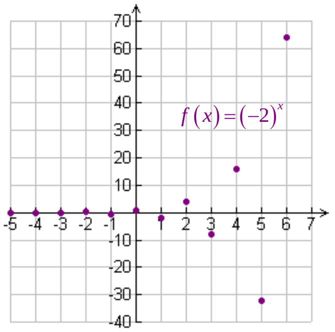
- $f(x)=(-2)^x$ is ***undefined*** at an infinite number of points
- $f(x)=(-2)^x$ “***jumps***” across the *x*-axis, between positive to negative values at an infinite number of points
- Functions like $f(x)=(-2)^x$ behave very erratically. They do not model natural processes such as radioactive decay or population growth.

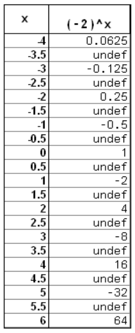


$y=(-2)^x$ is undefined for infinitely many values of $x\in\mathbb{R}$

$\underline{\text{e.g.}}$ $(-2)^{\frac{5}{2}}=\left(\sqrt{-2}\right)^5$
$=\left(\sqrt{(2)(-1)}\right)^5$
$=\left[\left(\sqrt{2}\right)\left(\sqrt{-1}\right)\right]^5$
$=\left(\sqrt{2}\right)^5 i^5$, where $i=\sqrt{-1}$
$=\sqrt{2^5}\, i$, since $i^2=-1$
$=\sqrt{32}\, i$ $\quad\therefore i^4=1$
$\quad\therefore i^5=i$
$=4\sqrt{2}\, i$
$=\underset{\text{real part}}{0}+\underbrace{4\sqrt{2}}_{\text{imaginary part}}i$

<u>The Function $f(x)=(-2)^x$ Plotted for Integer Values of *x* Between –5 and 6.</u>





x	(-2)^x
-4	0.0625
-3.5	undef
-3	-0.125
-2.5	undef
-2	0.25
-1.5	undef
-1	-0.5
-0.5	undef
0	1
0.5	undef
1	-2
1.5	undef
2	4
2.5	undef
3	-8
3.5	undef
4	16
4.5	undef
5	-32
5.5	undef
6	64

2. ***The bases 0 and 1 are not allowed***.

- When *a* = 1, $f(x)=1^x=1$ for all $x\in\mathbb{R}$. That is, the functions *f* is nothing more than a ***constant function***, that is, a linear function with a slope of zero.
- When *a* = 0, $f(x)=0^x=\begin{cases}0, \text{ if } x\neq 0\\ 1, \text{ if } x=0\end{cases}$. This is nothing more than a ***constant function*** with a strange “jump” at *x* = 0.
- means that their graphs are horizontal straight lines. To make matters even worse, their inverses are not even functions (graphs are vertical lines).

Summary

Functions that we call ***exponential must be*** of the form $f(x)=Aa^{b(x-h)}+k$, where $a>0$ and $a\neq 1$. If $a\leq 0$ or $a=1$, the resulting functions do not exhibit the behaviour of natural processes such as population growth and are ***not called*** exponential. (In addition, *A*, *b*, *k* and *h* must all be real numbers but $A\neq 0$ and $b\neq 0$.)

Furthermore, if an exponential function *A* is used as a ***mathematical model*** for a process that depends on time, then the ***amount or number present at time t*** is given by $A(t)=A_0\left(a^{\frac{t}{\tau}}\right)$,where A_0 represents the ***initial amount or number***, the base of the exponential function *a* corresponds to the ***growth/decay factor*** (**e.g.** if the amount doubles at regular intervals then *a* = 2), and *τ* represents how long it takes for the amount to increase/decrease by a factor of *a*.

Important Differences between Power Functions and Exponential Functions

Power Functions

$$f(x) = x^n, n \in \mathbb{Z}$$

e.g. $f(x) = x^2$

- The ***base*** is variable and the ***exponent*** is constant.
- Power functions grow more slowly than exponential functions.

x	$f(x) = x^2$
0	0
1	1
2	4
3	9
4	16
5	25
6	36

Exponential Functions

$$g(x) = a^x, a \in \mathbb{R}, a > 0, a \neq 1$$

e.g. $g(x) = 2^x$

- The ***base*** is constant and the ***exponent*** is variable.
- Exponential functions grow more quickly than power functions.

x	$g(x) = 2^x$
0	1
1	2
2	4
3	8
4	16
5	32
6	64

Homework 1

Precalculus (Ron Larson)
Do pp. 208 – 210: #7-12, 13-16, 65, 69-73, 75-78, 79

Problems 1

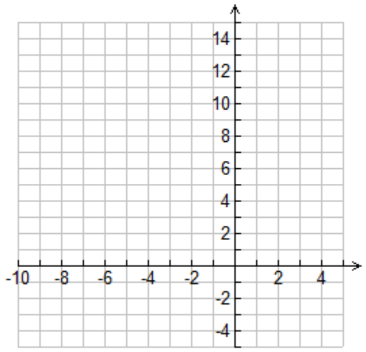
1. Given $f(x) = \left(\frac{1}{2}\right)^x$, sketch the graph of $g(x) = 3f(-2(x+1)) - 4$.

