Unit 3 – Trigonometric Functions

Essential Concepts of Trigonometry

Introduction

To a great extent, this unit is just an extension of the trigonometry that you studied in grade 10. Given below is a list of the additional topics and concepts that are covered in this course.

- For the most part, angles will be measured in ***radians*** instead of degrees.
- Angles of rotation will be introduced and extended beyond the range $0° \le \theta \le 360°$
- Your knowledge of transformations will be used extensively to help you understand trigonometric functions.
- The reciprocal trigonometric functions csc, sec and cot will be studied in detail.
- Trigonometric identities will be studied in detail.

What is Trigonometry?

Trigonometry (Greek *trigōnon* "triangle" + *metron* "measure") is a branch of mathematics that deals with the ***relationships among the interior angles and side lengths of triangles***, as well as with the study of trigonometric functions. Although the word "trigonometry" emerged in the mathematical literature only about 500 years ago, the origins of the subject can be traced back more than 4000 years to the ancient civilizations of Egypt, Mesopotamia and the Indus Valley. Trigonometry has evolved into its present form through important contributions made by, among others, the Greek, Chinese, Indian, Sinhalese, Persian and European civilizations.

Why Triangles?

Triangles are the basic building blocks from which any shape (with straight boundaries) can be constructed. A square, pentagon or any other polygon can be divided into triangles, for instance, using straight lines that radiate from one vertex to all the others.

Examples of Problems that can be solved using Trigonometry

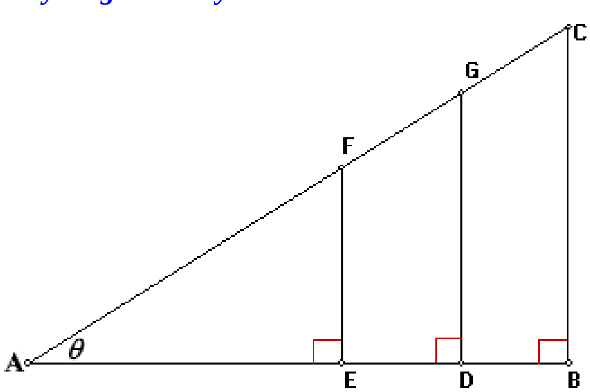
☺ How tall is Mount Everest? How tall is the CN Tower?
☺ What is the distance from the Earth to the sun? How far is the Alpha Centauri star system from the Earth?
☺ What is the diameter of Mars? What is the diameter of the sun?
☺ At what times of the day will the tide come in?

General Applications of Trigonometric Functions

Trigonometry is one of the most widely applied branches of mathematics. A small sample of its myriad uses is given below. ***The power of trigonometry is that it relates angles to distances***. Since it is much easier in general to measure angles than it is to measure distances, trigonometric relationships give us a method to calculate distances that are otherwise inaccessible.

Application	***Examples***
Modelling of ***cyclic*** processes	Orbits, Hours of Daylight, Tides
Measurement	Navigation, Engineering, Construction, Surveying
Electronics	Circuit Analysis (Modelling of Voltage Versus Time in AC Circuits, Fourier Analysis)

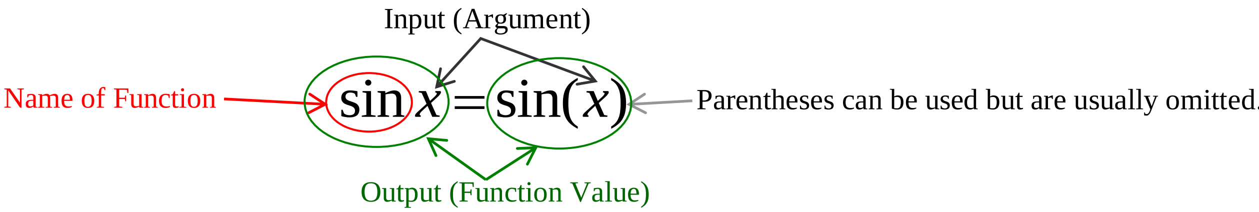
Why Trigonometry Works

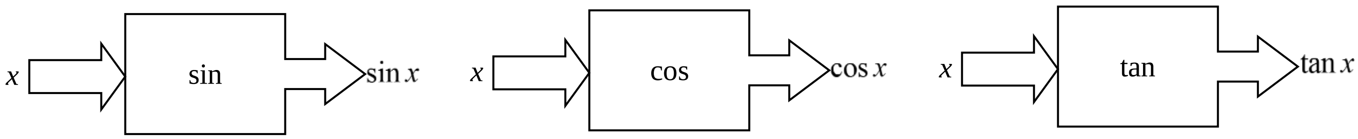


If you study the diagram at the left carefully, you will notice…

- $\Delta AFE \sim \Delta AGD \sim \Delta ACB$ (by AA similarity)
- Because of triangle similarity, the ratio of side lengths of corresponding sides is ***constant***! (**e.g.** $\frac{FE}{AE} = \frac{GD}{AD} = \frac{CB}{AB}$)
- Therefore, trigonometric ratios, which are nothing more than ratios of side lengths, depend ***only*** on the angles within a right triangle, ***NOT*** on the size of the triangle. Whether the right triangle is as miniscule as an atom or as vast as a galaxy, the ratios depend only on the angles.

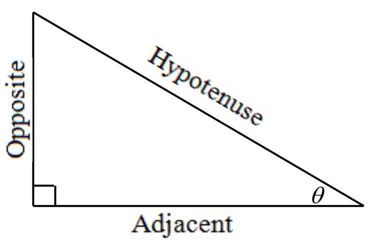
Extremely Important Note on the Notation of Trigonometry





Trigonometry of Right Triangles – Trigonometric Ratios of Acute Angles

Right triangles can be used to define the trigonometric ratios of ***acute*** angles (angles that measure less than 90º).



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

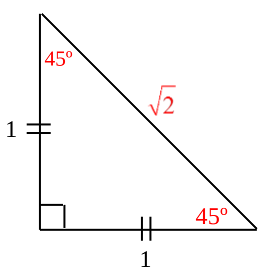
SOH CAH TOA

"**S**hout **O**ut **H**ey" "**C**anadians **A**re **H**ot" "**T**ight **O**iled **A**bs"

Have fun by creating your own mnemonic! BUT REMEMBER! THIS IS ***NOTHING MORE*** THAN A MNEMONIC. It is not a substitute for the understanding of concepts!

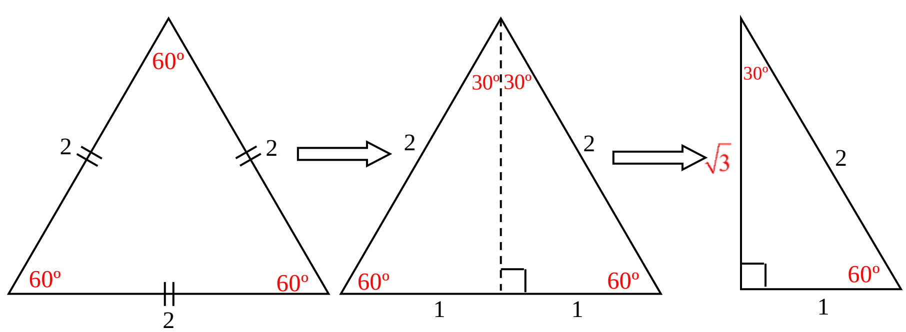
The Special Triangles – Trigonometric Ratios of Special Angles

For certain ***special angles***, it is possible to calculate the ***exact value*** of the trigonometric ratios. As I have mentioned on many occasions, it is not advisable to memorize without understanding! Instead, you can deduce the values that you need to calculate the trig ratios by ***understanding*** the following triangles!



- Isosceles Right Triangle
- Let the length of the equal sides be 1 unit.
- By the Pythagorean Theorem, the length of the hypotenuse must be $\sqrt{2}$.

$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{1}{\sqrt{2}} \qquad \tan 45° = \frac{1}{1} = 1$$



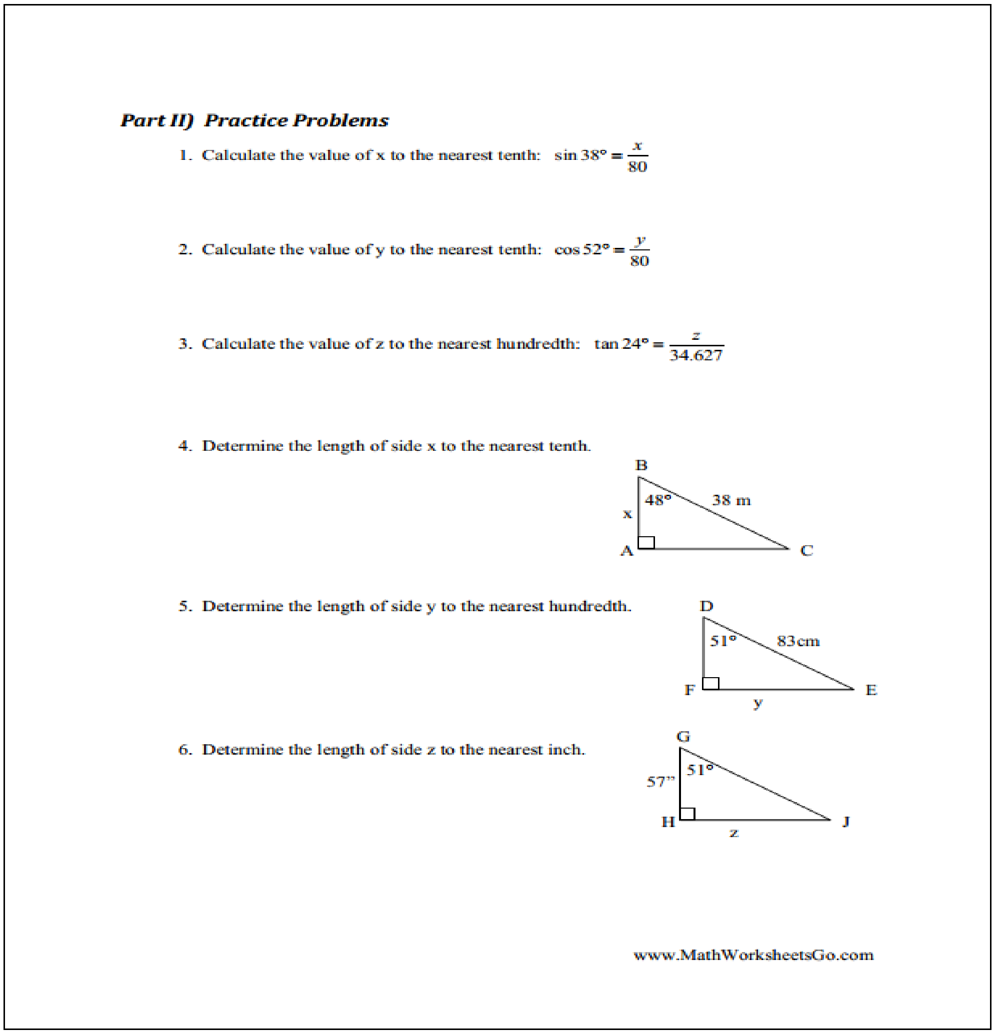
- Begin with an equilateral triangle having sides of length 2 units. Then cut it in half to form a 30º-60º-90º right triangle.
- Use the Pythagorean Theorem to calculate the height of the triangle. ($\sqrt{3}$)

$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}}$$

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \sqrt{3}$$

Homework

Click anywhere in the following box to open a PDF document that contains your homework.



Part II) Practice Problems

1. Calculate the value of x to the nearest tenth: $\sin 38° = \frac{x}{80}$

2. Calculate the value of y to the nearest tenth: $\cos 52° = \frac{y}{80}$

3. Calculate the value of z to the nearest hundredth: $\tan 24° = \frac{z}{34.627}$

4. Determine the length of side x to the nearest tenth.

5. Determine the length of side y to the nearest hundredth.

6. Determine the length of side z to the nearest inch.

www.MathWorksheetsGo.com

In-Class Practice

Use special triangles to complete the following table. The value of each trigonometric ratio must be ***exact***!

	30°	45°	60°
sin	$\sin 30° =$ ____________	$\sin 45° =$ ____________	$\sin 60° =$ ____________
cos	$\cos 30° =$ ____________	$\cos 45° =$ ____________	$\cos 60° =$ ____________
tan	$\tan 30° =$ ____________	$\tan 45° =$ ____________	$\tan 60° =$ ____________
$\frac{\sin}{\cos}$	$\frac{\sin 30°}{\cos 30°} =$ ____________	$\frac{\sin 45°}{\cos 45°} =$ ____________	$\frac{\sin 60°}{\cos 60°} =$ ____________

What did you notice about the last two rows of the table? Do you think it's true for all angles?

SUMMARY: ESSENTIAL IDEAS OF TRIGONOMETRY

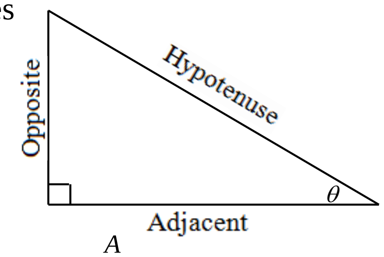
Trigonometry Relates Angles to Distances

- Trigonometry is powerful because it relates ***angles*** to ***distances***.
- Distances can be difficult to measure directly.
- Angles are generally easy to measure directly.
- Trigonometric relationships allow us to calculate distances that are difficult or impossible to measure directly.

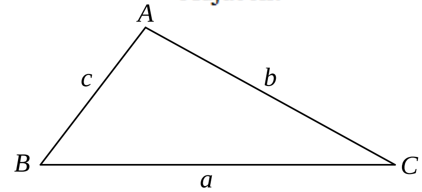
Trigonometric Relationships in Triangles

- ***Ratios of side lengths*** in triangles ***depend only on the interior angles*** of triangles not on the "size" of the triangle. This is the case because of ***triangle similarity***.
- Trigonometric relationships in ***right*** triangles:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



- Trigonometric relationships in ***all*** triangles (Law of Sines, Law of Cosines):

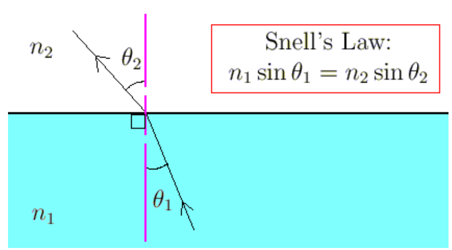


$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad\qquad c^2 = a^2 + b^2 - 2ab\cos C$$

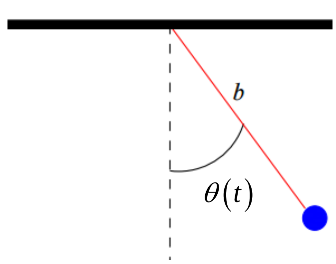
The Law of Cosines is a generalization of the Pythagorean Theorem. When angle C has a measure of 90°, the term $-2ab\cos C = 0$. The reason for this will become clear when we study angles of rotation.

Examples of Trigonometric Relationships in the Physical World

- **Snell's Law:** When a ray of light passes from one medium to another, its path changes direction. This "bending of light" is known as ***refraction***. The amount by which the path of the ray of light changes depends on the angle that the incident ray makes with the normal to the surface at the point of refraction and also on the media through which the light rays are travelling. This dependence is made explicit in Snell's Law via *refractive indices*, numbers which are constant for given media.



- All "well-behaved" periodic processes can be modelled using trigonometric functions or some combination of trigonometric functions. For example, consider a pendulum. Let t represent time and let $\theta(t)$ represent the angle between the current position of the pendulum and its rest position. The angle θ is taken to be positive if the pendulum is to the right of its rest position and negative otherwise. If the pendulum is initially held at a ***small angle*** α > 0 and then released, that is, $\theta(0) = \alpha$, then if friction is ignored, it can be shown that $\theta(t) = \alpha\cos\left(\sqrt{\frac{g}{b}}\,t\right)$, where g represents acceleration due to gravity and b represents the length of the pendulum.

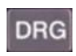
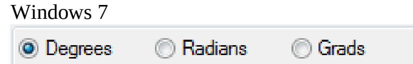


Radian Measure

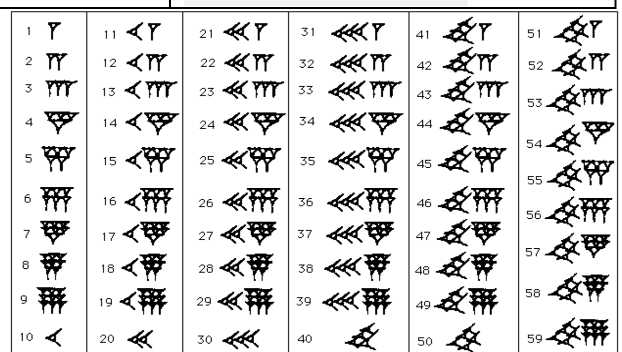
Summary of Various Units for Measuring Angles

Degree Measure	***Radian Measure***	***Grad (also known as Gon, Grade, Gradian) Measure***
• 360 degrees in one full revolution • Very well suited and widely used for practical applications because one degree is a small unit • For greater precision, one degree can be subdivided further into ***minutes*** (') and ***seconds*** (") • There are 60 *minutes* or *arcminutes* in a degree, 60 *seconds* or *arcseconds* in a minute **e.g.** Central Peel's location 43°, 41', 49" N (or 43.6969°) 79°, 44', 59" W (or –79.7496°)	• $2\pi \doteq 6.28$ radians in one full revolution • Not well-suited to practical applications because one radian is a rather large unit • Very well-suited to theory because the radian turns out to be dimensionless. Thus, the form of equations of trigonometric functions is simplified, especially for the purpose of differentiation and integration in calculus.	• 400 grads in one full revolution • Very well suited for practical applications because one grad is a small unit • The creation of the grad was an attempt to bring angle measure in line with the metric system (i.e. based on ten) • This idea never gained much momentum but most scientific calculators support the grad

Calculator Use

Scientific Calculator	***Windows Calculator***
 This key is used to switch among degrees, radians and grads mode. Whenever you are working with angles, make sure that your calculator is in the correct mode.	Windows 7  Windows 10 Click to Change Mode 

Why 360 Degrees in one Full Revolution?



The 59 symbols used by the Babylonians. These symbols are built from the two basic symbols 𒁹 and 𒌋, representing one and ten respectively.

The number 360 as the number of "degrees" in a circle, and hence the unit of a degree as a sub-arc of $\frac{1}{360}$ of the circle, was probably adopted because it approximates the number of days in a year. Its use is thought to originate from the methods of the ancient Babylonians, who used a sexagesimal number system (a number system with ***sixty*** as the base). Ancient astronomers noticed that the stars in the sky, which circle the celestial pole every day, seem to advance in that circle by approximately one-360th of a circle, that is, one degree, each day. Primitive calendars, such as the Persian Calendar used 360 days for a year. Its application to measuring angles in geometry can possibly be traced to Thales of Miletus, who popularized geometry among the Greeks and lived in Anatolia (modern western Turkey) among people who had dealings with Egypt and Babylon.

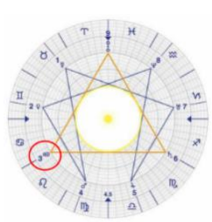
Another motivation for choosing the number 360 is that it is readily divisible: 360 has 24 divisors (including 1 and 360), including every number from 1 to 10 except 7. For the number of degrees in a circle to be divisible by every number from 1 to 10, there would need to be 2520 in a circle, which is a much less convenient number.

Divisors of 360: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360

The division of the circle into 360 parts also occurred in ancient Indian cosmology, as seen in the Rigveda:

Twelve spokes, one wheel, navels three.
Who can comprehend this?
On it are placed together
three hundred and sixty like pegs.
They shake not in the least.

(Dirghatama, Rig Veda 1.164.48)



"Dividing the circle (wheel) into 12 parts (spokes), and 360 degrees (pegs), we know that the circle in question is the ***ecliptic plane*** where Earth and all the planets travel in their respective orbits around the Sun. These are our 12 months of the year.

To locate where the number 3 falls we see that 3 is one of the angles of the inner triangle (gold-orange colour), and that it falls in the second quarter of the wheel, the Vital."

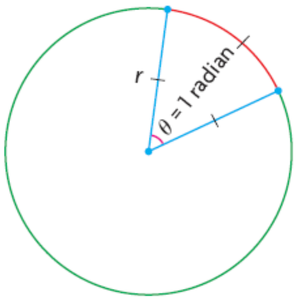
Source: http://www.aeoncentre.com/articles/conundrum-india-choice-destiny-4)

Warning: The site from which this information was taken relies a great deal on supernatural beliefs, mysticism, numerology and astrology, none of which has any basis in science.

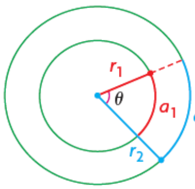
Definition of the Radian

radian

the size of an angle that is subtended at the centre of a circle by an arc with a length equal to the radius of the circle; both the arc length and the radius are measured in units of length (such as centimetres) and, as a result, the angle is a real number without any units



1 radian is defined as the angle subtended by an arc length, l, equal to the radius, r. It appears as though 1 radian should be a little less than 60°, since the sector formed resembles an equilateral triangle, with one side that is curved slightly.



It is important to note that the size of an angle in radians is not affected by the size of the circle. The diagram shows that a_1 and a_2 subtend the same angle θ, so $\theta = \frac{a_1}{r_1} = \frac{a_2}{r_2}$.

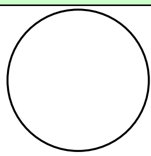
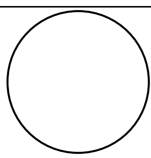
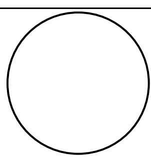
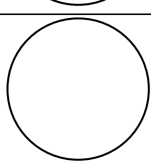
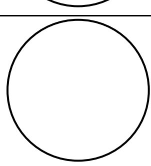
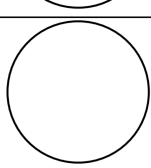
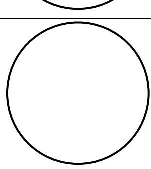
Verb: subtend sub'tend

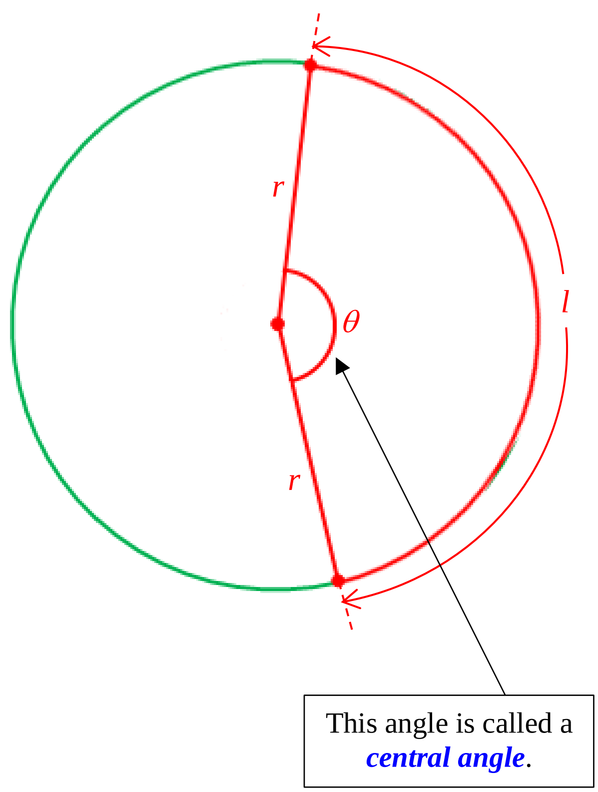
1. Be opposite to; of angles and sides, in geometry

> ***Important Note***
> Whenever the units of angle measure are not specified, the units are ***assumed to be radians***.

Investigation – The Relationship among θ, r and l

When θ is measured in radians, there is a very simple equation that relates r (the radius of the circle), θ (the angle at the centre of the circle) and l (the length of the arc that subtends the angle θ). The purpose of this investigation is to discover this relationship. Complete the table below and then answer the question at the bottom of the page.

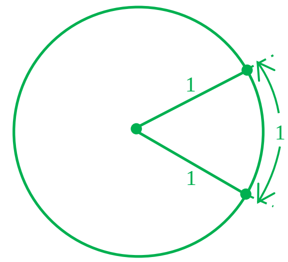
Diagram	***r***	***θ***	***l***
	1	1	
	1	2	
	1	3	
	2	1	
	2	2	
	2	3	
	3	1.5	

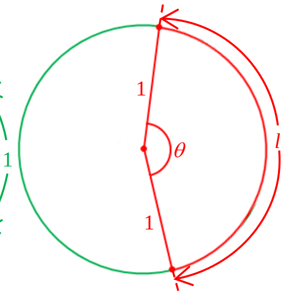


A more Analytical Approach to finding the Relationship among θ, r and l

How many Radians are there in One Full Revolution?

First, we need to establish the number of radians in one full revolution. We can accomplish this by considering a ***unit circle*** (a circle of radius 1). It is easy to see that for such a circle, $l=\theta$. For example, if $\theta=1$, then by the definition of a radian, $l=1$. Similarly, if $\theta=2$, then $l=2$. For an arc whose length is equal to the circumference of the circle, $l=C=2\pi r=2\pi(1)=2\pi$. Since $l=\theta$, for one complete revolution $\theta=2\pi$.



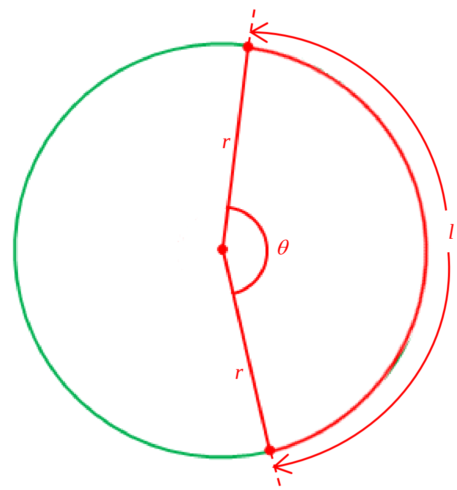


Therefore, one full revolution = 2π radians.

How is θ related to the "Amount of Rotation?"

It should be obvious that the angle θ determines the ***fraction*** of one full revolution. For instance, consider the examples in the table given below.

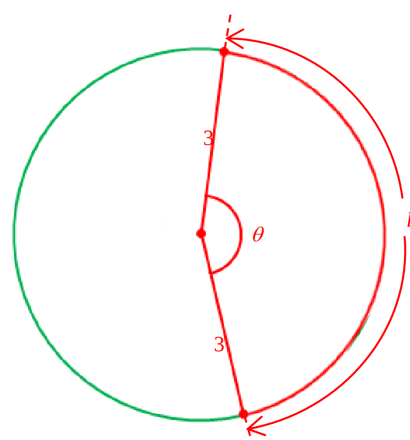
θ (rad)	*Fraction of One Revolution*	*How Fraction is Calculated*
$\frac{\pi}{4}$	$\frac{1}{8}$	$\frac{\theta}{2\pi}=\left(\frac{\pi}{4}\right)\Big/(2\pi)=\left(\frac{\pi}{4}\right)\left(\frac{1}{2\pi}\right)=\frac{1}{8}$
$\frac{\pi}{2}$	$\frac{1}{4}$	$\frac{\theta}{2\pi}=\left(\frac{\pi}{2}\right)\Big/(2\pi)=\left(\frac{\pi}{2}\right)\left(\frac{1}{2\pi}\right)=\frac{1}{4}$
π	$\frac{1}{2}$	$\frac{\theta}{2\pi}=(\pi)\Big/(2\pi)=\left(\frac{\pi}{2\pi}\right)=\frac{1}{2}$
$\frac{3\pi}{2}$	$\frac{3}{4}$	$\frac{\theta}{2\pi}=\left(\frac{3\pi}{2}\right)\Big/(2\pi)=\left(\frac{3\pi}{2}\right)\left(\frac{1}{2\pi}\right)=\frac{3}{4}$
θ	$\frac{\theta}{2\pi}$	$\frac{\theta}{2\pi}=\frac{\text{angle of rotation}}{\text{angle for one full rotation}}$



How is l related to the Circumference of a Circle?

It should also be obvious that l determines the ***fraction*** of the circumference of a circle. Consider the following table for a circle with $r = 3$ units and $C=2\pi r=2\pi(3)=6\pi$ units.

θ (rad)	*l*	*Fraction of the Circumference*
$\frac{\pi}{4}$ (1/8 of a circle)	$\frac{6\pi}{8}=\frac{3\pi}{4}=3\theta$	$\frac{l}{C}=\left(\frac{3\pi}{4}\right)\Big/(6\pi)=\left(\frac{3\pi}{4}\right)\left(\frac{1}{6\pi}\right)=\frac{1}{8}$
$\frac{\pi}{2}$ (1/4 of a circle)	$\frac{6\pi}{4}=\frac{3\pi}{2}=3\theta$	$\frac{l}{C}=\left(\frac{3\pi}{2}\right)\Big/(6\pi)=\left(\frac{3\pi}{2}\right)\left(\frac{1}{6\pi}\right)=\frac{1}{4}$
π (1/2 of a circle)	$\frac{6\pi}{2}=3\pi=3\theta$	$\frac{l}{C}=(3\pi)\Big/(6\pi)=\left(\frac{3\pi}{6\pi}\right)=\frac{1}{2}$
$\frac{3\pi}{2}$ (3/4 of a circle)	$3\left(\frac{3\pi}{2}\right)=\frac{9\pi}{2}=3\theta$	$\frac{l}{C}=\left(\frac{9\pi}{2}\right)\Big/(6\pi)=\left(\frac{9\pi}{2}\right)\left(\frac{1}{6\pi}\right)=\frac{3}{4}$
θ	3θ	$\frac{l}{C}=\frac{3\theta}{6\pi}=\frac{\theta}{2\pi}$



An important observation to make at this point is that $\frac{l}{C}=\frac{\theta}{2\pi}$. That is, the ratio of the length of the arc to the circumference of the circle is equal to the ratio of the angle subtended by the arc to the number of radians in one full revolution. Now, by recalling that $C=2\pi r$, we can write the above proportion as

$$\frac{l}{2\pi r}=\frac{\theta}{2\pi}$$

By multiplying both sides by $2\pi r$, we obtain $l=r\theta$.

Summary: Calculating the Length of an Arc

Let r represent the radius of a circle and θ represent the measure of an angle at the centre of the circle. If θ is subtended by an arc whose length is l, then

$$l = r\theta$$

$C = 2\pi r$ is a Special Case of $l = r\theta$

Note that the equation $l = r\theta$ is a generalized form of $C = 2\pi r$. In the case of $C = 2\pi r$, $l = C$ and $\theta = 2\pi$.

Converting between Radians and Degrees

We know that one full revolution is equal to 2π radians ***and*** that one full revolution is equal to 360°.

$$\therefore 2\pi \text{ rad} = 360°$$

$$\therefore \pi \text{ rad} = 180°$$

By remembering that π rad = 180°, you will be able to convert easily between radians and degrees

Radians to Degrees	***Degrees to Radians***
$\pi \text{ rad} = 180°$ $\therefore 1 \text{ rad} = \frac{180°}{\pi}$ $\therefore x \text{ rad} = \frac{x(180°)}{\pi}$	$180° = \pi \text{ rad}$ $\therefore 1° = \frac{\pi}{180} \text{rad}$ $\therefore x° = \frac{x\pi}{180} \text{rad}$

Examples

1. Convert 6 radians to degrees.

Solution

$$\pi \text{ rad} = 180°$$
$$\therefore 1 \text{ rad} = \frac{180°}{\pi}$$
$$\therefore 6 \text{ rad} = \frac{6(180°)}{\pi}$$
$$= \frac{1080°}{\pi}$$
$$\doteq 343.8°$$

We can be confident that this answer is correct because 6 radians is just short of one full revolution, as is 343.8°.

2. Convert 972° to radians.

Solution

$$180° = \pi \text{ rad}$$
$$\therefore 1° = \frac{\pi}{180} \text{rad}$$
$$\therefore 972° = \frac{972\pi}{180} \text{rad}$$
$$= \frac{27\pi}{5} \text{rad}$$
$$\doteq 16.96 \text{ rad}$$

We can be confident that this answer is correct because 972° falls short of 3 full revolutions by about 100°, as does 16.96 rad. (3 full revs $\doteq 18.85$ rad)

Special Angles

As shown in the following table, it is very easy to convert between degrees and radians for certain special angles.

Angle in Degrees	30°	45°	60°	90°
Angle in Radians	$\frac{180°}{6} = \frac{\pi}{6}$	$\frac{180°}{4} = \frac{\pi}{4}$	$\frac{180°}{3} = \frac{\pi}{3}$	$\frac{180°}{2} = \frac{\pi}{2}$

In addition, it is also very easy to convert between radians and degrees for multiples of the special angles. Examples are shown below.

$$150° = 5(30°) = \frac{5\pi}{6} \qquad 240° = 4(60°) = \frac{4\pi}{3} \qquad 315° = 7(45°) = \frac{7\pi}{4} \qquad 270° = 3(90°) = \frac{3\pi}{2}$$

Radians are Dimensionless

Since $l = \theta r$, it follows that $\theta = \frac{l}{r}$. Because both l and r are measured in units of distance, the units "divide out."

$$\theta = \frac{l}{r}$$

Both are measured in units of distance. Therefore, the units "divide out" and θ turns out to be dimensionless.

This means that when θ is measured in radians, it is a dimensionless number, that is, a pure real number. Because of this, the radian is very well-suited to theoretical purposes since functions operate on real numbers and not on angles measured in degrees or any other unit.

Angular Frequency (Angular Speed)

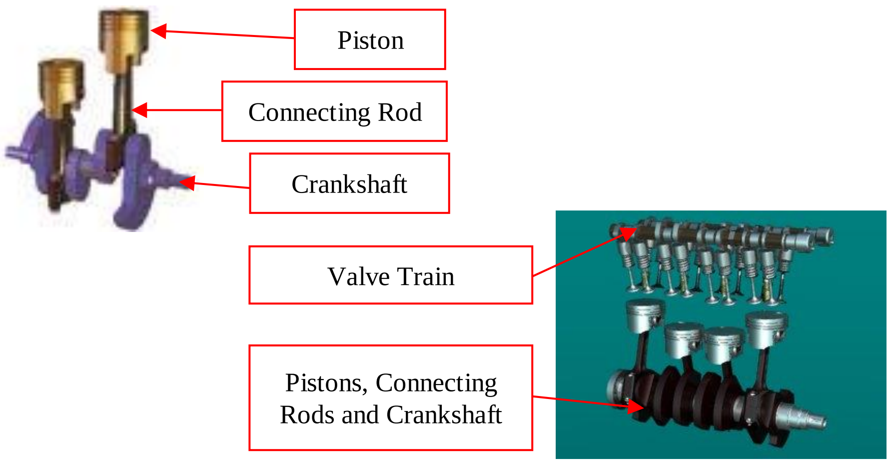
Angular frequency or ***angular speed*** is the rate at which an object rotates. The Greek letter ω (lowercase omega) is often used to denote angular frequency.

Example 1

The RPM gauge of a car measures the speed at which the crankshaft (see pictures below) rotates in ***revolutions per minute***. While Victor was driving through a school zone, his RPM gauge read 9500 RPM. Convert this value to radians/second.

Solution

$$\begin{aligned}\omega &= 9500 \text{ RPM} \\ &= \frac{9500 \text{ RPM}}{60 \text{ s/min}} \\ &= \frac{475}{3} \text{ rot/s} \\ &= \left(\frac{475}{3} \text{ rot/s}\right)(2\pi \text{ rad/rot}) \\ &= \frac{950\pi}{3} \text{ rad/s} \\ &\doteq 994.8 \text{ rad/s}\end{aligned}$$



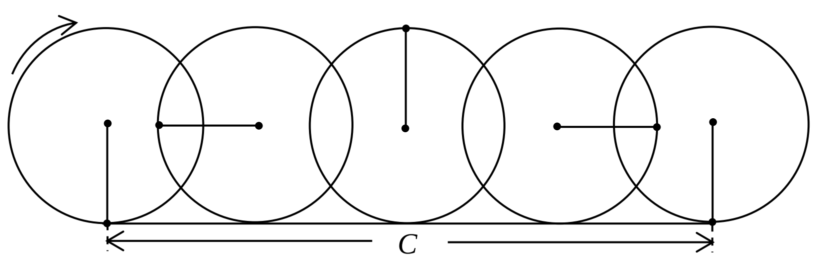
The angular frequency of Victor's crankshaft is about 994.8 rad/s.

Example 2

While Victor was driving his turbo-charged Volvo on the 410, the wheels of his car were spinning with an angular frequency of 200 rad/s. If the radius of each wheel is 40 cm, how far will Victor's Volvo travel in 15 minutes?

Solution

The ***crux*** of this problem is to make the ***connection*** between the ***circumference*** of the wheel and the ***distance travelled***.



As shown in the diagram at the right, the distance travelled after ***one rotation*** of the wheel ***is equal to*** the ***circumference*** of the wheel.

$$C = 2\pi r = 2\pi(0.4 \text{ m}) = 0.8\pi \text{ m}$$

In one second, the wheel moves through 200 rad. Therefore, the distance travelled in one second can be calculated easily using the relation $l = r\theta$:

$$l = r\theta = (0.4 \text{ m})(200 \text{ rad}) = 80 \text{ m}$$

Therefore, the car's speed is 80 m/s.

