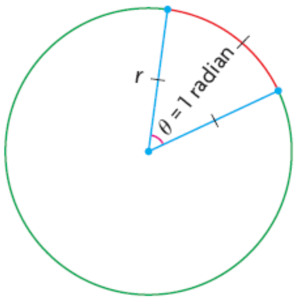
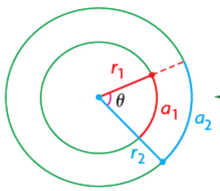
## *Definition of the Radian*

**radian**
the size of an angle that is subtended at the centre of a circle by an arc with a length equal to the radius of the circle; both the arc length and the radius are measured in units of length (such as centimetres) and, as a result, the angle is a real number without any units




1 radian is defined as the angle subtended by an arc length, $l$, equal to the radius, $r$. It appears as though 1 radian should be a little less than 60°, since the sector formed resembles an equilateral triangle, with one side that is curved slightly.




It is important to note that the size of an angle in radians is not affected by the size of the circle. The diagram shows that $a_1$ and $a_2$ subtend the same angle $\theta$, so $\theta = \frac{a_1}{r_1} = \frac{a_2}{r_2}$.

**Verb: subtend** sub'tend
1. Be opposite to; of angles and sides, in geometry

> ***Important Note***
> Whenever the units of angle measure are not specified, the units are ***assumed to be radians***.

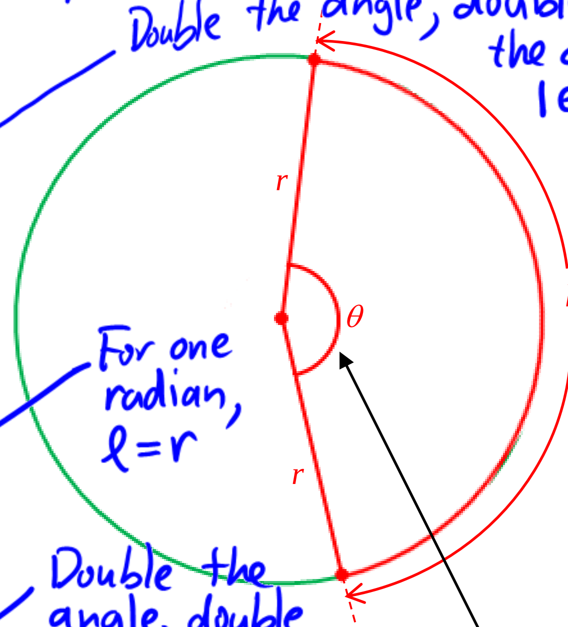
## *Investigation – The Relationship among θ, r and l*

When $\theta$ is measured in radians, there is a very simple equation that relates $r$ (the radius of the circle), $\theta$ (the angle at the centre of the circle) and $l$ (the length of the arc that subtends the angle $\theta$). The purpose of this investigation is to discover this relationship. Complete the table below and then answer the question at the bottom of the page.

| *Diagram* | *r* | *θ* | *l* |
|---|---|---|---|
| | 1 | 1 | 1 |
| | 1 | 2 | 2 |
| | 1 | 3 | 3 |
| | 2 | 1 | 2 |
| | 2 | 2 | 4 |
| | 2 | 3 | 6 |
| | 3 | 1.5 | 4.5 |

By the definition of 1 radian

Double the angle, double the arc length.




For one radian, $l = r$

Double the angle, double the arc length!

This angle is called a ***central angle***.

If $r = 3$ and $\theta = 1$ rad, then $l = 3$

If $\theta = 1.5$ and $r = 3$, then $l = 3(1.5) = 4.5$

$\therefore$ $l$ must be proportional to $\theta$

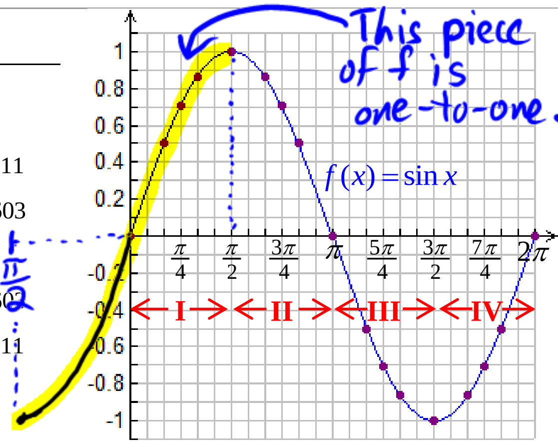
# Introduction to Trigonometric Functions

### *Overview*

Now that we have developed a thorough understanding of trigonometric ratios, we can proceed to our investigation of trigonometric functions. The old adage “a picture says a thousand words” is very fitting in the case of the graphs of trig functions. The curves summarize everything that we have learned about trig ratios. Answer the questions below to discover the details.

### *Graphs*

| $x$ | $y = \sin x$ |
|---|---|
| $0$ | $0$ |
| $\frac{\pi}{6}$ | $1/2 = 0.5$ |
| $\frac{\pi}{4}$ | $1/\sqrt{2} \doteq 0.70711$ |
| $\frac{\pi}{3}$ | $\sqrt{3}/2 \doteq 0.86603$ |
| $\frac{\pi}{2}$ | $1$ |
| $\frac{2\pi}{3}$ | $\sqrt{3}/2 \doteq 0.86603$ |
| $\frac{3\pi}{4}$ | $1/\sqrt{2} \doteq 0.70711$ |
| $\frac{5\pi}{6}$ | $1/2 = 0.5$ |
| $\pi$ | $0$ |
| $\frac{7\pi}{6}$ | $-1/2 = -0.5$ |
| $\frac{5\pi}{4}$ | $-1/\sqrt{2} \doteq -0.70711$ |
| $\frac{4\pi}{3}$ | $-\sqrt{3}/2 \doteq -0.86603$ |
| $\frac{3\pi}{2}$ | $-1$ |
| $\frac{5\pi}{3}$ | $-\sqrt{3}/2 \doteq -0.86603$ |
| $\frac{7\pi}{4}$ | $-1/\sqrt{2} \doteq -0.70711$ |
| $\frac{11\pi}{6}$ | $-1/2 = -0.5$ |
| $2\pi$ | $0$ |




***Questions about sin x***

1. State the domain and range of the sine function.

   See p. 28 of unit 2 notes

2. Is the sine function one-to-one or many-to-one? ✓

   It changes direction and ∴ cannot be one-to-one

3. State a suitable subset of the domain of $f(x) = \sin x$ over which $f^{-1}(x)$ is defined.

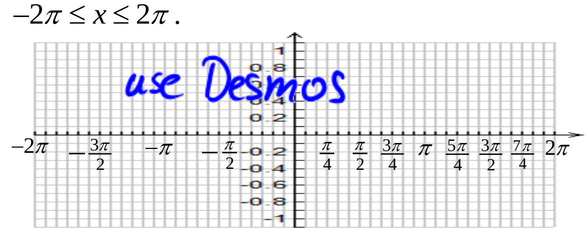
   $\{x \in \mathbb{R} \mid -\frac{\pi}{2} \le x \le \frac{\pi}{2}\}$

4. How can the graph of the sine function help you to remember the ***sign*** (i.e. + or –) of a sine ratio for any angle?

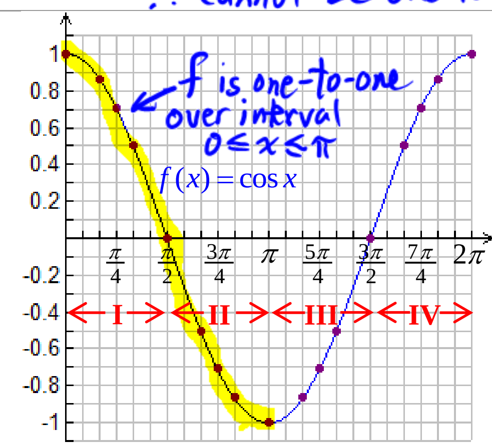
   above x-axis → sin positive
   below x-axis → sin negative

5. How can the graph of the sine function help you to remember the sine ratios of the special angles? The function increases from 0 to 1 over interval $0 \le x \le \frac{\pi}{2}$, passing through $\frac{1}{2}$, $\frac{1}{\sqrt{2}}$, $\frac{\sqrt{3}}{2}$

6. Sketch the graph of $f(x) = \sin x$ for $-2\pi \le x \le 2\pi$.




| $x$ | $y = \cos x$ |
|---|---|
| $0$ | $1$ |
| $\frac{\pi}{6}$ | $\sqrt{3}/2 \doteq 0.86603$ |
| $\frac{\pi}{4}$ | $1/\sqrt{2} \doteq 0.70711$ |
| $\frac{\pi}{3}$ | $1/2 = 0.5$ |
| $\frac{\pi}{2}$ | $0$ |
| $\frac{2\pi}{3}$ | $-1/2 = -0.5$ |
| $\frac{3\pi}{4}$ | $-1/\sqrt{2} \doteq -0.70711$ |
| $\frac{5\pi}{6}$ | $-\sqrt{3}/2 \doteq -0.86603$ |
| $\pi$ | $-1$ |
| $\frac{7\pi}{6}$ | $-\sqrt{3}/2 \doteq -0.86603$ |
| $\frac{5\pi}{4}$ | $-1/\sqrt{2} \doteq -0.70711$ |
| $\frac{4\pi}{3}$ | $-1/2 = -0.5$ |
| $\frac{3\pi}{2}$ | $0$ |
| $\frac{5\pi}{3}$ | $1/2 = 0.5$ |
| $\frac{7\pi}{4}$ | $1/\sqrt{2} \doteq 0.70711$ |
| $\frac{11\pi}{6}$ | $\sqrt{3}/2 \doteq 0.86603$ |
| $2\pi$ | $1$ |




***Questions about cos x***

1. State the domain and range of the cosine function.

   See p. 28 of notes

2. Is the cosine function one-to-one or many-to-one? ✓

3. State a suitable subset of the domain of $f(x) = \cos x$ over which $f^{-1}(x)$ is defined.

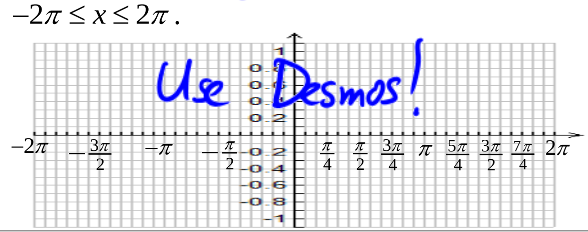
   $\{x \in \mathbb{R} \mid 0 \le x \le \pi\}$

4. How can the graph of the cosine function help you to remember the ***sign*** (i.e. + or –) of a cosine ratio for any angle?

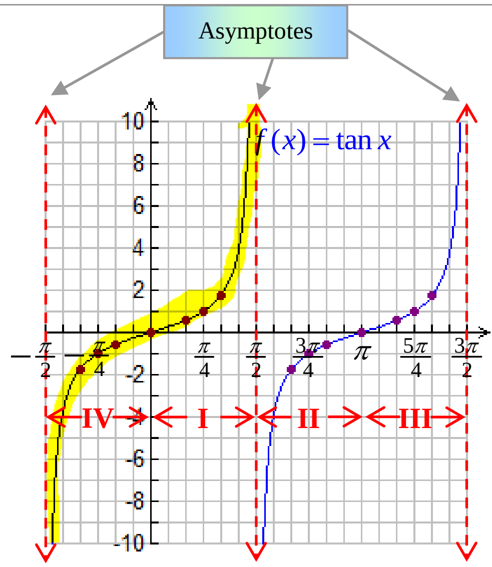
   Above x-axis → cos is positive
   Below x-axis → cos is negative

5. How can the graph of the cosine function help you to remember the cosine ratios of the special angles? The function decreases from 1 to 0 over interval $0 \le x \le \frac{\pi}{2}$, passing through $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$.