Equation of g

Transformations of f Expressed in Words

Transformations in Mapping Notation

Pre-image	Image



2. Lemmings are small rodents usually found in or near the Arctic. Contrary to popular belief, lemmings ***do not commit mass suicide*** when they migrate. Driven by strong biological urges, they will migrate in large groups when population density becomes too great. During such migrations, lemmings may choose to swim across bodies of water in search of a new habitat. Many lemmings drown during such treks, which may in part explain the myth of mass suicide.

What is true about lemmings is that they reproduce at a very fast rate, causing populations to increase dramatically over a very short time. Possibly because of limited resources and the life cycles of their predators, lemming populations tend to plummet every four years. These periodic "boom-and-bust" cycles may also contribute to the mass suicide myth.

Using the data in the table at the right, model the lemming population for a four-year cycle.

Time (Years)	Population Per Hectare
0	5
0.5	7.2
1	10.4
1.5	15
2	21.6
2.5	31.2
3	45
3.5	64.9
4	93.6

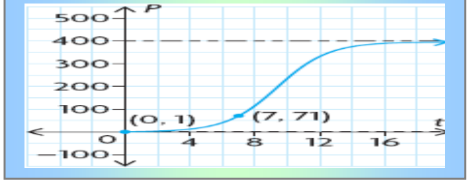
3. Since exponential growth is so fast, it usually cannot be sustained for very long. The rate of growth of any system is constrained by the availability of resources. Once the growth rate outstrips the rate of growth of resources, the system's growth is necessarily curbed. In such cases, a ***logistic function*** is likely a better mathematical model than an exponential function.

A town has a population of 5000 people. During a March Break trip, one of the town's residents contracted a virus. Ten days after her return to the town, 91 ***additional*** people had become infected with the same virus. Detailed scientific studies of the transmission of this virus determined that within three weeks, it infects approximately 10% of a given population.

(a) Try to construct both an ***exponential model*** and a ***logistic model*** for the above. Is it even possible to construct a good exponential model?

(b) According to each model, how long would it take to reach the maximum rate of infection? Which model describes the given situation more realistically? Explain.

The general equation of a ***logistic function*** is $f(x)=\dfrac{c}{1+ab^x}$, where c represents the ***carrying capacity*** (upper limit) of the function. The following is an example of a graph of a logistic function.



4. From Wikipedia: Carbon-14 or radiocarbon, is a radioactive isotope of carbon discovered on February 27, 1940, by Martin Kamen and Samuel Ruben at the University of California Radiation Laboratory in Berkeley, though its existence had been suggested already in 1934 by Franz Kurie. Radiocarbon dating is a radiometric dating method that uses carbon-14 to determine the age of carbonaceous materials up to about 60,000 years old. The technique was developed by Willard Libby and his colleagues in 1949 during his tenure as a professor at the University of Chicago. Libby estimated that the radioactivity of exchangeable carbon-14 would be about 14 disintegrations per minute (dpm) per gram. In 1960, he was awarded the Nobel Prize in chemistry for this work. One of the frequent uses of the technique is to date organic remains from archaeological sites.

Problem: A skull found at an archaeological excavation site contains 9% of the amount of ^{14}C present in a modern skull. Given that the half life of ^{14}C is 5730 years, how old is the skull?

5. Victoria invests $5000.00 at a rate of 2.4% per annum (per year) ***compounded monthly***.

(a) What does it ***mean*** for interest to be compounded?
(b) What is the ***meaning*** of the factor 1 + *i* in the equation shown below?
(c) How much will Victoria's investment be worth after 5 years?
(d) How long will it take for Victoria's investment to double in value?

Background Information that will Help you with this Problem

n = number of times interest is compounded per year = 12 (since interest is compounded monthly)

Annual Rate = *r* = 0.024 Periodic Rate (Monthly in this Problem) = $i=\dfrac{r}{n}=\dfrac{0.024}{12}=0.002$

t (months)	$V(t)$ Value ($) of investment after t months
0	5000
1	$5000(1.002)=5000(1.002)^1$
2	$5000(1.002)(1.002)=5000(1.002)^2$
3	$5000(1.002)(1.002)(1.002)=5000(1.002)^3$
4	$5000(1.002)(1.002)(1.002)(1.002)=5000(1.002)^4$
. . .	. . .
t	$5000(1.002)^t$

In general, $\boxed{V(t)=P(1+i)^t}$

- P represents the ***principal*** (the original amount invested). This is also called the ***present value***.
- t represents the ***time*** measured in units corresponding to the ***length of one compounding period***
- $V(t)$ represents the value of the investment after t units of time have elapsed. This is the ***future value***.
- i represents the ***periodic interest rate*** (the interest rate per compounding period)
- The periodic rate i is calculated by dividing the annual rate r by the number of compounding periods per year. That is, $i=\dfrac{r}{n}$.