Consequently, in 15 minutes, Victor's car travels (80 m/s)(15 min)(60 s/min) = 72000 m = 72 km.

Example 3

The London Eye Ferris wheel has a diameter of 135 m and completes one revolution in 30 min.

a) Determine the angular velocity, ω, in radians per second. (Think of the rider as a point on the circumference of the wheel.)

b) How far has a rider travelled at 10 min into the ride?

c) What is the ***linear speed*** of the rider?

Solution

a) $30\text{ min} = 30\,\cancel{\text{min}}^{1} \times \dfrac{60\text{ s}}{1\,\cancel{\text{min}}_{1}}$

$= 1800\text{ s}$

Since the question asks for angular velocity in radians per second, convert the time to seconds.

Angular velocity, $\omega = \dfrac{2\pi}{1800}\text{ radians/s}$

$= \dfrac{\pi}{900}\text{ radians/s}$

$\doteq 0.003\,49\text{ radians/s}$

Each revolution of the Ferris wheel represents an angular motion through an angle of 2π radians. Therefore, the Ferris wheel moves through 2π radians every 30 min.

b) Radius, $r = \dfrac{135}{2}\text{ m}$

$= 67.5\text{ m}$

The rider moves in a circular motion on the edge of a circle that has a radius of 67.5 m.

Number of revolutions, $n = \dfrac{10\,\cancel{\text{min}}^{1}}{30\,\cancel{\text{min}}_{1}}$

$= \dfrac{1}{3}\text{ revolution}$

The wheel turns through one revolution every 30 min, so the rider has gone through $\frac{1}{3}$ of a revolution at 10 min.

Distance travelled, $d = \dfrac{1}{3}(2\pi \times 67.5\text{ m})$

$= 45\pi\text{ m}$

$\doteq 141.4\text{ m}$

The rider travels $\frac{1}{3}$ of the circumference in 10 min.

c) $$\text{Linear speed} = \frac{d}{t} = \frac{r\theta}{t} = \frac{45\pi\text{ m}}{10\text{ min}} = \frac{4.5\pi\text{ m}}{1\text{ min}} = \frac{4.5\pi\text{ m}}{60\text{ s}} = \frac{9\pi}{120}\text{ m/s} = \frac{3\pi}{40}\text{ m/s} \doteq 0.24\text{ m/s}$$

In general, we can convert between angular speed ω and linear speed *v* as follows:

$$\textbf{linear speed} = v = \frac{d}{t} = \frac{r\theta}{t} = \frac{r}{1}\left(\frac{\theta}{t}\right) = r\omega = (\textbf{radius})(\textbf{angular speed})$$

$$\textbf{angular speed} = \omega = \frac{\theta}{t} = \frac{\theta}{t}\left(\frac{r}{r}\right) = \frac{\theta r}{t}\left(\frac{1}{r}\right) = \frac{d}{t}\left(\frac{1}{r}\right) = v\left(\frac{1}{r}\right) = \frac{v}{r} = \frac{\textbf{linear speed}}{\textbf{radius}}$$

Homework

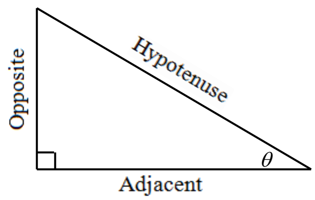
Precalculus (Ron Larson)

pp. 270 – 271: #35-46, 51-56, 62-68, 75, 77

Radian Measure and Angles on the Cartesian Plane

Trigonometry of Right Triangles – Trigonometric Ratios of Acute Angles

Right triangles can be used to define the ***trigonometric ratios*** of ***acute*** angles (angles that measure less than 90º).



$$\sin\theta=\frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta=\frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta=\frac{\text{opposite}}{\text{adjacent}}$$

$$\csc\theta=\frac{\text{hypotenuse}}{\text{opposite}}=\frac{1}{\sin\theta}$$

$$\sec\theta=\frac{\text{hypotenuse}}{\text{adjacent}}=\frac{1}{\cos\theta}$$

$$\cot\theta=\frac{\text{adjacent}}{\text{opposite}}=\frac{1}{\tan\theta}$$

csc=cosecant, sec=secant, cot=cotangent

SOH **C**AH **T**OA

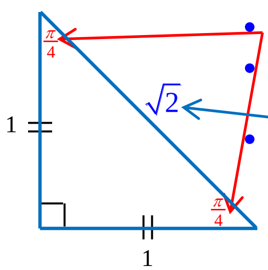
Shout **O**ut **H**ey" "**C**anadians **A**re **H**ot"
"**T**ight **O**iled **A**bs"

Have fun by creating your own mnemonic!

CHO **S**HA **COT**AO

The Special Triangles – Trigonometric Ratios of Special Angles

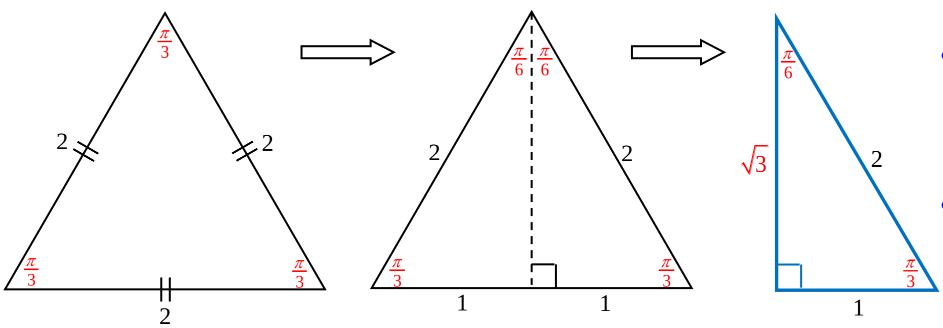
For certain ***special angles***, it is possible to calculate the ***exact value*** of the trigonometric ratios. As I have mentioned on many occasions, it is not advisable to memorize without understanding! Instead, you can ***deduce*** the values that you need to calculate the trig ratios by ***understanding*** the following triangles!



- Isosceles Right Triangle
- Let the length of the equal sides be 1 unit.
- By the Pythagorean Theorem, the length of the hypotenuse must be $\sqrt{2}$.

$$\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{4}=\frac{1}{\sqrt{2}} \qquad \tan\frac{\pi}{4}=\frac{1}{1}=1$$

$$\csc\frac{\pi}{4}=\frac{\sqrt{2}}{1}=\sqrt{2} \qquad \sec\frac{\pi}{4}=\frac{\sqrt{2}}{1}=\sqrt{2} \qquad \cot\frac{\pi}{4}=\frac{1}{1}=1$$

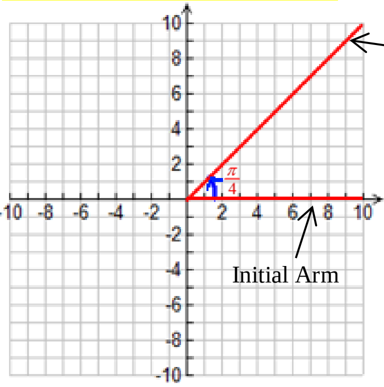


- Begin with an equilateral triangle having sides of length 2 units. Then cut it in half to form a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ right triangle.
- Use the Pythagorean Theorem to calculate the height of the triangle. ($\sqrt{3}$)

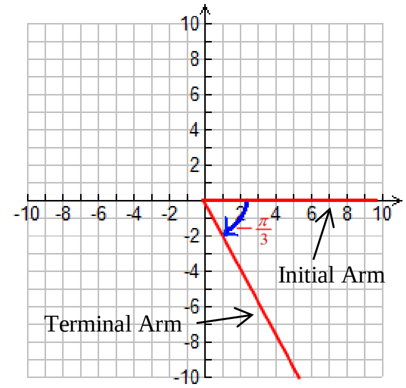
Trig Ratios of $\frac{\pi}{6}$			Trig Ratios of $\frac{\pi}{3}$		
$\sin\frac{\pi}{6}=\frac{1}{2}$	$\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$	$\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$	$\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$	$\cos\frac{\pi}{3}=\frac{1}{2}$	$\tan\frac{\pi}{3}=\sqrt{3}$
$\csc\frac{\pi}{6}=\frac{2}{1}=2$	$\sec\frac{\pi}{6}=\frac{2}{\sqrt{3}}$	$\cot\frac{\pi}{6}=\frac{\sqrt{3}}{1}=\sqrt{3}$	$\csc\frac{\pi}{3}=\frac{\sqrt{3}}{2}$	$\sec\frac{\pi}{3}=\frac{2}{1}=2$	$\cot\frac{\pi}{3}=\frac{1}{\sqrt{3}}$

Trigonometric Ratios of Angles of Rotation – Trigonometric Ratios of Angles of any Size

We can extend the idea of trigonometric ratios to angles of any size by introducing the concept of ***angles of rotation*** (also called ***angles of revolution***).



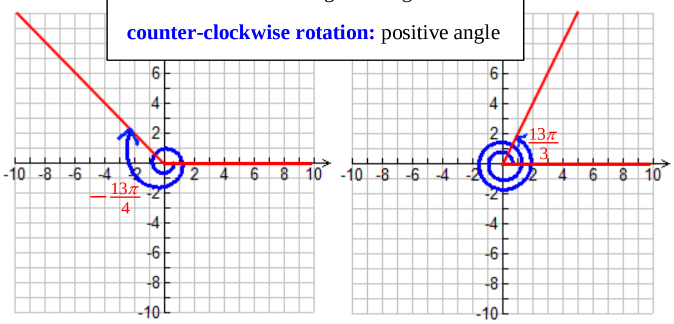
- A ***positive*** angle of revolution (rotation) in ***standard form***.
- "Standard form" means that the ***initial arm*** of the angle lies on the positive x-axis and the ***vertex*** of the angle is at the origin.
- A positive angle results from a ***counter-clockwise*** revolution. (The British say "anti-clockwise.")



- A ***negative*** angle of revolution (rotation) in ***standard form***.
- "Standard form" means that the ***initial arm*** of the angle lies on the positive x-axis and the ***vertex*** of the angle is at the origin.
- A negative angle results from a ***clockwise*** revolution.

clockwise rotation: negative angle

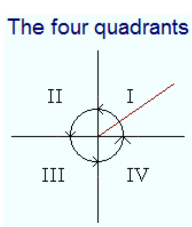
counter-clockwise rotation: positive angle

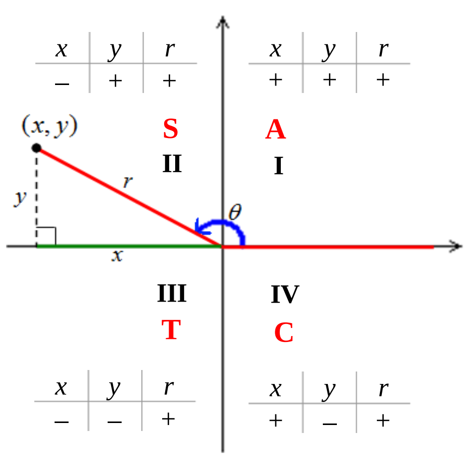


Why Angles of Rotation?

To describe motion that involves moving from one place to another, it makes sense to use units of distance. For instance, it is easy to find your destination if you are told that you need to move 2 km north and 1 km west of your current position.

Consider a spinning figure skater. It does not make sense to describe his/her motion using units of distance because he/she is fixed in one spot and rotating. However, it is very easy to describe the motion through angles of rotation.





$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{r}{y}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{r}{x}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$$

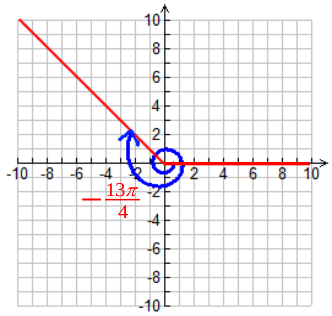
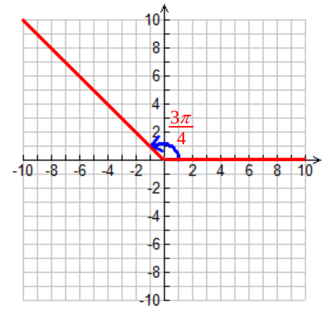
- Since r represents the length of the terminal arm, $r > 0$.
- In quadrant I, $x > 0$ and $y > 0$.
  Therefore **A**LL the trig ratios are positive.
- In quadrant II, $x < 0$ and $y > 0$.
  Therefore only **S**INE and cosecant are positive. The others are negative.
- In quadrant III, $x < 0$ and $y < 0$.
  Therefore only **T**ANGENT and cotangent are positive. The others are negative.
- In quadrant IV, $x > 0$ and $y < 0$.
  Therefore only **C**OSINE and secant are positive. The others are negative.
- Hence the ***mnemonic***,
  "**A**LL **S**TUDENTS **T**ALK on **C**ELLPHONES"

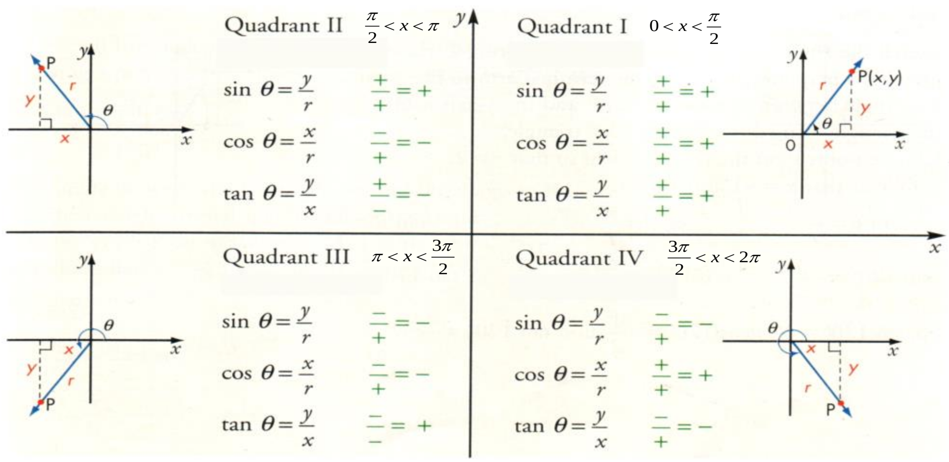
Coterminal Angles

Angles of revolution are called ***coterminal*** if, when in ***standard position***, they share the same terminal arm. For example, $-\frac{\pi}{2}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$ are coterminal angles. An angle coterminal to a given angle can be found by adding or subtracting any multiple of 2π.

Principal Angle

Any angle θ satisfying $0 \le \theta < 2\pi$ is called a ***principal angle***. Every angle of rotation has a principal angle. To find the principal angle of an angle α, simply find the angle θ that is coterminal with α and that also satisfies $0 \le \theta < 2\pi$. An example is given below.

Angle of Rotation $\alpha = -\dfrac{13\pi}{4}$	***Principal Angle of*** α, $\theta = \dfrac{3\pi}{4}$
	



Example – Evaluating Trig Ratios by using the Related First Quadrant Angle (Reference Angle)

Find the trigonometric ratios of $\dfrac{11\pi}{3}$.

> Every angle of rotation has a ***related (acute) first quadrant angle*** (often called the ***reference angle***). The related first quadrant angle is found by taking the ***acute angle between the terminal arm and the x-axis***.
>
> For $\dfrac{11\pi}{3}$, the ***reference angle*** is $\dfrac{\pi}{3}$

Solution

From the diagram at the right, we can see that the principal angle of $\dfrac{11\pi}{3}$ is $\dfrac{5\pi}{3}$. Furthermore, the terminal arm is in the fourth quadrant and we obtain a 30º-60º-90º right triangle in quadrant IV. By observing the ***acute angle*** between the ***terminal arm and the x-axis***, we find the ***related first quadrant angle (reference angle)***, $\dfrac{\pi}{3}$.

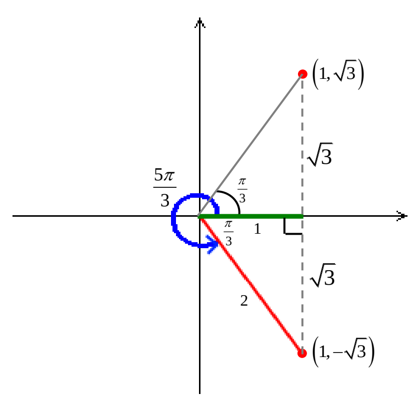
Therefore,

$$\sin\frac{11\pi}{3} = \sin\frac{5\pi}{3} = \frac{y}{r} = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$

$$\cos\frac{11\pi}{3} = \cos\frac{5\pi}{3} = \frac{x}{r} = \frac{1}{2}$$

$$\tan\frac{11\pi}{3} = \tan\frac{5\pi}{3} = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

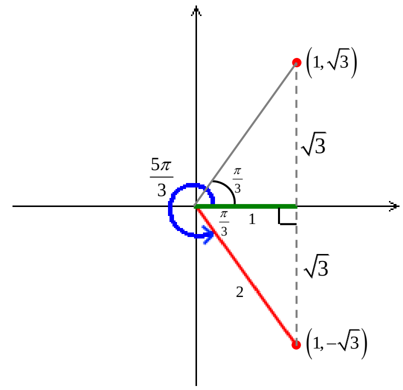
> Compare these answers to
>
> $$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
>
> $$\cos\frac{\pi}{3} = \frac{1}{2}$$
>
> $$\tan\frac{\pi}{3} = \sqrt{3}$$



Question

How are the trigonometric ratios of the principal angle $\frac{5\pi}{3}$ ***related to*** the trigonometric ratios of $\frac{\pi}{3}$?

Answer



Notice that the right triangle formed for $\frac{5\pi}{3}$ is ***congruent*** to the right triangle for $\frac{\pi}{3}$.

Therefore, the ***magnitudes*** of the trig ratios of $\frac{5\pi}{3}$ are equal to the ***magnitudes*** of those of the related first quadrant angle $\frac{\pi}{3}$. However, since the y-co-ordinate of any point in quadrant IV is negative, the ratios may differ in ***sign***. To determine the correct sign, use the ***ASTC*** rule. In case you forget how to apply the ASTC rule, just think about the ***signs*** of x and y in each quadrant. Keep in mind that r is always positive because it represents the length of the terminal arm. Thus, the above ratios could have been calculated as follows:

Angle of Rotation: $\frac{5\pi}{3}$ (quadrant IV) **Related First Quadrant (Reference) Angle:** $\frac{\pi}{3}$

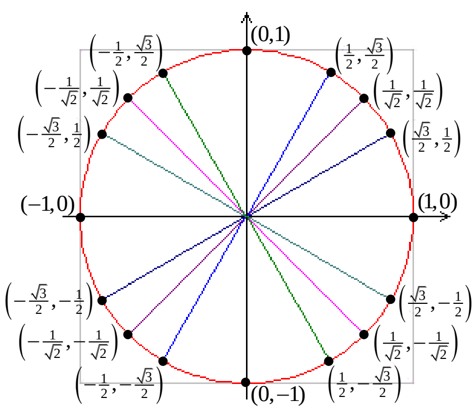
In quadrant IV, $\sin\theta=\frac{y}{r}<0$ because $\frac{-}{+}=-$, $\cos\theta=\frac{x}{r}>0$ because $\frac{+}{+}=+$ and $\tan\theta=\frac{y}{x}<0$ because $\frac{-}{+}=-$.

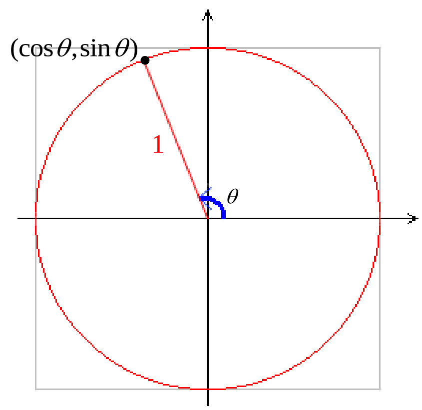
Hence, $\sin\frac{5\pi}{3}=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$, $\cos\frac{5\pi}{3}=\cos\frac{\pi}{3}=\frac{1}{2}$ and $\tan\frac{5\pi}{3}=-\tan\frac{\pi}{3}=-\sqrt{3}$.

Additional Tools for Determining Trig Ratios of Special Angles

The Unit Circle

A ***unit circle*** is any circle having a radius of ***one unit***. For any point (x, y) lying on the unit circle and for any angle θ, $r=1$. Therefore, $\cos\theta=\frac{x}{r}=\frac{x}{1}=x$ and $\sin\theta=\frac{y}{r}=\frac{y}{1}=y$. In other words, for any point (x, y) lying on the unit circle, the ***x-co-ordinate is equal to the cosine of θ*** and the ***y-co-ordinate is equal to the sine of θ***.





The Rule of Quarters (Beware of Blind Memorization!)

The rule of quarters makes it easy to remember the sine of special angles. ***Be aware, however, that this rule invites blind memorization!***

$$\sin(0^\circ)=\sqrt{\frac{0}{4}}=0$$

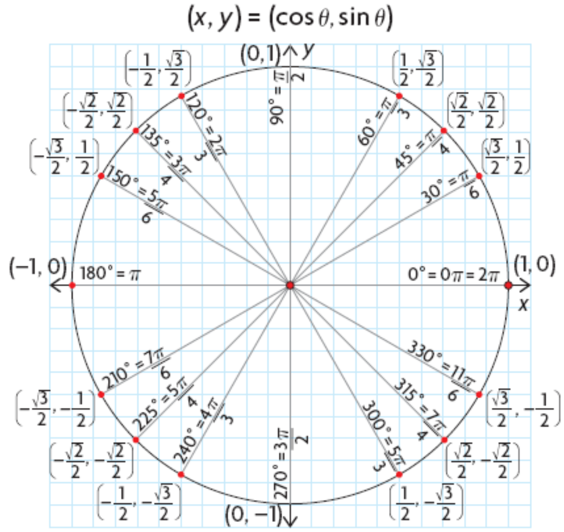
$$\sin(30^\circ)=\sqrt{\frac{1}{4}}=\frac{1}{2}$$

$$\sin(45^\circ)=\sqrt{\frac{2}{4}}=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$

$$\sin(60^\circ)=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$

$$\sin(90^\circ)=\sqrt{\frac{4}{4}}=1$$

Note that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. This follows by multiplying both the numerator and the denominator of $\frac{1}{\sqrt{2}}$ by $\sqrt{2}$.



By expressing $\frac{1}{\sqrt{2}}$ in the form $\frac{\sqrt{2}}{2}$, it is very easy to remember the unit circle.

Trigonometric Values of Common Angles

θ (degrees)	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.

$\frac{\sqrt{0}}{2}$

$\frac{\sqrt{1}}{2}$

$\frac{\sqrt{4}}{2}$

Notice the Pattern! $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$

This pattern can also be written as follows: $\sqrt{\frac{0}{4}}, \sqrt{\frac{1}{4}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{4}}$ (“Rule of Quarters” see page 17)

Homework

Precalculus (Ron Larson)

pp. 269 – 271: #1-46, 71-74, 78

p. 277: #1-22

p. 286: #5-12

p. 296 – 297: #1-22, 37-68, 75-90

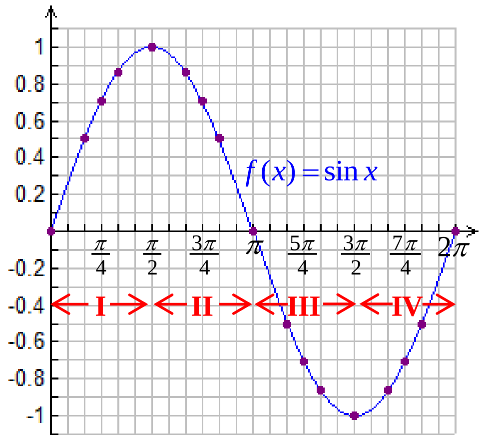
Introduction to Trigonometric Functions

Overview

Now that we have developed a thorough understanding of trigonometric ratios, we can proceed to our investigation of trigonometric functions. The adage "a picture says a thousand words" is very fitting in the case of the graphs of trig functions. The curves summarize everything that we have learned about trig ratios. Answer the questions below to discover the details.

Graphs

x	$y=\sin x$
0	0
$\frac{\pi}{6}$	$1/2=0.5$
$\frac{\pi}{4}$	$1/\sqrt{2} \doteq 0.70711$
$\frac{\pi}{3}$	$\sqrt{3}/2 \doteq 0.86603$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\sqrt{3}/2 \doteq 0.86603$
$\frac{3\pi}{4}$	$1/\sqrt{2} \doteq 0.70711$
$\frac{5\pi}{6}$	$1/2=0.5$
π	0
$\frac{7\pi}{6}$	$-1/2=-0.5$
$\frac{5\pi}{4}$	$-1/\sqrt{2} \doteq -0.70711$
$\frac{4\pi}{3}$	$-\sqrt{3}/2 \doteq -0.86603$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\sqrt{3}/2 \doteq -0.86603$
$\frac{7\pi}{4}$	$-1/\sqrt{2} \doteq -0.70711$
$\frac{11\pi}{6}$	$-1/2=-0.5$
2π	0



Questions about sin x

1. State the domain and range of the sine function.

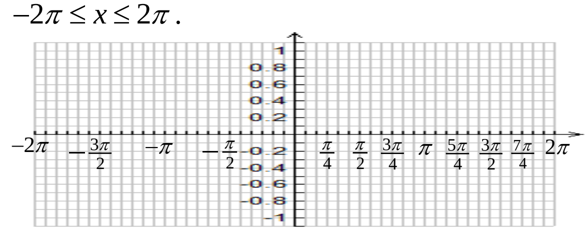
2. Is the sine function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x)=\sin x$ over which $f^{-1}(x)$ is defined.

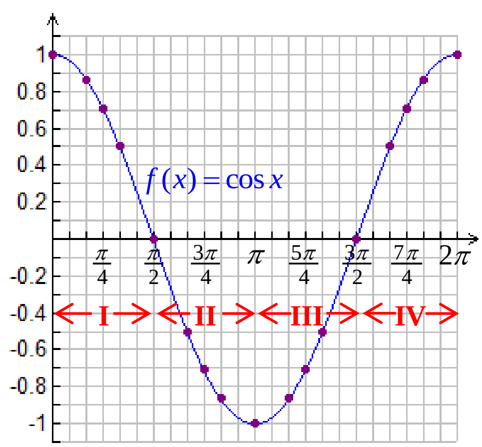
4. How can the graph of the sine function help you to remember the ***sign*** (i.e. + or –) of a sine ratio for any angle?

5. How can the graph of the sine function help you to remember the sine ratios of the special angles?

6. Sketch the graph of $f(x)=\sin x$ for $-2\pi \le x \le 2\pi$.



x	$y=\cos x$
0	1
$\frac{\pi}{6}$	$\sqrt{3}/2 \doteq 0.86603$
$\frac{\pi}{4}$	$1/\sqrt{2} \doteq 0.70711$
$\frac{\pi}{3}$	$1/2=0.5$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-1/2=-0.5$
$\frac{3\pi}{4}$	$-1/\sqrt{2} \doteq -0.70711$
$\frac{5\pi}{6}$	$-\sqrt{3}/2 \doteq -0.86603$
π	-1
$\frac{7\pi}{6}$	$-\sqrt{3}/2 \doteq -0.86603$
$\frac{5\pi}{4}$	$-1/\sqrt{2} \doteq -0.70711$
$\frac{4\pi}{3}$	$-1/2=-0.5$
$\frac{3\pi}{2}$	0
$\frac{5\pi}{3}$	$1/2=0.5$
$\frac{7\pi}{4}$	$1/\sqrt{2} \doteq 0.70711$
$\frac{11\pi}{6}$	$\sqrt{3}/2 \doteq 0.86603$
2π	1



Questions about cos x

1. State the domain and range of the cosine function.

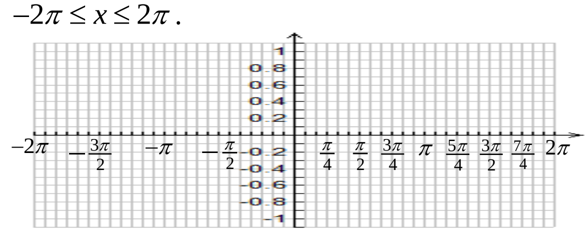
2. Is the cosine function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x)=\cos x$ over which $f^{-1}(x)$ is defined.

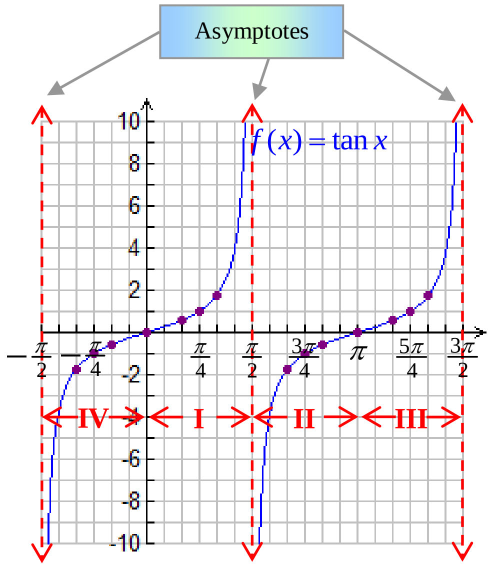
4. How can the graph of the cosine function help you to remember the ***sign*** (i.e. + or –) of a cosine ratio for any angle?

5. How can the graph of the cosine function help you to remember the cosine ratios of the special angles?

6. Sketch the graph of $f(x)=\cos x$ for $-2\pi \le x \le 2\pi$.



x	$y = \tan x$
$-\frac{\pi}{2}$	undefined
$-\frac{\pi}{3}$	$-\sqrt{3} \doteq -1.73205$
$-\frac{\pi}{4}$	-1
$-\frac{\pi}{6}$	$-1/\sqrt{3} \doteq -0.57735$
0	0
$\frac{\pi}{6}$	$1/\sqrt{3} \doteq 0.57735$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3} \doteq 1.73205$
$\frac{\pi}{2}$	undefined
$\frac{2\pi}{3}$	$-\sqrt{3} \doteq -1.73205$
$\frac{3\pi}{4}$	-1
$\frac{5\pi}{6}$	$-1/\sqrt{3} \doteq -0.57735$
π	0
$\frac{7\pi}{6}$	$1/\sqrt{3} \doteq 0.57735$
$\frac{5\pi}{4}$	1
$\frac{4\pi}{3}$	$\sqrt{3} \doteq 1.73205$
$\frac{3\pi}{2}$	undefined



Questions about tan x

1. State the domain and range of the tangent function.

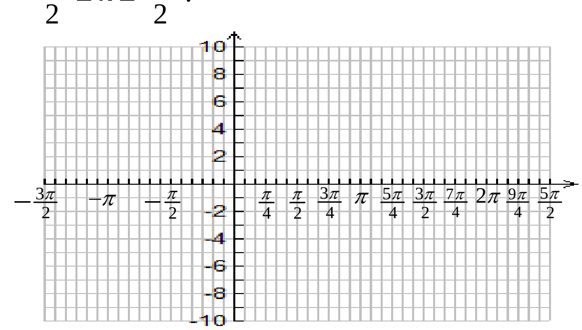
2. Is the tangent function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x) = \tan x$ over which $f^{-1}(x)$ is defined.

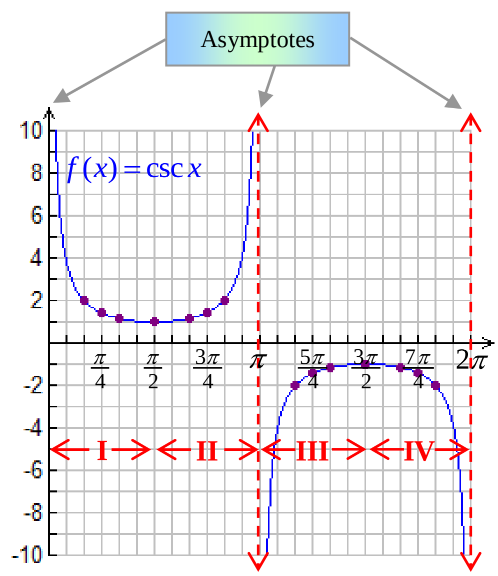
4. How can the graph of the tangent function help you to remember the sign (i.e. + or –) of a tangent ratio for any angle?

5. How can the graph of the tangent function help you to remember the tangent ratios of the special angles?

6. Sketch the graph of $f(x) = \tan x$ for $-\dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2}$.



x	$y = \csc x$
0	undefined
$\frac{\pi}{6}$	2
$\frac{\pi}{4}$	$\sqrt{2} \doteq 1.41421$
$\frac{\pi}{3}$	$2/\sqrt{3} \doteq 1.15470$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$2/\sqrt{3} \doteq 1.15470$
$\frac{3\pi}{4}$	$\sqrt{2} \doteq 1.41421$
$\frac{5\pi}{6}$	2
π	undefined
$\frac{7\pi}{6}$	-2
$\frac{5\pi}{4}$	$-\sqrt{2} \doteq -1.41421$
$\frac{4\pi}{3}$	$-2/\sqrt{3} \doteq -1.15470$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-2/\sqrt{3} \doteq -1.15470$
$\frac{7\pi}{4}$	$-\sqrt{2} \doteq -1.41421$
$\frac{11\pi}{6}$	-2
2π	undefined



Questions about csc x

1. State the domain and range of the cosecant function.

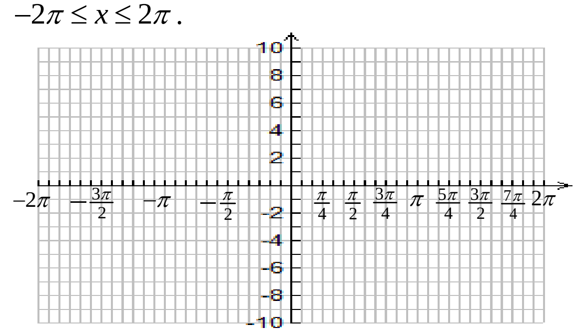
2. Is the cosecant function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x) = \csc x$ over which $f^{-1}(x)$ is defined.

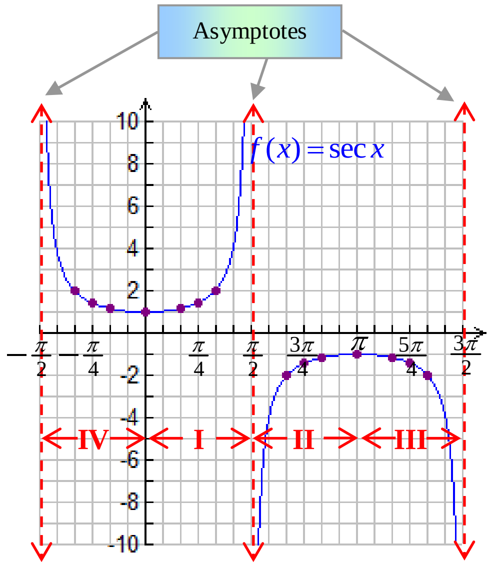
4. How can the graph of the cosecant function help you to remember the sign (i.e. + or –) of a cosecant ratio for any angle?

5. How can the graph of the cosecant function help you to remember the cosecant ratios of the special angles?

6. Sketch the graph of $f(x) = \csc x$ for $-2\pi \le x \le 2\pi$.



x	$y=\sec x$
$-\frac{\pi}{2}$	undefined
$-\frac{\pi}{3}$	2
$-\frac{\pi}{4}$	$\sqrt{2} \doteq 1.41421$
$-\frac{\pi}{6}$	$2/\sqrt{3} \doteq 1.15470$
0	1
$\frac{\pi}{6}$	$2/\sqrt{3} \doteq 1.15470$
$\frac{\pi}{4}$	$\sqrt{2} \doteq 1.41421$
$\frac{\pi}{3}$	2
$\frac{\pi}{2}$	undefined
$\frac{2\pi}{3}$	-2
$\frac{3\pi}{4}$	$-\sqrt{2} \doteq -1.41421$
$\frac{5\pi}{6}$	$-2/\sqrt{3} \doteq -1.15470$
π	-1
$\frac{7\pi}{6}$	$-2/\sqrt{3} \doteq -1.15470$
$\frac{5\pi}{4}$	$-\sqrt{2} \doteq -1.41421$
$\frac{4\pi}{3}$	-2
$\frac{3\pi}{2}$	undefined



Questions about sec x

1. State the domain and range of the secant function.

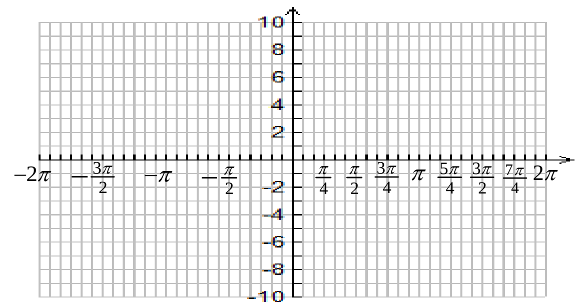
2. Is the secant function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x)=\sec x$ over which $f^{-1}(x)$ is defined.