6. Sketch the graph of $f(x) = \cos x$ for $-2\pi \le x \le 2\pi$.



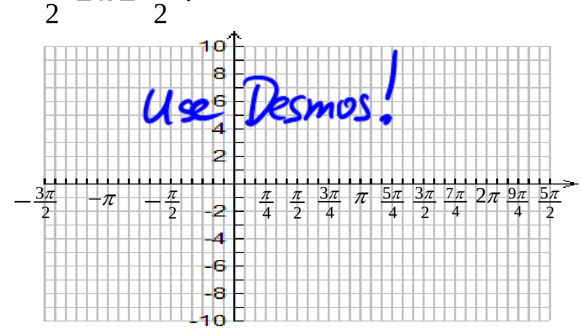
| $x$ | $y=\tan x$ |
|---|---|
| $-\frac{\pi}{2}$ | undefined |
| $-\frac{\pi}{3}$ | $-\sqrt{3} \doteq -1.73205$ |
| $-\frac{\pi}{4}$ | $-1$ |
| $-\frac{\pi}{6}$ | $-1/\sqrt{3} \doteq -0.57735$ |
| $0$ | $0$ |
| $\frac{\pi}{6}$ | $1/\sqrt{3} \doteq 0.57735$ |
| $\frac{\pi}{4}$ | $1$ |
| $\frac{\pi}{3}$ | $\sqrt{3} \doteq 1.73205$ |
| $\frac{\pi}{2}$ | undefined |
| $\frac{2\pi}{3}$ | $-\sqrt{3} \doteq -1.73205$ |
| $\frac{3\pi}{4}$ | $-1$ |
| $\frac{5\pi}{6}$ | $-1/\sqrt{3} \doteq -0.57735$ |
| $\pi$ | $0$ |
| $\frac{7\pi}{6}$ | $1/\sqrt{3} \doteq 0.57735$ |
| $\frac{5\pi}{4}$ | $1$ |
| $\frac{4\pi}{3}$ | $\sqrt{3} \doteq 1.73205$ |
| $\frac{3\pi}{2}$ | undefined |



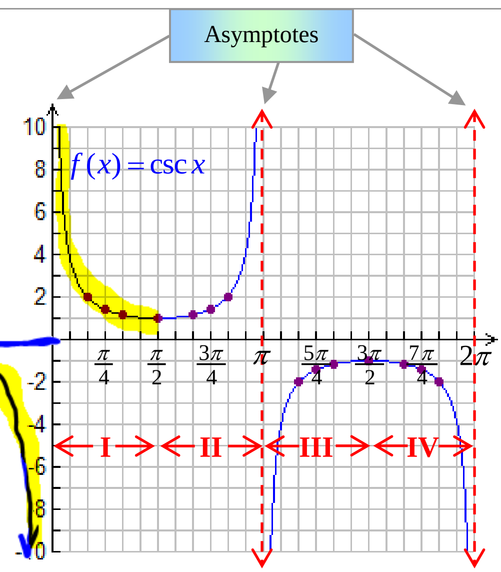

### *Questions about tan x*

1. State the domain and range of the tangent function.
   See p. 28 of notes
2. Is the tangent function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x)=\tan x$ over which $f^{-1}(x)$ is defined.
   $\{x \in \mathbb{R} \mid -\frac{\pi}{2} < x < \frac{\pi}{2}\}$
4. How can the graph of the tangent function help you to remember the sign (i.e. + or –) of a tangent ratio for any angle?
   above $x$-axis → tan positive
   below $x$-axis → tan negative
5. How can the graph of the tangent function help you to remember the tangent ratios of the special angles?
   Function increases over $0 \le x < \frac{\pi}{2}$ from 0 to $\infty$. $\tan\frac{\pi}{4}=1$ since $x=y$ for $45°$.
6. Sketch the graph of $f(x)=\tan x$ for $-\dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2}$.



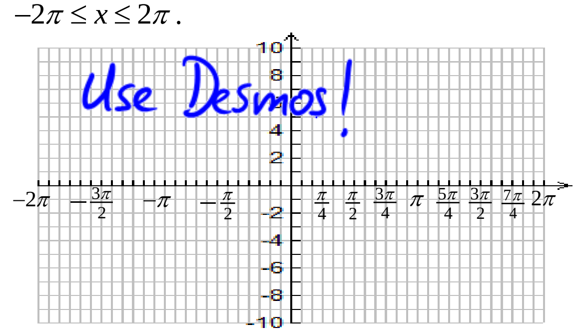

| $x$ | $y=\csc x$ |
|---|---|
| $0$ | undefined |
| $\frac{\pi}{6}$ | $2$ |
| $\frac{\pi}{4}$ | $\sqrt{2} \doteq 1.41421$ |
| $\frac{\pi}{3}$ | $2/\sqrt{3} \doteq 1.15470$ |
| $\frac{\pi}{2}$ | $1$ |
| $\frac{2\pi}{3}$ | $2/\sqrt{3} \doteq 1.15470$ |
| $\frac{3\pi}{4}$ | $\sqrt{2} \doteq 1.41421$ |
| $\frac{5\pi}{6}$ | $2$ |
| $\pi$ | undefined |
| $\frac{7\pi}{6}$ | $-2$ |
| $\frac{5\pi}{4}$ | $-\sqrt{2} \doteq -1.41421$ |
| $\frac{4\pi}{3}$ | $-2/\sqrt{3} \doteq -1.15470$ |
| $\frac{3\pi}{2}$ | $-1$ |
| $\frac{5\pi}{3}$ | $-2/\sqrt{3} \doteq -1.15470$ |
| $\frac{7\pi}{4}$ | $-\sqrt{2} \doteq -1.41421$ |
| $\frac{11\pi}{6}$ | $-2$ |
| $2\pi$ | undefined |



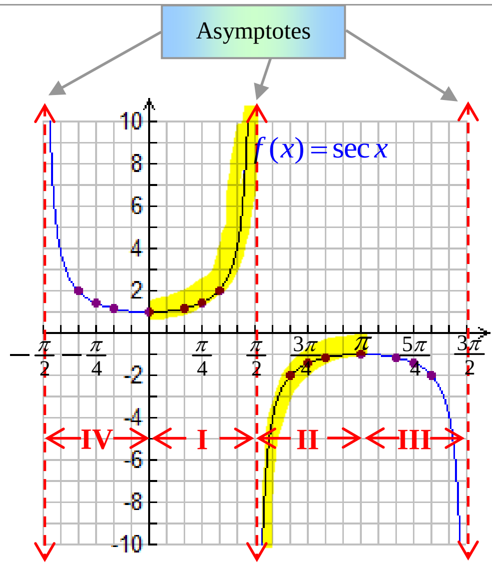

### *Questions about csc x*

1. State the domain and range of the cosecant function.
   See p. 28 of notes
2. Is the cosecant function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x)=\csc x$ over which $f^{-1}(x)$ is defined.
   $\{x \in \mathbb{R} \mid -\frac{\pi}{2} \le x < 0 \text{ or } 0 < x \le \frac{\pi}{2}\}$
4. How can the graph of the cosecant function help you to remember the sign (i.e. + or –) of a cosecant ratio for any angle?
   Same answer as all the others!
5. How can the graph of the cosecant function help you to remember the cosecant ratios of the special angles?
   Function decreases on $0 < x \le \frac{\pi}{2}$ from $\infty$ to 1. Reciprocal of sine ratio
6. Sketch the graph of $f(x)=\csc x$ for $-2\pi \le x \le 2\pi$.



| $x$ | $y=\sec x$ |
|---|---|
| $-\frac{\pi}{2}$ | undefined |
| $-\frac{\pi}{3}$ | 2 |
| $-\frac{\pi}{4}$ | $\sqrt{2} \doteq 1.41421$ |
| $-\frac{\pi}{6}$ | $2/\sqrt{3} \doteq 1.15470$ |
| 0 | 1 |
| $\frac{\pi}{6}$ | $2/\sqrt{3} \doteq 1.15470$ |
| $\frac{\pi}{4}$ | $\sqrt{2} \doteq 1.41421$ |
| $\frac{\pi}{3}$ | 2 |
| $\frac{\pi}{2}$ | undefined |
| $\frac{2\pi}{3}$ | $-2$ |
| $\frac{3\pi}{4}$ | $-\sqrt{2} \doteq -1.41421$ |
| $\frac{5\pi}{6}$ | $-2/\sqrt{3} \doteq -1.15470$ |
| $\pi$ | $-1$ |
| $\frac{7\pi}{6}$ | $-2/\sqrt{3} \doteq -1.15470$ |
| $\frac{5\pi}{4}$ | $-\sqrt{2} \doteq -1.41421$ |
| $\frac{4\pi}{3}$ | $-2$ |
| $\frac{3\pi}{2}$ | undefined |




***Questions about sec x***

1. State the domain and range of the secant function.

See p. 28 of notes

2. Is the secant function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x)=\sec x$ over which $f^{-1}(x)$ is defined.

$\{x \in \mathbb{R} \mid 0 \le x < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < x \le \pi\}$

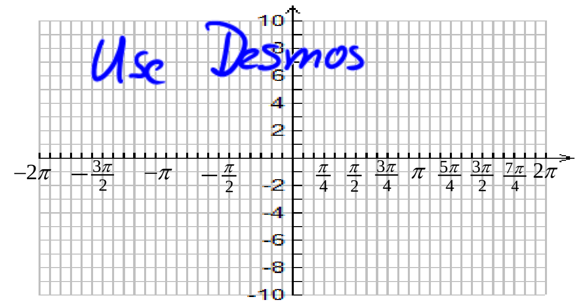
4. How can the graph of the secant function help you to remember the sign (i.e. + or –) of a secant ratio for any angle?

Same as all the others!!

5. How can the graph of the secant function help you to remember the secant ratios of the special angles?

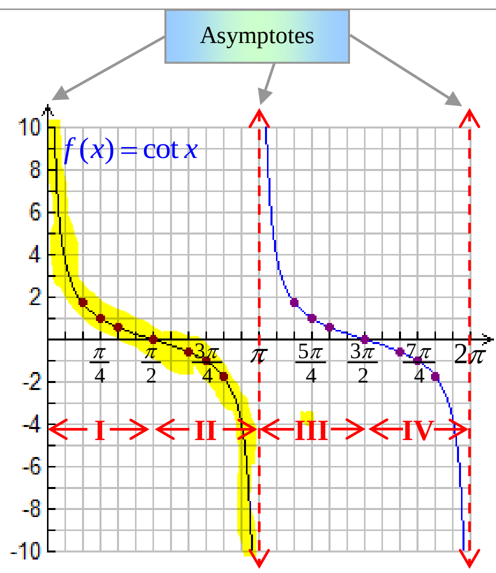
Same as the others!!

6. Sketch the graph of $f(x)=\sec x$ for $-2\pi \le x \le 2\pi$.




Use Desmos

| $x$ | $y=\cot x$ |
|---|---|
| 0 | undefined |
| $\frac{\pi}{6}$ | $\sqrt{3} \doteq 1.73205$ |
| $\frac{\pi}{4}$ | 1 |
| $\frac{\pi}{3}$ | $1/\sqrt{3} \doteq 0.57735$ |
| $\frac{\pi}{2}$ | 0 |
| $\frac{2\pi}{3}$ | $-1/\sqrt{3} \doteq -0.57735$ |
| $\frac{3\pi}{4}$ | $-1$ |
| $\frac{5\pi}{6}$ | $-\sqrt{3} \doteq -1.73205$ |
| $\pi$ | undefined |
| $\frac{7\pi}{6}$ | $\sqrt{3} \doteq 1.73205$ |
| $\frac{5\pi}{4}$ | 1 |
| $\frac{4\pi}{3}$ | $1/\sqrt{3} \doteq 0.57735$ |
| $\frac{3\pi}{2}$ | 0 |
| $\frac{5\pi}{3}$ | $-1/\sqrt{3} \doteq -0.57735$ |
| $\frac{7\pi}{4}$ | $-1$ |
| $\frac{11\pi}{6}$ | $-\sqrt{3} \doteq -1.73205$ |
| $2\pi$ | undefined |




***Questions about cot x***

1. State the domain and range of the cotangent function.

See p. 28 of notes

2. Is the cotangent function one-to-one or many-to-one? ✓

3. State a suitable subset of the domain of $f(x)=\cot x$ over which $f^{-1}(x)$ is defined.

$\{x \in \mathbb{R} \mid 0 < x < \frac{\pi}{2}\}$

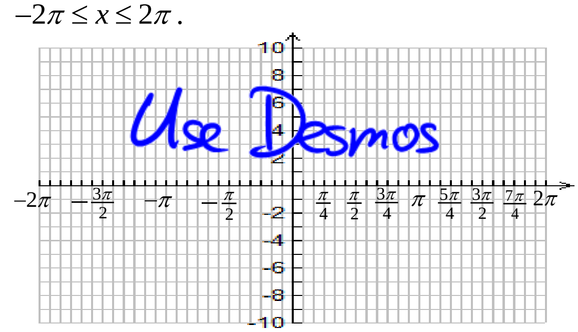
4. How can the graph of the cotangent function help you to remember the sign (i.e. + or –) of a cotangent ratio for any angle?