7. State the transformations of $f(x)=3^x$ that you could use to sketch the graph of each of the following functions. Use Desmos or some other graphing tool to confirm that your transformations are correct.

(a) $y=3^{-x}$

(b) $y=-3^{-x}$

(c) $y=-2\left(3^{-x}\right)-1$

(d) $y=-3\left(3^{-(x+4)}\right)$

(e) $y=4\left(3^{2(1-x)}\right)-6$

(f) $y=4\left(3^{\frac{1}{2}x-1}\right)-6$

Answers

1. $f(x)=3\left(\frac{1}{2}\right)^{-2(x+1)}-4$,

$(x,y)\rightarrow\left(-\frac{1}{2}x-1,3y-4\right)$

2. $P(t)=5\left(\frac{10.4}{5}\right)^t=5(2.08)^t$

3. $N(t)=\frac{500}{1+499\left(0.625^t\right)}$ answers may vary slightly

It's very difficult to create an exponential model that closely fits the data. A logistic model is more realistic.

4. Solve $0.09N_0=N_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$ for *t*.

Answer: $t \doteq 19900$ years (solved graphically)

5. (a) See https://en.wikipedia.org/wiki/Compound_interest
(b) 1 + *i* is 100% plus the periodic rate.
(c) $5636.81 (d) About 28.9 years
Also Google "Compound Interest Calculator"

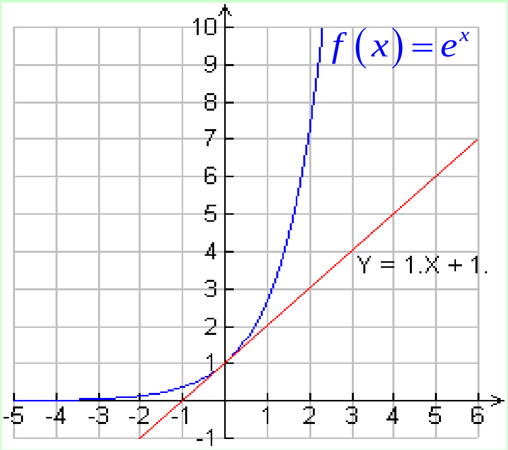
6. (a) $(x,y)\rightarrow(-x,y)$ (b) $(x,y)\rightarrow(-x,-y)$
(c) $(x,y)\rightarrow(-x,-2y-1)$ (d) $(x,y)\rightarrow(-x-4,-3y)$
(e) $(x,y)\rightarrow\left(-\frac{1}{2}x+1,4y-6\right)$ (f) $(x,y)\rightarrow(2(x+1),4y-6)$

Homework 2

Objective: Investigate the ***natural exponential function*** $f(x)=e^x$.

Background Information

The number *e*, whose approximate value is 2.718, is often referred to as "Euler's number" and sometimes also called "Napier's constant." It was discovered by the Swiss mathematician Jacob Bernoulli while he was studying compound interest.



Questions

1. How is the number *e* related to rates of change of exponential functions? The graph at the right gives you a small hint.
2. What is the relationship between *e* and compound interest?
3. What is the relationship between *e* and probability theory?
4. The number *e* is the unique base of the exponential function for which the inequality $a^x \geq x+1$ holds for all *x*. Explain what this means and suggest why this should be the case.

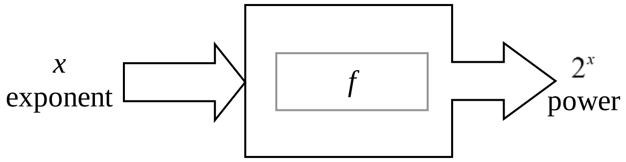
Questions from the Textbook

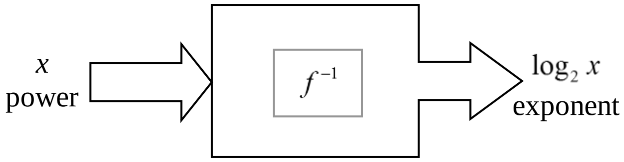
Precalculus (Ron Larson)
Do pp. 208 – 210: #35-50, 55-62, 63, 66-68, 81-82

INTRODUCTION TO LOGARITHMIC FUNCTIONS

Introduction

As shown in the following examples, ***logarithmic*** and ***exponential*** functions to the same base are ***inverses*** of each other.





x (exponent)	$f(x)=2^x$ (power)
–2	$\frac{1}{4}$
–1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

x (power)	$f^{-1}(x)=\log_2 x$ (exponent)
$\frac{1}{4}$	–2
$\frac{1}{2}$	–1
1	0
2	1
4	2
8	3

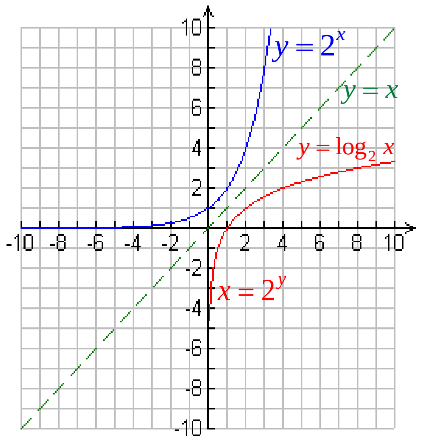
- 2^x is read “2 to the exponent *x*” or “the *x*th power of 2”
- $2^x \to 2$ is the ***base***, *x* is the ***exponent***, 2^x is the ***power***
- Sometimes, the word ***power*** is used as if it were synonymous with ***exponent***. This is not strictly correct. However, this mistake is so common that we are forced to accept that statements such as “2 to the power *x*” mean the same thing as “2 to the exponent *x*.”