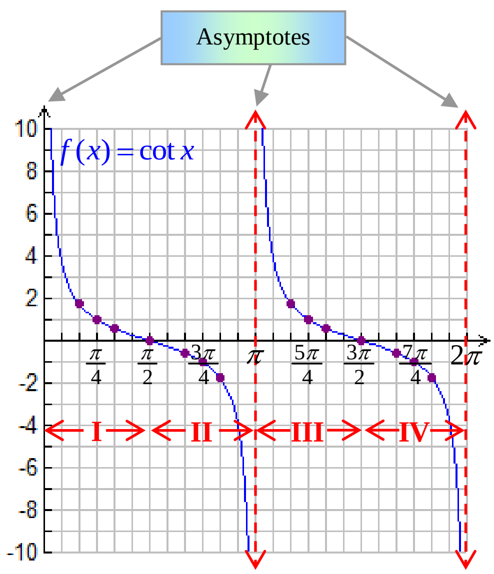
4. How can the graph of the secant function help you to remember the sign (i.e. + or –) of a secant ratio for any angle?

5. How can the graph of the secant function help you to remember the secant ratios of the special angles?

6. Sketch the graph of $f(x)=\sec x$ for $-2\pi \le x \le 2\pi$.



x	$y=\cot x$
0	undefined
$\frac{\pi}{6}$	$\sqrt{3} \doteq 1.73205$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$1/\sqrt{3} \doteq 0.57735$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-1/\sqrt{3} \doteq -0.57735$
$\frac{3\pi}{4}$	-1
$\frac{5\pi}{6}$	$-\sqrt{3} \doteq -1.73205$
π	undefined
$\frac{7\pi}{6}$	$\sqrt{3} \doteq 1.73205$
$\frac{5\pi}{4}$	1
$\frac{4\pi}{3}$	$1/\sqrt{3} \doteq 0.57735$
$\frac{3\pi}{2}$	0
$\frac{5\pi}{3}$	$-1/\sqrt{3} \doteq -0.57735$
$\frac{7\pi}{4}$	-1
$\frac{11\pi}{6}$	$-\sqrt{3} \doteq -1.73205$
2π	undefined



Questions about cot x

1. State the domain and range of the cotangent function.

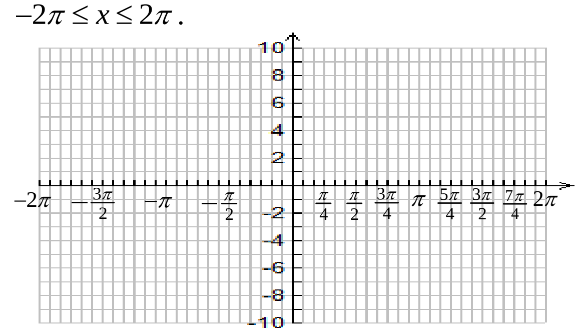
2. Is the cotangent function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x)=\cot x$ over which $f^{-1}(x)$ is defined.

4. How can the graph of the cotangent function help you to remember the sign (i.e. + or –) of a cotangent ratio for any angle?

5. How can the graph of the cotangent function help you to remember the cotangent ratios of the special angles?

6. Sketch the graph of $f(x)=\cot x$ for $-2\pi \le x \le 2\pi$.



TRANSFORMATIONS OF TRIGONOMETRIC FUNCTIONS

What on Earth is a Sinusoidal Function?

- A ***sinusoidal function*** is simply any function that can be obtained by ***stretching*** (compressing) and/or ***translating*** the function $f(x)=\sin x$. That is, a sinusoidal function is any function of the form $g(x)=A\sin\left(\omega(x-p)\right)+d$.
- Sinusoidal functions are very useful for modelling waves and wave-like phenomena.

Since we have already investigated transformations of functions in general, we can immediately state the following:

Transformation of $f(x)=\sin x$ ***expressed in Function Notation***	***Transformation of*** $f(x)=\sin x$ ***expressed in Mapping Notation***	***Why*** $\omega^{-1}x+p$ ***in Mapping Notation and not*** $\omega(x-p)$***?***
g's x-coordinate f's x-coordinate $g(x)=A\sin\left(\omega(x-p)\right)+d$ g's y-coordinate f's y-coordinate is multiplied by A, then d is added	$(x,y)\to\left(\omega^{-1}x+p,Ay+d\right)$ f: pre-image g: image	f's x-coordinate g's x-coordinate $\omega(x-p) \to \div\omega \to +p \to x$ $x \to \div\omega \to +p \to \omega^{-1}x+p$

Horizontal Transformations	***Vertical Transformations***
1. Stretch or compress horizontally by a factor of $\omega^{-1}=1/\omega$. If $\omega<0$, then this includes a reflection in the y-axis. **2.** Translate horizontally by p units. $(x,y)\to\left(\omega^{-1}x+p,y\right)$	**1.** Stretch or compress vertically by a factor of A. If $A<0$, then this includes a reflection in the x-axis. **2.** Translate vertically by d units. $(x,y)\to(x,Ay+d)$

Since sinusoidal functions look just like ***waves*** and are perfectly suited to modelling wave or wave-like phenomena, special names are given to the quantities A, d, p and ω.

- $|A|$ is called the ***amplitude*** (absolute value is needed because amplitude is a distance, which must be positive)
- d is called the ***vertical displacement***
- p is called the ***phase shift***
- ω (also, written as k) is called the ***angular frequency***

These quantities are described in detail on the next page.

Exercise

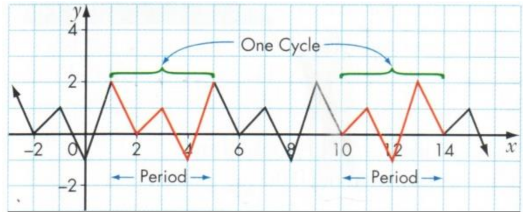
The purpose of this exercise is to emphasize that there is nothing magical about the symbols A, d, p and ω. Any symbols whatsoever can be used to represent the various transformations algebraically. Complete the following table:

Transformation of $f(x)=\sin x$ ***expressed in Mapping Notation***	***Transformation of*** $f(x)=\sin x$ ***expressed in Function Notation***
$(x,y)\to(ax+b,cy+d)$	
Horizontal Transformations	***Vertical Transformations***

Periodic Functions

There are many naturally occurring and artificially produced phenomena that undergo repetitive cycles. We call such phenomena ***periodic***. Examples of such processes include the following:

- orbits of planets, moons, asteroids, comets, etc.
- rotation of planets, moons, asteroids, comets, etc.
- phases of the moon
- the tides
- changing of the seasons
- hours of daylight on a given day
- light waves, radio waves, etc.
- alternating current (**e.g.** household alternating current has a frequency of 60 Hz, which means that it changes direction 60 times per second)



An example of a ***periodic*** function.

Intuitively, a function is said to be ***periodic*** if the graph consists of a "basic pattern" that is repeated over and over at ***regular intervals***. One complete pattern is called a ***cycle***.

Formally, if there is a number T such that $f(x+T) = f(x)$ for all values of x, then we say that f is ***periodic***. The smallest possible positive value of T is called the ***period*** of the function. The ***period*** of a periodic function is equal to the ***length of one cycle***.

Exercise

Suppose that the periodic function shown above is called f. Evaluate each of the following.

(a) $f(2)$ **(b)** $f(4)$ **(c)** $f(1)$ **(d)** $f(0)$ **(e)** $f(16)$ **(f)** $f(18)$ **(g)** $f(33)$ **(h)** $f(-16)$

(i) $f(-31)$ **(j)** $f(-28)$ **(k)** $f(-27)$ **(l)** $f(-11)$ **(m)** $f(-6)$ **(n)** $f(-9)$ **(o)** $f(-5)$ **(p)** $f(-101)$

Characteristics of Sinusoidal Functions

1. Sinusoidal functions have the general form $f(x) = A\sin(\omega(x-p)) + d$, where A, d, p and ω are as described above.

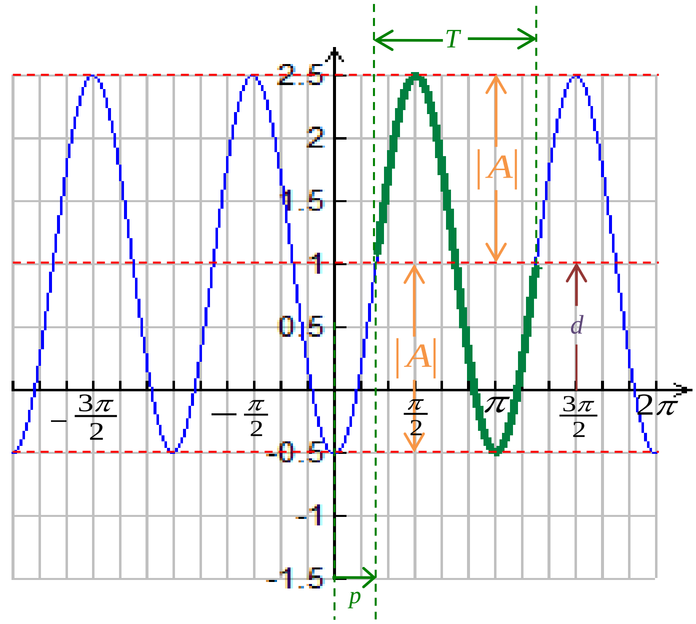
2. Sinusoidal functions are ***periodic***. This makes sinusoidal functions ideal for modelling ***periodic processes*** such as those described on page 22. The letter T is used to denote the ***period*** (also called ***primitive period*** or ***wavelength***) of a sinusoidal function.

3. Sinusoidal functions ***oscillate*** (vary continuously, back and forth) between a maximum and a minimum value. This makes sinusoidal functions ideal for modelling ***oscillatory*** or ***vibratory*** motions. (**e.g.** a pendulum swinging back and forth, a playground swing, a vibrating string, a tuning fork, alternating current, quartz crystal vibrating in a watch, light waves, radio waves, etc.)

4. There is a horizontal line called the ***horizontal axis*** that exactly "cuts" a sinusoidal function "in half." The vertical distance (maximum displacement) from this horizontal line to the peak of the curve is called the ***amplitude***.

Example

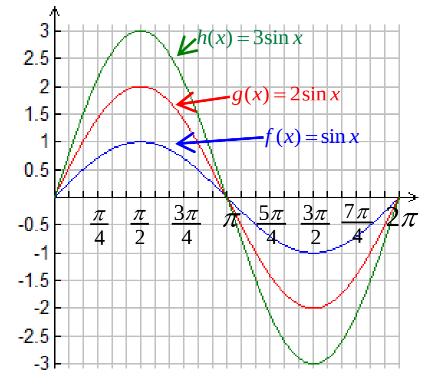
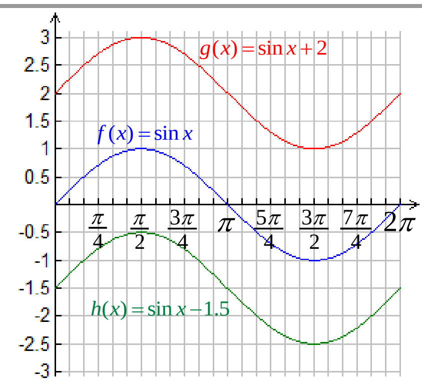
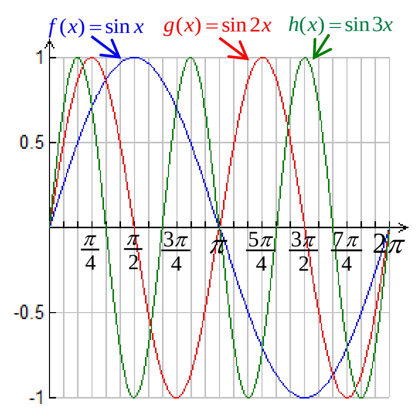
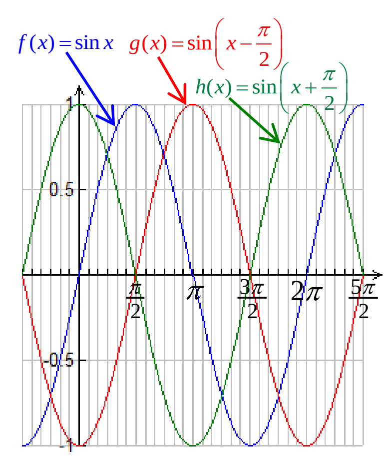
The graph at the right shows a few ***cycles*** of the function $f(x) = 1.5\sin\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$. One of the cycles is shown as a thick green curve to make it stand out among the others. Notice the following:



- The ***maximum*** value of *f* is 2.5.
- The ***minimum*** value of *f* is –0.5.
- The function *f* ***oscillates*** between –0.5 and 2.5.
- The horizontal line with equation $y = 1$ exactly "cuts" the function "in half." This line is called the ***horizontal axis***.
- The ***amplitude*** of this function is $|A| = 1.5$. This can be seen in several ways. Clearly, the vertical distance from the line $y = 1$ to the peak of the curve is 1.5. Also, the amplitude can be calculated by finding half the distance between the maximum and minimum values: $\dfrac{2.5-(-0.5)}{2} = \dfrac{3}{2} = 1.5$
- The ***period***, that is the length of one cycle, is $T = \pi$. This can be seen from the graph ($\dfrac{5\pi}{4} - \dfrac{\pi}{4} = \pi$) or it can be determined by applying your ***knowledge of transformations***. The period of $y = \sin x$ is 2π. Since *f* has undergone a horizontal compression by a factor of $1/2$, its period should be half of 2π, which is π. In general, **T = (period of base function) (absolute value of the horizontal compression factor)** or $\boldsymbol{T = 2\pi\left|\frac{1}{\omega}\right|}$. The reason that absolute value is needed here is that the period is a distance and hence, must be positive. (Those of you who prefer to use *k* to represent the angular frequency may write this as $\boldsymbol{T = 2\pi\left|\frac{1}{k}\right|}$.)
- The ***absolute value of the angular frequency*** determines the number of cycles in 2π radians. In this example, $|\omega| = 2$, which means that there are 2 cycles in 2π radians = 1 cycle in π radians = $\dfrac{1}{\pi}$ cycles in 1 radian = $\dfrac{1}{\pi}$ cycles/radian.
- The function $g(x) = 1.5\sin\left(2\left(x - \frac{\pi}{4}\right)\right)$ would be "cut in half" by the *x*-axis (i.e. the horizontal axis $y = 0$). The function *f* has the same shape as *g* except that it is ***shifted up by 1 unit***. This is the significance of the ***vertical displacement***. In this example, the vertical displacement $d = 1$.
- The function $g(x) = 1.5\sin\left(2\left(x - \frac{\pi}{4}\right)\right)$ has the same shape as $h(x) = 1.5\sin 2x$ but is shifted $\dfrac{\pi}{4}$ to the right. This horizontal shift is called the ***phase shift***.

Important Exercises

Complete the following table. The first one is done for you.

Function	A	d	p	ω=k	T	Description of Transformation	
f	1	0	0	1	2π	None	
g	2	0	0	1	2π	The graph of $f(x)=\sin x$ is stretched vertically by a factor of 2. The amplitude of g is 2.	
h	3	0	0	1	2π	The graph of $f(x)=\sin x$ is stretched vertically by a factor of 3. The amplitude of g is 3.	
f							
g							
h							
f							
g							
h							
f							
g							
h							

Example

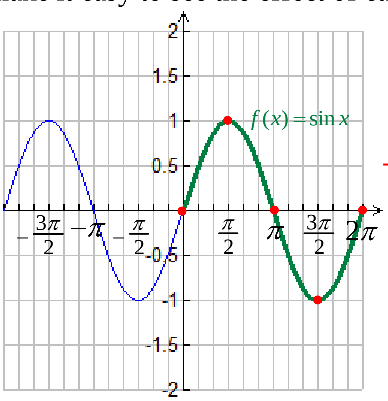
Sketch the graph of $f(x)=1.5\sin\left(2\left(x-\frac{\pi}{4}\right)\right)+1$

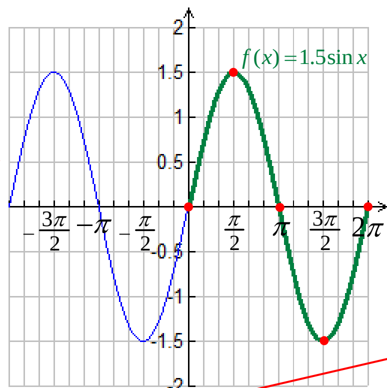
Solution

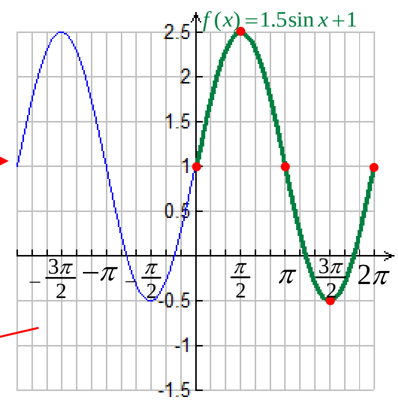
Transformations		Amplitude	Vertical Displacement	Phase Shift	Period	Angular Frequency
Vertical **1.** Stretch by a factor of 1.5. **2.** Translate 1 unit up.	***Horizontal*** **1.** Compress by a factor of $2^{-1}=\frac{1}{2}$. **2.** Translate $\frac{\pi}{4}$ to the right.	$\lvert A\rvert=1.5$	$d=1$	$p=\frac{\pi}{4}$	$T=2\pi\left\lvert\frac{1}{\omega}\right\rvert$ $=2\pi(1/2)$ $=\pi$	$\omega=2$ $\lvert\omega\rvert=2$, which means 2 cycles per 2π radians

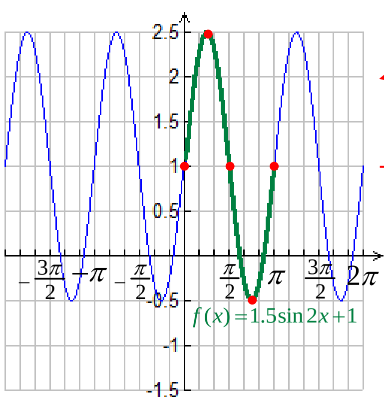
Method 1 – The Long Way

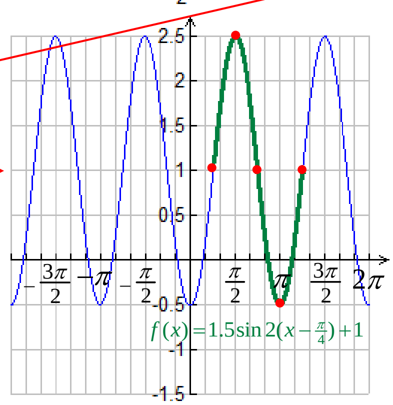
The following shows how the graph of $f(x)=1.5\sin\left(2\left(x-\frac{\pi}{4}\right)\right)+1$ is obtained by beginning with the base function $f(x)=\sin x$ and applying the transformations one-by-one in the correct order. One cycle of $f(x)=\sin x$ is highlighted in green to make it easy to see the effect of each transformation. In addition, ***five main points*** are displayed in red to make it easy to see the effect of each transformation.







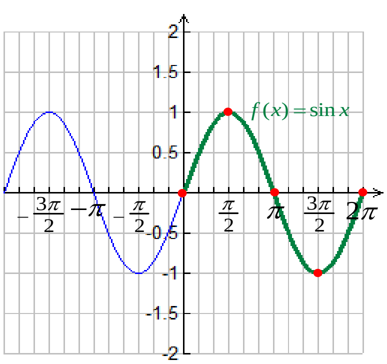




Note

- The five key points divide each cycle into quarters (four equal parts).
- Each quarter corresponds to one of the four quadrants.
- Dividing one cycle into four equal parts makes it very easy to sketch the graphs of sinusoidal functions.
- The transformations are very easy to apply to the five key points.

Method 2 – A Much Faster Approach



$f(x)=1.5\sin\left(2\left(x-\frac{\pi}{4}\right)\right)+1$, $A=1.5$, $d=1$, $\omega=2$, $p=\frac{\pi}{4}$

$(x,y)\to\left(\frac{1}{2}x+\frac{\pi}{4},1.5y+1\right)$

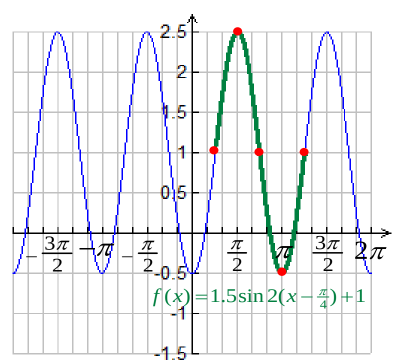
$(0,0)\to\left(\frac{1}{2}(0)+\frac{\pi}{4},1.5(0)+1\right)=\left(\frac{\pi}{4},1\right)$

$\left(\frac{\pi}{2},1\right)\to\left(\frac{1}{2}\left(\frac{\pi}{2}\right)+\frac{\pi}{4},1.5(1)+1\right)=\left(\frac{\pi}{2},2.5\right)$

$(\pi,0)\to\left(\frac{1}{2}\pi+\frac{\pi}{4},1.5(0)+1\right)=\left(\frac{3\pi}{4},1\right)$

$\left(\frac{3\pi}{2},-1\right)\to\left(\frac{1}{2}\left(\frac{3\pi}{2}\right)+\frac{\pi}{4},1.5(-1)+1\right)=(\pi,-0.5)$

$(2\pi,0)\to\left(\frac{1}{2}(2\pi)+\frac{\pi}{4},1.5(0)+1\right)=\left(\frac{5\pi}{4},1\right)$



Exercise 1

Using ***both*** approaches shown in the previous example, sketch a few cycles of the graph of $f(x) = -2\cos\left(-3\left(x+\frac{\pi}{3}\right)\right)-1$.

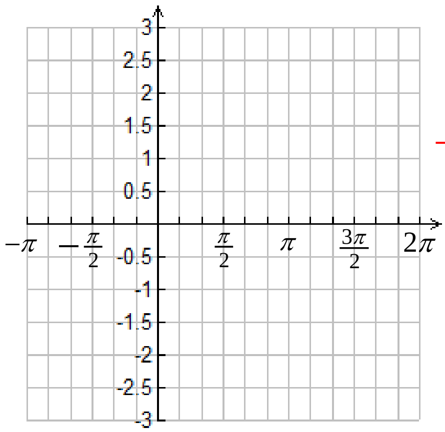
Solution

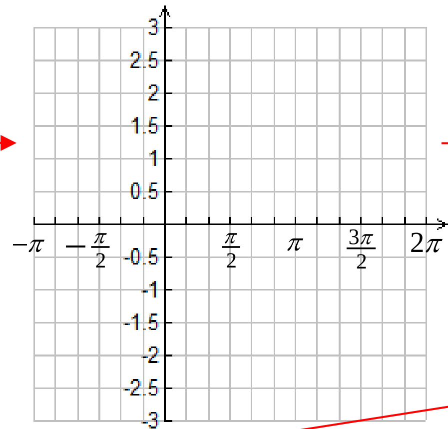
$A=$___ , $d=$___ , $\omega = k =$___ , $p=$___ $\qquad\qquad (x,y)\rightarrow$

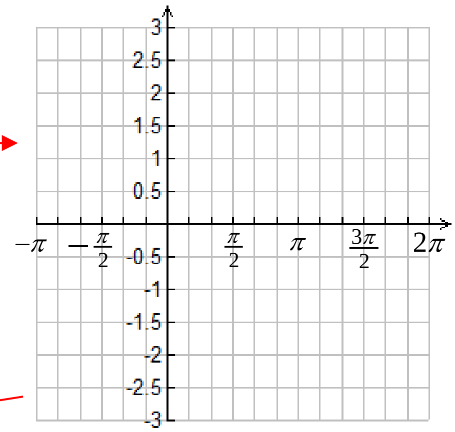
$T=$ (stretch/compression factor) (2π) = ($\qquad$) (2π) =

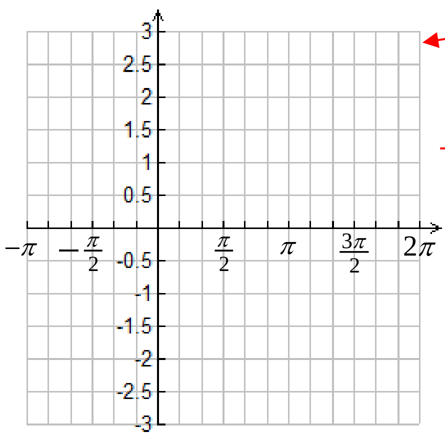
Transformations in Words	
Vertical	*Horizontal*

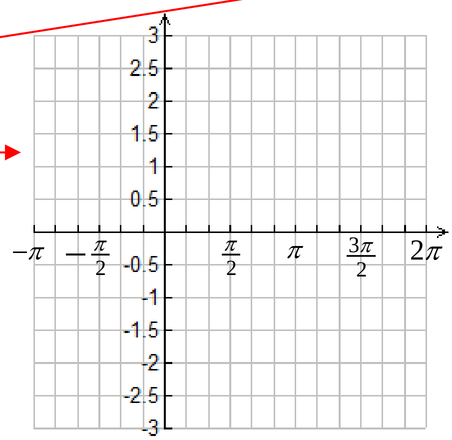
Method 1 – The Long Way







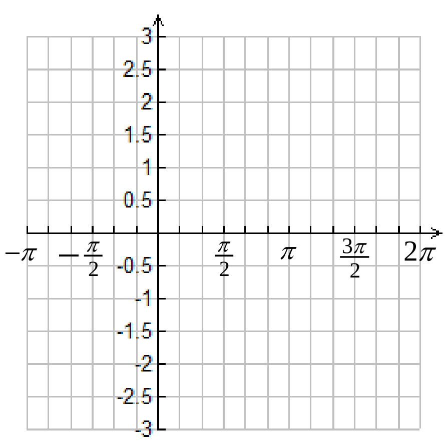




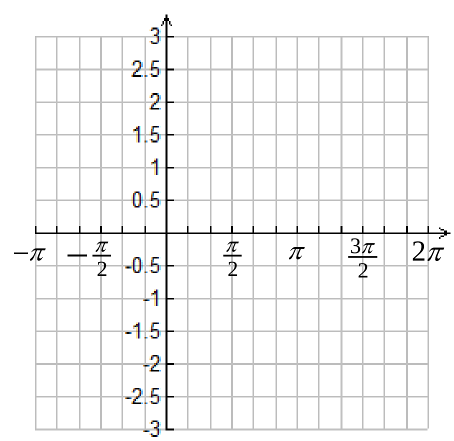
Meaning

Quan-tity	*Meaning*
A	
d	
ω	
p	
T	

Method 2 – A Much Faster Approach



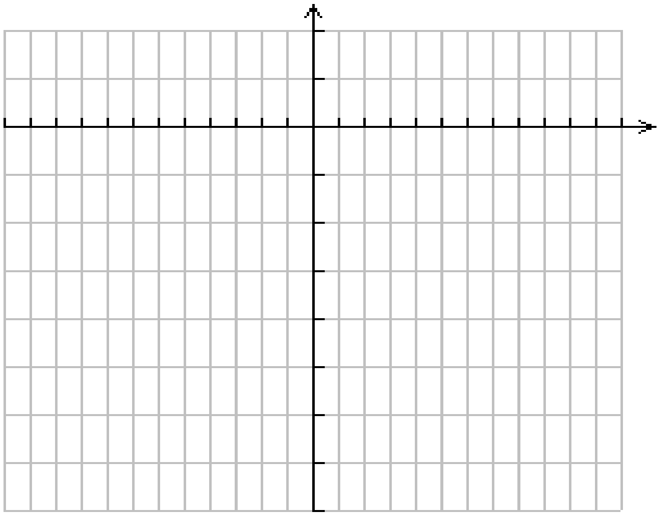
$(x,y)\rightarrow$



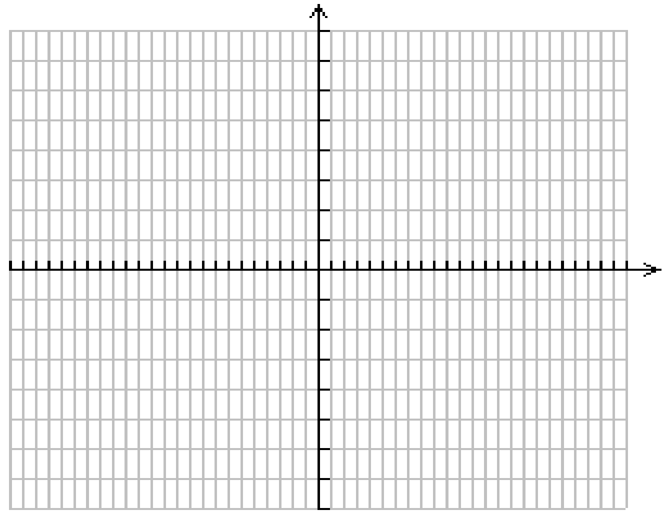
Homework Exercises

Sketch at least three cycles of each of the following functions. In addition, state the domain and range of each, as well as the amplitude, the vertical displacement, the phase shift and the period.

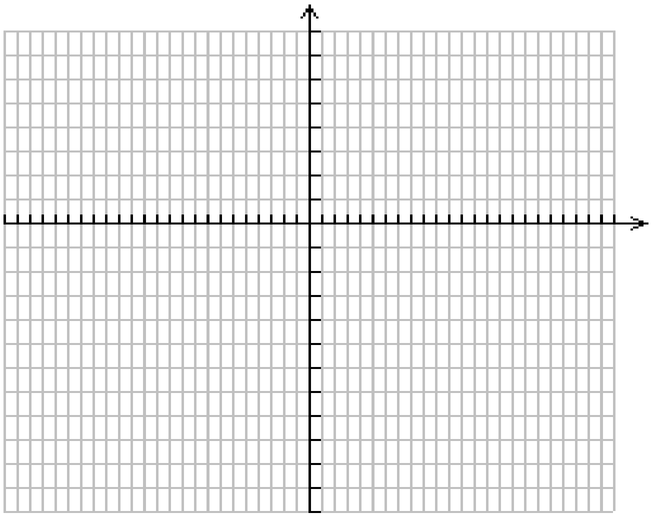
(a) $f(x) = -\cos 3x - 2$



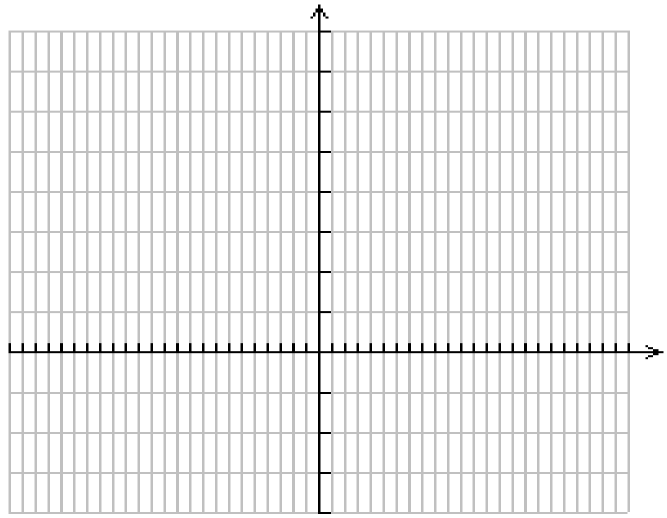
(b) $g(x) = 3\cos\left(x - \frac{\pi}{6}\right)$



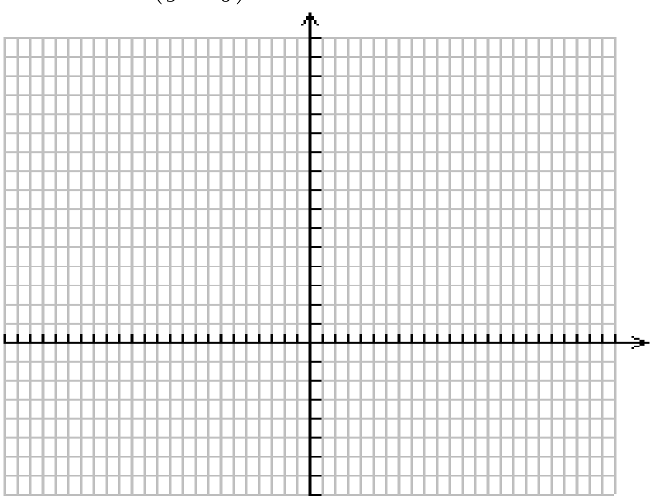
(c) $h(x) = 4\sin\left(2\left(x + \frac{\pi}{4}\right)\right) - 1$



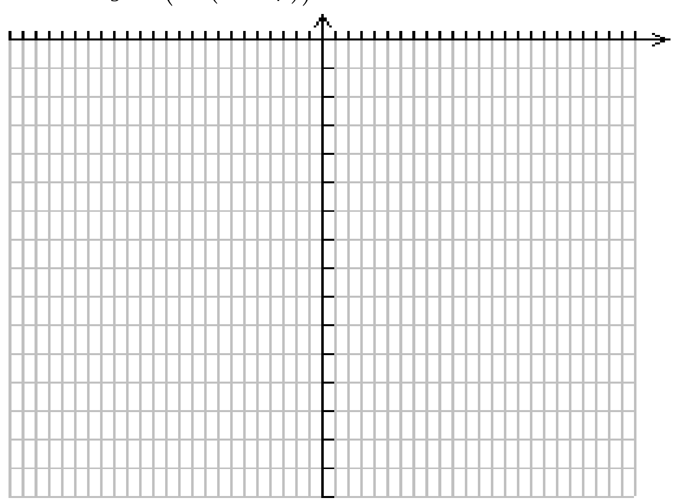
(d) $p(x) = -2\sin\left(2x + \frac{\pi}{3}\right) + 1$



(e) $q(t) = -5\cos\left(\frac{1}{3}t + \frac{\pi}{8}\right) + 2$

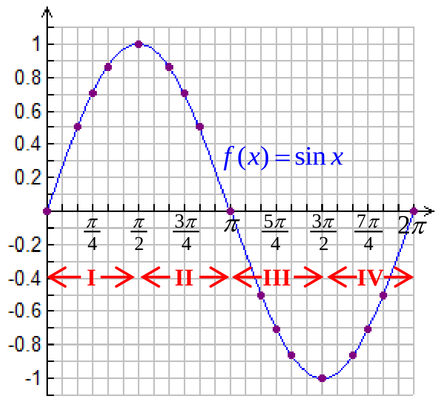


(f) $r(\theta) = -\frac{2}{3}\sin\left(-2\left(\theta - \frac{3\pi}{4}\right)\right) - 3$



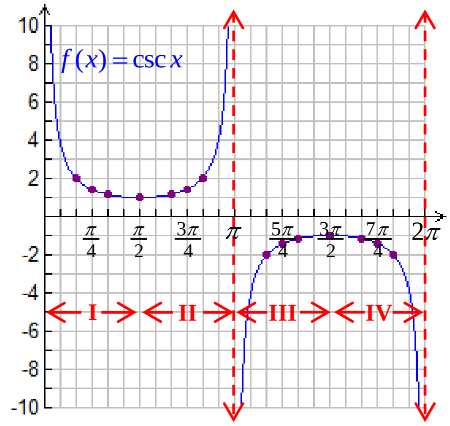
Graphing Trigonometric Functions

One Cycle of each of the Trigonometric Base/Parent/Mother Functions



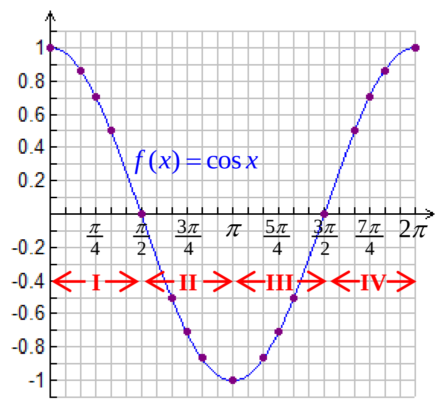
$D = \mathbb{R},\ R = \{y \in \mathbb{R} : -1 \le y \le 1\}$

$A = 1,\ d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$



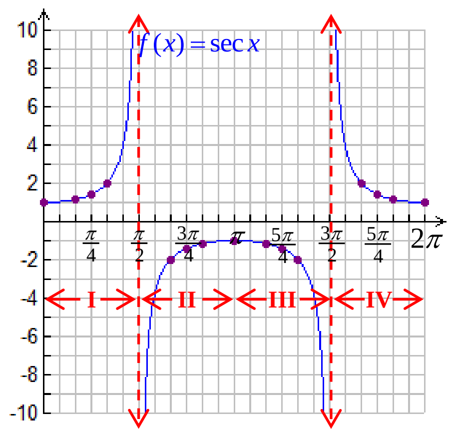
$D = \{x \in \mathbb{R} : x \ne n\pi, n \in \mathbb{Z}\},\ R = \{y \in \mathbb{R} : y \le -1 \text{ or } y \ge 1\}$

A = 1 but the amplitude is undefined, $d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$



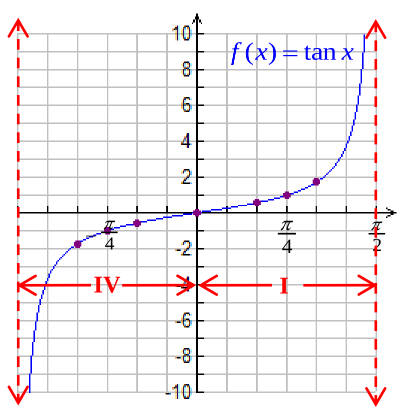
$D = \mathbb{R},\ R = \{y \in \mathbb{R} : -1 \le y \le 1\}$

$A = 1,\ d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$



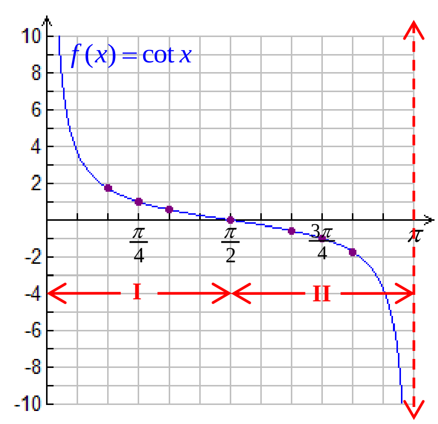
$D = \left\{x \in \mathbb{R} : x \ne \dfrac{(2n+1)\pi}{2}, n \in \mathbb{Z}\right\},\ R = \{y \in \mathbb{R} : y \le -1 \text{ or } y \ge 1\}$

A = 1 but the amplitude is undefined, $d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$



$D = \left\{x \in \mathbb{R} : x \ne \dfrac{(2n+1)\pi}{2}, n \in \mathbb{Z}\right\},\ R = \mathbb{R}$

A = 1 but the amplitude is undefined,
$d = 0,\ p = 0,\ \omega = 1,\ T = \pi$



$D = \{x \in \mathbb{R} : x \ne n\pi, n \in \mathbb{Z}\},\ R = \mathbb{R}$

A = 1 but the amplitude is undefined,
$d = 0,\ p = 0,\ \omega = 1,\ T = \pi$

Suggestions for Graphing Trigonometric Functions

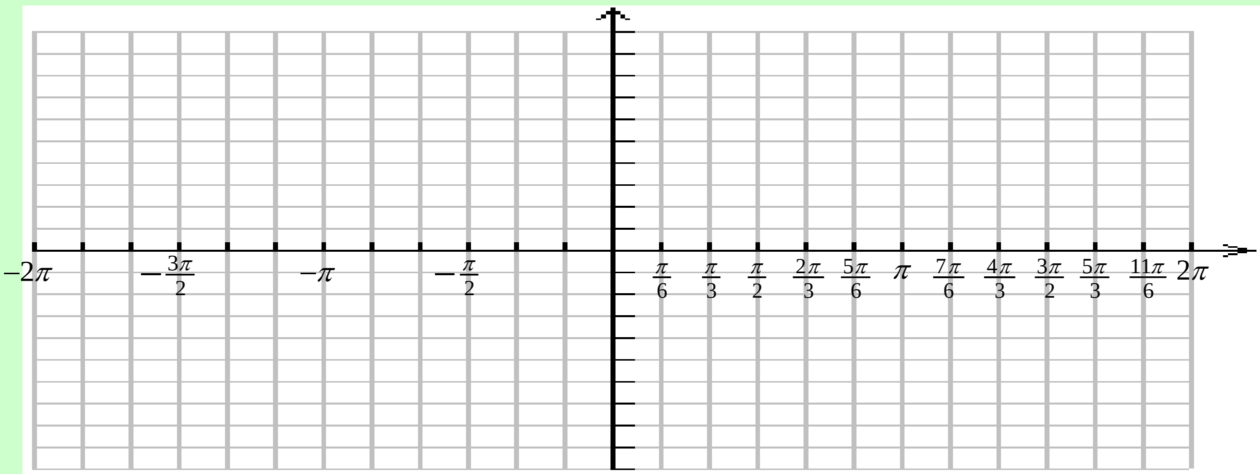
1. Identify the transformations and express using ***mapping notation***.