Same as all the others!

5. How can the graph of the cotangent function help you to remember the cotangent ratios of the special angles?

Same as all the others

6. Sketch the graph of $f(x)=\cot x$ for $-2\pi \le x \le 2\pi$.

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# TRANSFORMATIONS OF TRIGONOMETRIC FUNCTIONS

## *What on Earth is a Sinusoidal Function?*

A ***sinusoidal function*** is simply any function that can be obtained by ***stretching*** (compressing) and/or ***translating*** the function $f(x)=\sin x$. That is, a sinusoidal function is any function of the form $g(x)=A\sin\left(\omega(x-p)\right)+d$. Since we have already investigated transformations of logarithmic and exponential and logarithmic functions, we can immediately state the following:

| *Transformation of $f(x)=\sin x$ expressed in Function Notation* | *Transformation of $f(x)=\sin x$ expressed in Mapping Notation* |
|---|---|
| $g(x)=A\sin\left(\omega(x-p)\right)+d$ | $(x,y)\to\left(\omega^{-1}x+p, Ay+d\right)$ |

| ***Vertical Transformations*** ***(Apply Operations following Order of Operations)*** | ***Horizontal Transformations*** ***(Apply Inverse Operations opposite the Order of Operations)*** |
|---|---|
| **1.** Stretch or compress vertically by a factor of ***A***. If ***A*** <0, then this includes a reflection in the *x*-axis.<br>**2.** Translate vertically by ***d*** units.<br>$(x,y)\to(x, Ay+d)$ | **1.** Stretch or compress horizontally by a factor of $\omega^{-1}=1/\omega$. If $\omega$ <0, then this includes a reflection in the *y*-axis.<br>**2.** Translate horizontally by ***p*** units.<br>$(x,y)\to\left(\omega^{-1}x+p, y\right)$ |

Since sinusoidal functions look just like ***waves*** and are perfectly suited to modelling wave or wave-like phenomena, special names are given to the quantities *A*, *d*, *p* and *k*.

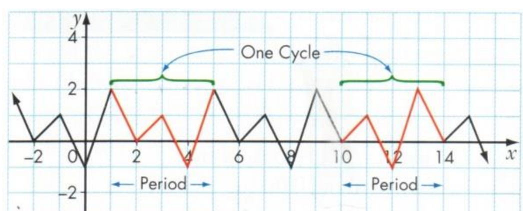
- $|A|$ is called the ***amplitude*** (absolute value is needed because amplitude is a distance, which must be positive)
- *d* is called the ***vertical displacement***
- *p* is called the ***phase shift***
- $\omega$ (also written as *k*) is called the ***angular frequency***

These quantities are described in detail on the next page.

## *Periodic Functions*

There are many naturally occurring and artificially produced phenomena that undergo repetitive cycles. We call such phenomena ***periodic***. Examples of such processes include the following:

- orbits of planets, moons, asteroids, comets, etc
- rotation of planets, moons, asteroids, comets, etc
- phases of the moon
- the tides
- changing of the seasons
- hours of daylight on a given day
- light waves, radio waves, etc
- alternating current (**e.g.** household alternating current has a frequency of 60 Hz, which means that it changes direction 60 times per second)




An example of a ***periodic*** function.

Intuitively, a function is said to be ***periodic*** if the graph consists of a "basic pattern" that is repeated over and over at ***regular intervals***. One complete pattern is called a ***cycle***.

Formally, if there is a number *T* such that $f(x+T)=f(x)$ for all values of *x*, then we say that *f* is ***periodic***. The smallest possible positive value of *T* is called the ***period*** of the function. The ***period*** of a periodic function is equal to the ***length of one cycle***.

## *Exercise*

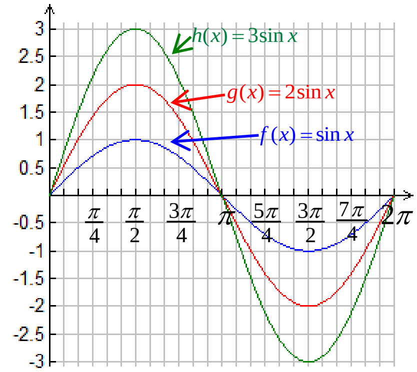
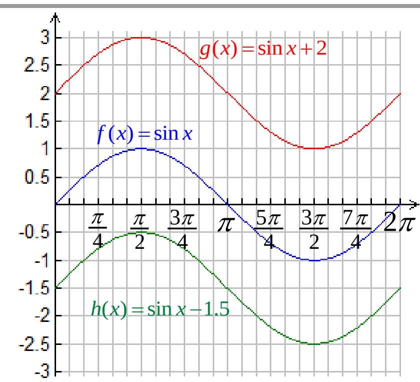
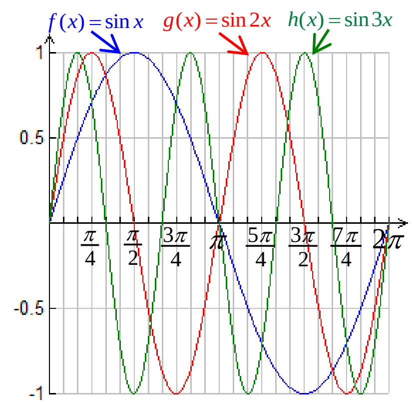
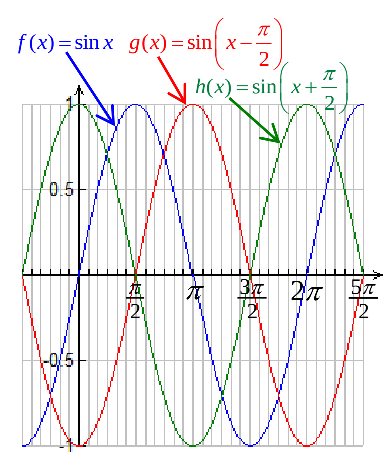
Suppose that the periodic function shown above is called *f*. Evaluate each of the following.

**(a)** $f(2)=0$ **(b)** $f(4)=-1$ **(c)** $f(1)=0$ **(d)** $f(0)=-1$ **(e)** $f(16)=-1$ **(f)** $f(18)=0$ **(g)** $f(33)=2$ **(h)** $f(-16)=-1$

**(i)** $f(-31)=2$ **(j)** $f(-28)=-1$ **(k)** $f(-27)=2$ **(l)** $f(-11)=2$ **(m)** $f(-6)=0$ **(n)** $f(-9)=1$ **(o)** $f(-5)=1$ **(p)** $f(-101)=1$

### *Important Exercises*

Complete the following table. The first one is done for you.

| *Function* | *A* | *d* | *p* | *ω=k* | *T* | *Description of Transformation* | |
|---|---|---|---|---|---|---|---|
| $f$ | 1 | 0 | 0 | 1 | $2\pi$ | None |  |
| $g$ | 2 | 0 | 0 | 1 | $2\pi$ | The graph of $f(x)=\sin x$ is stretched vertically by a factor of 2. The amplitude of $g$ is 2. | |
| $h$ | 3 | 0 | 0 | 1 | $2\pi$ | The graph of $f(x)=\sin x$ is stretched vertically by a factor of 3. The amplitude of $g$ is 3. | |
| $f$ | 1 | 0 | 0 | 1 | $2\pi$ | None |  |
| $g$ | 1 | 2 | 0 | 1 | $2\pi$ | Vertical translation 2 units upward | |
| $h$ | 1 | $-1.5$ | 0 | 1 | $2\pi$ | Vertical translation 1.5 units downward | |
| $f$ | 1 | 0 | 0 | 1 | $2\pi$ | None |  |
| $g$ | 1 | 0 | 0 | 2 | $\pi$ | Horizontal Compression by a factor of $\frac{1}{2}$ | |
| $h$ | 1 | 0 | 0 | 3 | $\frac{2\pi}{3}$ | Horizontal Compression by a factor of $\frac{1}{3}$ | |
| $f$ | 1 | 0 | 0 | 1 | $2\pi$ | None |  |
| $g$ | 1 | 0 | $\frac{\pi}{2}$ | 1 | $2\pi$ | Horizontal translation $\frac{\pi}{2}$ to the right | |
| $h$ | 1 | 0 | $-\frac{\pi}{2}$ | 1 | $2\pi$ | Horizontal translation $\frac{\pi}{2}$ to the left | |

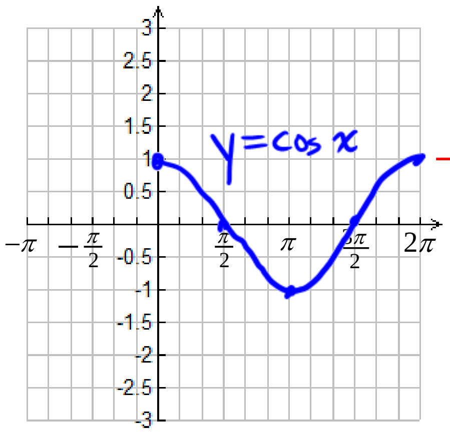
### *Exercise 1*

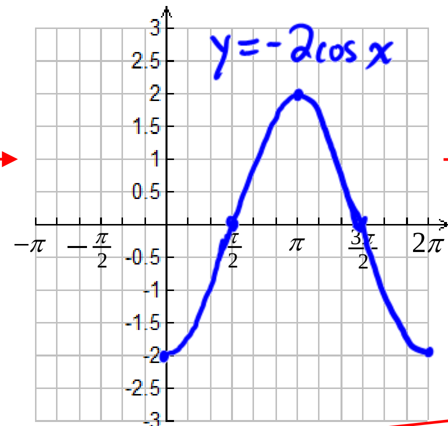
Using ***both*** of the approaches shown in the previous example, sketch a few cycles of the graph of $f(x) = -2\cos\left(-3\left(x + \frac{\pi}{3}\right)\right) - 1$.