- $\log_2 x$ is read “the logarithm of *x* to the base 2”
- If $y=\log_2 x$, then *y* is the ***exponent*** to which the ***base*** 2 must be raised to obtain the ***power*** *x*. Therefore, a logarithm is an ***exponent***.
- The functions $f(x)=2^x$ and $f^{-1}(x)=\log_2 x$ ***contain exactly the same information***. However, exponential functions grow very rapidly and can be hard to manage. Since logarithmic functions grow very slowly, they are often much easier to work with.

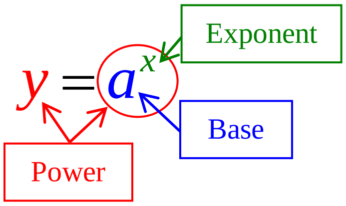
Definition of a Logarithmic Function

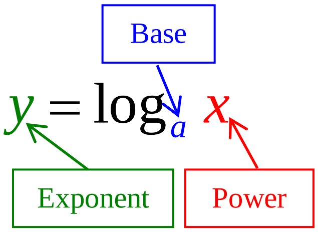
Consider the exponential function $y=a^x$, where $a>0$ and $a\neq 1$.

Since $y=a^x$ is one-to-one, its inverse (reflection in the line $y=x$) is also a function. We could write the equation of the inverse in the form $x=a^y$; however, it is preferable to write equations of functions in such a way that the ***value of the dependent variable*** is given by some expression that is specified ***only in terms of the independent variable***.



> ***Definition***
>
> Thus, we ***define*** $g(x)=\log_a x$ to be the ***inverse*** of $f(x)=a^x$.
>
> That is, $g(x)=\log_a x=f^{-1}(x)$. (Recall that the base *a* of an exponential function is a constant and that $a>0$ and $a\neq 1$.)





In words, if $y=\log_a x$, then *y* is the ***exponent*** to which *a* must be raised to obtain the power *x*.

Examples

1. Evaluate each logarithm.

(a) $\log_2 32 = 5$ because $2^5 = 32$ **(b)** $\log_2 \frac{1}{8} = -3$ because $2^{-3} = \frac{1}{8}$ **(c)** $\log_{\frac{1}{5}} 25 = -2$ because $\left(\frac{1}{5}\right)^{-2} = 25$

(d) $\log_{\sqrt{7}} 49 = 4$ because $\left(\sqrt{7}\right)^4 = 49$ **(e)** $\log_{10} 0.0001 = -4$ because $10^{-4} = 0.0001$

2. Express each exponential equation in logarithmic form.

(a) $2^y = 100$ (exponential form)
$\log_2 100 = y$ (logarithmic form)

(b) $a^y = x$ (exponential form)
$\log_a x = y$ (logarithmic form)

(c) $10^3 = 1000$ (exponential form)
$\log_{10} 1000 = 3$ (logarithmic form)

Important Note on Calculator Use

Scientific and graphing calculators usually have two buttons for computing logarithms, "log" and "ln." The "log" button computes the logarithm of a number to ***the base 10*** while the "ln" button evaluates the logarithm of a number ***to the base e***. (That is, $\log = \log_{10}$ and $\ln = \log_e$. See details below.) The function "ln" is called the ***natural logarithmic function*** (*logarithme naturel* in French, hence "ln" and not "nl") and is pounced "lawn."

log → This button computes the logarithm of a number ***to the base 10***.
The function $f(x) = \log_{10} x$ is sometimes called the ***common logarithmic function***.

ln → This button computes the logarithm of a number ***to the base e***.
The function $f(x) = \ln x = \log_e x$ is usually called the ***natural logarithmic function***.

Important Note on Notation

In most cases, whenever the base is omitted, it is understood to be 10. That is, it is usually the case that $\log = \log_{10}$. In advanced mathematics, however, "log" is used to mean "$\log_e$" because the base 10 has no special significance in mathematical theory.

Homework

Precalculus (Ron Larson)
Do pp. 218 – 220: #7-24, 49-64

LOGARITHMS–MECHANICAL EXERCISES

Name: Period: Date:

Practice Worksheet: Evaluating Logarithms

Rewrite the equation in exponential form.

1] $\log_7 49 = 2$

2] $\log_5 125 = 3$

3] $\log_4 \frac{1}{4} = -1$

4] $\log_2 16 = 4$

5] $\log_{16} 4 = \frac{1}{2}$

6] $\log_3 \frac{1}{9} = -2$

Rewrite the equation in logarithmic form.