2. Think carefully about the effect of the transformations on the features of the base graph
Horizontal stretches/compressions affect the ***period*** and the ***locations of the vertical asymptotes***.
Horizontal translations affect the ***locations of the vertical asymptotes*** and the ***phase shift***.
Vertical stretches/compressions affect the ***amplitude*** and the ***y-co-ordinates of maximum/minimum points***.
Vertical translations simply cause all the points on the graph to move up or down by some constant amount.

3. Apply the transformations to a few key points on the base function.

4. Sometimes it is easier to apply the stretches/compressions first to obtain the final "shape" of the curve. Then it is a simple matter to translate the curve into its final position.

5. To find a suitable scale for the *x*-axis, divide the period by a number that is divisible by 4. The number 12 works particularly well because it divides exactly into 360°, giving increments of 30° or $\frac{\pi}{6}$ radians (see diagram).



Graphing Exercises

Now sketch graphs of each of the following functions by applying appropriate transformations to one of the base functions given above. Once you are done, use Desmos or a graphing calculator to check whether your graphs are correct. Detailed solutions are also available at http:www.misternolfi.com/courses.htm under "Unit 3 - Trigonometric Functions."

a) $y = 18\cos\left(\frac{\pi x}{4}\right) - 14$

b) $y = -\frac{4}{5}\sin\left(\frac{2}{7}\left(x + \frac{3\pi}{4}\right)\right) + 10$

c) $y = 101\cos\left(x - \frac{7\pi}{4}\right) - \frac{9}{10}$

d) $y = 6\sin(\pi x + 13) + 22$

e) $y = -\cos\left(\frac{5\pi}{3}(x - 1)\right) + 1$

f) $y = 15\sin\left(\frac{x}{15}\right) - 5$

g) $y = -2\tan\left(3\left(x + \frac{\pi}{4}\right)\right) - 3$

h) $y = -2\sec\left(\frac{2}{\pi}\left(x + \frac{\pi}{6}\right)\right) + 5$

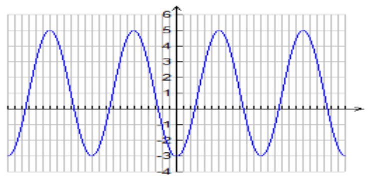
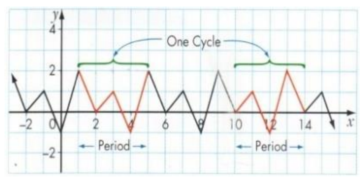
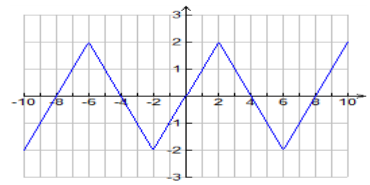
i) $y = 3\cot\left(\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right) + 2$

j) $y = \frac{5}{3}\csc\left(1.5x + \frac{\pi}{4}\right) + \frac{3}{2}$

Using Sinusoidal Functions to Model Periodic Phenomena

Summary

A sinusoidal function is any function of the form $f(x)=A\sin\left(\omega\left(x-p\right)\right)+d$. Such functions can be used to model ***periodic phenomena*** that involve some quantity alternately increasing and decreasing, "smoothly" and at regular intervals, between a maximum and minimum value. In addition to exhibiting this smooth and regular "up and down" behaviour, a sinusoidal function "spends" the same amount of "time" increasing as it does decreasing. Note also that the horizontal axis of a sinusoidal function is located exactly at the average of the maximum and minimum values.

A Sinusoidal Periodic Function	***Periodic Functions that are NOT Sinusoidal***
	 

Vertical Transformations

$A=$ the factor by which the base function $y=\sin x$ is ***stretched/compressed vertically***

$|A|=$ the ***amplitude*** of the sinusoidal function (must be positive since length/distance cannot be negative)

$=\dfrac{\text{max-min}}{2}$

$d=$ the amount by which the base function $y=\sin x$ is ***translated vertically***

= the ***vertical displacement***

$=\dfrac{\text{max+min}}{2}$ = the average of the max and the min

=the average value of the sinusoidal function

$y=d\rightarrow$ the equation of the ***horizontal axis*** of the sinusoidal function

= the image of the *x*-axis under the transformation

$|d|=$ absolute value of the vertical displacement

= distance from the *x*-axis to the horizontal axis

Horizontal Transformations

$\dfrac{1}{\omega}=$ the factor by which the base function $y=\sin x$ is ***stretched/compressed horizontally***

$\omega=$ the ***angular frequency***

= how fast an object rotates (positive value→counterclockwise, negative value→clockwise)

$|\omega|=$ the absolute value of the ***angular frequency***

= number of cycles in 2π radians

$T=$ ***period***

= length of one cycle (must be positive since length/distance cannot be negative)

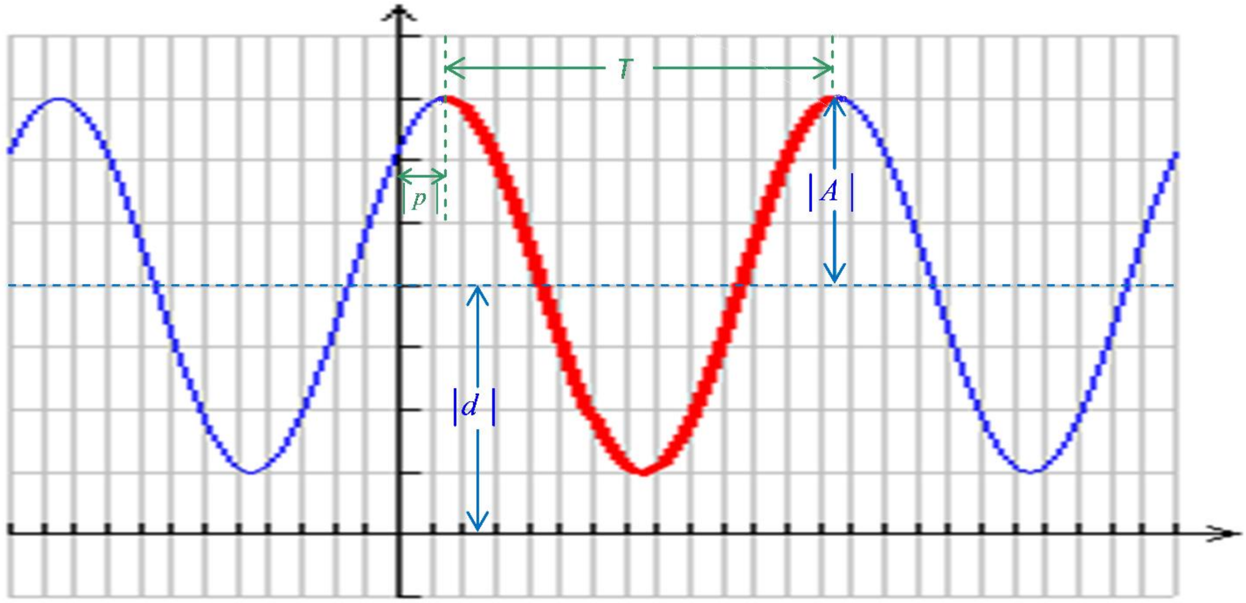
= (period of base function $y=\sin x$)(absolute value of the horizontal stretch/compression factor)

$=2\pi\left|\dfrac{1}{\omega}\right|$

$p=$ the amount by which the base function $y=\sin x$ is ***translated horizontally***

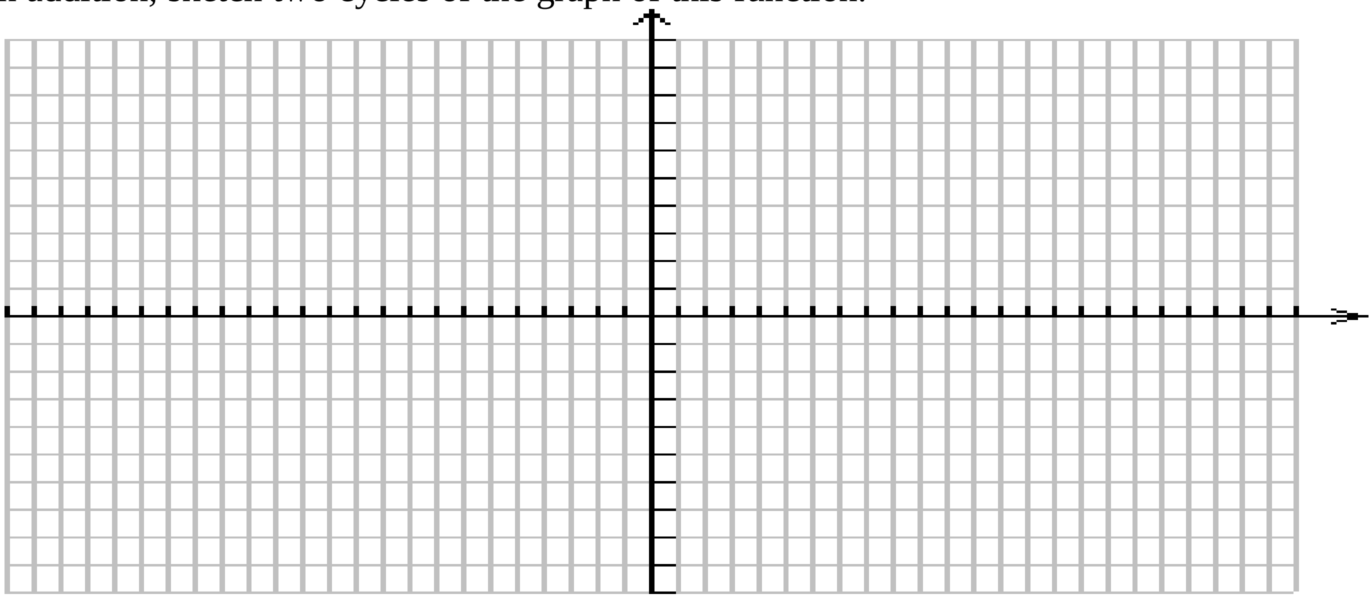
= the ***phase shift***

If it is difficult to determine the phase shift graphically, it can be ***calculated*** once *A*, *d* and ω are known. Simply choose a point known to be on the curve, substitute into the equation and solve for *p*. Alternatively, *p* can be determined by using the image of the point $(0,0)$, if $y=\sin x$ is used as the base function, or the image of the point $(0,1)$, if $y=\cos x$ is used as the base function. (See the example on page 37 for more details.)



Activity 1

1. A cosine curve has an amplitude of 3 units and a period of 3π radians. The equation of the axis is $y = 2$, and a horizontal shift of $\frac{\pi}{4}$ radians to the left has been applied. Write the equation of this function. In addition, sketch two cycles of the graph of this function.



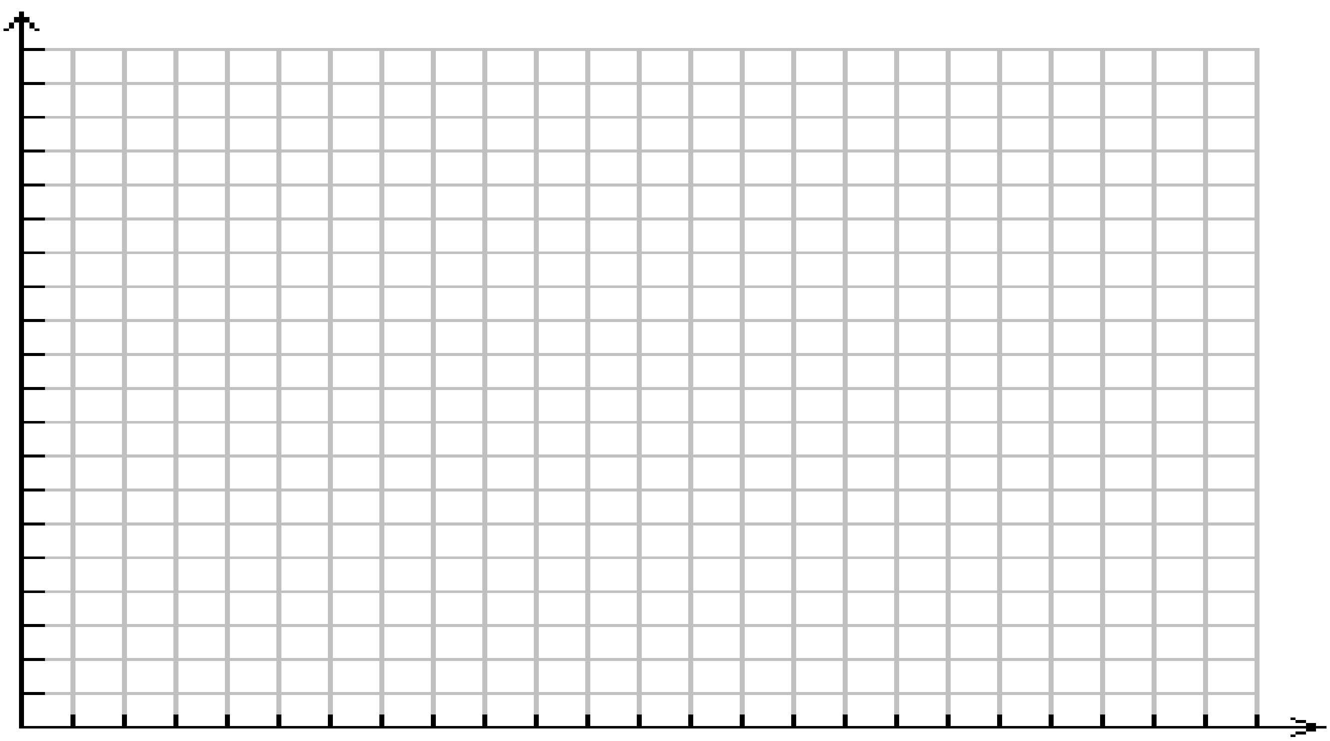
2. Determine the value of the function in question 1 if $x = \frac{\pi}{2}, \frac{3\pi}{4}$, and $\frac{11\pi}{6}$.

3. Use your graph to estimate the x-value(s) in the domain $0 < x < 2$, where $y = 2.5$, to one decimal place.

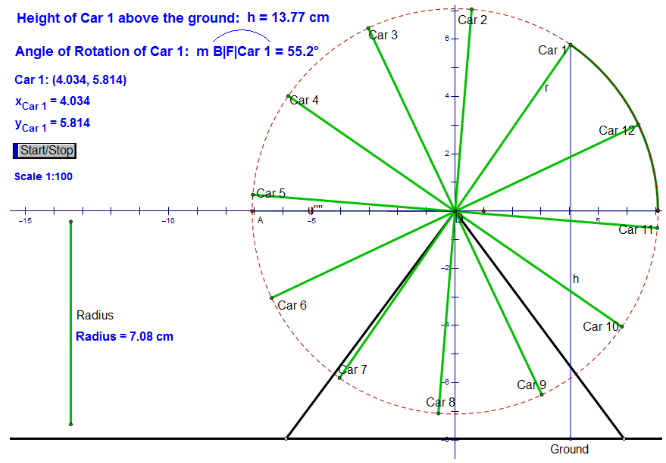
4. The number of hours of daylight in Vancouver can be modelled by a sinusoidal function of time, in days. The longest day of the year is June 21, with 15.7 h of daylight. The shortest day of the year is December 21, with 8.3 h of daylight.

a) Find an equation for $h(t)$, the number of hours of daylight on the tth day of the year. In addition, sketch one cycle of the graph of this function.

b) Use your equation to predict the number of hours of daylight in Vancouver on January 30th.



Activity 2 – Ferris Wheel Simulation



Download the Geometer's Sketchpad Ferris wheel simulation from www.misternolfi.com. Once you understand how to start and stop the animation and how to interpret the given information, answer the questions found below.

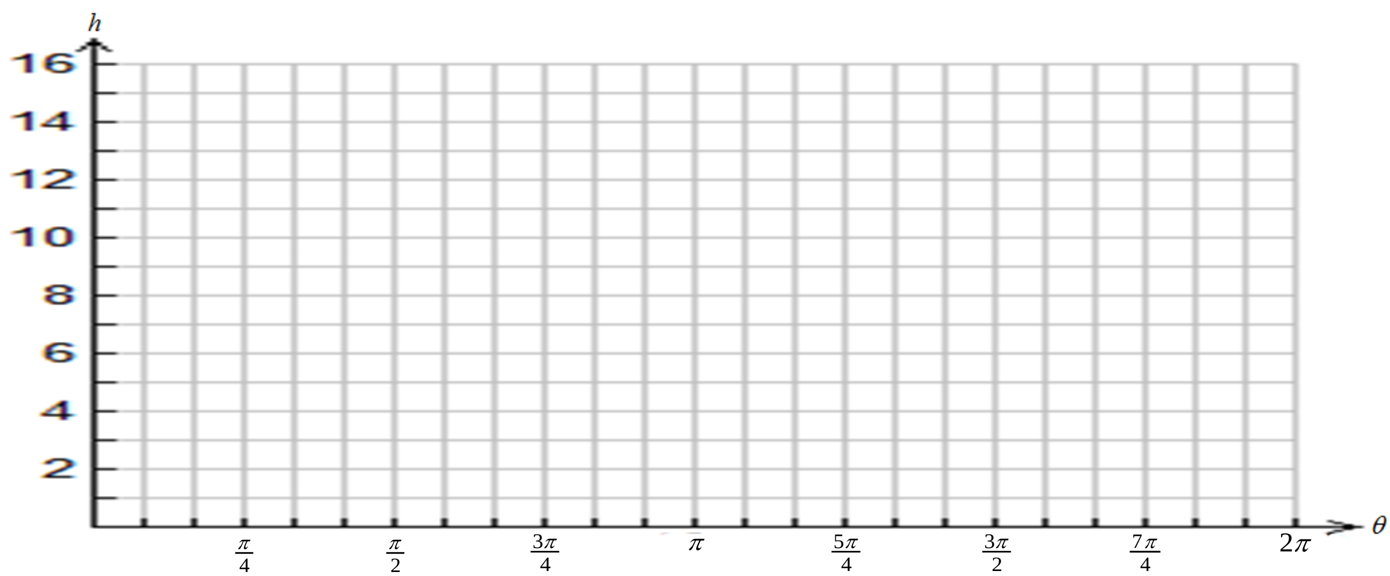
Questions

This simulation involves finding out how the height of "Car 1" above the ground ***is related to*** the angle of rotation of the line segment joining "Car 1" to the axis of rotation of the Ferris wheel.

1. Complete the table at the right. Stop the animation each time that a car reaches the x-axis (the car ***does not*** need to be exactly on the x-axis). Each time that you stop the animation, record the angle of rotation of "Car 1" and its height above the ground.

2. Now use the given grid to plot the data that you recorded in question 1. Once you have plotted all the points, join them by sketching a smooth curve that passes through all the points. Does your curve look familiar? Try to write an equation that describes the curve.

Angle of Rotation of Car 1	***Height of Car 1***



3. For this question, you may use either a graphing calculator or a spreadsheet. First, take the data from the above table and create two lists. Then perform a ***sinusoidal regression***. (Performing a regression means that the data are "fit" to a mathematical function. A ***sinusoidal regression*** finds the sinusoidal function that ***best fits*** the data.) How does the equation produced by the regression compare to the equation that you wrote in question 2?

4. Now use a graphing calculator or graphing program like Desmos to graph the function produced by the regression. How does it compare to the graph that you sketched in question 2?

5. Use the equation obtained in question 4 to predict the height of "Car 1" when its angle of rotation is 2 radians.

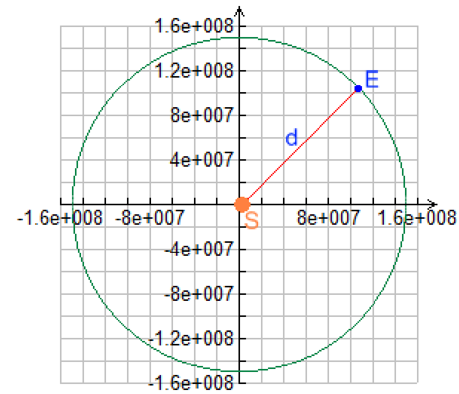
Activity 3 – Earth's Orbit

The table below gives the approximate distance from the Earth to the Sun on certain days of a year.

Date	*Day of the Year*	*Earth's Distance (d) from Sun (km)*
January 3	2	1.47098×10^{8}
February 2	32	1.47433×10^{8}
March 5	63	1.48349×10^{8}
April 4	93	1.49599×10^{8}
May 5	124	1.50848×10^{8}
June 4	154	1.51763×10^{8}
July 5	185	1.52098×10^{8}
August 4	215	1.51763×10^{8}
September 4	246	1.50848×10^{8}
October 4	276	1.49599×10^{8}
November 4	307	1.48349×10^{8}
December 4	337	1.47433×10^{8}

Perihelion is the point in the Earth's orbit at which it is closest to the sun. Perihelion occurs in early January.

Aphelion is the point in the Earth's orbit at which it is farthest from the sun. Aphelion occurs in early July.



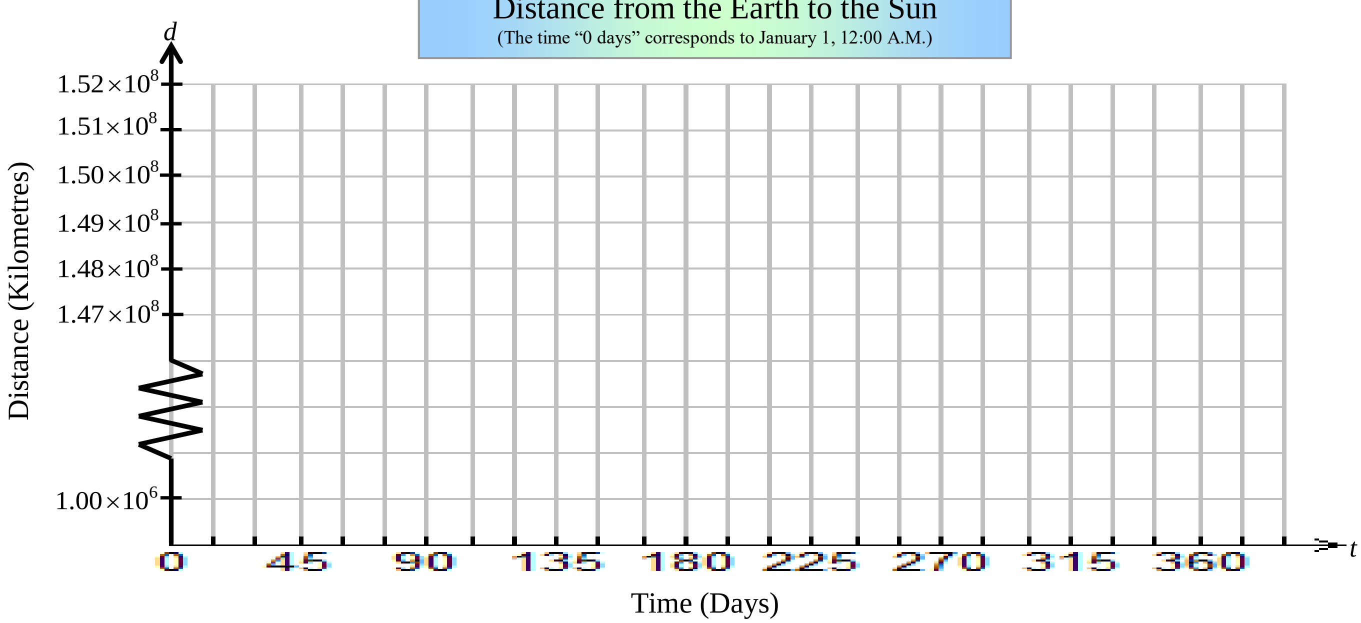
The Earth's orbit around the Sun is an ellipse that is very close to a perfect circle. The Sun is located at one of the two ***foci*** (singular ***focus***) of the ellipse.

Questions

1. Use the grid below to plot the data in the above table. Once you have done so, join the points with a smooth curve. Use your knowledge of trigonometric functions to write an equation of the curve.

Distance from the Earth to the Sun

(The time "0 days" corresponds to January 1, 12:00 A.M.)



2. Now use graphing calculator or a spreadsheet to perform a sinusoidal regression on the data in the above table. Compare the equation obtained by regression to the one that you wrote in question 1.

3. Are you surprised that perihelion occurs in early January and that aphelion occurs in early July? Explain.

4. Use the equation obtained in question 2 to predict the distance from the Earth to the Sun on Valentine's Day.

5. Suppose that the Earth's orbit were highly elliptical instead of being nearly a perfect circle. Do you think that life as we know it would still exist? Explain.

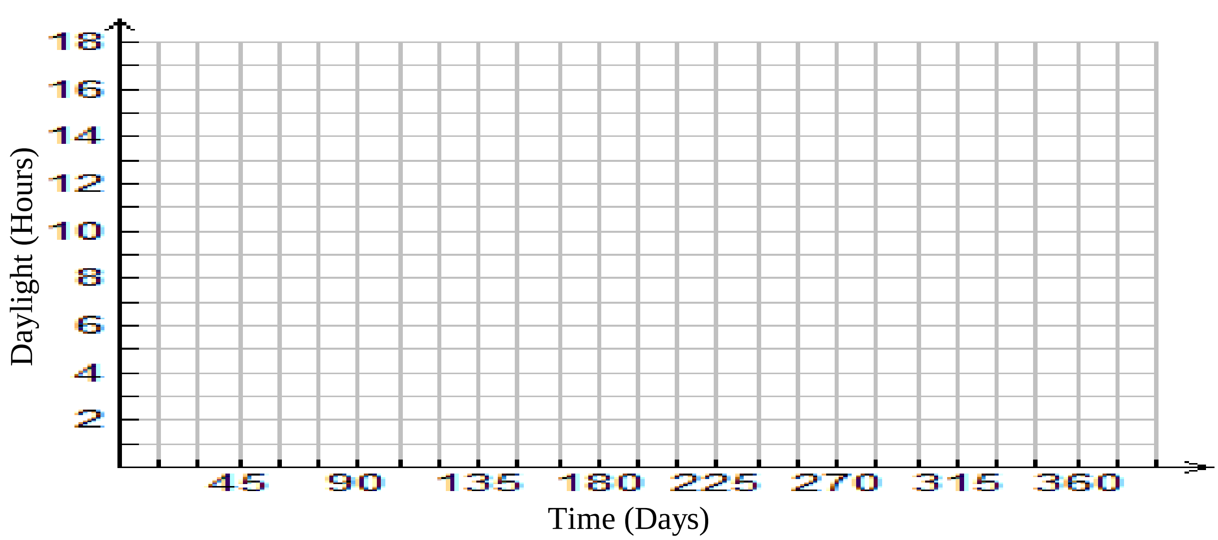
Activity 4 – Sunrise/Sunset

The table at the right contains sunrise and sunset data for Toronto, Ontario for the year 2007. (Data obtained from www.sunrisesunset.com.)

Questions

1. Complete the table.
2. Use the provided grid to plot a graph of ***number of daylight hours versus the day of the year***. First plot the points and then draw a smooth curve through the points.
3. Write an equation that describes the curve that you obtained in question 2.
4. Use graphing calculator or a spreadsheet to perform a sinusoidal regression. Compare the equation obtained by the regression to the one that you wrote in question 3.
5. Use your equation to predict the number of daylight hours on December 25.
6. Suppose that you lived in a town situated exactly on the equator. How would the graph of number of hours of daylight versus day of the year differ from the one for Toronto?

Date	*Day of the Year*	*Sunrise (hh:mm)*	*Sunset (hh:mm)*	*Daylight (hh:mm)*	*Number of daylight hours to the nearest 100th of an hour*
January 1	0	7:51am	4:51pm	9:00	9
January 15	14	7:48am	4:58pm	9:10	9.17
January 29		7:38am	5:23pm		
February 12		7:21am	5:42pm		
February 26		7:00am	6:01pm		
March 12		7:36am	7:19pm		
March 26		7:11am	7:36pm		
April 9		6:46am	7:52pm		
April 23		6:23am	8:09pm		
May 7		6:03am	8:25pm		
May 21		5:48am	8:40pm		
June 4		5:38am	8:53pm		
June 18		5:36am	9:01pm		
July 2		5:40am	9:02pm		
July 16		5:51am	8:56pm		
July 30		6:04am	8:44pm		
August 13		6:19am	8:26pm		
August 27		6:35am	8:04pm		
September 10		6:50am	7:39pm		
September 24		7:06am	7:14pm		
October 8		7:22am	6:48pm		
October 22		7:39am	6:25pm		
November 5		6:57am	5:05pm		
November 19		7:15am	4:50pm		
December 3		7:32am	4:42pm		
December 17		7:44am	4:42pm		
December 31		7:50am	4:50pm		



Example

On a Ferris wheel, the maximum height of a passenger above the ground is 35 m. The wheel takes 2 minutes to complete one revolution and the passengers board the Ferris wheel 2 m above the ground at the bottom of its rotation.

(a) Sketch two cycles of the graph of height of passenger (in metres) versus time (in seconds).

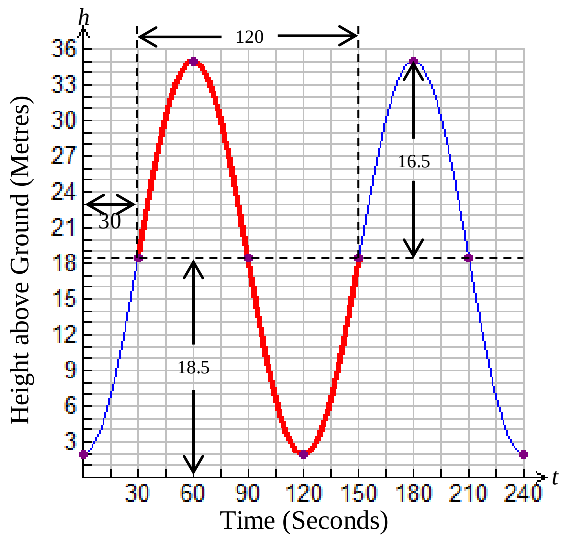
(b) Write ***an*** equation of the graph that you obtained in part (a).

(c) How high is the passenger after 25 s?

(d) If the ride lasts six minutes, at what times will the passenger be at the maximum height?

Solution

(a) For this question, we shall assume that the passenger is 2 m above the ground at time $t = 0$.



(b) Maximum Height = 35 m, Minimum Height = 2 m

$A=(35-2)\div 2=16.5$

$d=(35+2)\div 2=18.5$ (the average of the max and min)

Since it takes 120 seconds to complete one rotation, $T=120$.

But $T=2\pi\left|\frac{1}{\omega}\right|$, which implies that $2\pi\left(\frac{1}{\omega}\right)=120$. $\therefore \omega=\frac{\pi}{60}$

Finally, it's obvious from the graph that if we use $y=\sin x$ as the base function, $p=30$.

$\therefore h(t)=16.5\sin\left(\tfrac{\pi}{60}(t-30)\right)+18.5$

The value of p can be also be found in other ways:

Method 1

$\because h(t)=16.5\sin\left(\tfrac{\pi}{60}(t-p)\right)+18.5$ and $h(0)=2$

$\therefore 16.5\sin\left(\tfrac{\pi}{60}(0-p)\right)+18.5=2$

$\therefore \sin\left(\frac{-p\pi}{60}\right)=\frac{2-18.5}{16.5}$ $\qquad \therefore \sin\left(\frac{-p\pi}{60}\right)=-1$

$\therefore \frac{-p\pi}{60}=\frac{-\pi}{2}$ $\qquad \therefore p=30$

Method 2

In mapping notation, the transformation is expressed as $(x,y)\to(\omega^{-1}x+p, Ay+d)$. Since the image of $(0,0)$ is $(30,18.5)$.

$\therefore \omega^{-1}(0)+p=30 \Rightarrow p=30$

Method 3 (By Bobby B.)

The sine function is one-quarter cycle out of phase with the negative cosine function (see graph). Therefore, $p = 120 \div 4 = 30$.

(c) $h(25)=16.5\sin\left(\tfrac{\pi}{60}(25-30)\right)+18.5\doteq 14.2$

The passenger was about 14.2 m above the ground.

(d) The passenger is at the maximum height whenever $h(t)=35$. From the graph, we can see that this occurs at $t=60$ s and $t=180$ s. Since h is periodic, $h(180+120)=h(180)=35$ (the period is 120 radians). Therefore, the passenger is at the maximum height at 60 s, 180 s and 300 s.

Note on Angular Frequency

In the above problem, we determined that $\omega=\frac{\pi}{60}$. Since the angular frequency ω is equal to the number of cycles in 2π radians, the Ferris wheel completes $\frac{\pi}{60}$ cycles in a span of 2π radians. Since the independent variable is time and is measured in units of seconds, the Ferris wheel completes $\frac{\pi}{60}$ revolutions in 2π seconds or $\frac{1}{120}$ of a revolution in 1 second. Now $\frac{1}{120}$ revolutions/s = $\frac{1}{120}(2\pi)$ radians/s $=\frac{\pi}{60}$ radians/s. Therefore, the Ferris wheel turns at a rate of $\frac{\pi}{60}$ radians/s. Hence, the angular frequency $\omega=\frac{\pi}{60}$ determines the rate of rotation of the Ferris wheel.