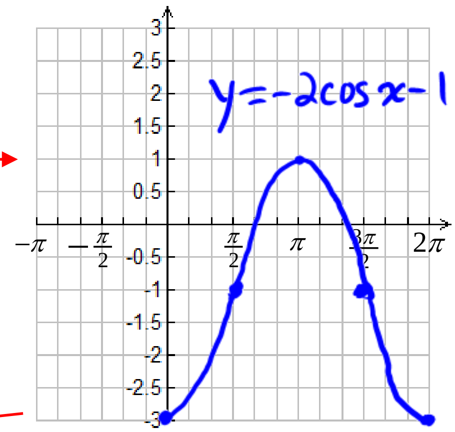
### *Solution*

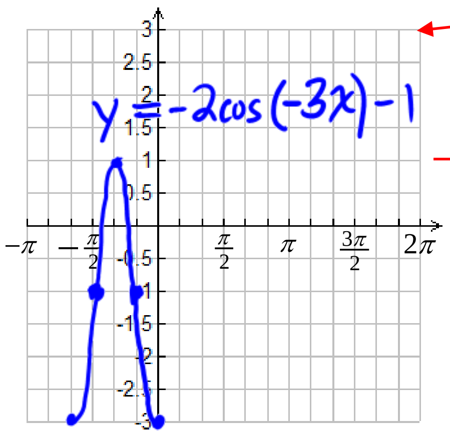
| *Amplitude* | *Vertical Displacement* | *Phase Shift* | *Period* | *Angular Frequency* | *Transformations* | |
|---|---|---|---|---|---|---|
| 2 | -1 | $-\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | -3 | *Vertical*<br>1. Stretch by -2<br>2. Shift down 1 | *Horizontal*<br>1. Compress by $-\frac{1}{3}$<br>2. Shift $\frac{\pi}{3}$ left |

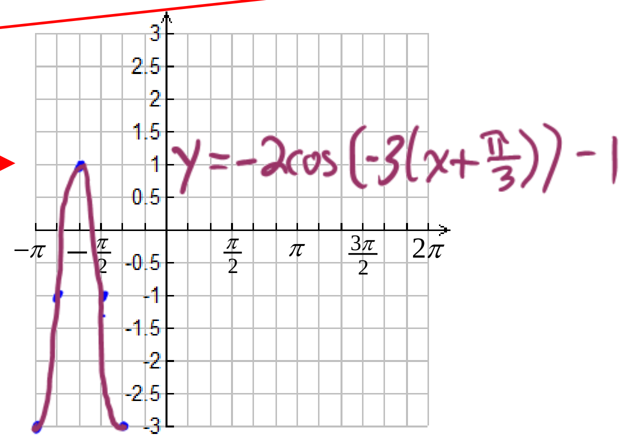
### *Method 1 – The Long Way*



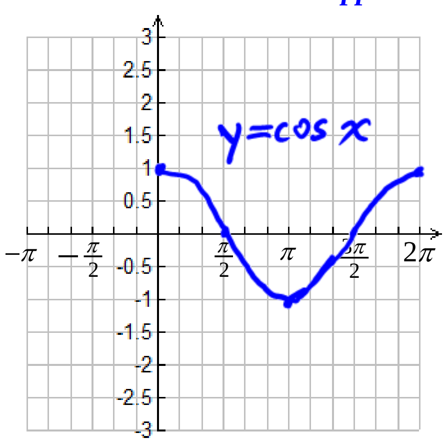




### *Method 2 – A Much Faster Approach*




$f(x) = -2\cos\left(-3\left(x + \frac{\pi}{3}\right)\right) - 1$

$A = \underline{-2}$, $d = \underline{-1}$, $\omega = k = \underline{-3}$, $p = -\frac{\pi}{3}$

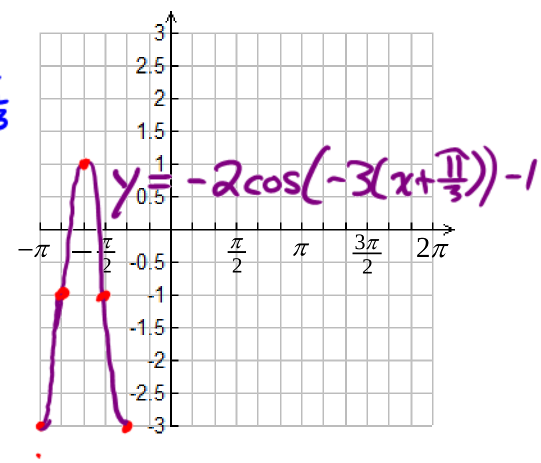
$(x, y) \to \left(-\frac{1}{3}x - \frac{\pi}{3}, -2y - 1\right)$

$(0, 1) \to \left(-\frac{\pi}{3}, -3\right)$

$\left(\frac{\pi}{2}, 0\right) \to \left(-\frac{\pi}{2}, -1\right)$

$(\pi, -1) \to \left(\frac{-2\pi}{3}, 1\right)$

$\left(\frac{3\pi}{2}, 0\right) \to \left(\frac{-5\pi}{6}, -1\right)$





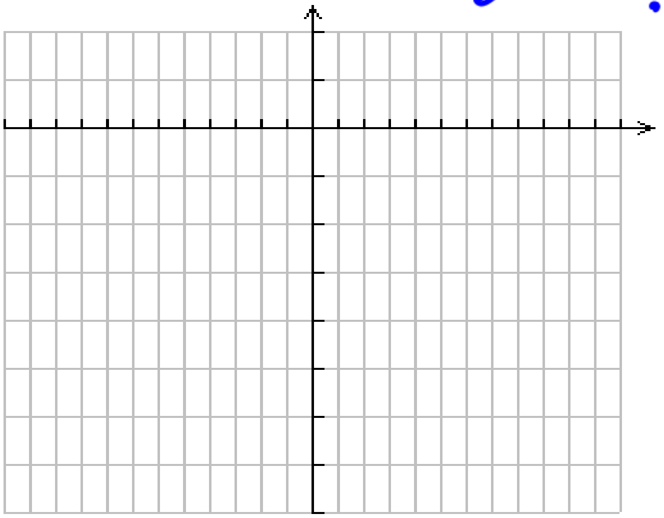

$(2\pi, 1) \to \left(-\frac{5\pi}{3}, -3\right)$

## *Homework Exercises*

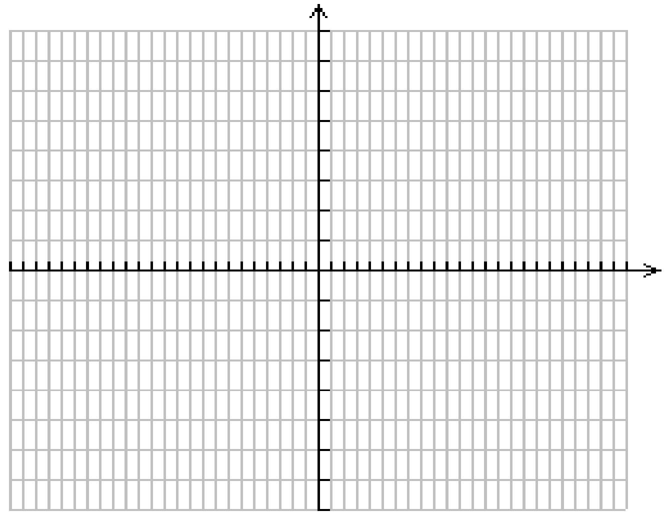
Sketch at least three cycles of each of the following functions. In addition, state the domain and range of each, as well as the amplitude, the vertical displacement, the phase shift and the period.

**(a)** $f(x) = -\cos 3x - 2$

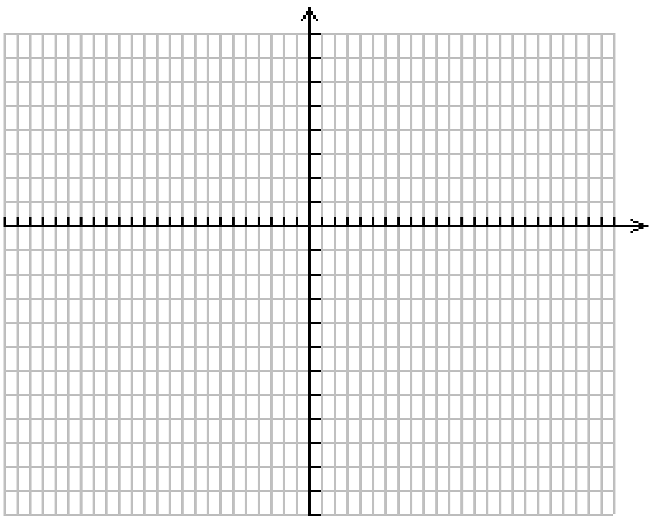
Use Desmos!!



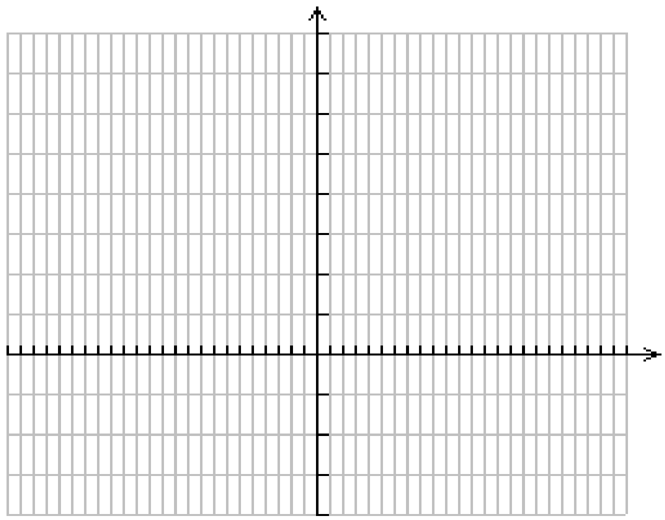
**(b)** $g(x) = 3\cos\left(x - \frac{\pi}{6}\right)$



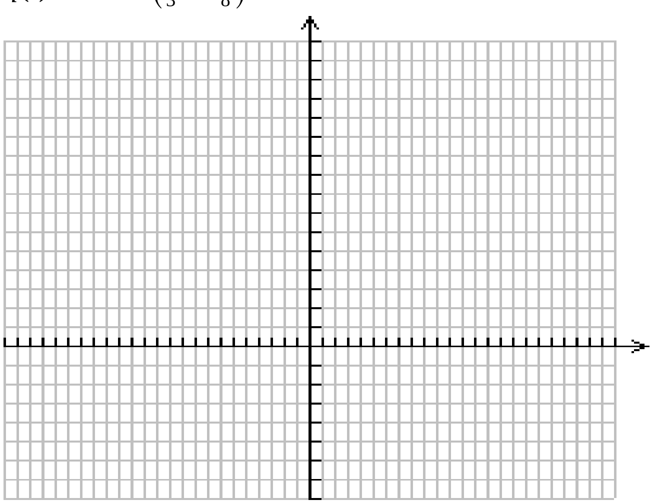
**(c)** $h(x) = 4\sin\left(2\left(x + \frac{\pi}{4}\right)\right) - 1$



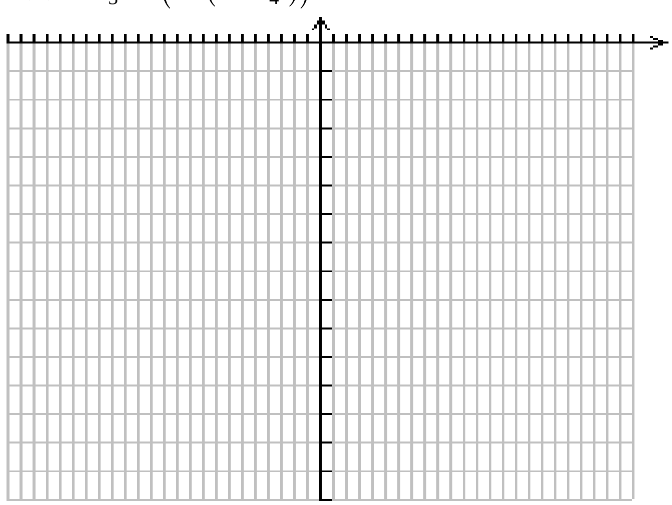
**(d)** $p(x) = -2\sin\left(2x + \frac{\pi}{3}\right) + 1$



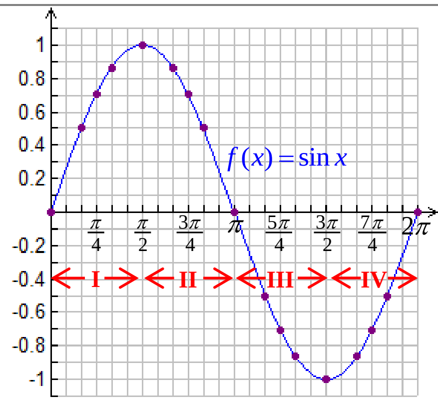
**(e)** $q(t) = -5\cos\left(\frac{1}{3}t + \frac{\pi}{8}\right) + 2$