7] $13^2 = 169$

8] $9^{3/2} = 27$

9] $4^{-3} = \frac{1}{64}$

10] $10^{-3} = 0.001$

11] $64^{\frac{1}{2}} = 8$

12] $9^{-2} = \frac{1}{81}$

13] $12^2 = 144$

14] $\left(\frac{1}{12}\right)^2 = \frac{1}{144}$

Evaluate the logarithm without using a calculator. Show work to support your answer.

15] $\log_9 81 =$	16] $\log_{27} 3 =$	17] $\log_4 32 =$
18] $\log_8 1 =$	19] $\ln e^4 =$	20] $\log_8 4 =$
21] $\log_3 \frac{1}{3} =$	22] $\log 1000 =$	23] $\log_{\frac{1}{2}} 128 =$
24] $\log_4 2 =$	25] $\log_{25} 125 =$	26] $\log_3 \frac{1}{243} =$
27$\log_4 64 =$	28] $\log_{64} 4 =$	29] $\log_6 \frac{1}{216} =$

Circle the points which are on the graph of the given logarithmic functions. Show your work.

30] $y = 2\log_3(x-4)+5$ (5, 3) (7, 7) (13, 9)

31] $y = -\log_{\frac{1}{2}}(2x)-1$ (4, 2) (8, 3) (16, 5)

32] $y = \log_2 2(x+1)-4$ (0, 3) (3, 1) (15, 1)

GRAPHS AND TRANSFORMATIONS OF LOGARITHMIC FUNCTIONS

General Form of a Logarithmic Function

Algebraic Form

$$f(x) = \log_a x$$

$$g(x) = Af\left(b(x-h)\right)+k$$

$$= A\log_a\left(b(x-h)\right)+k$$

Note

Since a is being used to denote the base of the logarithmic function, A is used to denote the vertical stretch/compression factor.

Transformations expressed in Words

Horizontal

1. Stretch/compress by a factor of $1/b = b^{-1}$ depending on whether $0<b<1$ or $b>1$. If b is negative, there is also a reflection in the y-axis.
2. Shift h units right if $h>0$ or h units left if $h<0$.

Vertical

1. Stretch/compress by a factor of A depending on whether $A>1$ or $0<A<1$. If A is negative, there is also a reflection in the x-axis.
2. Shift k units up/down depending on whether k is positive or negative.

Transf. in Mapping Notation

$$(x,y)\to\left(\frac{1}{b}x+h, Ay+k\right)$$

Example

$$f(x) = \log_2 x$$

$$g(x) = -\frac{1}{2}f\left(\tfrac{1}{4}(x-1)\right)+3$$

$$= -\frac{1}{2}\log_2\left(\tfrac{1}{4}(x-1)\right)+3$$

To obtain the graph of g from the graph of f, do the following:

Horizontal

1. ***Stretch*** horizontally by a factor of $1/\left(\tfrac{1}{4}\right)=4$.
2. Translate 1 unit to the right.

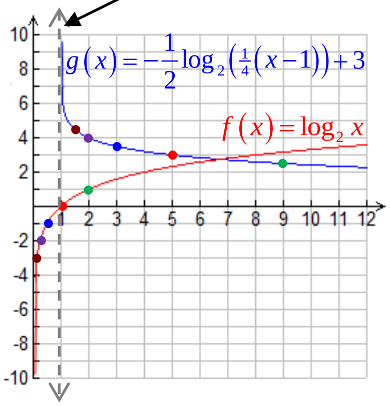
Vertical

1. ***Compress*** vertically by a factor of 1/2, reflect in the x-axis.
2. Translate 3 units up.

$$(x,y)\to\left(4x+1, -\frac{1}{2}y+3\right)$$

Pre-image	Image
$(1,0)$	$(5,3)$
$(2,1)$	$\left(9,\frac{5}{2}\right)$
$\left(\frac{1}{2},-1\right)$	$\left(3,\frac{7}{2}\right)$
$\left(\frac{1}{4},-2\right)$	$(2,4)$
$\left(\frac{1}{8},-3\right)$	$\left(\frac{3}{2},\frac{9}{2}\right)$

Vertical Asymptote
$x = 1$



Exercise

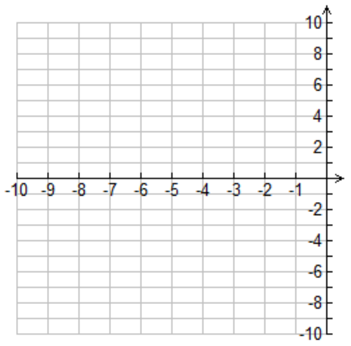
Given $f(x)=\log_2 x$, sketch the graph of $g(x)=3f\left(-2(x+1)\right)-4$.