Homework

Precalculus (Ron Larson)

pp. 306 – 309: #39-66, 73-80, 83-87, 95-98, 93

Modelling Periodic Phenomena – More Practice

Questions

5. **K** Mike is waving a sparkler in a circular motion at a constant speed. The tip of the sparkler is moving in a plane that is perpendicular to the ground. The height of the tip of the sparkler above the ground, as a function of time, can be modelled by a sinusoidal function. At $t = 0$, the sparkler is at its highest point above the ground.

a) What does the amplitude of the sinusoidal function represent in this situation?
b) What does the period of the sinusoidal function represent in this situation?
c) What does the equation of the axis of the sinusoidal function represent in this situation?
d) If no horizontal translations are required to model this situation, should a sine or cosine function be used?

6. **A** To test the resistance of a new product to temperature changes, the product is placed in a controlled environment. The temperature in this environment, as a function of time, can be described by a sine function. The maximum temperature is 120 °C, the minimum temperature is −60 °C, and the temperature at $t = 0$ is 30 °C. It takes 12 h for the temperature to change from the maximum to the minimum. If the temperature is initially increasing, what is the equation of the sine function that describes the temperature in this environment?

7. A person who was listening to a siren reported that the frequency of the sound fluctuated with time, measured in seconds. The minimum frequency that the person heard was 500 Hz, and the maximum frequency was 1000 Hz. The maximum frequency occurred at $t = 0$ and $t = 15$. The person also reported that, in 15, she heard the maximum frequency 6 times (including the times at $t = 0$ and $t = 15$). What is the equation of the cosine function that describes the frequency of this siren?

9. At one time, Maple Leaf Village (which no longer exists) had North America's largest Ferris wheel. The Ferris wheel had a diameter of 56 m, and one revolution took 2.5 min to complete. Riders could see Niagara Falls if they were higher than 50 m above the ground. Sketch three cycles of a graph that represents the height of a rider above the ground, as a function of time, if the rider gets on at a height of 0.5 m at $t = 0$ min. Then determine the time intervals when the rider could see Niagara Falls.

10. The number of hours of daylight in Vancouver can be modelled by a sinusoidal function of time, in days. The longest day of the year is June 21, with 15.7 h of daylight. The shortest day of the year is December 21, with 8.3 h of daylight.

a) Find an equation for $n(t)$, the number of hours of daylight on the nth day of the year.
b) Use your equation to predict the number of hours of daylight in Vancouver on January 30th.

11. **T** The city of Thunder Bay, Ontario, has average monthly temperatures that vary between −14.8 °C and 17.6 °C. The following table gives the average monthly temperatures, averaged over many years. Determine the equation of the sine function that describes the data, and use your equation to determine the times that the temperature is below 0 °C.

Month	Jan.	Feb.	Mar.	Apr.	May	June
Average Temperature (°C)	−14.8	−12.7	−5.9	2.5	8.7	13.9
Month	**July**	**Aug.**	**Sep.**	**Oct.**	**Nov.**	**Dec.**
Average Temperature (°C)	17.6	16.5	11.2	5.6	−2.7	−11.1

12. **C** A nail is stuck in the tire of a car. If a student wanted to graph a sine function to model the height of the nail above the ground during a trip from Kingston, Ontario, to Hamilton, Ontario, should the student graph the distance of the nail above the ground as a function of time or as a function of the total distance travelled by the nail? Explain your reasoning.

Extending

13. A clock is hanging on a wall, with the centre of the clock 3 m above the floor. Both the minute hand and the second hand are 15 cm long. The hour hand is 8 cm long. For each hand, determine the equation of the cosine function that describes the distance of the tip of the hand above the floor as a function of time. Assume that the time, t, is in minutes and that the distance, $D(t)$, is in centimetres. Also assume that $t = 0$ is midnight.

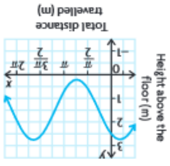
Answers

5. a) the radius of the circle in which the tip of the sparkler is moving
b) the time it takes Mike to make one complete circle with the sparkler
c) the height above the ground of the centre of the circle in which the tip of the sparkler is moving
d) cosine function

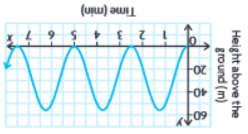
6. $y = 90 \sin\left(\frac{\pi}{12}x\right) + 30$

7. $y = 250 \cos\left(\frac{2\pi}{3}x\right) + 750$

8. $y = -1.25 \sin\left(\frac{4}{5}x\right) + 1.5$



9. $0.98 \text{ min} < t < 1.52 \text{ min}$, $3.48 \text{ min} < t < 4.02 \text{ min}$, $5.98 \text{ min} < t < 6.52 \text{ min}$



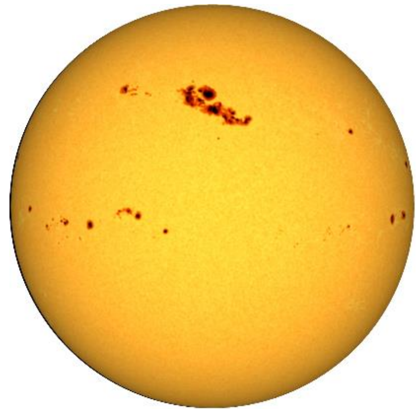
10. a) $y = 3.7 \sin\left(\frac{2\pi}{365}x\right) + 12$
b) $y \doteq 13.87$ hours

11. $T(t) = 16.2 \sin\left(\frac{2\pi}{365}(t - 116)\right) + 1.4$, $0 < t < 111$ and $304 < t < 365$

12. The student should graph the height of the nail above the ground as a function of the total distance travelled by the nail, because the nail would not be travelling at a constant speed. If the student graphed the height of the nail above the ground as a function of time, the graph would not be sinusoidal.

MODELLING PERIODIC PHENOMENA–EVEN MORE PRACTICE

1. For several hundred years, astronomers have kept track of the number of sunspots that occur on the surface of the sun. The number of sunspots counted in each year varies periodically from a minimum of about 10 per year to a maximum of about 110 per year. Between the maxima that occurred in the years 1750 and 1948, there were 18 complete cycles.



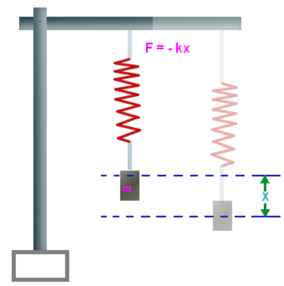
(a) What is the period of the sunspot cycle? Sketch two sunspot cycles, starting in 1948.

(b) Write ***four*** different equations (using base functions sin, cos, –sin, –cos) expressing the number of sunspots per year in terms of the year.

(c) How many sunspots would you expect this year?

(d) What was the first year after 2000 in which the number of sunspots was about 35? When will it be a maximum?

2. An object hangs from a spring in a stable (equilibrium) position. The spring is pulled 1.5 m downward and the object begins to oscillate, making one complete oscillation every 4 seconds.



(a) Write ***four*** different equations (using base functions sin, cos, –sin, –cos) that describe the motion of this object.

(b) At what two times within one cycle is the spring 1 m below the equilibrium position? Use these values to find the next two times it is in the same position.

3. It is a well-established fact that average temperatures on Earth vary over periods of thousands of years. Suppose that at one place, the highest ***average*** temperature is 25º C and the lowest is 15º C. Also, suppose that it takes 20,000 years for the average temperature to change from the maximum of 25º C to the minimum of 15º C. If in the year 2000 the average temperature was at a high point of 25º C, model this change in average temperature over time in ***four*** different ways (using base functions sin, cos, –sin, –cos).

4. A standard residential electrical outlet provides alternating current (AC) at a frequency of 60 Hertz (Hz) and a root-mean-square potential difference of 120 Volts (V).

The frequency of 60 Hz ***means*** that the flow of electric charge ***reverses direction*** at a rate of 60 ***cycles*** per second. Unlike a direct-current (DC) circuit, however, an AC circuit's voltage is not constant. It actually oscillates between a peak voltage of one polarity (e.g. 170 V) and the same peak voltage but of the opposite polarity (e.g. –170 V). Thus, the root-mean-square potential difference of 120 V ***does not mean*** that the current is delivered at a constant voltage of 120 V. Instead, it means that the ***time-averaged*** power delivered is ***equivalent*** to the power delivered by a DC voltage of 120 V.

For a sinusoidal waveform, the peak voltage of an AC current is equal to the product of $\sqrt{2}$ and the root-mean-square voltage. Use this to model the voltage change of the AC current supplied by a standard electrical outlet. Once again, write ***four*** different equations (using base functions sin, cos, –sin, –cos).

5. When you board a Ferris wheel, you are 1 m above the ground. At the highest point of the ride, you are 30 m above the ground.

(a) If it takes 30 seconds for the ride to complete one full rotation, write ***four*** different equations (using base functions sin, cos, –sin, –cos) that model your height above the ground at *t* seconds after the ride starts.

(b) Find at what two times within one cycle you are exactly 20 m above the ground.

(c) What is the linear speed at which you are travelling when you ride the Ferris wheel?

(d) How far have you travelled after the Ferris wheel has rotated through 10000 radians?

Answers

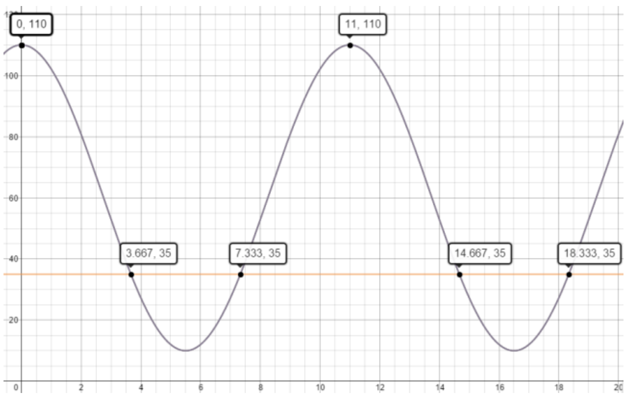
1. **(b)** $N(t)=50\cos\left(\frac{2\pi}{11}t\right)+60$, $N(t)=50\sin\left(\frac{2\pi}{11}\left(t+\frac{11}{4}\right)\right)+60$, $N(t)=-50\cos\left(\frac{2\pi}{11}\left(t+\frac{11}{2}\right)\right)+60$,

$N(t)=-50\sin\left(\frac{2\pi}{11}\left(t-\frac{11}{4}\right)\right)+60$ For all these equations, it is assumed that $t = 0$ corresponds to a year in which the number of sunspots ***is at a maximum***.

(c) Evaluate $N(\text{current year}-1992)$, assuming $t = 0$ corresponds to 1992. **e.g.** evaluate $N(2015-1992)$

(Of course, there is nothing terribly special about 1992. Any year in which there was a max could be used.)

(d) The last maximum before 2000 occurred in 1992 (1992 – 1948 = 44, which is divisible by 11). Solve $N(t)=35$, which has four solutions for $0 \le t \le 22$, $t \doteq 3.67$, $t \doteq 14.67$ and $t \doteq 18.33$. Using 1992 as the baseline, the correct solution is clearly $t \doteq 14.67$, which corresponds to the year 2007 (1992+14.67). The first maximum after 2000 occurred in 2003, 11 years after the maximum of 1992.

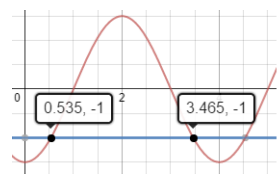


The time $t = 0$ corresponds to the year 1992.

The solutions of the equation $N(t)=35$ are equivalent to the points of intersection of $y=N(t)$ and $y=35$. The points of intersection shown at the left correspond to the years (after rounding) 1996, 1999, 2007 and 2010.

Thus, the first year after 2000 in which the number of sunspots was about 35 must have been 2007.

2. **(a)** $h(t)=-1.5\cos\left(\frac{\pi}{2}t\right)=-1.5\sin\left(\frac{\pi}{2}(t+1)\right)=1.5\cos\left(\frac{\pi}{2}(t+2)\right)=1.5\sin\left(\frac{\pi}{2}(t-1)\right)$



(b) Use Desmos to solve the equation $h(t)=-1$.

3. $A(t)=5\cos\left(\frac{\pi}{20000}t\right)+20$, $A(t)=5\sin\left(\frac{\pi}{20000}(t+10000)\right)+20$,

$A(t)=-5\cos\left(\frac{\pi}{20000}(t+20000)\right)$, $A(t)=-1.5\sin\left(\frac{\pi}{20000}(t-10000)\right)+20$

4. Peak voltage = $120\sqrt{2} \doteq 170$, $V(t)=170\sin(120\pi t)$, $V(t)=170\cos\left(120\pi\left(t-\frac{1}{240}\right)\right)$,

$V(t)=-170\sin\left(120\pi\left(t-\frac{1}{120}\right)\right)$, $V(t)=-170\cos\left(120\pi\left(t+\frac{1}{240}\right)\right)$

5. **(a)** $h(t)=-14.5\cos\left(\frac{\pi}{15}t\right)+15.5$, $h(t)=-14.5\sin\left(\frac{\pi}{15}(t+7.5)\right)+15.5$

$h(t)=14.5\cos\left(\frac{\pi}{15}(t-15)\right)+15.5$, $h(t)=14.5\sin\left(\frac{\pi}{15}(t-7.5)\right)+15.5$

(b) Use Desmos to solve $h(t)=20$. **(c)** $v=\frac{d}{t}=\frac{C}{T}=\frac{2\pi(14.5)}{30}=\frac{29\pi}{30}$ m/s $\doteq 3$ m/s

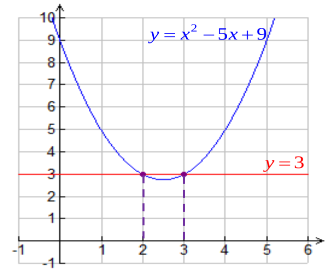
(d) $d=r\theta=14.5(10000)=145000$ m = 145 km

TRIGONOMETRIC IDENTITIES

Important Prerequisite Information – Different Types of Equations

☺ *Equations that are Solved for the Unknown*

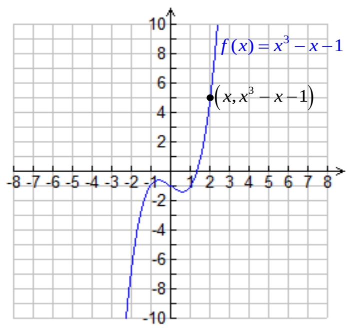
e.g. Solve $x^2 - 5x + 9 = 3$



- This means that we need to ***find*** the value(s) of x that make the left-hand-side equal to the right-hand-side.
- Geometrically, this equation describes the ***x-co-ordinates*** of the ***points of intersection*** of the graphs of $y = x^2 - 5x + 9$ and $y = 3$.
- As can be seen in the graph at the right, there are only two points of intersection and hence, only two solutions $x = 2$ and $x = 3$.

☺ *Equations that Express Mathematical Relationships (i.e. Functions, Relations)*

e.g. $f(x) = x^3 - x - 1$, $\quad x^2 + y^2 = 16$, $\quad c^2 = a^2 + b^2$ (Pythagorean Theorem)



- Such equations express a ***relationship*** between an ***independent variable*** (or a group of independent variables) and a ***dependent variable***. For instance, in the graph of $f(x) = x^3 - x - 1$ shown at the right, any point lying on the curve must have co-ordinates $(x, x^3 - x - 1)$. Once the value of the independent variable x is chosen, the dependent variable y ***must*** have a value of $x^3 - x - 1$.
- Equations that express relationships are not solved in the same sense as equations such as $x^2 - 5x + 9 = 3$ are because there are generally an infinite number of solutions. However, it does make sense to rewrite them in a different form.
- For relationships between two variables in which the independent variable varies continuously, the equations usually describe (piecewise) continuous curves.
- If the independent variable is restricted to integral or rational values (i.e. whole numbers or fractions), the graphs of such functions are a discrete collection of points in the Cartesian plane.

☺ *Identities*

An ***identity*** is an ***equation that expresses the equivalence of two expressions***.

e.g. $(a+b)^2 = a^2 + 2ab + b^2$ $\quad \cos^2\theta + \sin^2\theta = 1$, $\quad \tan\theta = \dfrac{\sin\theta}{\cos\theta}$

- The given equations are identities. ***For all values of the unknown(s) for which both sides of the equation are defined***, the left-hand-side ***equals*** the right-hand-side. That is, the expression on the left side ***is equivalent to*** the expression on the right side.
- For the identity $(a+b)^2 = a^2 + 2ab + b^2$, there are no restrictions on the values of a and b.
- For the identity $\cos^2\theta + \sin^2\theta = 1$, there are no restrictions on the value of θ.
- For the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, θ cannot take on values that make $\cos\theta = 0$ because this would lead to division by zero, which is ***undefined***.

Note

- Identities ***need not*** involve trigonometry!
- To discourage the mistaken notion that θ is the only symbol that can be used to represent the independent variable of a trigonometric function, other symbols will often be used in place of θ.
- Once an equation is proved to be an identity, it can be used to construct proofs of other identities.

List of Basic Identities

Pythagorean Identities	*Quotient Identities*	*Reciprocal Identities*
For all $x \in \mathbb{R}$, $\cos^2 x + \sin^2 x = 1$ The following can be derived very easily from the above identity. In mathematical terms, we say that they are ***corollaries*** of $\cos^2 x + \sin^2 x = 1$. $\sin^2 x = 1 - \cos^2 x$ $\cos^2 x = 1 - \sin^2 x$ $1 + \tan^2 x = \sec^2 x$ $1 + \cot^2 x = \csc^2 x$	For all $x \in \mathbb{R}$ such that $\cos x \neq 0$, $\tan x = \dfrac{\sin x}{\cos x}$ The following identity can be derived very easily from the above identity. For all $x \in \mathbb{R}$ such that $\sin x \neq 0$, $\cot x = \dfrac{\cos x}{\sin x}$	The following identities can be derived easily from the definitions of csc, sec and cot. For all $x \in \mathbb{R}$ such that $\sin x \neq 0$, $\csc x = \dfrac{1}{\sin x}$ For all $x \in \mathbb{R}$ such that $\cos x \neq 0$, $\sec x = \dfrac{1}{\cos x}$ For all $x \in \mathbb{R}$ such that $\tan x \neq 0$, $\cot x = \dfrac{1}{\tan x}$

Important Note about Notation

- $\sin^2 x$ is a shorthand notation for $(\sin x)^2$, which means that ***first*** $\sin x$ is evaluated, ***then*** the result is squared

  e.g. $\sin^2 \dfrac{\pi}{4} = \left(\sin\dfrac{\pi}{4}\right)^2 = \left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1}{2}$ $\qquad$ $\cos^2 \dfrac{\pi}{3} = \left(\cos\dfrac{\pi}{3}\right)^2 = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$
- This notation is used to avoid the excessive use of parentheses
- $\sin x^2 \neq \sin^2 x$ $\qquad$ The expression $\sin x^2$ means $\sin\left(x^2\right)$, that is, ***first*** x is squared and ***then*** "sin" is applied

Proofs of the Pythagorean and Quotient Identities

Prove the following identities. (Here L.S. means "left side" and R.S. means "right side.")

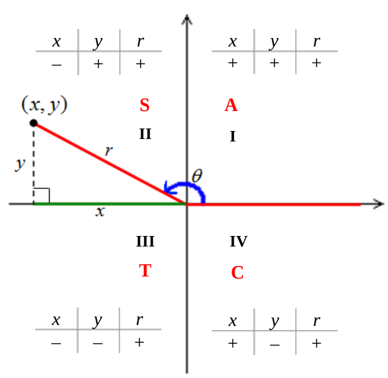
1. $\sin^2\theta + \cos^2\theta = 1$ (Assertion)

Proof (Justification)

$$\text{L.S.} = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 \quad \text{(definitions of sin, cos)}$$
$$= \frac{y^2}{r^2} + \frac{x^2}{r^2}$$
$$= \frac{y^2 + x^2}{r^2}$$
$$= \frac{r^2}{r^2} \quad \text{(Pyth. Theorem)}$$
$$= 1$$
$$= \text{R.S.}$$

$\therefore$ L.S. = R.S.

$\therefore \sin^2\theta + \cos^2\theta = 1$ is an identity



CRUX of the problem

crux: a vital, basic, decisive or pivotal point

Once you have identified the crux of a problem and found a way to proceed, the rest of the solution is usually straightforward.

2. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (Assertion)

Proof (Justification)

$$\text{L.S.} = \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)} \quad \text{(definitions of sin, cos)}$$
$$= \left(\frac{y}{r}\right)\left(\frac{r}{x}\right)$$
$$= \frac{y}{x}$$
$$= \tan\theta \quad \text{(definition of tan)}$$
$$= \text{R.S.}$$

$\therefore$ L.S. = R.S.

$\therefore \tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is an identity

Examples

Prove that each of the following equations is an identity.

Assertions

1. $\cot x = \dfrac{\cos x}{\sin x}$

Proof (Justification)

Assertions

$$\text{L.S.} = \cot x$$
$$= \frac{1}{\tan x} \quad \text{(reciprocal identity)}$$
$$= \frac{1}{\left(\frac{\sin x}{\cos x}\right)} \quad \text{(quotient identity)}$$

Justifications

$$= \frac{1}{1} \times \left(\frac{\cos x}{\sin x}\right)$$
$$= \frac{\cos x}{\sin x}$$
$$\text{R.S.} = \frac{\cos x}{\sin x}$$

$\therefore$ L.S. = R.S.

$$\therefore \cot x = \frac{\cos x}{\sin x}$$

2. $1 + \tan^2 x = \sec^2 x$

Proof (Justification)

$\because \cos^2 x + \sin^2 x = 1$ (proved above)

$$\therefore \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$
$$\therefore \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$
$$\therefore 1 + \tan^2 x = \sec^2 x$$

Note

- The final step is justified by one of the quotient identities and one of the reciprocal identities.
- The nature of this proof is a little different from the others because it does not begin with the left side or right side of the given equation. Instead, it begins with an identity that has already been proved and through algebraic manipulations, the desired equation is derived!

3. $\dfrac{\sin^2 x}{1 - \cos x} = 1 + \cos x$

Proof (Justification)

$$\text{L.S.} = \frac{\sin^2 x}{1 - \cos x}$$
$$= \frac{1 - \cos^2 x}{1 - \cos x} \quad \text{(Pyth identity)}$$
$$= \frac{(1 - \cos x)(1 + \cos x)}{1 - \cos x}$$
$$= 1 + \cos x$$
$$\text{R.S.} = 1 + \cos x$$

$\therefore$ L.S. = R.S

$$\therefore \frac{\sin^2 x}{1 - \cos x} = 1 + \cos x$$

Exercises

1. Prove the rest of the Pythagorean identities (i.e. the ones that have not been proved on pages 43-44).

2. Prove the reciprocal identities by using the definitions of sin, cos, tan, csc, sec and cot (i.e. $\sin\theta = \dfrac{y}{r}$, $\cos\theta = \dfrac{x}{r}$, etc.).

Proofs that make use solely of definitions are known as ***proofs from first principles*** because they do not rely upon any "facts" that are derived.

Logical and Notational Pitfalls – Please Avoid Absurdities!

1. The purpose of a proof is to ***establish*** the "truth" of a mathematical statement. ***Therefore, you must never assume what you are trying to prove!*** A common error is shown at the right. ***The series of steps shown is wrong and would be assigned a mark of zero!*** To write a correct proof, the left and right sides of the equation must be treated separately. Only once you have demonstrated that the left side is equal to the right side are you allowed to declare their equality.

"Proof"

$$\cot x = \frac{\cos x}{\sin x}$$
$$\therefore \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$
$$\therefore \frac{1}{\left(\frac{\sin x}{\cos x}\right)} = \frac{\cos x}{\sin x}$$
$$\therefore \frac{1}{1} \times \left(\frac{\cos x}{\sin x}\right) = \frac{\cos x}{\sin x}$$
$$\therefore \frac{\cos x}{\sin x} = \frac{\cos x}{\sin x}$$

2. Keep in mind that words like "sin," "cos" and "tan" ***are function names, not numerical values!*** Therefore, you must not treat them as numbers. For example, it makes sense to write $\dfrac{\sin 2x}{\sin x}$ but it makes no sense whatsoever to "cancel" the sines. Many students will write statements such as $\cancel{\dfrac{\sin 2x}{\sin x} = \dfrac{2x}{x}} = 2$, which are completely nonsensical. First, dividing the numerator and the denominator by "sin" is invalid because "sin" is not a number.

Furthermore, a simple test reveals that $\dfrac{\sin 2x}{\sin x} \neq 2$: $\dfrac{\sin 2(\frac{\pi}{4})}{\sin\frac{\pi}{4}} = \dfrac{\sin\frac{\pi}{2}}{\sin\frac{\pi}{4}} = \dfrac{1}{1/\sqrt{2}} = \sqrt{2}$. Clearly, $\sqrt{2} \neq 2$. Therefore, the assertion that was made is false!

Suggestions for Proving that Equations are Trig Identities

1. Write the given expressions in terms of sin and cos.

2. Begin with the more complicated side and try to simplify it.

3. Keep a list of important identities in plain view while working.

4. Expect to make mistakes! If one approach seems to lead to a dead end, try another. Don't give up!

Homework

Do a representative selection of questions 1 to 20.

1. State an equivalent expression for each.

a) $\cos x \tan x$ b) $\sin^2 x$

c) $\cos^2 x$ d) $\tan^2 x$

e) $\tan x \sin x$ f) $1 - \sin^2 x$

g) $\sin x \tan x \cos x$ h) $1 - \cos^2 x$

i) $\sin^2 x + \cos^2 x$

2. Prove each identity.

a) $\dfrac{\sin x}{\tan x} = \cos x$

b) $\sin x \cos x \tan x = 1 - \cos^2 x$

c) $\dfrac{1 - \cos^2 x}{\sin x} = \sin x$

d) $\sin^2 x + \dfrac{\sin x \cos x}{\tan x} = 1$

e) $1 + \dfrac{1}{\tan^2 x} = \dfrac{1}{\sin^2 x}$

f) $2\sin^2 x - 1 = \sin^2 x - \cos^2 x$

g) $\dfrac{1}{\cos x} - \cos x = \sin x \tan x$

h) $\sin x + \tan x = \tan x\,(1 + \cos x)$

i) $\dfrac{1}{1 - \sin^2 x} = 1 + \tan^2 x$

j) $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$

k) $\sin^2 x + \cos^2 x + \tan^2 x = \dfrac{1}{\cos^2 x}$

l) $\dfrac{\sin x}{\sin x + \cos x} = \dfrac{\tan x}{1 + \tan x}$

m) $\dfrac{1 + \tan^2 x}{1 - \tan^2 x} = \dfrac{1}{\cos^2 x - \sin^2 x}$

3. Use a graphing calculator to show that each equation appears to be an identity. Then, prove that the equation is an identity.

a) $\cos x \tan x = \sin x$

b) $\sin x + \tan x = \tan x\,(1 + \cos x)$

c) $1 + \tan^2 x = \dfrac{1}{\cos^2 x}$

d) $\cos^2 x = \sin^2 x + 2\cos^2 x - 1$

e) $\dfrac{1}{1 + \sin x} + \dfrac{1}{1 - \sin x} = \dfrac{2}{\cos^2 x}$

f) $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$

4. Prove each identity.

a) $\dfrac{1}{\sin^2 x} + \dfrac{1}{\cos^2 x} = \dfrac{1}{\sin^2 x \cos^2 x}$

b) $\tan x + \dfrac{1}{\tan x} = \dfrac{1}{\sin x \cos x}$

c) $\dfrac{1}{1 - \cos x} + \dfrac{1}{1 + \cos x} = \dfrac{2}{\sin^2 x}$

d) $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

e) $(1 - \cos^2 x)\left(1 + \dfrac{1}{\tan^2 x}\right) = 1$

f) $\dfrac{1 + 2\sin x \cos x}{\sin x + \cos x} = \sin x + \cos x$

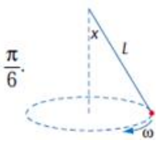
g) $\dfrac{\sin x}{1 - \cos x} - \dfrac{1 + \cos x}{\sin x} = 0$

h) $\sin^2 x - \sin^4 x = \cos^2 x - \cos^4 x$

i) $(1 + \tan^2 x)(1 - \cos^2 x) = \tan^2 x$

j) $\dfrac{\sin x - 1}{\sin x + 1} = \dfrac{-\cos^2 x}{(\sin x + 1)^2}$

5. Conical pendulum A conical pendulum is so named because of the cone-shaped path traced by the bob and the wire. The length of the pendulum wire, L, is related to the angle, x, that the wire makes with the vertical by the formula $L = \dfrac{g}{\omega^2 \cos x}$, where g is the acceleration due to gravity and ω is the angular velocity of the bob about the vertical, in radians per second. Another way of expressing the relationship is the formula $L = \dfrac{g \tan x}{\omega^2 \sin x}$.



a) Verify that the two formulas are equivalent when $x = \dfrac{\pi}{6}$.

b) Prove that $\dfrac{g}{\omega^2 \cos x} = \dfrac{g \tan x}{\omega^2 \sin x}$ is an identity.

6. Kicking a ball When a ball is kicked from the ground, the time of flight of the ball can be determined by the formula $t = \dfrac{2v_0 \sin x}{g}$.

In this formula, t seconds is the time of flight, v_0 metres per second is the initial velocity of the ball, x is the angle that the path of the ball makes with the ground when the ball is kicked, and g is the acceleration due to gravity.

a) Write another formula that determines the time of flight of the ball.

b) Equate the trigonometric expressions from the given formula and the formula you found in part a) to write an equation.

c) Use a graphing calculator to check if the equation appears to be an identity.

d) If the equation appears to be an identity, prove that it is an identity.

e) The formula for the horizontal distance, d metres, travelled by a ball kicked from the ground is $d = \dfrac{2v_0^2 \sin x \cos x}{g}$.

Write another formula for the horizontal distance.

f) Equate the trigonometric expressions from the two formulas in part e) to write an equation. Use a graphing calculator to check if the equation appears to be an identity.

g) If the equation appears to be an identity, prove that it is an identity.

7. Prove each identity.

a) $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

b) $\sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x = 1$

c) $\dfrac{4}{\cos^2 x} - 5 = 4\tan^2 x - 1$

d) $\dfrac{\cos x - \sin x - \cos^3 x}{\cos x} = \sin^2 x - \tan x$

e) $\dfrac{\sin^2 x - 6\sin x + 9}{\sin^2 x - 9} = \dfrac{\sin x - 3}{\sin x + 3}$

8. Find a counterexample to show that each equation is not an identity.

a) $\sin x = \sqrt{\sin^2 x}$ b) $\cos x = \sqrt{\cos^2 x}$

9. Technology Use radian measure for the following.

a) In the same viewing window, graph $y = \sin x$ and $y = x$ for $-0.2 \le x \le 0.2$ and $-0.2 \le y \le 0.2$. Do the graphs suggest that $\sin x = x$ is an identity?

b) Repeat part a) for $-2 \le x \le 2$ and $-2 \le y \le 2$.

c) Write a conclusion about verifying identities graphically.

10. Application Use the x, y, and r definitions of $\sin x$ and $\cos x$ to prove the following identity.

$$\frac{\cos\theta}{1 - \sin\theta} = \frac{1 + \sin\theta}{\cos\theta}$$

11. Inquiry/Problem Solving Determine if each of the following equations is an identity or not.

a) $\dfrac{1}{\tan x} + \cos x = \dfrac{\cos x(1 + \sin x)}{\sin x}$ b) $\dfrac{1}{\tan x} + \cos x = \tan x + \sin x$

c) $\dfrac{1}{\tan x} + \cos x = \dfrac{2\cos x}{\sin x}$

12. Communication Explain why you think that the equation $(a + b)^2 = a^2 + 2ab + b^2$ can be called an algebraic identity.

13. Algebra If $x = a\cos\theta - b\sin\theta$ and $y = a\sin\theta + b\cos\theta$, show that $x^2 + y^2 = a^2 + b^2$.

14. Since $1 - \cos^2 x = \sin^2 x$ is an identity, is $\sqrt{1 - \cos^2 x} = \sin x$ also an identity? Explain how you know.

15. a) Show graphically that $\sin^2 x + \cos^2 x = (\sin x + \cos x)^2$ is not an identity. Explain your reasoning.

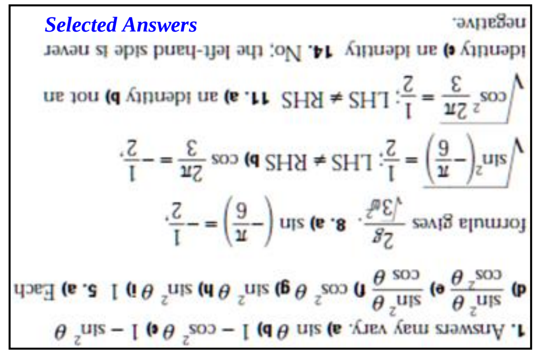
b) Explain how the graph shows if there any values of x for which the equation is true.

16. Write a list of helpful strategies for proving trigonometric identities, and describe situations in which you would try each strategy. Compare your list with your classmates'.

17. **Formulating problems** **a)** Create a trigonometric identity that has not appeared in this section.
b) Have a classmate check graphically that your equation may be an identity. If so, have your classmate prove your identity.

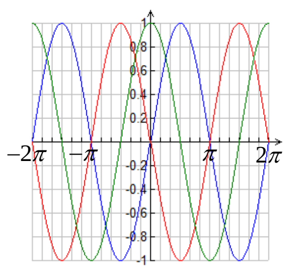
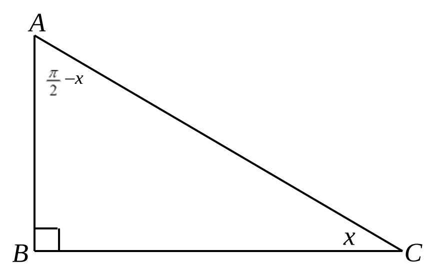
18. **Technology** **a)** Use a graph to show that the equation $\frac{\cos^2 x - 1}{\cos x + 1} = \cos x - 1$ appears to be an identity.
b) Compare the functions defined by each side of the equation by displaying a table of values. Find a value of x for which the values of the two functions are not the same. Have you shown that the equation is not an identity? Explain.

20. Prove that $\frac{\tan x \sin x}{\tan x + \sin x} = \frac{\tan x - \sin x}{\tan x \sin x}$.



Exercises on Equivalence of Trigonometric Expressions

Complete the following table. The first row is done for you.

Identity	*Graphical Justification*	*Justification using Right Triangle or Angle of Rotation*
$\sin(\frac{\pi}{2} - x) = \cos x$	Since $\sin\left(\frac{\pi}{2} - x\right) = \sin\left(-1\left(x - \frac{\pi}{2}\right)\right)$, the graph of $y = \sin\left(\frac{\pi}{2} - x\right)$ can be obtained by reflecting $y = \sin x$ in the y-axis, followed by a shift to the right by $\frac{\pi}{2}$. Once these transformations are applied, lo and behold, the graph of $y = \cos x$ is obtained! 	 $\cos x = \frac{BC}{AC}$ $\sin\left(\frac{\pi}{2} - x\right) = \frac{BC}{AC}$ $\therefore \cos x = \sin\left(\frac{\pi}{2} - x\right)$
$\cos(\frac{\pi}{2} - x) = \sin x$		
$\cos(\frac{\pi}{2} + \theta) = -\sin\theta$		

Identity	*Graphical Justification*	*Justification using Angles of Rotation*
$\sin(\pi-\theta)=\sin\theta$		
$\cos(\pi-\theta)=-\cos\theta$		
$\sin(-\theta)=-\sin\theta$		
$\cos(-\theta)=\cos\theta$		

List of Important Identities that can be Discovered/Justified using Transformations

1. Read the summary on page 48 (i.e. the next page).

2. Do the questions on page 49 for homework.

In Summary

Key Ideas

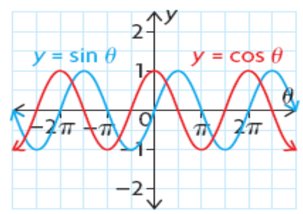
- Because of their periodic nature, there are many equivalent trigonometric expressions.
- Two expressions may be equivalent if the graphs created by a graphing calculator of their corresponding functions coincide, producing only one visible graph over the entire domain of both functions. To demonstrate equivalency requires additional reasoning about the properties of both graphs.

Need to Know

- Horizontal translations that involve multiples of the period of a trigonometric function can be used to obtain two equivalent functions with the same graph. For example, the sine function has a period of 2π, so the graphs of $f(\theta) = \sin\theta$ and $f(\theta) = \sin(\theta + 2\pi)$ are the same. Therefore, $\sin\theta = \sin(\theta + 2\pi)$.

- Horizontal translations of $\frac{\pi}{2}$ that involve both a sine function and a cosine function can be used to obtain two equivalent functions with the same graph. Translating the cosine function $\frac{\pi}{2}$ to the right $\left(f(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)\right)$ results in the graph of the sine function, $f(\theta) = \sin\theta$.

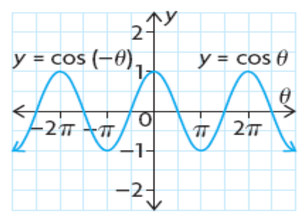
  Similarly, translating the sine function $\frac{\pi}{2}$ to the left $\left(f(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)\right)$ results in the graph of the cosine function, $f(\theta) = \cos\theta$.