**(f)** $r(\theta) = -\frac{2}{3}\sin\left(-2\left(\theta - \frac{3\pi}{4}\right)\right) - 3$

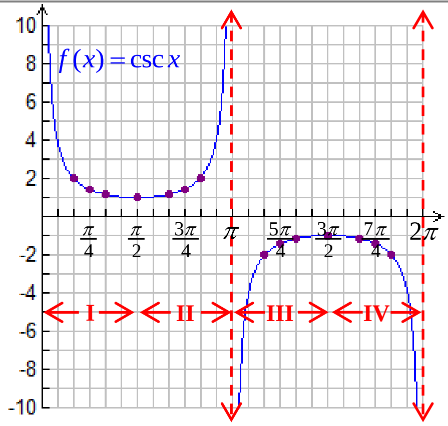


## *One Cycle of each of the Trigonometric Base/Parent/Mother Functions*




$D = \mathbb{R},\ R = \{y \in \mathbb{R} : -1 \le y \le 1\}$

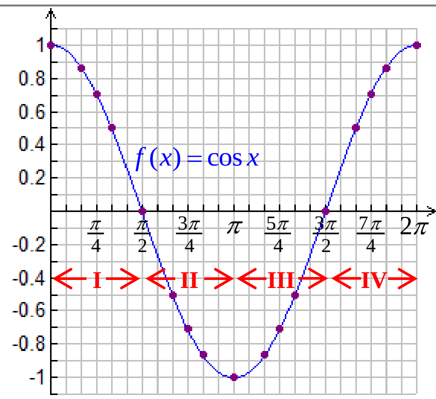
$A = 1,\ d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$




$D = \{x \in \mathbb{R} : x \ne n\pi, n \in \mathbb{Z}\},\ R = \{y \in \mathbb{R} : y \le -1 \text{ or } y \ge 1\}$

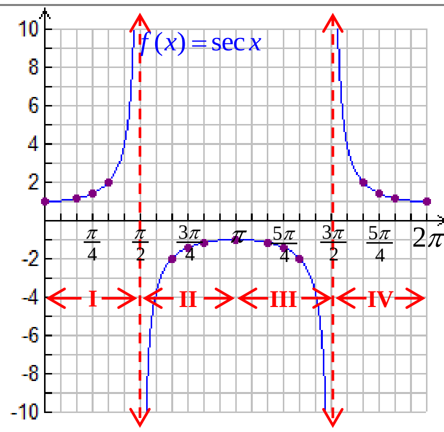
$A$ = 1 but the amplitude is undefined, $d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$

(The vertical stretch factor is 1 but there is no feature corresponding to amplitude.)




$D = \mathbb{R},\ R = \{y \in \mathbb{R} : -1 \le y \le 1\}$

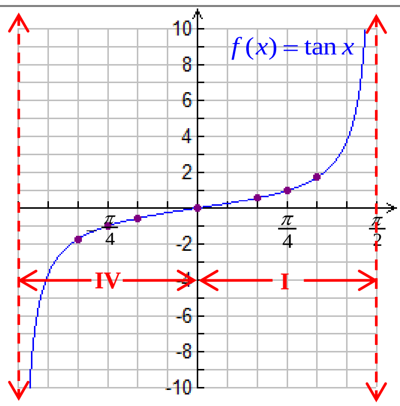
$A = 1,\ d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$




$D = \left\{x \in \mathbb{R} : x \ne \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}\right\},\ R = \{y \in \mathbb{R} : y \le -1 \text{ or } y \ge 1\}$

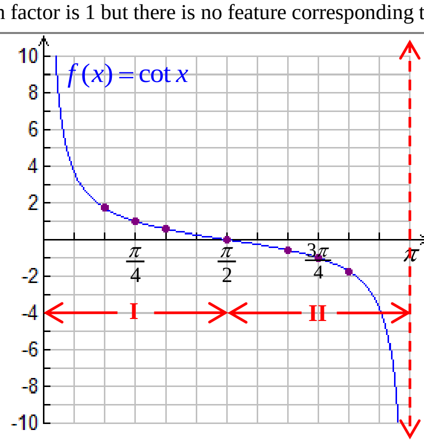
$A$ = 1 but the amplitude is undefined, $d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$

(The vertical stretch factor is 1 but there is no feature corresponding to amplitude.)




$D = \left\{x \in \mathbb{R} : x \ne \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}\right\},\ R = \mathbb{R}$

$A$ = 1 but the amplitude is undefined,
$d = 0,\ p = 0,\ \omega = 1,\ T = \pi$




$D = \{x \in \mathbb{R} : x \ne n\pi, n \in \mathbb{Z}\},\ R = \mathbb{R}$

$A$ = 1 but the amplitude is undefined,
$d = 0,\ p = 0,\ \omega = 1,\ T = \pi$

### *Suggestions for Graphing Trigonometric Functions*

**1.** Identify the transformations and express using ***mapping notation***.

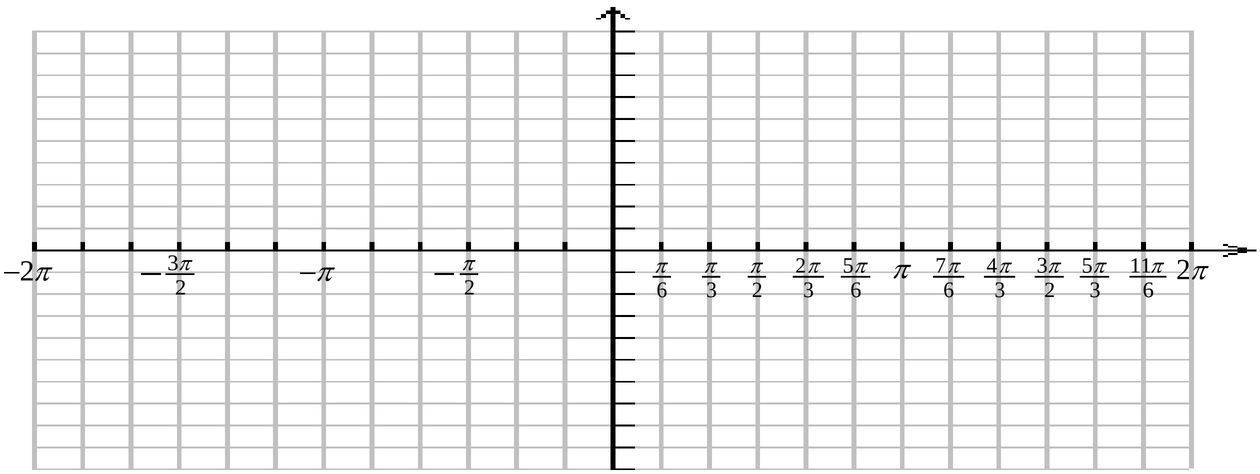
**2.** Think carefully about the effect of the transformations on the features of the base graph

- **(a)** ***Horizontal stretches/compressions*** affect the ***period*** and the ***locations of the vertical asymptotes***.
- **(b)** ***Horizontal translations*** affect the ***locations of the vertical asymptotes*** and the ***phase shift***.
- **(c)** ***Vertical stretches/compressions*** affect the ***amplitude*** (if applicable) and the ***y-co-ordinates of maximum/minimum points***.
- **(d)** ***Vertical translations*** simply cause all the points on the graph to move up or down by some constant amount.

**3.** Apply the transformations to a few special points on the base function.

**4.** Sometimes it is easier to apply the stretches/compressions first to obtain the final "shape" of the curve. Then it is a simple matter to translate the curve into its final position.

**5.** To find a suitable scale for the $x$-axis, divide the period by a number that is divisible by 4. The number 12 works particularly well because it divides evenly into 360°, giving increments of 30° or $\frac{\pi}{6}$ radians (see diagram).




### *Graphing Exercises*

Now sketch graphs of each of the following functions by applying appropriate transformations to one of the base functions given above. Once you are done, use TI-Interactive or a graphing calculator to check whether your graphs are correct. Detailed solutions are also available at http:www.misternolfi.com/courses.htm under "Unit 2 - Trigonometric Functions."

a) $y = 18\cos\left(\frac{\pi x}{4}\right) - 14$

b) $y = -\frac{4}{5}\sin\left(\frac{2}{7}\left(x + \frac{3\pi}{4}\right)\right) + 10$

c) $y = 101\cos\left(x - \frac{7\pi}{4}\right) - \frac{9}{10}$

d) $y = 6\sin(\pi x + 13) + 22$

e) $y = -\cos\left(\frac{5\pi}{3}(x - 1)\right) + 1$

f) $y = 15\sin\left(\frac{x}{15}\right) - 5$

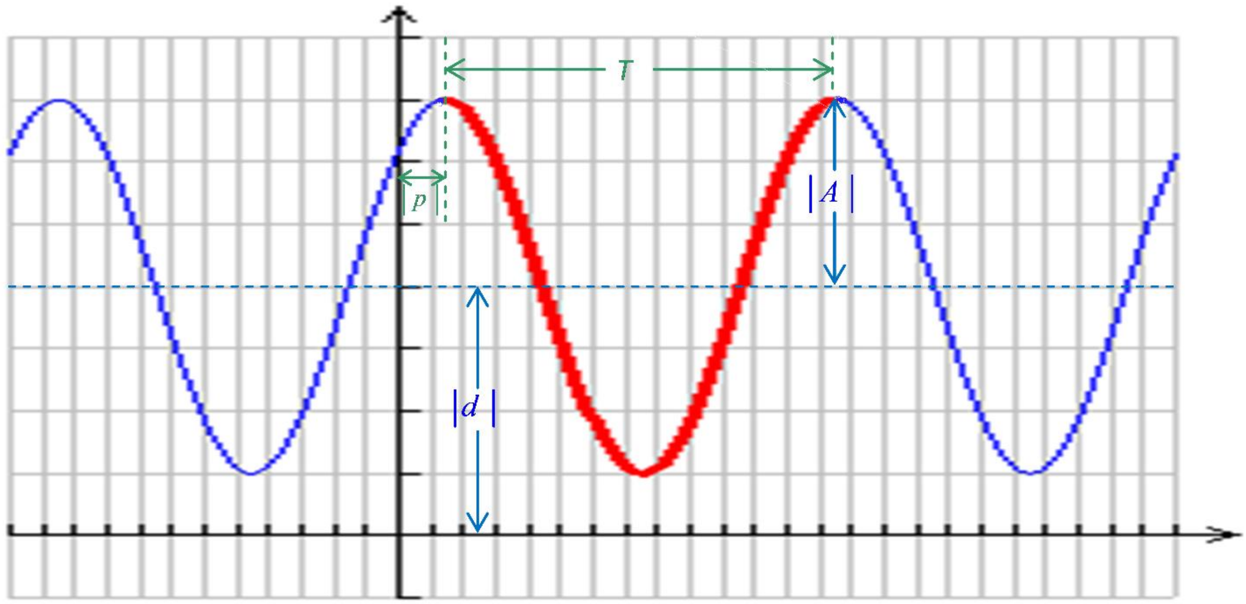
g) $y = -2\tan\left(3\left(x + \frac{\pi}{4}\right)\right) - 3$

h) $y = -2\sec\left(\frac{2}{\pi}\left(x + \frac{\pi}{6}\right)\right) + 5$

i) $y = 3\cot\left(\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right) + 2$