Equation of g

Transformations of f expressed in Words

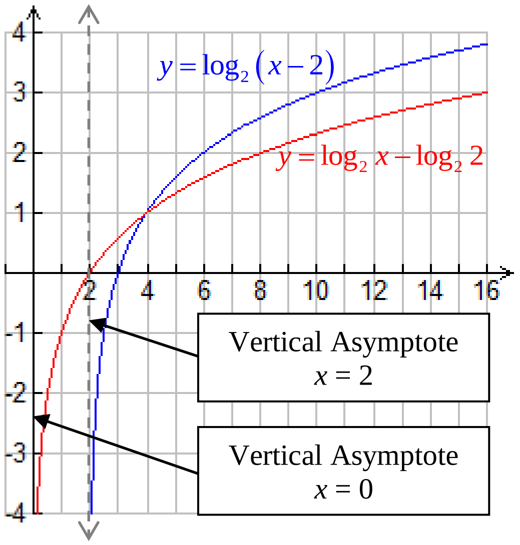
Transformations in Mapping Notation

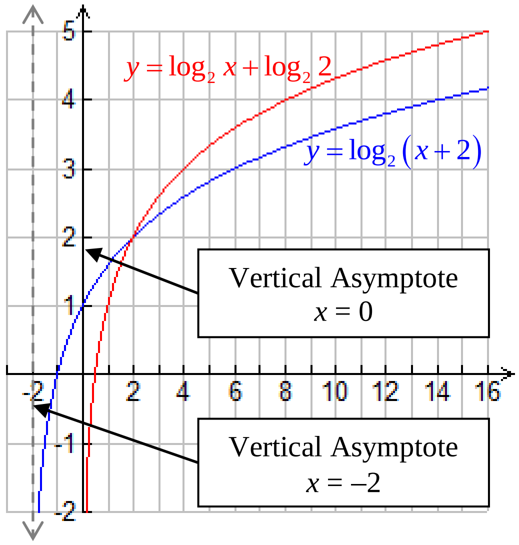
Pre-image	Image



Using Transformations to avoid Careless Mistakes

Law of Logarithms Each of these equations is an identity	***Common Errors*** Each of these equations is ***not*** an identity.	***Explanation using Counterexample***	***Graphical Explanation***
$\log_a(xy)=\log_a x+\log_a y$	$\xcancel{\log_a(x+y)=\log_a x+\log_a y}$	Let $x = y = 1$, $a = 2$. $\text{L.S.}=\log_2(1+1)$ $=\log_2 2$ $=1$ $\text{R.S.}=\log_2 1+\log_2 1$ $=0+0$ $=0$ $\therefore \text{L.S.} \neq \text{R.S.}$	**e.g.** $y=\log_2(x+2)$ The graph of $y=\log_2 x$ is shifted two units to the left. **e.g.** $y=\log_2 x+\log_2 2$ $=\log_2 x+1$ The graph of $y=\log_2 x$ is shifted one up. $\therefore \log_2(x+2)\neq\log_2 x+\log_2 2$
$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$	$\xcancel{\log_a(x-y)=\log_a x-\log_a y}$	Let $x = 16$, $y = 8$, $a = 2$. $\text{L.S.}=\log_2(16-8)$ $=\log_2 8$ $=3$ $\text{R.S.}=\log_2 16-\log_2 8$ $=4-3$ $=1$ $\therefore \text{L.S.} \neq \text{R.S.}$	$y=\log_2(x-2)$ The graph of $y=\log_2 x$ is shifted two units to the right. $y=\log_2 x-\log_2 2$ $=\log_2 x-1$ The graph of $y=\log_2 x$ is shifted one down. $\therefore \log_2(x-2)\neq\log_2 x-\log_2 2$





As can be seen from the graphs, $\log_2(x-2)$ and $\log_2 x-\log_2 2$ ***are not equivalent*** expressions. In addition, $\log_2(x+2)$ and $\log_2 x+\log_2 2$ ***are not equivalent*** expressions.

Homework

Precalculus (Ron Larson)
Do pp. 218 – 220: #37-48, 65-72, 82, 85-89

PROBLEMS: REVIEW AND EXTEND

1. Shown below are the laws of exponents that you learned in grade 9 as well as corresponding laws of logarithms.

Exponent Law	$a^xa^y=a^{x+y}$	$\dfrac{a^x}{a^y}=a^{x-y}$	$\left(a^x\right)^y=a^{xy}$	$\left(ab\right)^x=a^xb^x$
Exponent Law in Words	The product of two powers with the same base is equal to the base raised to the sum of the exponents on each power.	The quotient of two powers with the same base is equal to the base raised to the difference of the exponents on each power.	A power raised to an exponent is equal to the base of the power raised to the product of the exponents.	A product raised to an exponent is equal to the product of each factor raised to the same exponent.
Example that Makes Reasoning Behind Law Very Clear	2^32^4 $=\left[2(2)(2)\right]\left[2(2)(2)(2)\right]$ $=2^7$	$\dfrac{2^5}{2^3}$ $=\dfrac{2(2)(2)(2)(2)}{2(2)(2)}$ $=2^2$	$\left(3^2\right)^4$ $=\left(3^2\right)\left(3^2\right)\left(3^2\right)\left(3^2\right)$ $=3^8$	$\left(3^22^3\right)^4$ $=\left(3^22^3\right)\left(3^22^3\right)\left(3^22^3\right)\left(3^22^3\right)$ $=\left(3^23^23^23^2\right)\left(2^32^32^32^3\right)$ $=3^82^{12}$
Corresponding Law of Logarithms	$\log_a xy=\log_a x+\log_a y$	$\log_a \dfrac{x}{y}=\log_a x-\log_a y$	$\log_a x^y=y\log_a x$	N/A
Law of Logarithms in Words	The exponent on the product of two powers with the same base is equal to the sum of the exponents on each power.	The exponent on the quotient of two powers with the same base is equal to the difference of the exponents on each power.	The exponent on a power raised to an exponent is equal to the product of the exponent and the exponent on the power.	N/A