$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$

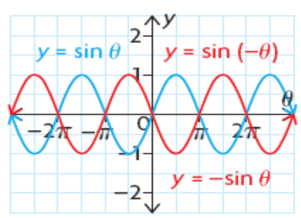
$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$

- Since $f(\theta) = \cos\theta$ is an even function, reflecting its graph across the y-axis results in two equivalent functions with the same graph.

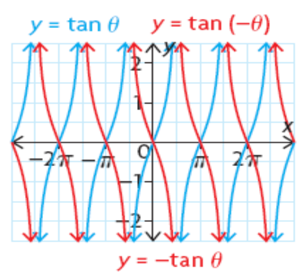


$\cos\theta = \cos(-\theta)$

- $f(\theta) = \sin\theta$ and $f(\theta) = \tan\theta$ are odd and have the property of rotational symmetry about the origin. Reflecting these functions across both the x-axis and the y-axis produces the same effect as rotating the function through 180° about the origin. Thus, the same graph is produced.

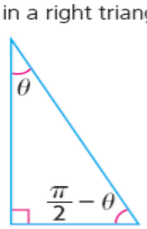


$\sin(-\theta) = -\sin\theta$



$\tan(-\theta) = -\tan\theta$

- The cofunction identities describe trigonometric relationships between the complementary angles θ and $\left(\frac{\pi}{2} - \theta\right)$ in a right triangle.



$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

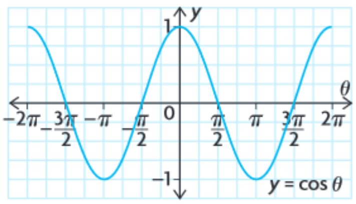
$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

- You can identify equivalent trigonometric expressions by comparing principal angles drawn in standard position in quadrants II, III, and IV with their related acute angle, θ, in quadrant I.

Principal Angle in Quadrant II	Principal Angle in Quadrant III	Principal Angle in Quadrant IV
$\sin(\pi - \theta) = \sin\theta$	$\sin(\pi + \theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
$\cos(\pi - \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
$\tan(\pi - \theta) = -\tan\theta$	$\tan(\pi + \theta) = \tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$

Homework

1. **a)** Use transformations and the cosine function to write three equivalent expressions for the following graph.



b) Use transformations and a different trigonometric function to write three equivalent expressions for the graph.

2. **a)** Classify the reciprocal trigonometric functions as odd or even, and then write the corresponding equation.
b) Use transformations to explain why each equation is true.

3. Use the cofunction identities to write an expression that is equivalent to each of the following expressions.

a) $\sin\frac{\pi}{6}$ **c)** $\tan\frac{3\pi}{8}$ **e)** $\sin\frac{\pi}{8}$

b) $\cos\frac{5\pi}{12}$ **d)** $\cos\frac{5\pi}{16}$ **f)** $\tan\frac{\pi}{6}$

4. **a)** Write the cofunction identities for the reciprocal trigonometric functions.
b) Use transformations to explain why each identity is true.

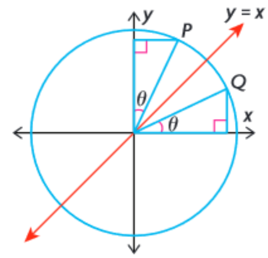
5. Write an expression that is equivalent to each of the following expressions, using the related acute angle.

a) $\sin\frac{7\pi}{8}$ **c)** $\tan\frac{5\pi}{4}$ **e)** $\sin\frac{13\pi}{8}$

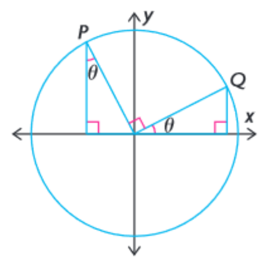
b) $\cos\frac{13\pi}{12}$ **d)** $\cos\frac{11\pi}{6}$ **f)** $\tan\frac{5\pi}{3}$

6. Show that each equation is true, using the given diagram.

a) $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$



b) $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$



7. State whether each of the following are true or false. For those that are false, justify your decision.

a) $\cos(\theta + 2\pi) = \cos\theta$ **d)** $\tan(\pi - \theta) = \tan\theta$

b) $\sin(\pi - \theta) = -\sin\theta$ **e)** $\cot\left(\frac{\pi}{2} + \theta\right) = \tan\theta$

c) $\cos\theta = -\cos(\theta + 4\pi)$ **f)** $\sin(\theta + 2\pi) = \sin(-\theta)$

Answers

1. **a)** Answers may vary. For example: $y = \cos(\theta + 2\pi)$, $y = \cos(\theta + 4\pi)$, $y = \cos(\theta - 2\pi)$
b) $y = \sin\left(\theta + \frac{\pi}{2}\right)$, $y = \sin\left(\theta - \frac{3\pi}{2}\right)$, $y = \sin\left(\theta + \frac{5\pi}{2}\right)$

2. **a)** $y = \csc\theta$ is odd, $\csc(-\theta) = -\csc\theta$; $y = \sec\theta$ is even, $\sec(-\theta) = \sec\theta$; $y = \cot\theta$ is odd, $\cot(-\theta) = -\cot\theta$
b) $y = \cot(-\theta)$ is the graph of $y = \cot\theta$ reflected across the y-axis; $y = -\cot\theta$ is the graph of $y = \cot\theta$ reflected across the x-axis. Both of these transformations result in the same graph. $y = \csc(-\theta)$ is the graph of $y = \csc\theta$ reflected across the y-axis; $y = -\csc\theta$ is the graph of $y = \csc\theta$ reflected across the x-axis. Both of these transformations result in the same graph. $y = \sec(-\theta)$ is the graph of $y = \sec\theta$ reflected across the y-axis. This results in the same graph as $y = \sec\theta$.

3. **a)** $\cos\frac{\pi}{3}$ **c)** $\cot\frac{\pi}{8}$ **e)** $\cos\frac{3\pi}{8}$
b) $\sin\frac{\pi}{12}$ **d)** $\sin\frac{3\pi}{16}$ **f)** $\cot\frac{\pi}{3}$

4. **a)** $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$; $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$; $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$
b) $y = \tan\left(\frac{\pi}{2} - \theta\right) = \tan\left(-\left(\theta - \frac{\pi}{2}\right)\right)$; This is the graph of $y = \tan\theta$ reflected across the y-axis and translated $\frac{\pi}{2}$ to the right, which is identical to the graph of $y = \cot\theta$.
$y = \csc\left(\frac{\pi}{2} - \theta\right) = \csc\left(-\left(\theta - \frac{\pi}{2}\right)\right)$; This is the graph of $y = \csc\theta$ reflected across the y-axis and translated $\frac{\pi}{2}$ to the right, which is identical to the graph of $y = \sec\theta$.
$y = \sec\left(\frac{\pi}{2} - \theta\right) = \sec\left(-\left(\theta - \frac{\pi}{2}\right)\right)$; This is the graph of $y = \sec\theta$ reflected across the y-axis and translated $\frac{\pi}{2}$ to the right, which is identical to the graph of $y = \csc\theta$.

5. **a)** $\sin\frac{\pi}{8}$ **d)** $\cos\frac{\pi}{6}$
b) $-\cos\frac{\pi}{12}$ **e)** $-\sin\frac{3\pi}{8}$
c) $\tan\frac{\pi}{4}$ **f)** $-\tan\frac{\pi}{3}$

6. **a)** Assume the circle is a unit circle. Let the coordinates of Q be (x, y). Since P and Q are reflections of each other in the line $y = x$, the coordinates of P are (y, x). Draw a line from P to the positive x-axis. The hypotenuse of the new right triangle makes an angle of $\left(\frac{\pi}{2} - \theta\right)$ with the positive x-axis. Since the x-coordinate of P is y, $\cos\left(\frac{\pi}{2} - \theta\right) = y$. Also, since the y-coordinate of Q is y, $\sin\theta = y$. Therefore, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.
b) Assume the circle is a unit circle. Let the coordinates of the vertex on the circle of the right triangle in the first quadrant be (x, y). Then $\sin\theta = y$, so $-\sin\theta = -y$. The point on the circle that results from rotating the vertex by $\frac{\pi}{2}$ counterclockwise about the origin has coordinates $(-y, x)$, so $\cos\left(\frac{\pi}{2} + \theta\right) = -y$. Therefore, $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$.

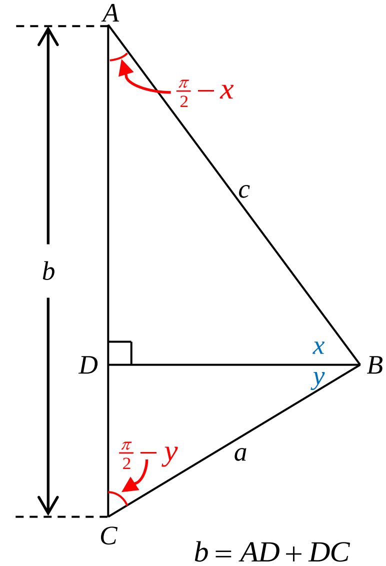
7. **a)** true
b) false; Answers may vary. For example: Let $\theta = \frac{\pi}{2}$. Then the left side is $\sin\frac{\pi}{2}$, or 1. The right side is $-\sin\frac{\pi}{2}$, or -1.
c) false; Answers may vary. For example: Let $\theta = \pi$. Then the left side is $\cos\pi$, or -1. The right side is $-\cos 5\pi$, or 1.
d) false; Answers may vary. For example: Let $\theta = \frac{\pi}{4}$. Then the left side is $\tan\frac{3\pi}{4}$, or $-\frac{\sqrt{2}}{2}$. The right side is $\tan\frac{\pi}{4}$, or $\frac{\sqrt{2}}{2}$.
e) false; Answers may vary. For example: Let $\theta = \pi$. Then the left side is $\cot\frac{3\pi}{4}$, or -1. The right side is $\tan\frac{\pi}{4}$, or 1.
f) false; Answers may vary. For example: Let $\theta = \frac{\pi}{2}$. Then the left side is $\sin\frac{5\pi}{2}$, or 1. The right side is $\sin\left(-\frac{\pi}{2}\right)$, or -1.

Compound-Angle Identities

Question

If we know how to evaluate the trig ratios of the angles x and y, can we use these values to evaluate quantities such as $\sin(x+y)$ and $\sin(x-y)$? With a little hard work, we shall see that the answer to this question is "yes!"

Expressing sin(x+y) in terms of sin x, sin y, cos x and cos y



By the Law of Sines,

$$\frac{\sin(x+y)}{b} = \frac{\sin(\pi/2-x)}{a} \quad \text{and} \quad \frac{\sin(x+y)}{b} = \frac{\sin(\pi/2-y)}{c}$$

Using the cofunction identity $\sin(\pi/2-\theta)=\cos\theta$, the above equations can be written

$$\frac{\sin(x+y)}{b} = \frac{\cos x}{a} \quad (1) \qquad \text{and} \qquad \frac{\sin(x+y)}{b} = \frac{\cos y}{c} \quad (2)$$

By multiplying both sides of equations (1) and (2) by b, we obtain

$$\sin(x+y) = \frac{b\cos x}{a} \qquad (3)$$

$$\sin(x+y) = \frac{b\cos y}{c} \qquad (4)$$

Adding equations (3) and (4), we obtain $2\sin(x+y) = \dfrac{b\cos x}{a} + \dfrac{b\cos y}{c}$

$$\therefore \sin(x+y) = \frac{b\cos x}{2a} + \frac{b\cos y}{2c}$$

$$\therefore \sin(x+y) = \frac{bc\cos x}{2ac} + \frac{ab\cos y}{2ac} \qquad \text{(expressing with a common denominator)}$$

$$\therefore \sin(x+y) = \frac{b(c\cos x)}{2ac} + \frac{b(a\cos y)}{2ac}$$

$$\therefore \sin(x+y) = \frac{1}{2}\left(\frac{(AD+DC)DB}{ac} + \frac{(AD+DC)DB}{ac}\right) \qquad (\text{since } b = AD+DC \text{ and } c\cos x = a\cos y = DB)$$

$$\therefore \sin(x+y) = \frac{1}{2}\left(\frac{(AD)(DB)}{ac} + \frac{(DC)(DB)}{ac} + \frac{(AD)(DB)}{ac} + \frac{(DC)(DB)}{ac}\right)$$

$$\therefore \sin(x+y) = \frac{1}{2}\left[2\left(\frac{(AD)(DB)}{ac}\right) + 2\left(\frac{(DC)(DB)}{ac}\right)\right]$$

$$\therefore \sin(x+y) = \frac{(AD)(DB)}{ac} + \frac{(DC)(DB)}{ac}$$

$$\therefore \sin(x+y) = \left(\frac{AD}{c}\right)\left(\frac{DB}{a}\right) + \left(\frac{DB}{c}\right)\left(\frac{DC}{a}\right)$$

$$\therefore \sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$\boxed{\sin(x+y) = \sin x\cos y + \cos x\sin y}$$

Using sin(x+y)=sin x cos y + cos x sin y to Derive many other Compound-Angle Identities

$$\begin{aligned}&\sin(x-y)\\&=\sin(x+(-y))\\&=\sin x\cos(-y)+\cos x\sin(-y)\\&=\sin x\cos y+\cos x(-\sin y)\\&=\sin x\cos y-\cos x\sin y\end{aligned}$$

$$\begin{aligned}&\cos(x+y)\\&=\sin(\pi/2-(x+y))\\&=\sin((\pi/2-x)-y)\\&=\sin(\pi/2-x)\cos(-y)+\cos(\pi/2-x)\sin(-y)\\&=\cos x\cos y+\sin x(-\sin y)\\&=\cos x\cos y-\sin x\sin y\end{aligned}$$

$$\begin{aligned}&\cos(x-y)\\&=\cos(x+(-y))\\&=\cos x\cos(-y)-\sin x\sin(-y)\\&=\sin x\cos y-\sin x(-\sin y)\\&=\cos x\cos y+\sin x\sin y\end{aligned}$$

$$\begin{aligned}&\tan(x+y)\\&=\frac{\sin(x+y)}{\cos(x+y)}\\&=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}\\&=\frac{\left(\dfrac{\sin x\cos y+\cos x\sin y}{\cos x\cos y}\right)}{\left(\dfrac{\cos x\cos y-\sin x\sin y}{\cos x\cos y}\right)}\\&=\frac{\left(\dfrac{\sin x\cos y}{\cos x\cos y}+\dfrac{\cos x\sin y}{\cos x\cos y}\right)}{\left(\dfrac{\cos x\cos y}{\cos x\cos y}-\dfrac{\sin x\sin y}{\cos x\cos y}\right)}\\&=\frac{\left(\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}\right)}{\left(1-\left(\dfrac{\sin x}{\cos x}\right)\left(\dfrac{\sin y}{\cos y}\right)\right)}\\&=\frac{\tan x+\tan y}{1-\tan x\tan y}\end{aligned}$$

$$\begin{aligned}&\tan(x-y)\\&=\tan(x+(-y))\\&=\frac{\tan x+\tan(-y)}{1-\tan x\tan(-y)}\\&=\frac{\tan x+(-\tan y)}{1-\tan x(-\tan y)}\\&=\frac{\tan x-\tan y}{1+\tan x\tan y}\end{aligned}$$

$$\begin{aligned}&\cot(x+y)\\&=\frac{1}{\tan(x+y)}\\&=\frac{1}{\left(\dfrac{\tan x+\tan y}{1-\tan x\tan y}\right)}\\&=\frac{1-\tan x\tan y}{\tan x+\tan y}\\&=\frac{\left(1-\dfrac{1}{\cot x\cot y}\right)}{\left(\dfrac{1}{\cot x}+\dfrac{1}{\cot y}\right)}\\&=\frac{\left(\dfrac{\cot x\cot y-1}{\cot x\cot y}\right)}{\left(\dfrac{\cot x+\cot y}{\cot x\cot y}\right)}\\&=\frac{\cot x\cot y-1}{\cot x+\cot y}\end{aligned}$$

$$\begin{aligned}&\cot(x-y)\\&=\cot(x+(-y))\\&=\frac{\cot x\cot(-y)-1}{\cot x+\cot(-y)}\\&=\frac{\cot x(-\cot y)-1}{\cot x+(-\cot y)}\\&=\frac{-\cot x\cot y-1}{\cot x-\cot y}\\&=\frac{-1(\cot x\cot y+1)}{\cot x-\cot y}\\&=\frac{\cot x\cot y+1}{-1(\cot x-\cot y)}\\&=\frac{\cot x\cot y+1}{\cot y-\cot x}\end{aligned}$$

Summary

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$

$$\sin(x-y)=\sin x\cos y-\cos x\sin y$$

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$

$$\cos(x-y)=\cos x\cos y+\sin x\sin y$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

$$\cot(x+y)=\frac{\cot x\cot y-1}{\cot x+\cot y}$$

$$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x}$$

Using Compound-Angle Identities to Derive Double-Angle Identities

$$\begin{aligned}&\sin 2x\\&=\sin(x+x)\\&=\sin x\cos x+\cos x\sin x\\&=2\sin x\cos x\end{aligned}$$

$$\begin{aligned}&\cos 2x\\&=\cos(x+x)\\&=\cos x\cos x-\sin x\sin x\\&=\cos^2 x-\sin^2 x\end{aligned}$$

$$\begin{aligned}&\cos 2x\\&=\cos^2 x-\sin^2 x\\&=(1-\sin^2 x)-\sin^2 x\\&=1-2\sin^2 x\end{aligned}$$

$$\begin{aligned}&\cos 2x\\&=\cos^2 x-\sin^2 x\\&=\cos^2 x-(1-\cos^2 x)\\&=2\cos^2 x-1\end{aligned}$$

$$\begin{aligned}&\tan 2x\\&=\tan(x+x)\\&=\frac{\tan x+\tan x}{1-\tan x\tan x}\\&=\frac{2\tan x}{1-\tan^2 x}\end{aligned}$$

$$\begin{aligned}&\cot 2x\\&=\cot(x+x)\\&=\frac{\cot x\cot x-1}{\cot x+\cot x}\\&=\frac{\cot^2 x-1}{2\cot x}\end{aligned}$$

Summary

$\sin 2x=2\sin x\cos x$	$\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$
$\cos 2x=\cos^2 x-\sin^2 x$	
$\cos 2x=1-2\sin^2 x$	$\cot 2x=\dfrac{\cot^2 x-1}{2\cot x}$
$\cos 2x=2\cos^2 x-1$	

Examples

1. Use compound-angle identities to evaluate each of the following. Exact values are required. Do not use calculators!

(a) $\sin 75°$

$$\begin{aligned}&\sin 75°\\&=\sin(45°+30°)\\&=\sin(\pi/4+\pi/6)\\&=\sin(\pi/4)\cos(\pi/6)+\cos(\pi/4)\sin(\pi/6)\\&=\frac{1}{\sqrt2}\left(\frac{\sqrt3}{2}\right)+\frac{1}{\sqrt2}\left(\frac12\right)\\&=\frac{\sqrt3+1}{2\sqrt2}\end{aligned}$$

(b) $\cos 255°$

$$\begin{aligned}&\cos 255°\\&=\cos(315°-60°)\\&=\cos(7\pi/4-\pi/3)\\&=\cos(7\pi/4)\cos(\pi/3)+\sin(\pi/3)\sin(7\pi/4)\\&=\cos(\pi/4)\cos(\pi/3)+\sin(\pi/3)(-\sin(\pi/4))\\&=\frac{1}{\sqrt2}\left(\frac12\right)+\frac{\sqrt3}{2}\left(-\frac{1}{\sqrt2}\right)\\&=\frac{1-\sqrt3}{2\sqrt2}\end{aligned}$$

(c) $\tan 105°$

$$\begin{aligned}&\tan 105°\\&=\tan(45°+60°)\\&=\tan(\pi/4+\pi/3)\\&=\frac{\tan(\pi/4)+\tan(\pi/3)}{1-\tan(\pi/4)\tan(\pi/3)}\\&=\frac{1+\sqrt3}{1-1(\sqrt3)}\\&=\frac{1+\sqrt3}{1-\sqrt3}\end{aligned}$$

Quick Check

$\dfrac{\sqrt3+1}{2\sqrt2}>0$ as we would expect for $\sin 75°$	$\dfrac{\sqrt3+1}{2\sqrt2}\doteq 0.9659$	$\sin 75°\doteq 0.9659$
$\dfrac{1-\sqrt3}{2\sqrt2}<0$ as we would expect for $\cos 255°$	$\dfrac{1-\sqrt3}{2\sqrt2}\doteq -0.2588$	$\cos 255°\doteq -0.2588$
$\dfrac{1+\sqrt3}{1-\sqrt3}<0$ as we would expect for $\tan 105°$	$\dfrac{1+\sqrt3}{1-\sqrt3}\doteq -3.732$	$\tan 105°\doteq -3.732$

2. Use double-angle identities to evaluate each of the following. Exact values are required. Do not use calculators!

(a) $\sin 15°$

Setting $\theta = \frac{x}{2}$ in $\cos 2\theta = 1 - 2\sin^2\theta$, we obtain

$$\cos x = 1 - 2\sin^2\frac{x}{2}$$

Solving for $\sin\frac{x}{2}$, we obtain

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

Therefore,

$$\begin{aligned}\sin 15° &= \pm\sqrt{\frac{1-\cos 30°}{2}}\\ &= \pm\sqrt{\frac{1-\sqrt{3}/2}{2}}\\ &= \pm\sqrt{\frac{1}{2}\left(\frac{2-\sqrt{3}}{2}\right)}\\ &= \pm\sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \pm\frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Since $15°$ is in quadrant I,

$$\sin 15° = \frac{\sqrt{2-\sqrt{3}}}{2} = \frac{1}{2}\sqrt{2-\sqrt{3}}$$

(b) $\cos 22.5°$

Setting $\theta = \frac{x}{2}$ in $\cos 2\theta = 2\cos^2\theta - 1$, we obtain

$$\cos x = 2\cos^2\frac{x}{2} - 1$$

Solving for $\cos\frac{x}{2}$, we obtain

$$\cos\frac{x}{2} = \pm\sqrt{\frac{\cos x + 1}{2}}$$

Therefore,

$$\begin{aligned}\cos 22.5° &= \pm\sqrt{\frac{\cos 45° + 1}{2}}\\ &= \pm\sqrt{\frac{1/\sqrt{2}+1}{2}}\\ &= \pm\sqrt{\frac{1}{2}\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)}\\ &= \pm\sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}\end{aligned}$$

Since $22.5°$ is in quadrant I, $\cos 22.5° = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$

3. Prove that the following equations are identities.

(a) $\frac{\sin 2x}{1+\cos 2x} = \tan x$

Proof

$$\begin{aligned}\text{L.S.} &= \frac{\sin 2x}{1+\cos 2x}\\ &= \frac{2\sin x\cos x}{1+2\cos^2 x - 1}\\ &= \frac{2\sin x\cos x}{2\cos^2 x}\\ &= \frac{\sin x}{\cos x}\\ &= \tan x\end{aligned}$$

There are 3 different identities for $\cos 2x$. $\cos 2x = 2\cos^2 x - 1$ was chosen because it leads to the most convenient simplification of the denominator.

Divide top and bottom by $2\cos x$.

$\text{R.S.} = \tan x$

$\therefore$ L.S. = R.S.

$\therefore \frac{\sin 2x}{1+\cos 2x} = \tan x$ ***is*** an identity.

(b) $\cos^4\theta - \sin^4\theta = \cos 2\theta$

Proof

$$\begin{aligned}\text{L.S.} &= \cos^4\theta - \sin^4\theta\\ &= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) \quad \text{(Factor diff. of squares)}\\ &= 1(\cos^2\theta - \sin^2\theta) \quad \text{(Pythagorean identity)}\\ &= \cos^2\theta - \sin^2\theta\\ &= \cos 2\theta\end{aligned}$$

$\text{R.S.} = \cos 2\theta$

$\therefore$ L.S. = R.S.

$\therefore \cos^4\theta - \sin^4\theta = \cos 2\theta$ ***is*** an identity.

4. Use counterexamples to prove that the following equations are ***not*** identities.

(a) $\sin(x+y)=\sin x+\sin y$

Proof

Let $x=y=\dfrac{\pi}{4}$

$$\begin{aligned}\text{L.S.}&=\sin(x+y)\\&=\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right)\\&=\sin\left(\frac{\pi}{2}\right)\\&=1\end{aligned}$$

$$\begin{aligned}\text{R.S.}&=\sin x+\sin y\\&=\sin\frac{\pi}{4}+\sin\frac{\pi}{4}\\&=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\\&=\frac{2}{\sqrt{2}}\\&=\sqrt{2}\end{aligned}$$

$\because 1\neq\sqrt{2}$

$\therefore \text{L.S.}\neq\text{R.S.}$

$\therefore \sin(x+y)=\sin x+\sin y$ is ***not*** an identity.

(b) $\cos 4\theta-\cos\theta=\cos 3\theta$

Proof

Let $\theta=\dfrac{\pi}{2}$

$$\begin{aligned}\text{L.S.}&=\cos 4\theta-\cos\theta\\&=\cos\left(4\left(\frac{\pi}{2}\right)\right)-\cos\frac{\pi}{2}\\&=\cos 2\pi-\cos\frac{\pi}{2}\\&=\cos 2\pi-\cos\frac{\pi}{2}\\&=1-0\\&=1\end{aligned}$$

$$\begin{aligned}\text{R.S.}&=\cos 3\theta\\&=\cos\left(3\left(\frac{\pi}{2}\right)\right)\\&=\cos\frac{3\pi}{2}\\&=0\end{aligned}$$

$\because 1\neq 0$

$\therefore \text{L.S.}\neq\text{R.S.}$

$\therefore \cos 4\theta-\cos\theta=\cos 3\theta$ is ***not*** an identity.

5. Use identities that we have learned to derive an identity for $\sin 3\theta$ that is expressed entirely in terms of $\sin\theta$.

$$\begin{aligned}&\sin 3\theta\\&=\sin(2\theta+\theta)\\&=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta\\&=(2\sin\theta\cos\theta)\cos\theta+(1-2\sin^2\theta)\sin\theta\\&=2\sin\theta\cos^2\theta+\sin\theta-2\sin^3\theta\\&=2\sin\theta(1-\sin^2\theta)+\sin\theta-2\sin^3\theta\\&=2\sin\theta-2\sin^3\theta+\sin\theta-2\sin^3\theta\\&=3\sin\theta-4\sin^3\theta\end{aligned}$$

$\therefore \sin 3\theta=3\sin\theta-4\sin^3\theta$ is an identity.

In Summary

Key Ideas

- A trigonometric identity states the equivalence of two trigonometric expressions. It is written as an equation that involves trigonometric functions and the solution set is all real numbers for which the expressions on both sides of the equation are defined. As a result, the equation has an infinite number of solutions.
- Some trigonometric identities are the result of a definition, while others are derived from relationships that exist among trigonometric ratios.

Need to Know

- The following trigonometric identities are important for you to remember:

Identities Based on Definitions

Reciprocal Identities

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Identities Derived from Relationships

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Addition and Subtraction Formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- You can verify the truth of a given trigonometric identity by graphing each side separately and showing that the two graphs are the same.
- To prove that a given equation is an identity, the two sides of the equation must be shown to be equivalent. This can be accomplished using a variety of strategies, such as
  - simplifying the more complicated side until it is identical to the other side, ormanipulating both sides to get the same expression
  - rewriting expressions using any of the identities stated above
  - using a common denominator or factoring, where possible

Homework #1

4. Determine the exact value of each trigonometric ratio.

a) $\sin 75^\circ$ c) $\tan \frac{5\pi}{12}$ e) $\cos 105^\circ$

b) $\cos 15^\circ$ d) $\sin\left(-\frac{\pi}{12}\right)$ f) $\tan \frac{23\pi}{12}$

PRACTISING

5. **K** Use the appropriate compound angle formula to determine the exact value of each expression.

a) $\sin\left(\pi + \frac{\pi}{6}\right)$ c) $\tan\left(\frac{\pi}{4} + \pi\right)$ e) $\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$

b) $\cos\left(\pi - \frac{\pi}{4}\right)$ d) $\sin\left(-\frac{\pi}{2} + \frac{\pi}{3}\right)$ f) $\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right)$

6. Use the appropriate compound angle formula to create an equivalent expression.

a) $\sin(\pi + x)$ c) $\cos\left(x + \frac{\pi}{2}\right)$ e) $\sin(x - \pi)$

b) $\cos\left(x + \frac{3\pi}{2}\right)$ d) $\tan(x + \pi)$ f) $\tan(2\pi - x)$

7. Use transformations to explain why each expression you created in question 6 is equivalent to the given expression.

8. Determine the exact value of each trigonometric ratio.

a) $\cos 75^\circ$ c) $\cos \frac{11\pi}{12}$ e) $\tan \frac{7\pi}{12}$

b) $\tan(-15^\circ)$ d) $\sin \frac{13\pi}{12}$ f) $\tan \frac{-5\pi}{12}$

9. If $\sin x = \frac{4}{5}$ and $\sin y = -\frac{12}{13}$, $0 < x < \frac{\pi}{2}$, $\frac{3\pi}{2} < y < 2\pi$, evaluate

a) $\cos(x + y)$ c) $\cos(x - y)$ e) $\tan(x + y)$

b) $\sin(x + y)$ d) $\sin(x - y)$ f) $\tan(x - y)$

10. **A** α and β are acute angles in quadrant I, with $\sin \alpha = \frac{7}{25}$ and $\cos \beta = \frac{5}{13}$. Without using a calculator, determine the values of $\sin(\alpha + \beta)$ and $\tan(\alpha + \beta)$.

11. Use compound angle formulas to verify each of the following cofunction identities.

a) $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ b) $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

12. Simplify each expression.

a) $\sin(\pi + x) + \sin(\pi - x)$ b) $\cos\left(x + \frac{\pi}{3}\right) - \sin\left(x + \frac{\pi}{6}\right)$

13. **T** Simplify $\frac{\sin(f + g) + \sin(f - g)}{\cos(f + g) + \cos(f - g)}$.

14. **C** Create a flow chart to show how you would evaluate $\cos(a + b)$, given the values of $\sin a$ and $\sin b$, if both a and $b \in \left[0, \frac{\pi}{2}\right]$.

15. List the compound angle formulas you used in this lesson, and look for similarities and differences. Explain how you can use these similarities and differences to help you remember the formulas.

Extending

16. Prove $\sin C + \sin D = 2 \sin\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$.

17. Determine $\cot(x + y)$ in terms of $\cot x$ and $\cot y$.

18. Prove $\cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$.

19. Prove $\cos C - \cos D = -2 \sin\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)$.

Selected Answers

4. a) $\frac{\sqrt{2} + \sqrt{6}}{4}$ d) $\frac{\sqrt{2} - \sqrt{6}}{4}$
b) $\frac{\sqrt{2} + \sqrt{6}}{4}$ e) $\frac{\sqrt{2} - \sqrt{6}}{4}$
c) $2 + \sqrt{3}$ f) $-2 + \sqrt{3}$

5. a) $-\frac{1}{2}$ d) $-\frac{1}{2}$
b) $-\frac{\sqrt{2}}{2}$ e) $\frac{\sqrt{3}}{3}$
c) 1 f) $-\frac{\sqrt{3}}{2}$

6. a) $-\sin x$ d) $\tan x$
b) $\sin x$ e) $-\sin x$
c) $-\sin x$ f) $-\tan x$

7. a) $\sin(\pi + x)$ is equivalent to $\sin x$ translated π to the left, which is equivalent to $-\sin x$.
b) $\cos\left(x + \frac{3\pi}{2}\right)$ is equivalent to $\cos x$ translated $\frac{3\pi}{2}$ to the left, which is equivalent to $\sin x$.
c) $\cos\left(x + \frac{\pi}{2}\right)$ is equivalent to $\cos x$ translated $\frac{\pi}{2}$ to the left, which is equivalent to $-\sin x$.
d) $\tan(x + \pi)$ is equivalent to $\tan x$ translated π to the left, which is equivalent to $\tan x$.
e) $\sin(x - \pi)$ is equivalent to $\sin x$ translated π to the right, which is equivalent to $-\sin x$.
f) $\tan(2\pi - x)$ is equivalent to $\tan(-x)$, which is equivalent to $\tan x$ reflected in the y-axis, which is equivalent to $-\tan x$.

9. a) $\frac{63}{65}$ d) $\frac{56}{65}$
b) $-\frac{16}{65}$ e) $-\frac{16}{63}$
c) $-\frac{33}{65}$ f) $-\frac{56}{33}$

10. $\frac{323}{325}; \frac{323}{36}$

11. a) $\cos\left(\frac{\pi}{2} - x\right)$
$= \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$
$= (0)(\cos x) + (1)(\sin x)$
$= 0 + \sin x$
$= \sin x$
b) $\sin\left(\frac{\pi}{2} - x\right)$
$= \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$
$= (1)(\cos x) - (0)(\sin x)$
$= \cos x - 0$
$= \cos x$

12. a) 0 b) $-\sqrt{3}\sin x$

13. $\tan f$, $\cos f \neq 0$, $\cos g \neq 0$

Homework #2

1. Express each of the following as a single trigonometric ratio.
   a) $2 \sin 5x \cos 5x$
   b) $\cos^2 \theta - \sin^2 \theta$
   c) $1 - 2 \sin^2 3x$
   d) $\dfrac{2 \tan 4x}{1 - \tan^2 4x}$
   e) $4 \sin \theta \cos \theta$
   f) $2 \cos^2 \dfrac{\theta}{2} - 1$

2. Express each of the following as a single trigonometric ratio and then evaluate.
   a) $2 \sin 45° \cos 45°$
   b) $\cos^2 30° - \sin^2 30°$
   c) $2 \sin \dfrac{\pi}{12} \cos \dfrac{\pi}{12}$
   d) $\cos^2 \dfrac{\pi}{12} - \sin^2 \dfrac{\pi}{12}$
   e) $1 - 2 \sin^2 \dfrac{3\pi}{8}$
   f) $2 \tan 60° \cos^2 60°$

3. Use a double angle formula to rewrite each trigonometric ratio.
   a) $\sin 4\theta$
   b) $\cos 3x$
   c) $\tan x$
   d) $\cos 6\theta$
   e) $\sin x$
   f) $\tan 5\theta$

5. Determine the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given $\tan \theta = -\frac{7}{24}$ and $\frac{\pi}{2} \le \theta \le \pi$.

7. Determine the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given $\cos \theta = -\frac{4}{5}$ and $\frac{\pi}{2} \le \theta \le \pi$.

8. **A** Determine the value of a in the following equation:
   $2 \tan x - \tan 2x + 2a = 1 - \tan 2x \tan^2 x$.

9. Jim needs to find the sine of $\frac{\pi}{8}$. If he knows that $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, how can he use this fact to find the sine of $\frac{\pi}{8}$? What is his answer?

10. Marion needs to find the cosine of $\frac{\pi}{12}$. If she knows that $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, how can she use this fact to find the cosine of $\frac{\pi}{12}$? What is her answer?

11. **T** a) Use a double angle formula to develop a formula for $\sin 4x$ in terms of x.
    b) Use the formula you developed in part a) to verify that $\sin \frac{2\pi}{3} = \sin \frac{8\pi}{3}$.

12. Use the appropriate compound angle formula and double angle formula to develop a formula for
    a) $\sin 3\theta$ in terms of $\cos \theta$ and $\sin \theta$
    b) $\cos 3\theta$ in terms of $\cos \theta$ and $\sin \theta$
    c) $\tan 3\theta$ in terms of $\tan \theta$

13. The angle x lies in the interval $\frac{\pi}{2} \le x \le \pi$, and $\sin^2 x = \frac{8}{9}$. Without using a calculator, determine the value of
    a) $\sin 2x$
    b) $\cos 2x$
    c) $\cos \dfrac{x}{2}$
    d) $\sin 3x$

15. Describe how you could use your knowledge of double angle formulas to sketch the graph of each function. Include a sketch with your description.
    a) $f(x) = \sin x \cos x$
    b) $f(x) = 2 \cos^2 x$
    c) $f(x) = \dfrac{\tan x}{1 - \tan^2 x}$

16. Eliminate A from each pair of equations to find an equation that relates x to y.
    a) $x = \tan 2A, y = \tan A$
    b) $x = \cos 2A, y = \cos A$
    c) $x = \cos 2A, y = \csc A$
    d) $x = \sin 2A, y = \sec 4A$

Selected Answers

1. a) $\sin 10x$ b) $\cos 2\theta$ c) $\cos 6x$ d) $\tan 8x$ e) $2 \sin 2\theta$ f) $\cos \theta$

2. a) $\sin 90°; 1$ b) $\cos 60°; \frac{1}{2}$ c) $\sin \frac{\pi}{6}; \frac{1}{2}$ d) $\cos \frac{\pi}{6}; \frac{\sqrt{3}}{2}$ e) $\cos \frac{3\pi}{4}; -\frac{\sqrt{2}}{2}$ f) $\sin 120°; \frac{\sqrt{3}}{2}$

3. a) $2 \sin 2\theta \cos 2\theta$
   b) $2 \sin^2 (1.5x) - 1$
   c) $\dfrac{2 \tan (0.5x)}{1 - \tan^2 (0.5x)}$
   d) $\cos^2 3\theta - \sin^2 3\theta$
   e) $2 \sin (0.5x) \cos (0.5x)$
   f) $\dfrac{2 \tan (2.5\theta)}{1 - \tan^2 (2.5\theta)}$

4. $\sin 2\theta = \frac{24}{25}$, $\cos 2\theta = -\frac{7}{25}$, $\tan 2\theta = -\frac{24}{7}$

7. $\sin 2\theta = -\frac{24}{25}$, $\cos 2\theta = \frac{7}{25}$, $\tan 2\theta = -\frac{24}{7}$

8. $a = \frac{1}{2}$

9. Jim can find the sine of $\frac{\pi}{8}$ by using the formula $\cos 2x = 1 - 2 \sin^2 x$ and isolating $\sin x$ on one side of the equation. When he does this, the formula becomes $\sin x = \pm\sqrt{\frac{1 - \cos 2x}{2}}$. The cosine of $\frac{\pi}{4}$ is $\frac{\sqrt{2}}{2}$, so $\sin \frac{\pi}{8} = \pm\sqrt{\frac{1 - \cos \frac{\pi}{4}}{2}}$ $= \pm\frac{\sqrt{2 - \sqrt{2}}}{2}$. Since $\frac{\pi}{8}$ is in the first quadrant, the sign of $\sin \frac{\pi}{8}$ is positive.