j) $y = \frac{5}{3}\csc\left(1.5x + \frac{\pi}{4}\right) + \frac{3}{2}$

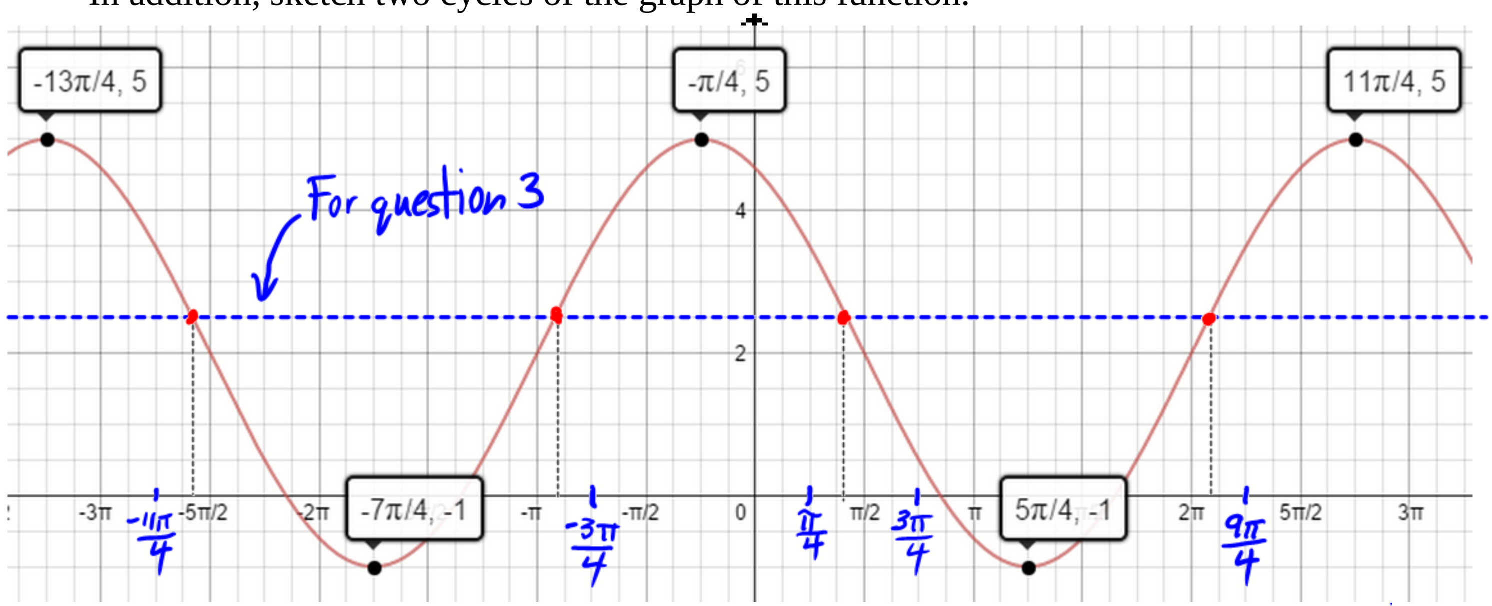
See separate document on www.misternolfi.com for detailed solutions!




*Activity 1*

$A=3,\ T=3\pi,\ p=-\frac{\pi}{4}\ \therefore\ \omega=\frac{2\pi}{3\pi}=\frac{2}{3},\ d=2$

**1.** A cosine curve has an amplitude of 3 units and a period of $3\pi$ radians. The equation of the axis is $y = 2$, and a horizontal shift of $\frac{\pi}{4}$ radians to the left has been applied. Write the equation of this function. $f(x)=3\cos\left(\frac{2}{3}\left(x+\frac{\pi}{4}\right)\right)+2$
In addition, sketch two cycles of the graph of this function.




**2.** Determine the value of the function in question 1 if $x = \frac{\pi}{2}, \frac{3\pi}{4}$, and $\frac{11\pi}{6}$.

Simply evaluate $f\left(\frac{\pi}{2}\right)$, $f\left(\frac{3\pi}{4}\right)$ and $f\left(\frac{11\pi}{6}\right)$.
Check the graph to see if your answers make sense.

**3.** Use your graph to estimate the $x$-value(s) in the domain $0 < x < 2$, where $y = 2.5$, to one decimal place.

$x$ is slightly more than $\frac{3\pi}{8}$

$x \doteq \frac{3\pi}{8} \doteq 1.2$

**4.** The number of hours of daylight in Vancouver can be modelled by a sinusoidal function of time, in days. The longest day of the year is June 21, with 15.7 h of daylight. The shortest day of the year is December 21, with 8.3 h of daylight. $p = -11$ (11 days before Jan. 1)

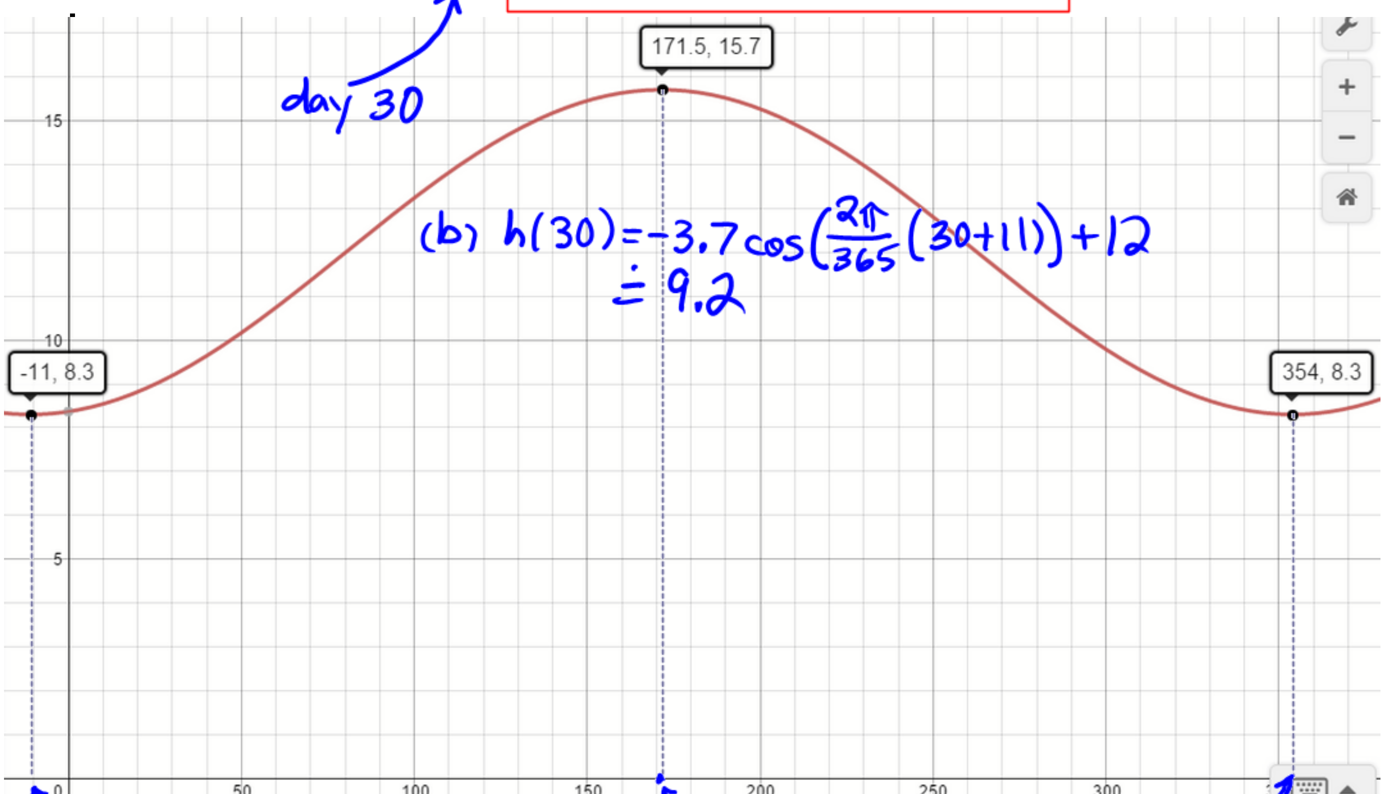
min $= 8.3$
max $= 15.7$
$A = \frac{15.7 - 8.3}{2} = \frac{7.4}{2} = 3.7$
$d = \frac{8.3 + 15.7}{2} = 12$
$T = 365$
$\omega = \frac{2\pi}{365}$

**a)** Find an equation for $h(t)$, the number of hours of daylight on the $t$th day of the year. In addition, sketch one cycle of the graph of this function.

**b)** Use your equation to predict the number of hours of daylight in Vancouver on January 30th.

$\boxed{h(t) = -3.7\cos\left(\frac{2\pi}{365}(t+11)\right) + 12}$

day 30



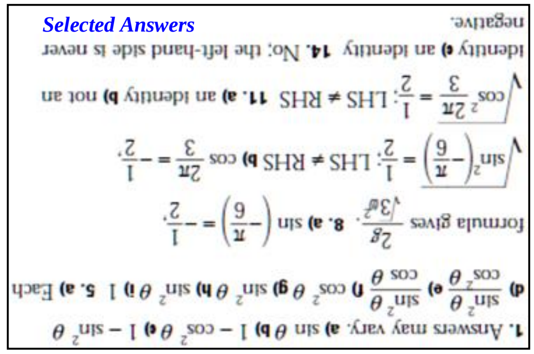

(b) $h(30) = -3.7\cos\left(\frac{2\pi}{365}(30+11)\right) + 12$
$\doteq 9.2$

Dec. 21    July 21 is day 171 of the year    Dec. 21

**16.** Write a list of helpful strategies for proving trigonometric identities, and describe situations in which you would try each strategy. Compare your list with your classmates'.

**17. Formulating problems** **a)** Create a trigonometric identity that has not appeared in this section.
**b)** Have a classmate check graphically that your equation may be an identity. If so, have your classmate prove your identity.

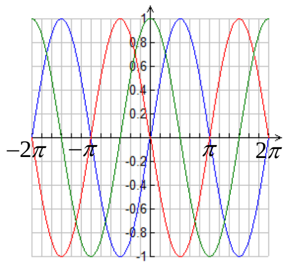
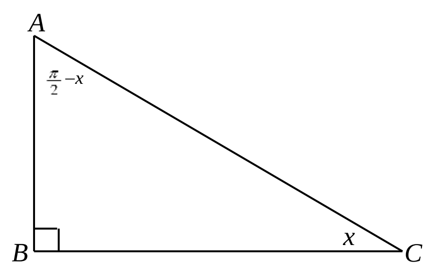
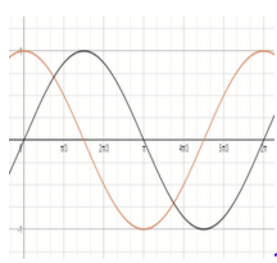
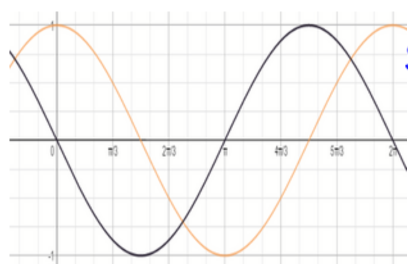
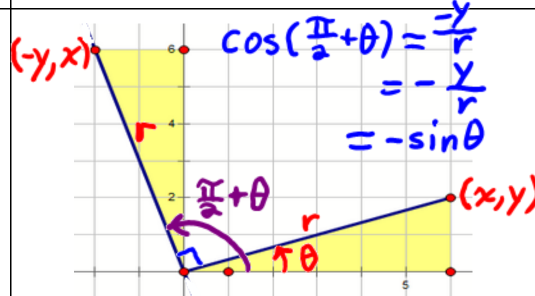
**18. Technology** **a)** Use a graph to show that the equation $\dfrac{\cos^2 x - 1}{\cos x + 1} = \cos x - 1$ appears to be an identity.
**b)** Compare the functions defined by each side of the equation by displaying a table of values. Find a value of $x$ for which the values of the two functions are not the same. Have you shown that the equation is not an identity? Explain.