Example

Prove that $\log_a xy=\log_a x+\log_a y$.

Proof:

Let $x=a^w$ and $y=a^z$.

Then, $\log_a x=\log_a a^w$ and $\log_a y=\log_a a^z$.

Since $f\left(x\right)=a^x$ and $g\left(x\right)=\log_a x$ are inverses of each other, $\log_a a^x=x$ (and $a^{\log_a x}=x$).

Therefore, $\log_a x=w$ and $\log_a y=z$.

Now,

$$xy=a^wa^z=a^{w+z}$$

$$\therefore \log_a xy=\log_a a^{w+z}$$

$$\therefore \log_a xy=w+z$$

$$\therefore \log_a xy=\log_a x+\log_a y$$

Questions

(a) Using the example at the left as a model, prove that

$$\log_a \frac{x}{y}=\log_a x-\log_a y \text{ and } \log_a x^y=y\log_a x$$

(b) Prove that $\log_a \dfrac{x}{y}=\log_a x-\log_a y$ as a ***corollary*** of

$\log_a xy=\log_a x+\log_a y$ and $\log_a x^y=y\log_a x$.

Hint: Write $\log_a \dfrac{x}{y}$ as $\log_a xy^{-1}$.

(c) Prove that for all real numbers *x*, *a* and *b* such that $x>0$, $a>0$, $b>0$, $a\neq 1$ and $b\neq 1$, $\log_a x=\dfrac{\log_b x}{\log_b a}$. (This is often called the “change of base” formula).

(d) Why are the restrictions in (c) necessary?

2. Assuming that the laws of exponents hold for all real-number exponents, prove each of the following:

(a) $a^0 = 1$ for all $a \in \mathbb{R}$ **(b)** $a^{-x} = \dfrac{1}{a^x}$ for all $a \in \mathbb{R}$ such that $a \neq 0$ and all $x \in \mathbb{R}$

(c) $x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$ for all $m \in \mathbb{Z}$, $n \in \mathbb{Z}$ such $n \neq 0$ and for all $x \in \mathbb{R}$ such that each expression is defined.

3. The half-life of ${}^{14}\text{C}$ is about 5730 years. What is the "third-life" of ${}^{14}\text{C}$? That is, how long would it take for a sample of ${}^{14}\text{C}$ to decay such that only one-third of the initial number of ${}^{14}\text{C}$ atoms remains?

4. In Canada, mortgage payments are typically made monthly. However, Canadian law governing mortgages does not allow interest to be compounded more frequently than semi-annually. Suppose that the annual interest rate for a mortgage is 4%. Given that payments are made monthly and that interest is compounded semi-annually, what is the monthly rate of interest?

5. Use the fact that $f(x) = \log_a x$ and $g(x) = a^x$ are inverses of each other to solve the following equations. Use Desmos or some other graphing software to check your answers.

(a) $10^x = 1347$ **(b)** $e^x = 1347$ **(c)** $10^{x^2} = 1390147$ **(d)** $e^{2x-1} = 56876$

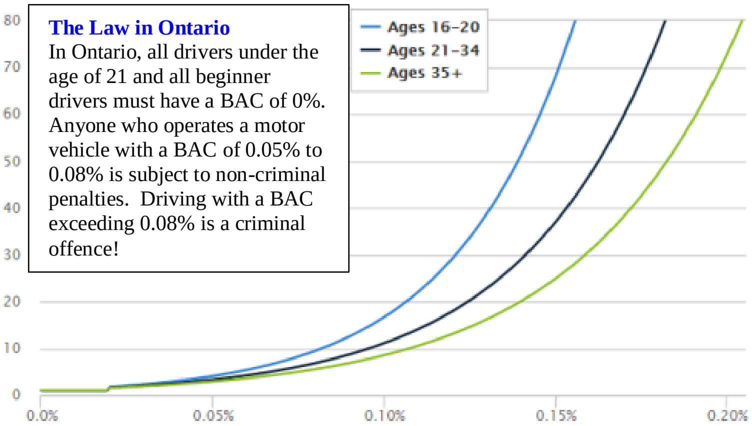
(e) $5\log_2\left(x^2 - 10x + 21\right) - 9 = 1000000$ **(f)** $5\log_2\left(x^3 - 1\right) - 9 = 1000000$

6. When alcohol is consumed, blood-alcohol concentration (BAC) increases at a rate that depends on a variety of factors. Metabolic processes break down alcohol, eliminating it from the body at a rate that varies from one individual to another. On average, this elimination of alcohol causes BAC to decrease at a rate of about 0.016% per hour.