10. Marion can find the cosine of $\frac{\pi}{12}$ by using the formula $\cos 2x = 2 \cos^2 x - 1$ and isolating $\cos x$ on one side of the equation. When she does this, the formula becomes $\cos x = \pm\sqrt{\frac{1 + \cos 2x}{2}}$. The cosine of $\frac{\pi}{6}$ is $\frac{\sqrt{3}}{2}$, so $\cos \frac{\pi}{12} = \pm\sqrt{\frac{1 + \cos \frac{\pi}{6}}{2}}$ $= \pm\frac{\sqrt{2 + \sqrt{3}}}{2}$. Since $\frac{\pi}{12}$ is in the first quadrant, the sign of $\cos \frac{\pi}{12}$ is positive.

11. a) $\sin 4x$
$$\begin{aligned} &= (2)(2 \sin x \cos x)(\cos 2x) \\ &= (2)(2 \sin x \cos x)(1 - 2 \sin^2 x) \\ &= (4 \sin x \cos x)(1 - 2 \sin^2 x) \\ &= 4 \sin x \cos x - 8 \sin^3 x \cos x \end{aligned}$$

13. a) $-\frac{4\sqrt{2}}{9}$ b) $-\frac{7}{9}$ c) $\frac{\sqrt{3}}{3}$ d) $-\frac{10\sqrt{2}}{27}$

16. a) $\dfrac{\tan^{-1} x}{2} = \tan^{-1} y$
    b) $\dfrac{\cos^{-1} x}{2} = \cos^{-1} y$
    c) $\dfrac{\cos^{-1} x}{2} = \csc^{-1} y$ or $\dfrac{\cos^{-1} x}{2} = \sin^{-1}\left(\dfrac{1}{y}\right)$
    d) $\dfrac{\sin^{-1} x}{2} = \dfrac{\sec^{-1} y}{4}$ or $\dfrac{\sin^{-1} x}{2} = \dfrac{\cos^{-1}\left(\frac{1}{y}\right)}{4}$

Trig Identities – Summary and Extra Practice

1. Complete the following statements:

(a) An ***equation*** is an ***identity*** if ____

____.

(b) There are many ways to confirm whether an equation is an identity. List ***at least three*** such ways.

____

____

____

(c) There is a very simple way to confirm that an equation is ***not*** an identity. In fact, this method can be used to show the falsity of any invalid mathematical statement. Describe the method and use it to demonstrate that the equation $\sqrt{x+y}=\sqrt{x}+\sqrt{y}$ is ***not*** an identity.

2. Mr. Nolfi asked Uday to prove that the equation $\dfrac{\sin 2x}{1+\cos 2x}=\tan x$ is an identity. What mark would Uday receive for the following response? Explain.

$$\frac{\sin 2x}{1+\cos 2x}=\tan x$$

$$\therefore \frac{2\sin x\cos x}{1+2\cos^2 x-1}=\tan x$$

$$\therefore \frac{2\sin x\cos x}{2\cos^2 x}=\tan x$$

$$\therefore \left(\frac{2}{2}\right)\left(\frac{\sin x}{\cos x}\right)\left(\frac{\cos x}{\cos x}\right)=\tan x$$

$$\therefore 1(\tan x)(1)=\tan x$$

$$\therefore \tan x=\tan x$$

3. List several strategies that can help you to prove that an equation is an identity.

____

____

____

____

____

____

____

4. Justify each of the following identities by using transformations and by using angles of rotation.

(a) $\sin(-x) = -\sin x$ **(b)** $\sin(\pi/2 - x) = \cos x$ **(c)** $\sin(x+\pi) = -\sin x$

(d) $\cos(-x) = \cos x$ **(e)** $\cos(\pi/2 - x) = \sin x$ **(f)** $\cos(x+\pi) = -\cos x$

(g) $\tan(-x) = -\tan x$ **(h)** $\tan(\pi/2 - x) = \cot x$ **(i)** $\tan(x+\pi) = \tan x$

5. Prove that each of the following equations is an identity:

a) $\dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin\theta\cos\theta} = 1 - \tan\theta$

b) $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$

c) $\tan^2 x - \cos^2 x = \dfrac{1}{\cos^2 x} - 1 - \cos^2 x$

d) $\dfrac{1}{1+\cos\theta} + \dfrac{1}{1-\cos\theta} = \dfrac{2}{\sin^2\theta}$

6. Prove that each of the following equations is an identity:

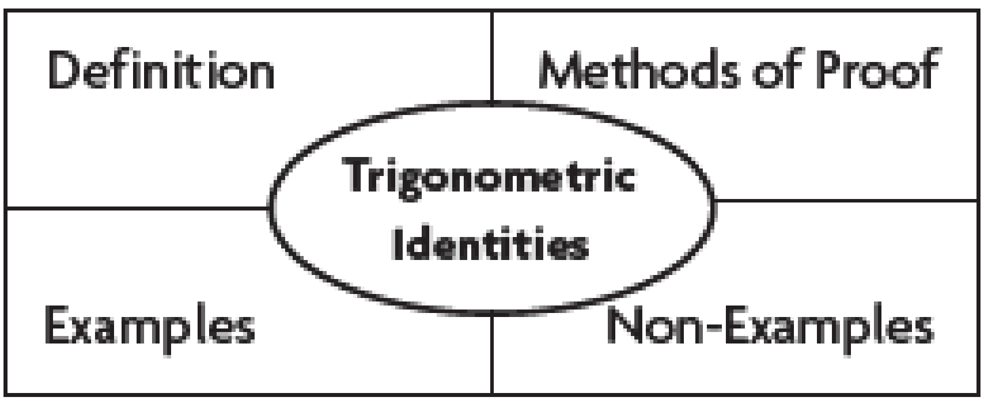
a) $\cos x \tan^3 x = \sin x \tan^2 x$

b) $\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta$

c) $(\sin x + \cos x)\left(\dfrac{\tan^2 x + 1}{\tan x}\right) = \dfrac{1}{\cos x} + \dfrac{1}{\sin x}$

d) $\tan^2\beta + \cos^2\beta + \sin^2\beta = \dfrac{1}{\cos^2\beta}$

7. Copy and complete the following Frayer diagram:



8. Express $8\cos^4 x$ in the form $a\cos 4x + b\cos 2x + c$. State the values of the constants *a*, *b* and *c*.

9. Give a counterexample to demonstrate that each of the following equations is not an identity.

a) $\cos x = \dfrac{1}{\cos x}$ **c)** $\sin(x+y) = \cos x \cos y + \sin x \sin y$

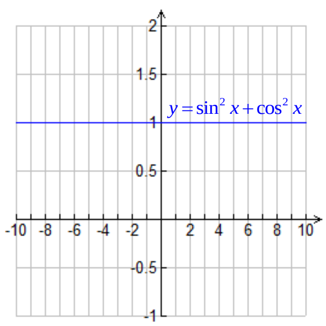
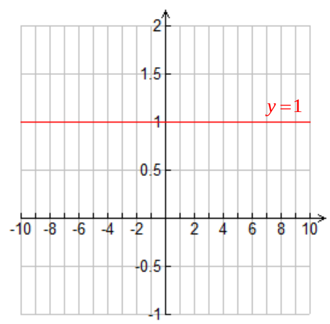
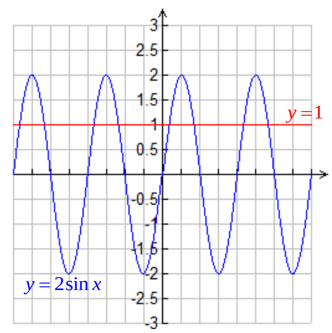
b) $1 - \tan^2 x = \sec^2 x$ **d)** $\cos 2x = 1 + 2\sin^2 x$

10. Demonstrate graphically that each of the equations in 9 is not an identity.

Solving Trigonometric Equations

Introduction – A Graphical Look at Equations that are not Identities

Let "L.S." represent the expression on the left side of an equation and let "R.S." represent the expression on the right side.

An Equation that is an Identity: $\sin^2 x + \cos^2 x = 1$	***An Equation that is not an Identity:*** $2\sin x = 1$
• If an equation is an identity, the expression on the L.S. is ***equivalent*** to the expression on the R.S. of the equation. That is, the equation is satisfied for all real numbers for which the expressions are defined. • If an equation is an identity, then the graph of "y = L.S." is ***identical*** to the graph of "y = R.S." The graphs intersect at all real values for which the expressions are defined. In other words, every such value is a solution to the equation.  	• If an equation is ***not*** an identity, the expression on the L.S. is ***not equivalent*** to the expression on the R.S. The expressions may agree for some real values but they ***do not agree*** for ***all*** values. • If an equation is ***not*** an identity, then the graph of "y = L.S." is ***not identical*** to the graph of "y = R.S." The expressions agree only at the point(s) of intersection of the two graphs. The number of points of intersection is equal to the number of solutions of the equation. 

Examples

1. Use an algebraic method to solve the trigonometric equation $2\sin x - 1 = 0$. State all solutions in the interval $-4\pi \le x \le 4\pi$. Verify the solutions graphically. (**Note:** An alternative notation for writing the interval $-4\pi \le x \le 4\pi$ is $[-4\pi, 4\pi]$. The square brackets indicate that the endpoints are included in the interval.)

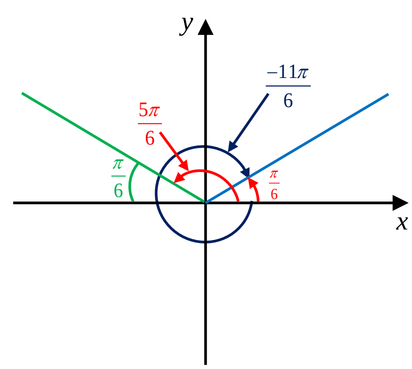
Solution

$$2\sin x - 1 = 0$$
$$\therefore 2\sin x = 1$$
$$\therefore \sin x = \frac{1}{2}$$
$$\therefore x = \sin^{-1}\left(\frac{1}{2}\right)$$
$$\therefore x = \frac{\pi}{6}$$

> If your calculator is in "degrees mode," this will produce an answer of 30°. In "radians mode," an answer of about 0.5326 is obtained. Naturally, since $^1/_2$ is a trig ratio of a special angle, you should be able to state the exact answer $x = {}^{\pi}/_6$. To state the other solutions in the interval $[-4\pi, 4\pi]$, use the concept of related angles and a graph.

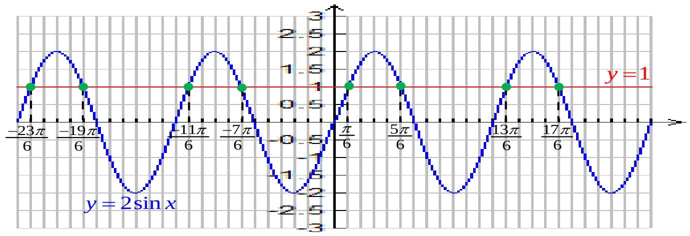
As shown in the diagram at the right,

- $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ are the ***principal-angle*** solutions to the equation (since the sine function is positive in quadrants I and II)
- All the other solutions in $[-4\pi, 4\pi]$ are found by taking all angles in this interval that are ***coterminal*** with $\frac{\pi}{6}$ and $\frac{5\pi}{6}$



Therefore, the solutions in the interval $[-4\pi, 4\pi]$ are $\frac{-23\pi}{6}, \frac{-19\pi}{6}, \frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$ and $\frac{17\pi}{6}$.

The following is a graphical verification of the solutions given above.



2. Solve for x given that $x \in [0, 2\pi]$.

(a) $2\sec^2 x - 3 + \tan x = 0$

Solution

$\therefore 2(\tan^2 x + 1) - 3 + \tan x = 0$ (Pyth. identity)

$\therefore 2\tan^2 x + \tan x - 1 = 0$

$\therefore (2\tan x - 1)(\tan x + 1) = 0$

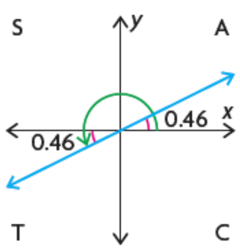
$\therefore 2\tan x - 1 = 0$ or $\tan x + 1 = 0$

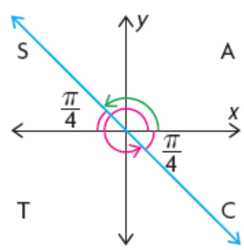
$\therefore \tan x = \frac{1}{2}$ or $\tan x = -1$

$\therefore x = \tan^{-1}\left(\frac{1}{2}\right)$ or $x = \tan^{-1}(-1)$

$\therefore x \doteq 0.46$ or $x = \frac{3\pi}{4}$ (calculator gives $-\frac{\pi}{4}$)

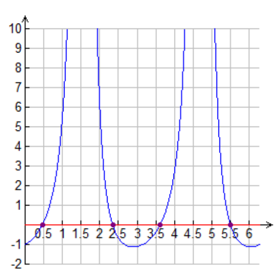
These solutions are in quadrants I and II. There are also solutions in quadrants III and IV:





$\therefore x \doteq 0.46 + \pi \doteq 3.6$ or $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$\therefore x \doteq 0.46$, $x \doteq 3.6$, $x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$



The points of intersection of $y = 2\sec^2 x - 3 + \tan x$ and $y = 0$ agree with the solutions obtained above.

(b) $3\sin x + 3\cos 2x = 2$

Solution

$\therefore 3\sin x + 3(1 - 2\sin^2 x) = 2$ (double-angle identity)

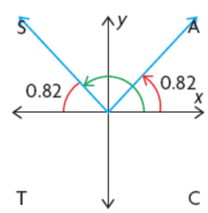
$\therefore 6\sin^2 x - 3\sin x - 1 = 0$

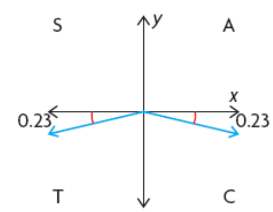
A quick check of the discriminant of this quadratic equation in $\sin x$ demonstrates that it ***does not*** factor: $b^2 - 4ac = (-3)^2 - 4(6)(-1) = 33$, which is not a perfect square. Therefore, the quadratic formula must be used.

$$\sin x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(6)(-1)}}{2(6)} = \frac{3 \pm \sqrt{33}}{12}$$

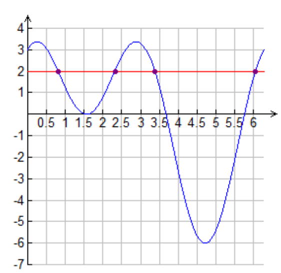
$\therefore x = \sin^{-1}\left(\frac{3+\sqrt{33}}{12}\right)$ or $x = \sin^{-1}\left(\frac{3-\sqrt{33}}{12}\right)$

$\therefore x \doteq 0.82$ or $x \doteq -0.23$ (solutions given by calculator)





Therefore, the solutions in the interval $[0, 2\pi]$ are $x \doteq 0.82$, $x \doteq \pi - 0.82 \doteq 2.32$, $x \doteq \pi + 0.23 \doteq 3.37$ and $x \doteq 2\pi - 0.23 \doteq 6.05$



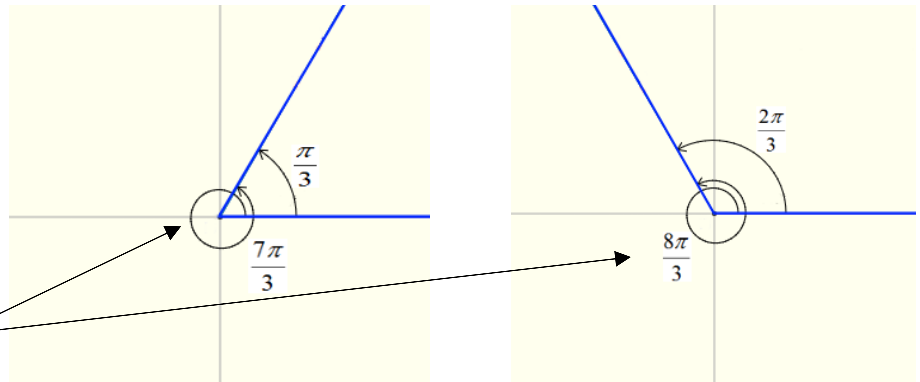
The points of intersection of $y = 3\sin x + 3\cos 2x$ and $y = 2$ agree with the solutions obtained above.

3. Use an algebraic method to solve the trigonometric equation $2\sin 2x - \sqrt{3} = 0$. State all solutions in the interval $0 \le x \le 2\pi$. Verify the solutions graphically.

Solution

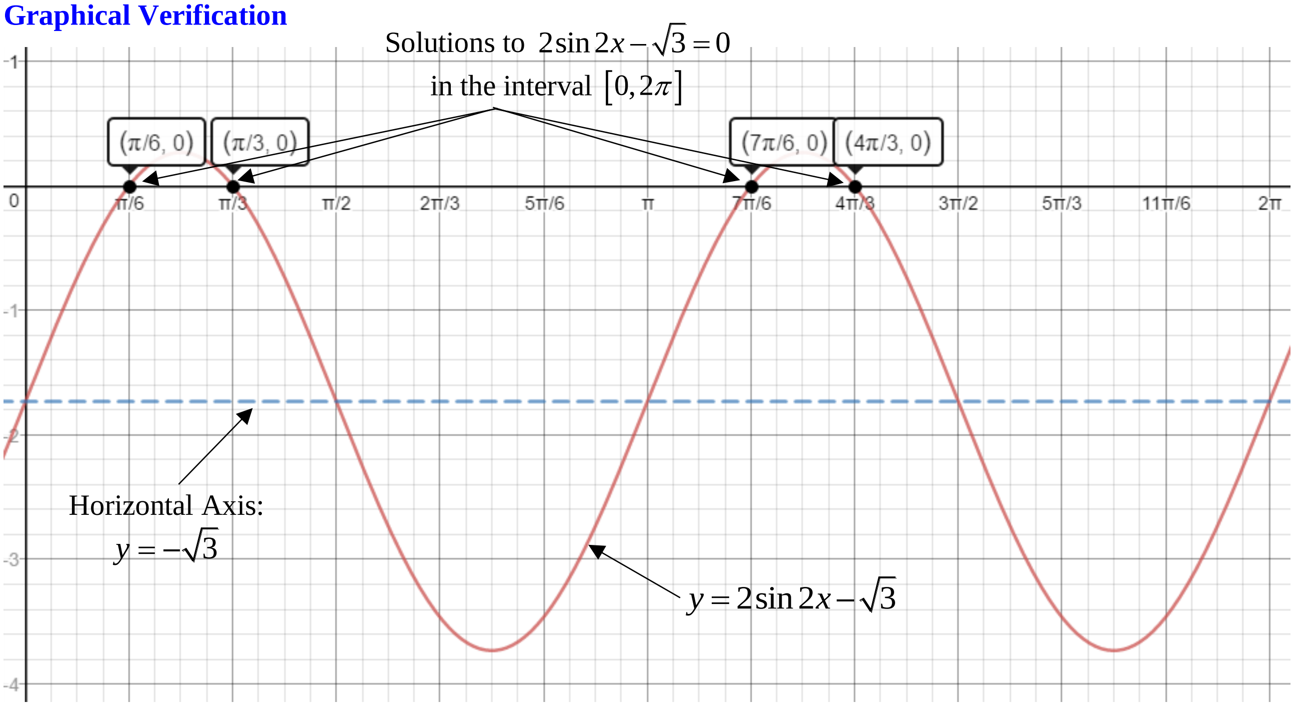
First, observe that since $0 \le x \le 2\pi$, then $0 \le 2x \le 4\pi$. Thus, we must find all solutions for $2x$ in $[0, 4\pi]$. This will give us all solutions for x in $[0, 2\pi]$.

$$2\sin 2x - \sqrt{3} = 0$$
$$\therefore 2\sin 2x = \sqrt{3}$$
$$\therefore \sin 2x = \frac{\sqrt{3}}{2}$$
$$\therefore 2x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$\therefore 2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3} \text{ in } [0, 4\pi]$$
$$\therefore x = \frac{\pi}{6}, \frac{2\pi}{6}, \frac{7\pi}{6}, \frac{8\pi}{6} \text{ in } [0, 2\pi]$$
$$\therefore x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3} \text{ in } [0, 2\pi]$$



The solutions for $2x$ of the equation $\sin 2x = \frac{\sqrt{3}}{2}$ in $[0, 4\pi]$

Graphical Verification



4. How many solutions would you expect each of the following equations to have in $[0, 2\pi]$?

(a) $\cos 3x = \frac{1}{2}$ **(b)** $\cos 3x = 1$ **(c)** $\tan 3x = \sqrt{3}$ **(d)** $\cot 5x = 1$

Solution

To be discussed in class.

In Summary

Key Idea

- The same strategies can be used to solve linear trigonometric equations when the variable is measured in degrees or radians.

Need to Know

- Because of their periodic nature, trigonometric equations have an infinite number of solutions. When we use a trigonometric model, we usually want solutions within a specified interval.
- To solve a linear trigonometric equation. use special triangles, a calculator, a sketch of the graph, and/or the ASTC rule.
- A scientific or graphing calculator provides very accurate estimates of the value for an inverse trigonometric function. The inverse trigonometric function of a positive ratio yields the related angle. Use the related acute angle and the period of the corresponding function to determine all the solutions in the given interval.
- You can use a graphing calculator to verify the solutions for a linear trigonometric equation by
  - graphing the appropriate functions on the graphing calculator and determining the points of intersection
  - graphing an equivalent single function and determining its zeros

In Summary

Key Ideas

- In some applications, the formula contains a square of a trigonometric ratio. This leads to a quadratic trigonometric equation that can be solved algebraically or graphically.
- A quadratic trigonometric equation may have multiple solutions in the interval $0 \le x \le 2\pi$. Some of the solutions may be inadmissible, however, in the context of the problem.

Need to Know

- You can often factor a quadratic trigonometric equation and then solve the resulting two linear trigonometric equations. In cases where the equation cannot be factored, use the quadratic formula and then solve the resulting linear trigonometric equations.

Note: The solutions to $ax^2 + bx + c = 0$ are determined by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

- You may need to use a Pythagorean identity, compound angle formula, or double angle formula to create a quadratic equation that contains only a single trigonometric function whose arguments all match.

Homework #1

4. Solve $\cos x = -0.8667$, where $0° \le x \le 360°$.
- **a)** How many solutions are possible?
- **b)** In which quadrants would you find the solutions?
- **c)** Determine the related angle for the equation, to the nearest degree.
- **d)** Determine all the solutions for the equation, to the nearest degree.

5. Solve $\tan\theta = 2.7553$, where $0 \le \theta \le 2\pi$.
- **a)** How many solutions are possible?
- **b)** In which quadrants would you find the solutions?
- **c)** Determine the related angle for the equation, to the nearest hundredth.
- **d)** Determine all the solutions for the equation, to the nearest hundredth.

6. **K** Determine the solutions for each equation, where $0 \le \theta \le 2\pi$.
- **a)** $\tan\theta = 1$
- **b)** $\sin\theta = \frac{1}{\sqrt{2}}$
- **c)** $\cos\theta = \frac{\sqrt{3}}{2}$
- **d)** $\sin\theta = -\frac{\sqrt{3}}{2}$
- **e)** $\cos\theta = -\frac{1}{\sqrt{2}}$
- **f)** $\tan\theta = \sqrt{3}$

8. Using a calculator, determine the solutions for each equation, to two decimal places, on the interval $0 \le x \le 2\pi$.
- **a)** $3\sin x = \sin x + 1$
- **b)** $5\cos x - \sqrt{3} = 3\cos x$
- **c)** $\cos x - 1 = -\cos x$
- **d)** $5\sin x + 1 = 3\sin x$

9. Using a calculator, determine the solutions for each equation, to two decimal places, on the interval $0 \le x \le 2\pi$.
- **a)** $2 - 2\cot x = 0$
- **b)** $\csc x - 2 = 0$
- **c)** $7\sec x = 7$
- **d)** $2\csc x + 17 = 15 + \csc x$
- **e)** $2\sec x + 1 = 6$
- **f)** $8 + 4\cot x = 10$

10. Using a calculator, determine the solutions for each equation, to two decimal places, on the interval $0 \le x \le 2\pi$.
- **a)** $\sin 2x = \frac{1}{\sqrt{2}}$
- **b)** $\sin 4x = \frac{1}{2}$
- **c)** $\sin 3x = -\frac{\sqrt{3}}{2}$
- **d)** $\cos 4x = -\frac{1}{\sqrt{2}}$
- **e)** $\cos 2x = -\frac{1}{2}$
- **f)** $\cos\frac{x}{2} = \frac{\sqrt{3}}{2}$

11. **A** A city's daily high temperature, in degrees Celsius, can be modelled by the function $t(d) = -28\cos\frac{2\pi}{365}d + 10$, where d is the day of the year and 1 = January 1. On days when the temperature is approximately 32 °C or above, the air conditioners at city hall are turned on. During what days of the year are the air conditioners running at city hall?

12. The height, in metres, of a nail in a water wheel above the surface of the water, as a function of time, can be modelled by the function $h(t) = -4\sin\frac{\pi}{4}(t - 1) + 2.5$, where t is the time in seconds. During what periods of time is the nail below the water in the first 24 s that the wheel is rotating?

13. **T** Solve $\sin\left(x + \frac{\pi}{4}\right) = \sqrt{2}\cos x$ for $0 \le x \le 2\pi$.

15. Explain why the value of the function $f(x) = 25\sin\frac{\pi}{50}(x + 20) - 55$ at $x = 3$ is the same as the value of the function at $x = 7$.

17. Solve the trigonometric equation $2\sin x\cos x + \sin x = 0$. (*Hint:* You may find it helpful to factor the left side of the equation.)

18. Solve each equation for $0 \le x \le 2\pi$.
- **a)** $\sin 2x - 2\cos^2 x = 0$
- **b)** $3\sin x + \cos 2x = 2$

Selected Answers

4. **a)** 2 **b)** quadrants II and III **c)** 30° **d)** $x = 150°$ or $210°$

5. **a)** 2 **b)** quadrants I and III **c)** 1.22 **d)** $\theta = 1.22$ or 4.36

6. **a)** $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ **b)** $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ **c)** $\theta = \frac{\pi}{6}$ or $\frac{11\pi}{6}$ **d)** $\theta = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$ **e)** $\theta = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$ **f)** $\theta = \frac{\pi}{3}$ or $\frac{4\pi}{3}$

8. **a)** $x = 0.52$ or 2.62 **b)** $x = 0.52$ or 5.76 **c)** $x = 1.05$ or 5.24 **d)** $x = 3.67$ or 5.76

9. **a)** $x = 0.79$ or 3.93 **b)** $x = 0.52$ or 2.62 **c)** $x = 0$ or 6.28 **d)** $x = 3.67$ or 5.76 **e)** $x = 1.16$ or 5.12 **f)** $x = 1.11$ or 4.25

10. **a)** $x = 0.39, 1.18, 3.53$, or 4.32 **b)** $x = 0.13, 0.65, 1.70, 2.23, 3.27, 3.80, 4.84$, or 5.37 **c)** $x = 1.40, 1.75, 3.49, 3.84, 5.59$, or 5.93 **d)** $x = 0.59, 0.985, 2.16, 2.55, 3.73, 4.12, 5.304$, or 5.697 **e)** $x = 1.05, 2.09, 4.19$, or 5.24 **f)** $x = 1.05$

11. from about day 144 to about day 221

12. $1.86\text{ s} < t < 4.14\text{ s}$; $9.86\text{ s} < t < 12.14\text{ s}$; $17.86\text{ s} < t < 20.14\text{ s}$

13. $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

17. $x = 0 + n\pi, \frac{2\pi}{3} + 2n\pi$, and $\frac{4\pi}{3} + n\pi$, where $n \in \mathbf{I}$

18. **a)** $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$, or $\frac{3\pi}{4}$ **b)** $x = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$

Homework #2

6. Solve each equation for x, where $0 \le x \le 2\pi$.

a) $(2\sin x - 1)\cos x = 0$

b) $(\sin x + 1)^2 = 0$

c) $(2\cos x + \sqrt{3})\sin x = 0$

d) $(2\cos x - 1)(2\sin x + \sqrt{3}) = 0$

e) $(\sqrt{2}\cos x - 1)(\sqrt{2}\cos x + 1) = 0$

f) $(\sin x + 1)(\cos x - 1) = 0$

7. Solve for θ to the nearest hundredth, where $0 \le \theta \le 2\pi$.

a) $2\cos^2\theta + \cos\theta - 1 = 0$

b) $2\sin^2\theta = 1 - \sin\theta$

c) $\cos^2\theta = 2 + \cos\theta$

d) $2\sin^2\theta + 5\sin\theta - 3 = 0$

e) $3\tan^2\theta - 2\tan\theta = 1$

f) $12\sin^2\theta + \sin\theta - 6 = 0$

8. Solve each equation for x, where $0 \le x \le 2\pi$.

a) $\sec x\csc x - 2\csc x = 0$

b) $3\sec^2 x - 4 = 0$

c) $2\sin x\sec x - 2\sqrt{3}\sin x = 0$

d) $2\cot x + \sec^2 x = 0$

e) $\cot x\csc^2 x = 2\cot x$

f) $3\tan^3 x - \tan x = 0$

9. Solve each equation in the interval $0 \le x \le 2\pi$. Round to two decimal places, if necessary.

a) $5\cos 2x - \cos x + 3 = 0$

b) $10\cos 2x - 8\cos x + 1 = 0$

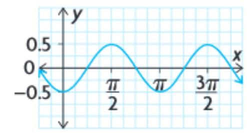
c) $4\cos 2x + 10\sin x - 7 = 0$

d) $-2\cos 2x = 2\sin x$

10. Solve the equation $8\sin^2 x - 8\sin x + 1 = 0$ in the interval $0 \le x \le 2\pi$.

11. The quadratic trigonometric equation $\cot^2 x - b\cot x + c = 0$ has the solutions $\frac{\pi}{6}, \frac{\pi}{4}, \frac{7\pi}{6}$, and $\frac{5\pi}{4}$ in the interval $0 \le x \le 2\pi$. What are the values of b and c?

12. The graph of the quadratic trigonometric function $f(x) = \sin^2 x - c$ is shown. What is the value of c?



13. **A** Natasha is a marathon runner, and she likes to train on a 2π km stretch of rolling hills. The height, in kilometres, of the hills above sea level, relative to her home, can be modelled by the function $h(d) = 4\cos^2 d - 1$, where d is the distance travelled in kilometres. At what intervals in the stretch of rolling hills is the height above sea level, relative to Natasha's home, less than zero?

14. **T** Solve the equation $6\sin^2 x = 17\cos x + 11$ for x in the interval $0 \le x \le 2\pi$.

15. a) Solve the equation $\sin^2 x - \sqrt{2}\cos x = \cos^2 x + \sqrt{2}\cos x + 2$ for x in the interval $0 \le x \le 2\pi$.

b) Write a general solution for the equation in part a).

16. **C** Explain why it is possible to have different numbers of solutions for quadratic trigonometric equations. Give examples to illustrate your explanation.

18. Solve the equation $2\cos 3x + \cos 2x + 1 = 0$.

20. Solve $\sqrt{2}\sin\theta = \sqrt{3} - \cos\theta$, $0 \le \theta \le 2\pi$.

Selected Answers

6. a) $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$

b) $x = \frac{3\pi}{2}$

c) $x = 0, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}$, or 2π

d) $x = \frac{\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$

e) $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$

f) $x = 0, \frac{3\pi}{2}$, or 2π

7. a) $\theta = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3}$

b) $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$

c) $\theta = \pi$

d) $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

e) $\theta = \frac{\pi}{4}$, 2.82, $\frac{5\pi}{4}$, or 5.96

f) $\theta = 0.73, 2.41, 3.99$, or 5.44

8. a) $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

b) $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$

c) $x = 0, 0.96\,\pi, 5.33$, or 2π

d) $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

e) $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}$, or $\frac{7\pi}{4}$

f) $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$, or 2π

9. a) $x = \frac{\pi}{3}$, 1.98, 4.30, or $\frac{5\pi}{3}$

b) $x = 0.45, \frac{2\pi}{3}, \frac{4\pi}{3}$, or 5.83

c) $x = \frac{\pi}{6}$, 0.85, $\frac{5\pi}{6}$, or 2.29

d) $x = \frac{\pi}{2}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$

10. $x = 0.15, 1.02, 2.12$, or 2.99

11. $b = 1 + \sqrt{3}, c = \sqrt{3}$

12. $c = \frac{1}{2}$

13. $\frac{\pi}{3}\text{km} < d < \frac{2\pi}{3}\text{km}$, $\frac{4\pi}{3}\text{km} < d < \frac{5\pi}{3}\text{km}$

14. $x = 1.91$ or 4.37

15. a) $x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$

b) $x = \frac{3\pi}{4} + 2n\pi$ or $\frac{5\pi}{4} + 2n\pi$, where $n \in \mathbf{I}$

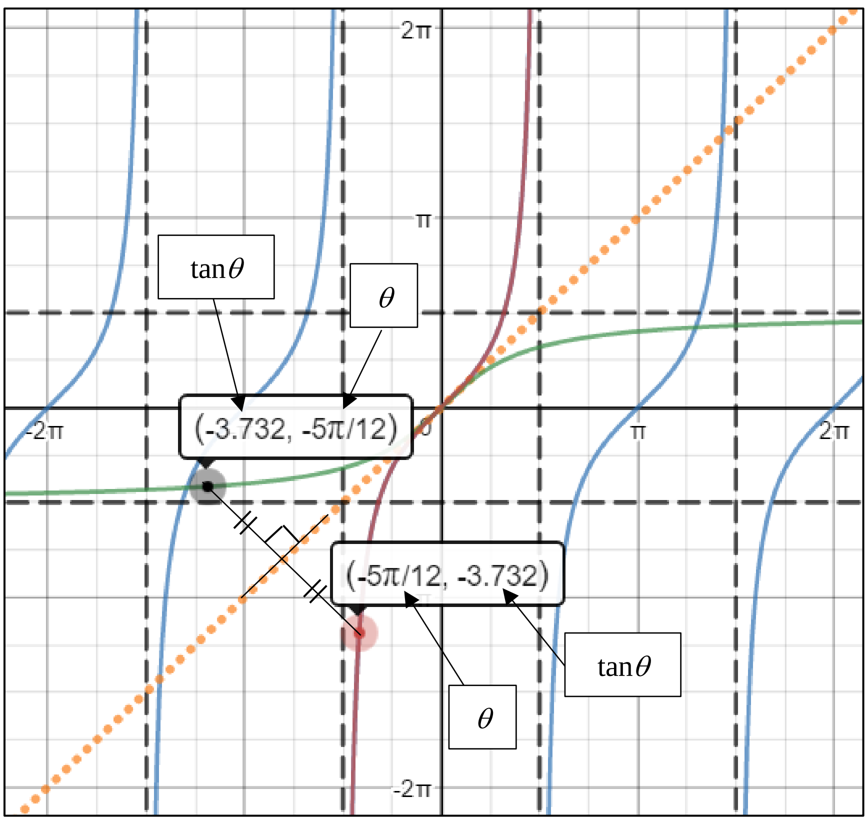
18. $x = 0.72, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, or 5.56

20. $\theta = 0.96$

Review – Inverses of Functions

- ***Inverse of a Function*** $\rightarrow$ Think "**Opposite**" **e.g.** x^2 and $\sqrt{x}$ are inverses
- The inverse of *f* is written f^{-1} $\rightarrow$ This is ***NOT*** the reciprocal of *f*
- Perform transformation $(x, y) \rightarrow (y, x)$ to form f^{-1} from *f*
- Graphically $\rightarrow$ Reflect *f* in the line $y = x$ to obtain the graph of f^{-1}
- To form f^{-1} when *f* is not one-to-one, the domain of *f* must be restricted to an interval on which *f* is one-to-one
- Domain of f^{-1} = Range of *f* Range of f^{-1} = Domain of *f*

Example: The Inverse of the* tan *Function



Legend

$f(x) = \tan x,\ x \in \mathbb{R},\ x \neq \pm\frac{n\pi}{2}$ for odd *n*

$g(x) = \tan x,\ \frac{-\pi}{2} < x < \frac{\pi}{2}$

$h(x) = \tan^{-1} x = \arctan x$

$y = x$

The function $f(x) = \tan x$ is ***not one-to-one*** so ***its inverse is not a function***. However, *f* is one-to-one on the interval $\frac{-\pi}{2} < x < \frac{\pi}{2}$. Thus, the inverse of *f* on this interval ***is a function***.