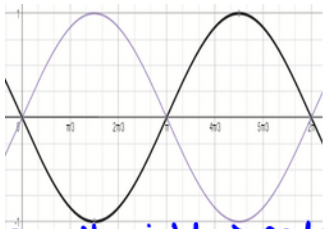
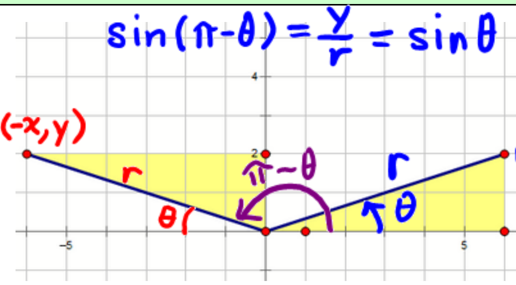
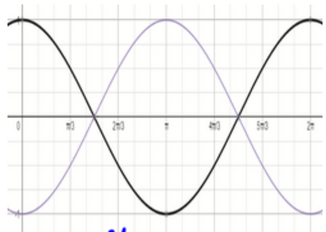
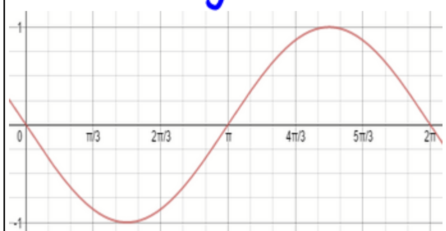
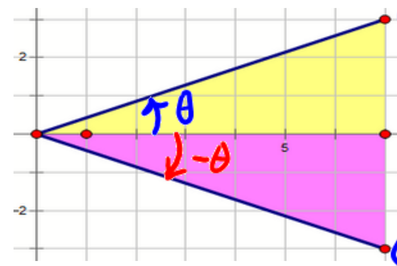
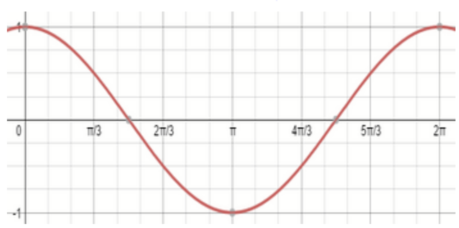



**20.** Prove that $\dfrac{\tan x \sin x}{\tan x + \sin x} = \dfrac{\tan x - \sin x}{\tan x \sin x}$.

## *Exercises on Equivalence of Trigonometric Expressions*

Complete the following table. The first row is done for you.

| *Identity* | *Graphical Justification* | *Justification using Right Triangle or Angle of Rotation* |
|---|---|---|
| $\sin(\frac{\pi}{2} - x) = \cos x$ | Since $\sin\left(\frac{\pi}{2} - x\right) = \sin\left(-1\left(x - \frac{\pi}{2}\right)\right)$, the graph of $y = \sin\left(\frac{\pi}{2} - x\right)$ can be obtained by reflecting $y = \sin x$ in the $y$-axis, followed by a shift to the right by $\frac{\pi}{2}$. Once these transformations are applied, lo and behold, the graph of $y = \cos x$ is obtained!  |  $\cos x = \dfrac{BC}{AC}$ <br> $\sin\left(\frac{\pi}{2} - x\right) = \dfrac{BC}{AC}$ <br> $\therefore \cos x = \sin\left(\frac{\pi}{2} - x\right)$ |
| $\cos(\frac{\pi}{2} - x) = \sin x$ |  $y = \cos(-1(x - \frac{\pi}{2}))$ <br> Reflect $y = \cos x$ in the $y$-axis, then translate $\frac{\pi}{2}$ to the right <br> $\rightarrow$ produces graph of $y = \sin x$ | Same diagram as above <br> $\sin x = \dfrac{AB}{AC}$, $\cos(\frac{\pi}{2} - x) = \dfrac{AB}{AC}$ <br> $\therefore \sin x = \cos(\frac{\pi}{2} - x)$ |
| $\cos(\frac{\pi}{2} + \theta) = -\sin\theta$ |  When the graph of $y = \cos\theta$ is shifted $\frac{\pi}{2}$ to the left, the graph of $y = -\sin\theta$ is obtained. |  $\cos(\frac{\pi}{2} + \theta) = \dfrac{-y}{r}$ <br> $= -\dfrac{y}{r}$ <br> $= -\sin\theta$ |

| *Identity* | *Graphical Justification* | *Justification using Angles of Rotation* |
|---|---|---|
| $\sin(-1(\theta-\pi))$<br>$\sin(\pi-\theta)=\sin\theta$<br>$y=\sin(-1(\theta-\pi))$<br>Reflect $y=\sin\theta$ in $y$-axis, then shift $\pi$ units right → produces | <br>→ graph of $y=\sin\theta$ | $\sin(\pi-\theta)=\frac{y}{r}=\sin\theta$<br><br> |
| $\cos(-1(\theta-\pi))$<br>$\cos(\pi-\theta)=-\cos\theta$<br>$y=\cos(-1(\theta-\pi))$<br>Reflect $y=\cos\theta$ in $y$-axis, then shift $\pi$ units right → produces | <br>→ graph of $y=-\cos\theta$ | Same diagram as previous row.<br>$\cos(\pi-\theta)=\frac{-x}{r}$<br>$=-\frac{x}{r}$<br>$=-\cos\theta$ |
| $\sin(-\theta)=-\sin\theta$<br>$y=\sin(-1\theta)$<br>Reflect $y=\sin\theta$ in $y$-axis → produces graph of $y=-\sin\theta$ |  | <br><br>$\sin(-\theta)$<br>$=\frac{-y}{r}$<br>$=-\sin\theta$ |
| $\cos(-\theta)=\cos\theta$<br>$y=\cos(-1\theta)$<br>Reflect $y=\cos\theta$ in $y$-axis → produces graph of $y=\cos\theta$ |  | Same diagram as previous row.<br>$\cos(-\theta)=\frac{x}{r}$<br>$=\cos\theta$ |

| *List of Important Identities that can be Discovered/Justified using Transformations* |
|---|
| **1.** Read the summary on page 43 (i.e. the next page). |
| **2.** Do the questions on page 44 for homework. |

# Trig Identities – Summary and Extra Practice

**1.** Complete the following statements:

**(a)** An ***equation*** is an ***identity*** if ______________________________________________

______________________________________________________________________.

**(b)** There are many different ways to confirm whether an equation is an identity. List ***at least three*** such ways.

______________________________________________________________________

______________________________________________________________________

______________________________________________________________________

**(c)** There is a very simple way to confirm that an equation is ***not*** an identity. In fact, this method can be used to show the falsity of any invalid mathematical statement. Describe the method and use it to demonstrate that the equation $\sqrt{x+y}=\sqrt{x}+\sqrt{y}$ is ***not*** an identity.

**2.** Mr. Nolfi asked Uday to prove that the equation $\dfrac{\sin 2x}{1+\cos 2x}=\tan x$ is an identity. What mark would Uday receive for the following response? Explain.

$$\frac{\sin 2x}{1+\cos 2x}=\tan x$$
$$\therefore \frac{2\sin x\cos x}{1+2\cos^2 x-1}=\tan x$$
$$\therefore \frac{2\sin x\cos x}{2\cos^2 x}=\tan x$$
$$\therefore \left(\frac{2}{2}\right)\left(\frac{\sin x}{\cos x}\right)\left(\frac{\cos x}{\cos x}\right)=\tan x$$
$$\therefore 1(\tan x)(1)=\tan x$$
$$\therefore \tan x=\tan x$$

**3.** List several strategies that can help you to prove that an equation is an identity.

______________________________________________________________________

______________________________________________________________________

______________________________________________________________________

______________________________________________________________________

______________________________________________________________________

______________________________________________________________________

______________________________________________________________________

**4.** Justify each of the following identities by using transformations and by using angles of rotation.

**(a)** $\sin(-x) = -\sin x$ **(b)** $\sin(\pi/2 - x) = \cos x$ **(c)** $\sin(x+\pi) = -\sin x$

**(d)** $\cos(-x) = \cos x$ **(e)** $\cos(\pi/2 - x) = \sin x$ **(f)** $\cos(x+\pi) = -\cos x$

**(g)** $\tan(-x) = -\tan x$ **(h)** $\tan(\pi/2 - x) = \cot x$ **(i)** $\tan(x+\pi) = \tan x$

**5.** Prove that each of the following equations is an identity:

**a)** $\dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin\theta\cos\theta} = 1 - \tan\theta$

**b)** $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$

**c)** $\tan^2 x - \cos^2 x = \dfrac{1}{\cos^2 x} - 1 - \cos^2 x$

**d)** $\dfrac{1}{1+\cos\theta} + \dfrac{1}{1-\cos\theta} = \dfrac{2}{\sin^2\theta}$

**6.** Prove that each of the following equations is an identity:

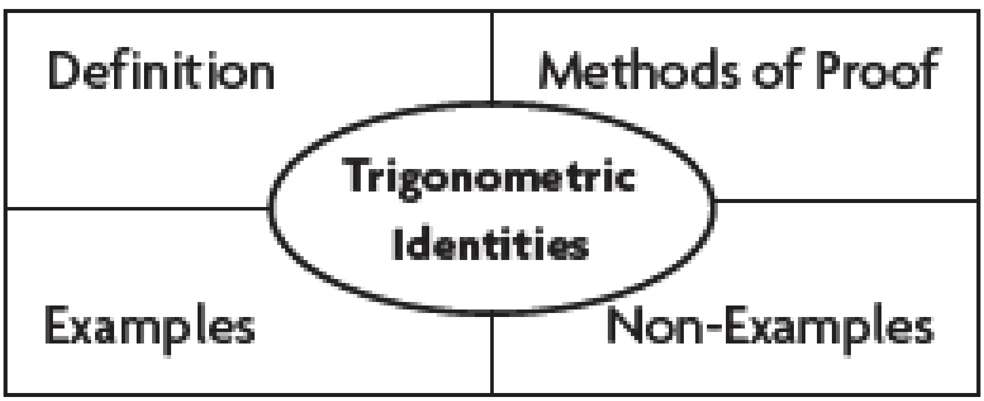
**a)** $\cos x \tan^3 x = \sin x \tan^2 x$

**b)** $\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta$

**c)** $(\sin x + \cos x)\left(\dfrac{\tan^2 x + 1}{\tan x}\right) = \dfrac{1}{\cos x} + \dfrac{1}{\sin x}$

**d)** $\tan^2\beta + \cos^2\beta + \sin^2\beta = \dfrac{1}{\cos^2\beta}$

**7.** Copy and complete the following Frayer diagram:




**8.** Express $8\cos^4 x$ in the form $a\cos 4x + b\cos 2x + c$. State the values of the constants *a*, *b* and *c*.

**9.** Give a counterexample to demonstrate that each of the following equations is not an identity.

**a)** $\cos x = \dfrac{1}{\cos x}$ **c)** $\sin(x+y) = \cos x \cos y + \sin x \sin y$

**b)** $1 - \tan^2 x = \sec^2 x$ **d)** $\cos 2x = 1 + 2\sin^2 x$

**10.** Demonstrate graphically that each of the equations in 9 is not an identity.

# Rates of Change in Trigonometric Functions

## *Introductory Investigation*

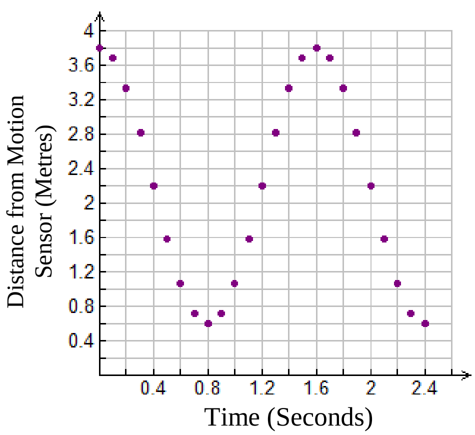
Vyshna was walking through a playground minding his own business when all of a sudden, he felt little Anshul tugging at his pants. "Vyshna, Vyshna!" little Anshul exclaimed. "Please push me on a swing!" Being in a hurry, Vyshna was a little reluctant to comply with little Anshul's request at first. Upon reflection, however, Vyshna remembered that he had to collect some data for his math homework. He reached into his knapsack and pulled out his very handy portable motion sensor. "Get on the swing Anshul!" Vyshna bellowed. "I'll set up the motion sensor in front of you and it will take some measurements as I push." Gleefully, little Anshul hopped into the seat of the swing and waited for Vyshna to start pushing.