(a) Against his/her better judgment and contrary to Mr. Nolfi's advice, a student consumed enough alcohol to cause his/her BAC to rise to 0.2%. Assuming that the student did not consume any additional alcohol, how long would it take for the student's BAC to fall to 0.05%?

(b) Suppose that the same student became so impaired that he/she continued to drink at rate that would cause BAC to rise at 0.009% per hour in the absence of metabolic processes that eliminate alcohol from the body. Assuming once again that metabolic processes cause BAC to decrease at a rate of 0.016% per hour, how long would it take for the student's BAC to fall to 0.05%?

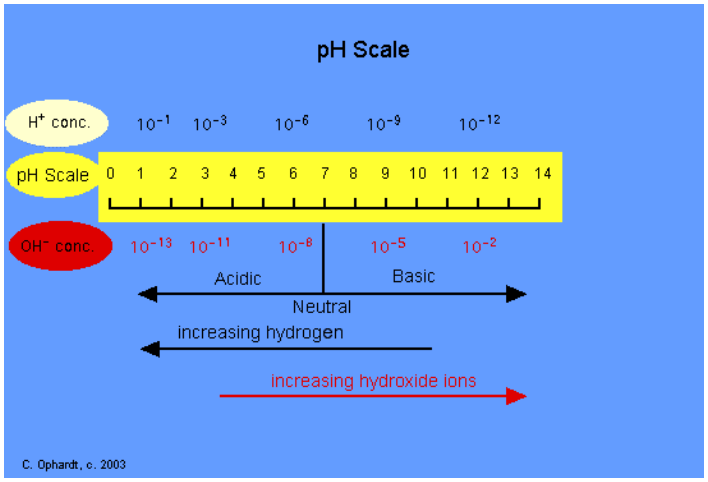
7. The greater the blood-alcohol concentration, the greater the risk of being involved in a fatal crash. At a BAC of 0.05%, a driver's risk of crashing begins to increase rapidly. At a BAC of 0.08%, the crash risk increases ***exponentially***. The 16-20-year-old age group has the highest crash risk at all BAC levels. At a BAC of .05%, this age group's crash risk is 4.09% and it rises to 9.51% at the 0.08% BAC level compared to a sober driver. The 21-34-year-old age group has the second highest crash risk by BAC level. At a 0.05% level, this age group's crash risk is 3.33% and it rises to 6.87% at the 0.08% level compared to a sober driver. The 35-plus age group has the lowest crash risk at all BAC levels. At a BAC of 0.05%, this age group's crash risk is 2.92%, and it rises to 5.56% at the 0.08% BAC level compared to a sober driver.



The Law in Ontario
In Ontario, all drivers under the age of 21 and all beginner drivers must have a BAC of 0%. Anyone who operates a motor vehicle with a BAC of 0.05% to 0.08% is subject to non-criminal penalties. Driving with a BAC exceeding 0.08% is a criminal offence!

Let *c* represent %-blood-alcohol concentration and let *R*(*c*) represent the %-risk of crashing. Use the information given above to write equations for *R*(*c*) for each of the given age groups.

8. The pH scale measures how acidic or basic a substance is. The scale ranges from 0 to 14, with a pH of 7 being neutral, a pH less than 7 being acidic and a pH greater than 7 being basic. The pH scale is logarithmic and thus, each whole pH value below 7 is ten times more acidic than the next higher value. For example, pH 4 is ten times more acidic than pH 5 and 100 times (10 times 10) more acidic than pH 6. The same holds true for pH values above 7, each of which is ten times more alkaline (another way to say basic) than the next lower whole value. For example, pH 10 is ten times more alkaline than pH 9 and 100 times (10 times 10) more alkaline than pH 8. Pure water is neutral. But when chemicals are mixed with water, the mixture can become either acidic or basic.



C. Ophardt, c. 2003

In any solution in which water is the solvent, H^+ ions (hydrogen ions) and OH^- ions (hydroxide ions) are always present and exist in equilibrium with the water molecules. Experimentally it has been determined that in any solution in any such solution, the product of the concentrations of hydrogen ions and hydroxide ions is always 10^{-14}. That is, $[H^+][OH^-] = 10^{-14}$.

(a) The pH of a solution is defined as $\text{pH} = -\log_{10}\left[\text{H}^+\right]$. Explain why this makes sense.

(b) An acidic solution has an H^+ concentration of 0.0001 M. What is the pH of the solution? What does the "M" mean?

(c) An acidic solution has a pH of 5. How much of a basic solution with a pH of 8 must be added to the acidic solution to neutralize it?

9. Find at least two scales (other then pH) that are logarithmic. Explain how each of the scales works.

10. For both logarithmic and exponential functions, the base *a* cannot be negative, zero or one. Explain why this is the case.