The data collected by Vyshna's motion sensor are shown in the following tables. Time is measured in seconds and the distance, in metres, is measured from the motion sensor to little Anshul on the swing.

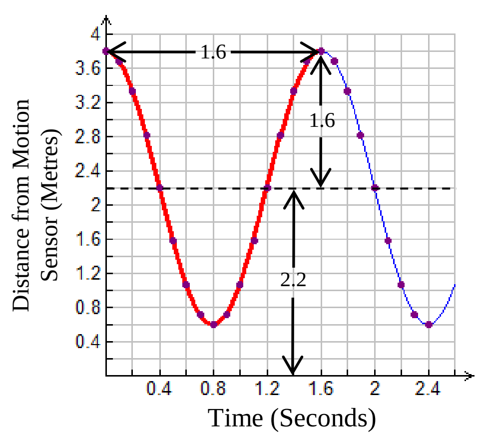
| Time (s) | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 | 1.1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Distance (m) | 3.8 | 3.68 | 3.33 | 2.81 | 2.2 | 1.59 | 1.07 | 0.72 | 0.6 | 0.72 | 1.07 | 1.59 |

| Time (s) | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1.8 | 1.9 | 2.0 | 2.1 | 2.2 | 2.3 | 2.4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Distance (m) | 2.2 | 2.81 | 3.33 | 3.68 | 3.8 | 3.68 | 3.33 | 2.81 | 2.2 | 1.59 | 1.07 | 0.72 | 0.6 |

### *Scatter Plot*




### *Curve of Best Fit*




### *Mathematical Model*

$|A| = \text{Amplitude} = 1.6$

$T = \text{Period} = 1.6 = \left|\frac{1}{\omega}\right| 2\pi$

$\omega = \text{Angular Frequency}$

$= \left(\frac{1}{T}\right)2\pi = \left(\frac{1}{1.6}\right)2\pi = \frac{5\pi}{4}$

$d = \text{vertical displacement} = 2.2$

$p = \text{phase shift} = 0$

$\therefore d(t) = 1.6\cos\left(\frac{5\pi}{4}t\right) + 2.2$

(Alternatively, we could write $d(t) = 1.6\sin\left(\frac{5\pi}{4}(t-1.2)\right) + 2.2$ )

## *Questions*

**1.** What quantity is measured by

**(a)** the slope of the secant line through the points $(t_1, d(t_1))$ and $(t_2, d(t_2))$?

**(b)** the slope of the tangent line at $(t, d(t))$?

**2.** Complete the following table.

| *Intervals of Time over which Anshul approaches the Motion Sensor* | *Intervals of Time over which Anshul recedes from the Motion Sensor* | *Intervals of Time over which Anshul's Speed Increases* | *Intervals of Time over which Anshul's Speed Decreases* |
|---|---|---|---|
| | | | |

**3.** Explain the difference between speed and velocity.

**4.** Describe the "shape" of the curve over the intervals of time during which

**(a)** Anshul's velocity is increasing

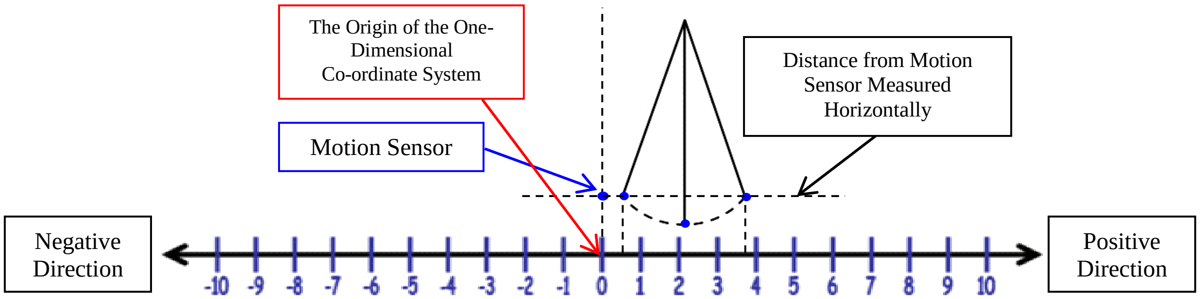
**(b)** Anshul's velocity is decreasing

**5.** Use the function given above to ***calculate*** the ***average*** rate of change of distance from the motion sensor with respect to time between 0.2 s and 1.0 s. Is your answer negative or positive? Interpret your result geometrically (i.e. as a slope) and physically (i.e. as a velocity).

**6.** Use the function given above to ***estimate*** the ***instantaneous*** rate of change of distance from the motion sensor with respect to time at 0.6 s. Is your answer negative or positive? Interpret your result geometrically (i.e. as a slope) and physically (i.e. as a velocity).

### *Rectilinear (Linear) Motion*

- ***Rectilinear*** or ***linear*** motion is motion that occurs along a ***straight line***.
- Rectilinear motion can be described fully using a ***one-dimensional co-ordinate system***.
- Strictly speaking, Anshul's swinging motion is not rectilinear because he moves along a curve (see diagram).
- However, since only the horizontal distance to the motion sensor is measured, we can imagine that Anshul is moving along the horizontal line that passes through the motion sensor (see diagram). A more precise interpretation is that the equation given above models the position of Anshul's *x*-co-ordinate with respect to time.

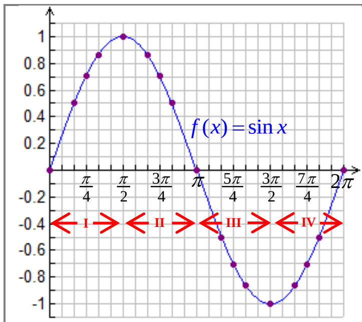



The table below lists the meanings of various quantities that are used to describe one-dimensional motion.

| *Quantity* | *Meaning and Description* | *Properties* |
|---|---|---|
| Position | The ***position*** of an object measures ***where*** the object is located at any given time. In linear motion, the position of an object is simply a number that indicates ***where it is*** with respect to a number line like the one shown above. Usually, the position function of an object is written as $s(t)$. | At any time *t*, if the object is located<br>**(a)** ***at*** the origin, then $s(t)=0$<br>**(b)** to the ***right*** of the origin, then $s(t)>0$<br>**(c)** to the ***left*** of the origin, then $s(t)<0$<br>Also, $\lvert s(t)\rvert$ is the distance from the object to the origin. |
| Displacement | The ***displacement*** of an object between the times $t_1$ and $t_2$ is equal to its ***change in position*** between $t_1$ and $t_2$. That is, **displacement** $=\Delta s = s(t_2)-s(t_1)$. | If $\Delta s>0$, the object is to the ***right*** of its initial position.<br>If $\Delta s<0$, the object is to the ***left*** of its initial position.<br>If $\Delta s=0$, the object is ***at*** its initial position. |
| Distance | ***Distance*** measures ***how far*** an object has travelled. Since an object undergoing linear motion can change direction, the distance travelled is found by summing (adding up) the absolute values of all the displacements for which there is a ***change in direction***. | The position of an object undergoing linear motion is tracked between times $t_0$ and $t_n$. In addition, the object changes direction at times $t_1, t_2, \ldots, t_{n-1}$ (and at no other times), where $t_0 < t_1 < \ldots < t_{n-1} < t_n$. If $\Delta s_i$ represents the displacement from time $t_{i-1}$ to time $t_i$, then the total distance travelled is equal to<br>$d = \lvert\Delta s_1\rvert + \lvert\Delta s_2\rvert + \cdots + \lvert\Delta s_{n-1}\rvert + \lvert\Delta s_n\rvert$ |
| Velocity | ***Velocity*** is the ***instantaneous rate of change of position with respect to time***. Velocity measures ***how fast*** an object moves as well as its ***direction*** of travel. In one-dimensional rectilinear motion, velocity can be negative or positive, depending on the direction of travel. | At any time *t*, if the object is<br>**(a)** moving in the ***positive*** direction, then $v(t)>0$<br>**(b)** moving in the ***negative*** direction, then $v(t)<0$<br>**(c)** at ***rest***, then $v(t)=0$<br>Also, $\lvert v(t)\rvert$ is the ***speed*** of the object. |
| Speed | ***Speed*** is simply a measure of how fast an object moves ***without regard to its direction of travel***. | **speed** $=\lvert v(t)\rvert$ |

## *Exercises*

Determine the tendency of $f(x)$.




As $x \to 0^+$, $f(x) \to$ ______

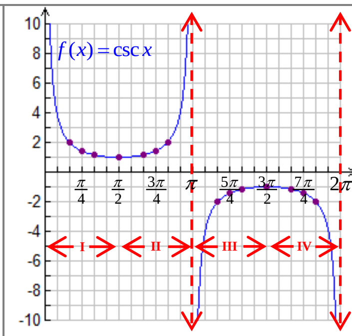
As $x \to 0^-$, $f(x) \to$ ______

As $x \to 0$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^+$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}$, $f(x) \to$ ______




As $x \to 0^+$, $f(x) \to$ ______

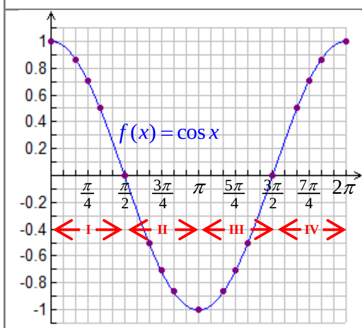
As $x \to 0^-$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}$, $f(x) \to$ ______

As $x \to \frac{3\pi}{2}$, $f(x) \to$ ______

As $x \to \pi^-$, $f(x) \to$ ______

As $x \to \pi^+$, $f(x) \to$ ______




As $x \to 0^+$, $f(x) \to$ ______

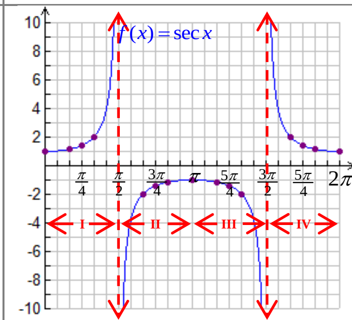
As $x \to 0^-$, $f(x) \to$ ______

As $x \to 0$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^+$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}$, $f(x) \to$ ______




As $x \to 0$, $f(x) \to$ ______

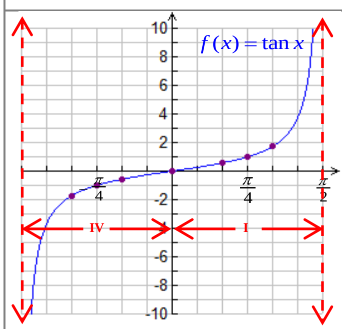
As $x \to \pi$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^+$, $f(x) \to$ ______

As $x \to \frac{3\pi}{2}^-$, $f(x) \to$ ______

As $x \to \frac{3\pi}{2}^+$, $f(x) \to$ ______




As $x \to 0$, $f(x) \to$ ______

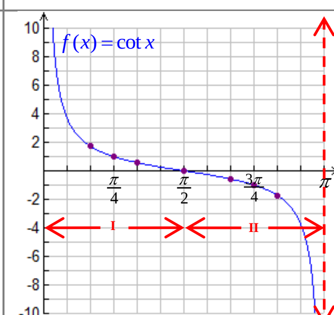
As $x \to \frac{\pi}{4}$, $f(x) \to$ ______

As $x \to \frac{\pi}{6}$, $f(x) \to$ ______

As $x \to -\frac{\pi}{2}^+$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ ______

As $x \to \frac{3\pi}{2}^+$, $f(x) \to$ ______




As $x \to 0^+$, $f(x) \to$ ______

As $x \to \frac{\pi}{4}$, $f(x) \to$ ______

As $x \to \frac{\pi}{6}$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}$, $f(x) \to$ ______

As $x \to \pi^-$, $f(x) \to$ ______

As $x \to \pi^+$, $f(x) \to$ ______