Domain of *g*: $\left\{x \in \mathbb{R} : \frac{-\pi}{2} < x < \frac{\pi}{2}\right\}$

Range of *g*: $\mathbb{R}$

Domain of $g^{-1} = h$: $\mathbb{R}$

Range of $g^{-1} = h$: $\left\{y \in \mathbb{R} : \frac{-\pi}{2} < y < \frac{\pi}{2}\right\}$

Questions

1. When you use a calculator to evaluate $\tan^{-1} x$ for any value of *x*, the result is always between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$. Why is this the case?
2. Solve the equation $\tan x = 2,\ 2\pi \le x \le 4\pi$.

Activity

Create a table like the one shown below. Use Desmos to create the graphs (as shown on the previous page).

Trigonometric Function f	***Interval on which f is One-to-One***	***Domain and Range of f and f^{-1}***	***Graph of f and f^{-1} on the same Grid***
$f(x)=\sin x$			
$f(x)=\cos x$			
$f(x)=\cot x$			
$f(x)=\csc x$			
$f(x)=\sec x$			

OPTIONAL TOPICS

Applications of Simple Geometry and Trigonometric Ratios

1. As shown very crudely in the diagram at the right, the data stored on a compact disc are arranged along a continuous spiral that begins near the centre of the disc. Because of this, a CD must spin at different rates to read data from different parts of the disc. For instance, audio CDs spin at a rate of about 28000º per minute when data are read near the centre, decreasing gradually to about 12000º per minute when data are read near the edge.

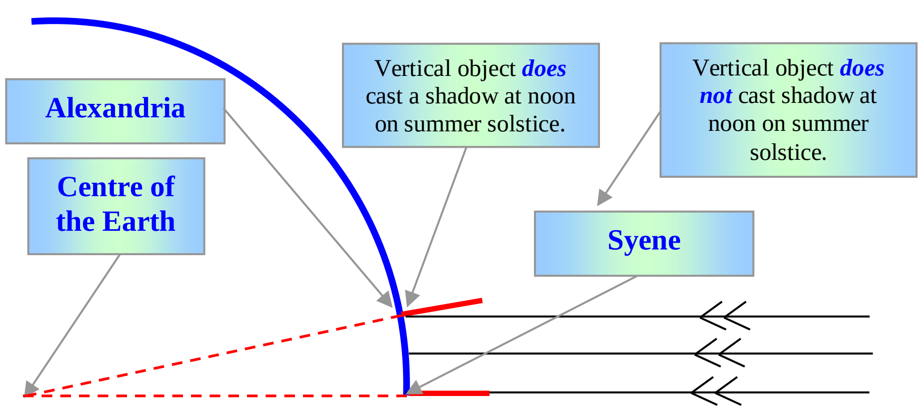
(a) Through how many degrees per second does an audio CD spin when data are read near the centre of the disc? Through how many degrees per second does an audio CD spin when data are read near the edge of the disc?

(b) Computer CD-ROM drives rotate CDs at multiples of the values used in audio CD players. A 1× drive has the same angular velocity as a music player and an n× drive is n times as fast. Through how many degrees per second does a CD rotate in a 52× drive when data are read near the centre of the disc?

2. It is a popular misconception that until the time of Christopher Columbus, people believed that the Earth was a flat plate. In reality, the spherical shape of the Earth was known to the ancient Greeks and quite likely, even to earlier civilizations. In the fourth century B.C., Aristotle put forth two strong arguments for supporting the theory that the Earth was a sphere. First, he observed that during lunar eclipses, the Earth's shadow on the moon was always circular. Second, he remarked that the North Star appeared at different ***angles of elevation*** in the sky, depending on whether the observer viewed the star from northerly or southerly locations. These two observations, being entirely inconsistent with the "flat Earth" hypothesis, led Aristotle to conclude that the Earth's surface must be curved. Even sailors in ancient Greece realized that the Earth's surface was curved. They noticed that the sails of a ship were always visible before the hull as the ship emerged over the horizon and that the sails appeared to "dip" into the ocean as the ship would retreat beyond the horizon.

A Greek named Eratosthenes (born: 276 BC in Cyrene, North Africa, which is now Shahhat, Libya, died: 194 BC, Alexandria, Egypt) took these observations one step further. He was the chief librarian in the great library of Alexandria in Egypt and a leading all-round scholar. At his disposal was the latest scientific knowledge of his day. One day, he read that a deep vertical well near Syene, in southern Egypt, was entirely lit up by the sun at noon once a year (on the summer solstice). This seemingly mundane fact probably would not have captured the attention of someone of ordinary intellectual abilities, but it instantly piqued Eratosthenes' curiosity. He reasoned that at this time, the sun must be directly overhead, with its rays shining directly into the well. Upon further investigation, he learned that in Alexandria, almost due north of Syene, the sun was not directly overhead at noon on the summer solstice because a vertical object would cast a shadow. He deduced, therefore, that the Earth's surface must be curved or the sun would be directly overhead in both places at the same time of day. By adding two simple assumptions to his deductions, Eratosthenes could calculate the Earth's circumference to a high degree of accuracy! First, he knew that the Earth's surface was curved so he assumed that it was a sphere. This assumption was strongly supported by Aristotle's observations in the fourth century B.C. Second, he assumed that the sun's rays are parallel to each other. This was also a very reasonable assumption because he knew that since the sun was so distant from the Earth, its rays would be virtually parallel as they approached the Earth.

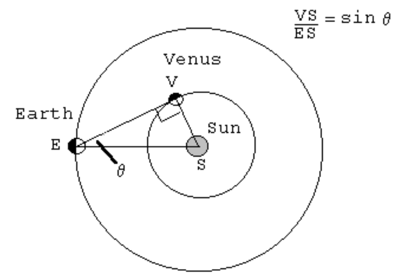
Now it's your turn to reproduce Eratosthenes' calculation. The diagram below summarizes all the required information.



Eratosthenes hired a member of a camel-powered trade caravan to "pace out" the distance between Syene and Alexandria. This distance was found to be about 5000 "stadia." The length of one "stadion" varied from ancient city to ancient city so there is some debate concerning how to convert Eratosthenes' measurement in stadia to a modern value in kilometres. However, it is usually assumed that Eratossthenes' stadion measured 184.98 m. Eratosthenes measured the "shadow angle" at Alexandria and found that it was approximately 7.2°.

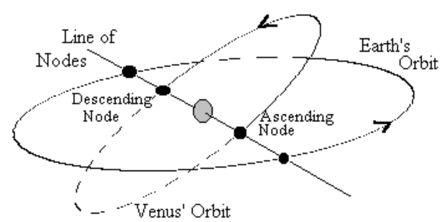
3. Have you ever wondered how astronomers calculate distances from the Earth to celestial bodies? In this problem, you will learn about how the distance from the Earth to the Sun was first calculated in 1882.

(a) Using the ***angle of separation*** (as measured from the Earth) between a body in our solar system and the Sun, its distance from the sun can be determined in terms of the distance from the Earth to the Sun. To facilitate this process, the "AU" (astronomical unit) was created, the distance from the Earth to the sun being defined as 1 AU. As a result of astronomical measurements made prior to 1882, the distances from the Sun of all the planets known at the time had been calculated ***in terms of*** the AU. Unfortunately, however, it was not possible to convert these distances into kilometres because the distance from the Earth to the Sun was not known. The diagram at the right shows how astronomers calculated the distance from Venus to the sun in terms of the AU. When Venus and the sun were conveniently positioned in the sky to make the angle of separation as large as possible, the angle θ was measured to be approximately 46.054°.

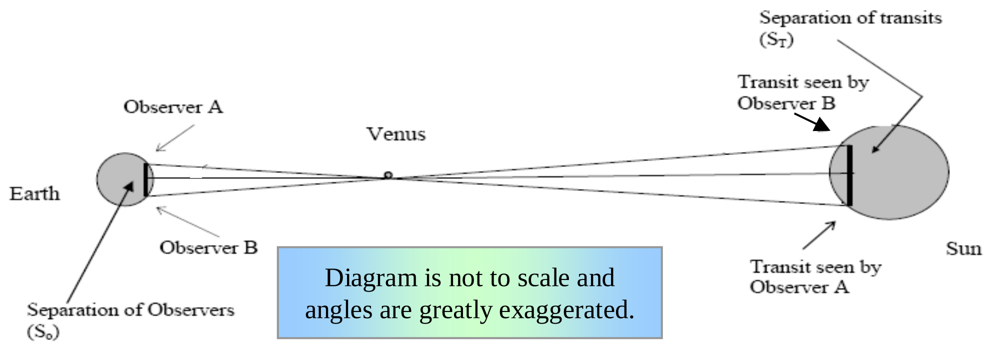


Use this information to calculate the distance from Venus to the Sun in terms of the astronomical unit. In addition, explain why ΔSVE must be a right triangle.

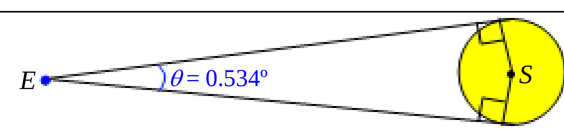
(b) As shown in the diagram at the right, the plane of Venus' orbit is inclined to that of the Earth. (The actual orbital inclination of Venus is only 3.39° but it is highly exaggerated in the illustration for the sake of clarity.) Notice that Venus passes through the Earth's orbital plane twice per orbit. When Venus intersects both the orbital plane of the Earth and the line connecting the Earth to the Sun, ***a transit of Venus*** occurs. From the Earth, Venus is observed as a small black disk that slowly makes its way across the face of the Sun. Transits of Venus are exceedingly rare, usually occurring in pairs spaced apart by 121.5±8 years (4 transits per 243-year cycle).



In 1882, a transit of Venus occurred. This opportunity was seized by astronomers to calculate, once and for all, the Earth-Sun distance. Observers on Earth separated by a North-South distance S_O (separation of observers) would observe transits separated by a North-South distance S_T (separation of transits). Observer *A* in the northern hemisphere would see Venus lower on the face of the Sun than observer *B* in the southern hemisphere. This is due to an effect called ***parallax***, the same phenomenon that causes an apparent shift in the position of objects when viewed successively with one eye and then with the other.



Use the information in the following table and your answer from 3(a) to express 1 AU in kilometres.

Symbols	Measured Values	RTF	Strategies / Hints
S_O = separation of observers S_T = separation of transits	S_O = 2000 km	S_T = ?	Use similar triangles to calculate S_T. Your answer from 3(a) is important!
S_{16} = separation of transits on circle of radius 16 cm r_S = radius of the sun	S_{16} = 0.059198 cm	r_S = ?	$\frac{r_S}{S_T} = \frac{0.5(16)}{S_{16}}$
θ = angle of separation (see diagram at the right)	$\theta = 0.534°$ This is the average of many measurements made by astronomers.	ES = ?	

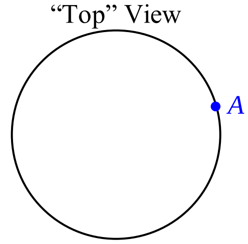
4. The circumference of the Earth at the equator is approximately 40074 km. A ***sidereal day*** is defined as the time required for the Earth to make one complete rotation relative to its axis of rotation. Careful scientific measurements have shown that the length of a sidereal day is about 23.9344696 hours (23 hours, 56 minutes, 4.09056 s) and that the circumference of the Earth at the equator is approximately 40074 km.

Therefore, a point on the equator moves in a very large circle at a speed of $\dfrac{40074\text{ km}}{23.9344696\text{ h}} \doteq 1674.3\text{ km/h}$.

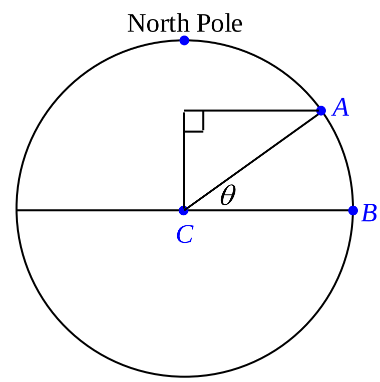
(a) Use the given information to calculate the rotational speed of a point on any line of latitude, relative to the Earth's axis of rotation.
Hint: Take any point A located at θ degrees north or θ degrees south. Keeping in mind that a line of latitude is nothing more than a large circle, you should be able to calculate its circumference in terms of the Earth's radius and the angle θ.

(b) Central Peel is located at 43.6964 degrees north. How fast is CPSS moving in a large circle about the Earth's axis of rotation?

"Top" View



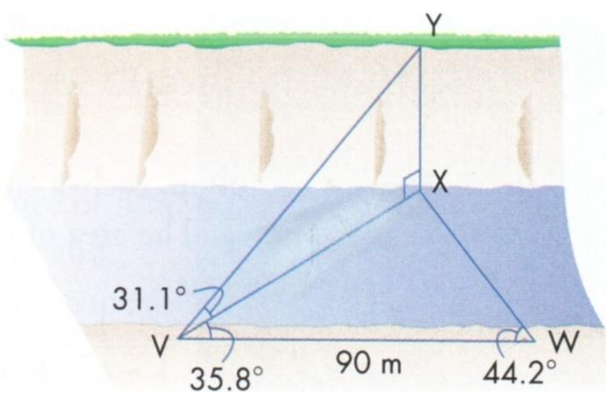
The great circle at latitude θ degrees north as seen when looking "down" from a point directly above the North Pole



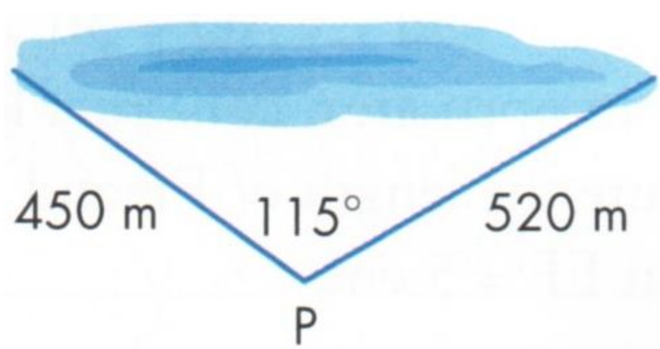
"Side" View

(c) What is the speed of the North Pole relative to the Earth's axis of rotation? What is the speed of the South Pole relative to the Earth's axis of rotation?

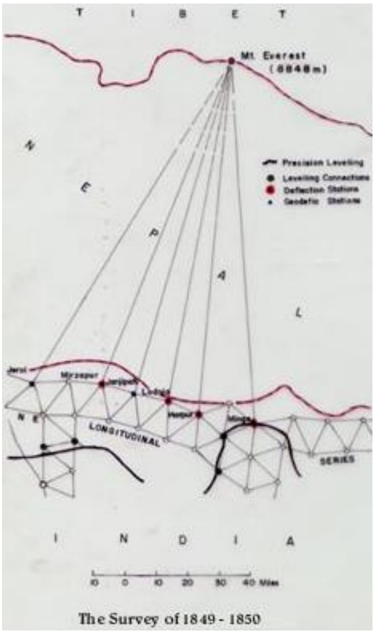
5. To measure the height XY of an inaccessible cliff, a surveyor recorded the data shown in the diagram at the right. If the theodolite used to take the measurement was 1.7 m above the ground, find the height of the cliff.

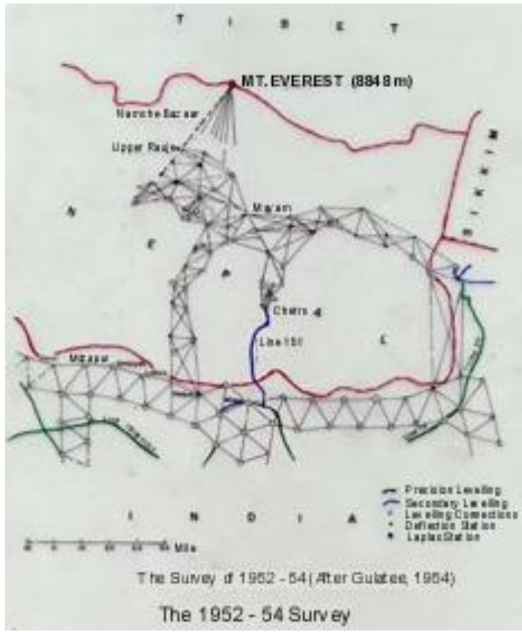


6. From point P, the distance to one end of a pond is 450 m and the distance to the other end is 520 m. The angle formed by the line of sight is 115º. Find the length of the pond



7. In 1852, as part of the Great Trigonometric Survey of India, Radhanath Sikhdar, an Indian mathematician and surveyor from Bengal, was the first to identify Mount Everest (called ***Sagarmatha*** by the Nepalese people) as the world's highest peak. Sikhdar used ***theodolites*** to make measurements of "Peak XV," as it was then known, from a distance of about 240 km. He then used trigonometry to calculate the height of Sagarmatha. Based on the average of measurements made from six different observation stations, Peak XV was found to be exactly 29,000 feet (8,839 m) high but was publicly declared to have a height of 29,002 feet (8,840 m). The arbitrary addition of 2 feet (0.6 m) was to avoid the impression that an exact height of 29,000 feet was nothing more than a rounded estimate.

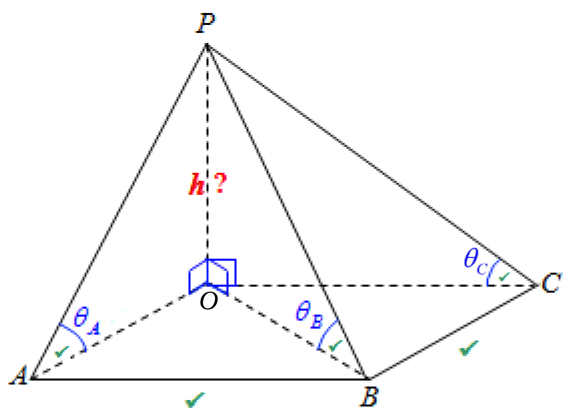




Radhanath Sikhdar

A ***theodolite*** is an instrument for measuring both horizontal and vertical angles,

(a) In the diagram at the right, the point *P* represents the peak of a mountain. Points *A*, *B* and *C* represent points on the ground that are at the same elevation above sea level. From these points, the ***angles of elevation*** θ_A, θ_B and θ_C are measured. As indicated in the diagram, the lengths of *AB* and *BC* are known, as are the measures of the angles θ_A, θ_B and θ_C. Does this give us enough information to calculate *h*, the height of the mountain above the surface of the plane through quadrilateral *ABCO*? If so, use an example to outline a method for calculating the height *h* in terms of the given information. If not, describe what other information would be needed.



(b) Is it necessary for the points *A*, *B* and *C* to be at the same elevation above sea level? Explain.

(c) Explain why it would be impractical to measure directly the lengths of *AO*, *BO* and *CO*. (Obviously, the length of *PO* = *h* cannot be measured directly.)

(d) Describe the orientation of quadrilateral *ABCO* relative to the surface of the Earth.

(e) Once *h* is calculated, how would the height of the mountain above sea level be determined?

(f) What is the currently accepted height above sea level of Mount Everest? Are you surprised that Sikhdar's measurement from 1852 is so close to the modern value?

Rates of Change in Trigonometric Functions

Introductory Investigation

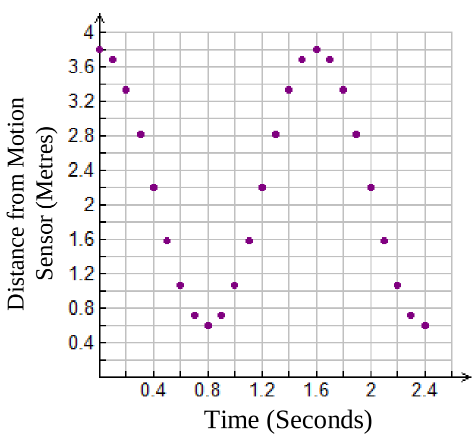
Vyshna was walking through a playground minding his own business when suddenly, he felt little Anshul tugging at his pants. “Vyshna, Vyshna!” little Anshul exclaimed. “Please push me on a swing!” Being in a hurry, Vyshna was a little reluctant to comply with little Anshul’s request at first. Upon reflection, however, Vyshna remembered that he had to collect some data for his math homework. He reached into his knapsack and pulled out his very handy portable motion sensor. “Get on the swing Anshul!” Vyshna bellowed. “I’ll set up the motion sensor in front of you and it will take some measurements as I push.” Gleefully, little Anshul hopped into the seat of the swing and waited for Vyshna to start pushing.

The data collected by Vyshna’s motion sensor are shown in the following tables. Time is measured in seconds and the distance, in metres, is measured from the motion sensor to little Anshul on the swing.

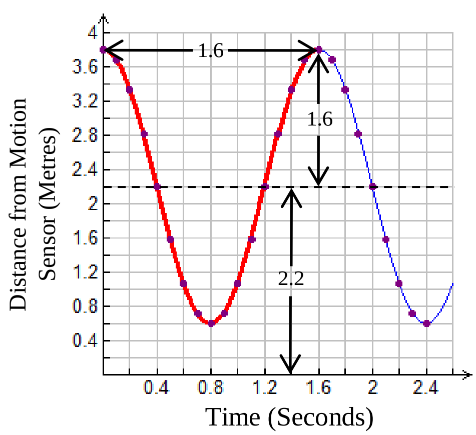
Time (s)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1
Distance (m)	3.8	3.68	3.33	2.81	2.2	1.59	1.07	0.72	0.6	0.72	1.07	1.59

Time (s)	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4
Distance (m)	2.2	2.81	3.33	3.68	3.8	3.68	3.33	2.81	2.2	1.59	1.07	0.72	0.6

Scatter Plot



Curve of Best Fit



Mathematical Model

$|A| = \text{Amplitude} = 1.6$

$T = \text{Period} = 1.6 = \left|\dfrac{1}{\omega}\right| 2\pi$

$\omega = \text{Angular Frequency}$

$= \left(\dfrac{1}{T}\right)2\pi = \left(\dfrac{1}{1.6}\right)2\pi = \dfrac{5\pi}{4}$

$d = \text{vertical displacement} = 2.2$

$p = \text{phase shift} = 0$

$\therefore d(t) = 1.6\cos\left(\dfrac{5\pi}{4}t\right) + 2.2$

(Alternatively, we could write $d(t) = 1.6\sin\left(\dfrac{5\pi}{4}(t-1.2)\right) + 2.2$)

Questions

1. What quantity is measured by

(a) the slope of the secant line through the points $(t_1, d(t_1))$ and $(t_2, d(t_2))$?

(b) the slope of the tangent line at $(t, d(t))$?

2. Complete the following table.

Intervals of Time over which Anshul approaches the Motion Sensor	*Intervals of Time over which Anshul recedes from the Motion Sensor*	*Intervals of Time over which Anshul’s Speed Increases*	*Intervals of Time over which Anshul’s Speed Decreases*

3. Explain the difference between speed and velocity.

4. Describe the "shape" of the curve over the intervals of time during which

(a) Anshul's velocity is increasing

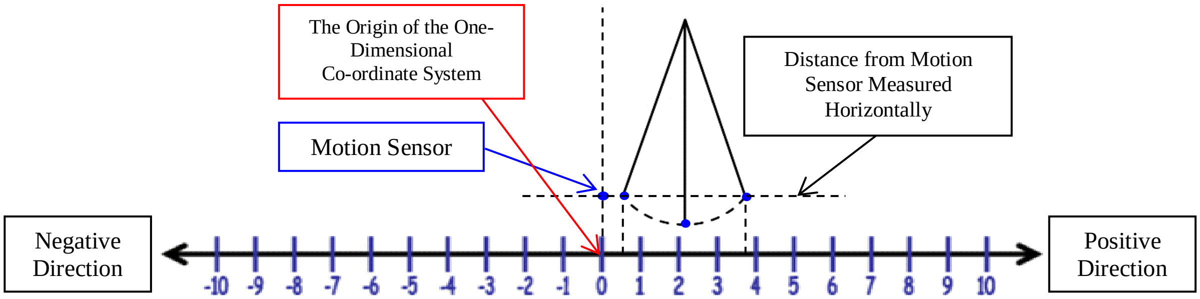
(b) Anshul's velocity is decreasing

5. Use the function given above to ***calculate*** the ***average*** rate of change of distance from the motion sensor with respect to time between 0.2 s and 1.0 s. Is your answer negative or positive? Interpret your result geometrically (i.e. as a slope) and physically (i.e. as a velocity).

6. Use the function given above to ***estimate*** the ***instantaneous*** rate of change of distance from the motion sensor with respect to time at 0.6 s. Is your answer negative or positive? Interpret your result geometrically (i.e. as a slope) and physically (i.e. as a velocity).

Rectilinear (Linear) Motion

- ***Rectilinear*** or ***linear*** motion is motion that occurs along a ***straight line***.
- Rectilinear motion can be described fully using a ***one-dimensional co-ordinate system***.
- Strictly speaking, Anshul's swinging motion is not rectilinear because he moves along a curve (see diagram).
- However, since only the horizontal distance to the motion sensor is measured, we can imagine that Anshul is moving along the horizontal line that passes through the motion sensor (see diagram). A more precise interpretation is that the equation given above models the position of Anshul's *x*-co-ordinate with respect to time.



The table below lists the meanings of various quantities that are used to describe one-dimensional motion.

Quantity	*Meaning and Description*	*Properties*
Position	The ***position*** of an object measures ***where*** the object is located at any given time. In linear motion, the position of an object is simply a number that indicates ***where it is*** with respect to a number line like the one shown above. Usually, the position function of an object is written as $s(t)$.	At any time *t*, if the object is located **(a)** ***at*** the origin, then $s(t)=0$ **(b)** to the ***right*** of the origin, then $s(t)>0$ **(c)** to the ***left*** of the origin, then $s(t)<0$ Also, $\lvert s(t)\rvert$ is the distance from the object to the origin.
Displacement	The ***displacement*** of an object between the times t_1 and t_2 is equal to its ***change in position*** between t_1 and t_2. That is, **displacement** $=\Delta s=s(t_2)-s(t_1)$.	If $\Delta s>0$, the object is to the ***right*** of its initial position. If $\Delta s<0$, the object is to the ***left*** of its initial position. If $\Delta s=0$, the object is ***at*** its initial position.
Distance	***Distance*** measures ***how far*** an object has travelled. Since an object undergoing linear motion can change direction, the distance travelled is found by summing (adding up) the absolute values of all the displacements for which there is a ***change in direction***.	The position of an object undergoing linear motion is tracked between times t_0 and t_n. In addition, the object changes direction at times $t_1, t_2, \ldots, t_{n-1}$ (and at no other times), where $t_0 < t_1 < \ldots < t_{n-1} < t_n$. If Δs_i represents the displacement from time t_{i-1} to time t_i, then the total distance travelled is equal to $d=\lvert\Delta s_1\rvert+\lvert\Delta s_2\rvert+\cdots+\lvert\Delta s_{n-1}\rvert+\lvert\Delta s_n\rvert$
Velocity	***Velocity*** is the ***instantaneous rate of change of position with respect to time***. Velocity measures ***how fast*** an object moves as well as its ***direction*** of travel. In one-dimensional rectilinear motion, velocity can be negative or positive, depending on the direction of travel.	At any time *t*, if the object is **(a)** moving in the ***positive*** direction, then $v(t)>0$ **(b)** moving in the ***negative*** direction, then $v(t)<0$ **(c)** at ***rest***, then $v(t)=0$ Also, $\lvert v(t)\rvert$ is the ***speed*** of the object.
Speed	***Speed*** is simply a measure of how fast an object moves ***without regard to its direction of travel***.	**speed** $=\lvert v(t)\rvert$

Table continued from previous page…

Quantity	***Meaning and Description***	***Properties***
Acceleration	***Acceleration*** is the ***instantaneous rate of change of velocity with respect to time***. In one-dimensional rectilinear motion, acceleration can be negative or positive, depending on the direction of the force causing the acceleration.	At any time *t*, if the object is **(a)** moving in the ***positive*** direction and ***speeding up*** or moving in the ***negative*** direction and ***slowing down***, then $a(t)>0$ **(b)** moving in the ***positive*** direction and ***slowing down*** or moving in the ***negative*** direction and ***speeding up***, then $a(t)<0$ **(c)** moving with a ***constant*** velocity, then $a(t)=0$

Example

Determine the quantities listed in the following table. (All times are specified in seconds.)

(a) Anshul's ***position*** at $t=2$

(b) Anshul's ***displacement*** over the interval $[0,2.6]$

(c) Anshul's ***average velocity*** over the interval $[0,2.6]$

(d) Anshul's ***average speed*** over the interval $[0,2.6]$

(e) The total ***distance*** travelled by Anshul over the interval $[0,2.6]$

(f) An estimate of Anshul's ***instantaneous velocity*** at $t=2$

Solution

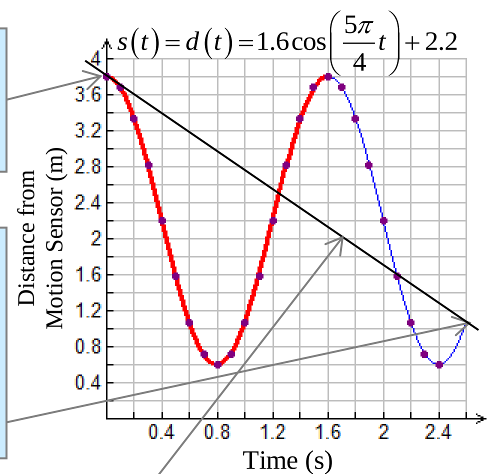
(a) $s(2)=1.6\cos\left(\frac{5\pi}{4}(2)\right)+2.2=1.6\cos\left(\frac{5\pi}{2}\right)+2.2=1.6(0)+2.2=2.2$

Anshul's position at $t=2$ is 2.2 m to the right of the origin.

(b)
$$
\begin{aligned}
\Delta s &= s(2.6)-s(0)\\
&=1.6\cos\left(\frac{5\pi}{4}(2.6)\right)+2.2-\left[1.6\cos\left(\frac{5\pi}{4}(0)\right)+2.2\right]\\
&=1.6\cos\frac{13\pi}{4}-1.6\cos 0\\
&=1.6\left(-\frac{1}{\sqrt{2}}\right)-1.6(1)\\
&=-1.6\left(\frac{1}{\sqrt{2}}+1\right)\\
&=-1.6\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)\\
&\doteq -2.73
\end{aligned}
$$

Position at time 0
$=s(0)$
$=3.8$

Position at time 2.6
$=s(2.6)$
$=1.6\left(-\frac{1}{\sqrt{2}}\right)+2.2$
$\doteq 1.07$



Over the interval $[0,2.6]$, Anshul's ***displacement*** is –2.73 m. This means that at 2.6 s, his position was 2.73 m to the ***left*** of his initial position (i.e. his position at 0 s).

(c)
$$
\begin{aligned}
v_{\text{avg}} &= \frac{\Delta s}{\Delta t}\\
&=\frac{s(2.6)-s(0)}{2.6-0}\\
&=\frac{-1.6\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)}{2.6}\\
&\doteq -1.05
\end{aligned}
$$

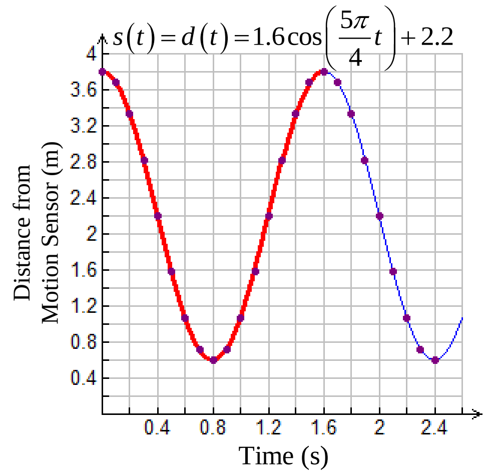
Average velocity over the interval $[0,2.6]$
$=$ slope of secant line between $(0,s(0))$ and $(2.6,s(2.6))$
$\doteq -1.05$ m/s

Over the interval $[0,2.6]$, Anshul's ***average velocity*** is –1.05 m/s. This means that Anshul's position ***decreases*** at an average rate of 1.05 m/s over the interval $[0,2.6]$. That is, Anshul ***moves to the left*** with an average speed of 1.05 m/s.

(d) average speed $= |v_{\text{avg}}| = \left|\frac{\Delta s}{\Delta t}\right| = |-1.05| = 1.05$ m/s.

Over the interval $[0, 2.6]$, Anshul's average speed was 1.05 m/s.

(e) Examine the graph at the right, showing Anshul's position over time. Notice that over the interval $[0, 2.6]$, Anshul changes direction at 0.8 s, 1.6 s and 2.4 s. Therefore,



$$\begin{aligned}
d &= |\Delta s_1| + |\Delta s_2| + |\Delta s_3| + |\Delta s_4| \\
&= |s(0.8) - s(0)| + |s(1.6) - s(0.8)| + |s(2.4) - s(1.6)| + |s(2.6) - s(2.4)| \\
&\doteq |0.6 - 3.8| + |3.8 - 0.6| + |0.6 - 3.8| + |1.07 - 0.6| \\
&= 3.2 + 3.2 + 3.2 + 0.47 \\
&= 10.07
\end{aligned}$$

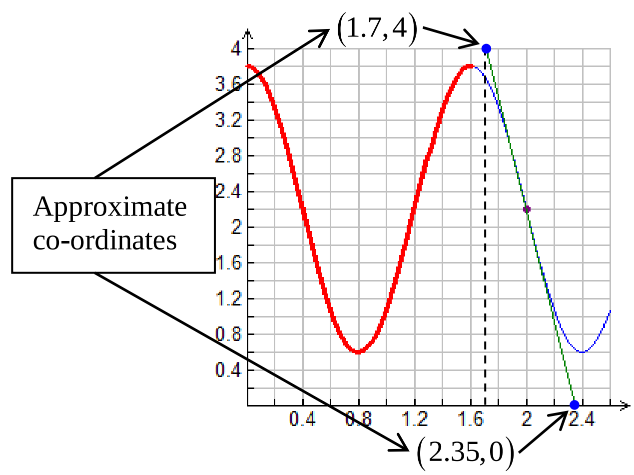
Over the interval $[0, 2.6]$, Anshul travelled about 10.07 m.

(f) $v(2)$ = Instantaneous velocity at 2 seconds

= slope of tangent of position-time graph at 2 seconds (see graph at right)

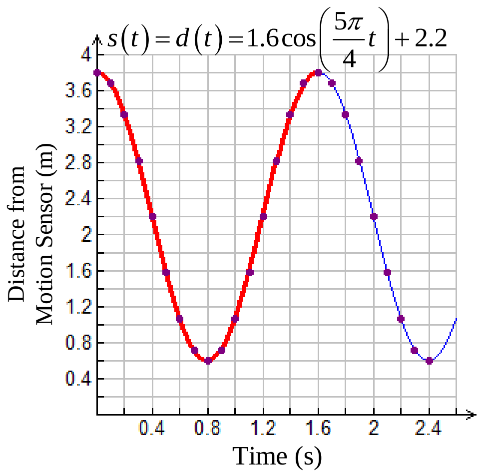
Since we do not yet have the tools of calculus at our disposal, the best we can do is to approximate the slope of the required tangent line by using a secant line that passes through two points that are very close to $t = 2$. If we use the centered interval $[1.99, 2.01]$, then

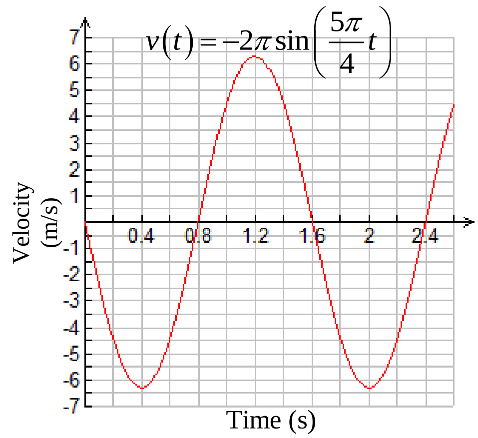
$$\begin{aligned}
v(2) &\doteq \frac{\Delta s}{\Delta t} \quad \text{(if } \Delta t \text{ is very small)} \\
&= \frac{s(2.01) - s(1.99)}{2.01 - 1.99} \\
&= \frac{1.6\cos\left(\frac{5\pi}{4}(2.01)\right) + 2.2 - \left(1.6\cos\left(\frac{5\pi}{4}(1.99)\right) + 2.2\right)}{0.02} \\
&= \frac{1.6\cos\left(\frac{10.05\pi}{4}\right) - 1.6\cos\left(\frac{9.95\pi}{4}\right)}{0.02} \\
&\doteq -6.28
\end{aligned}$$

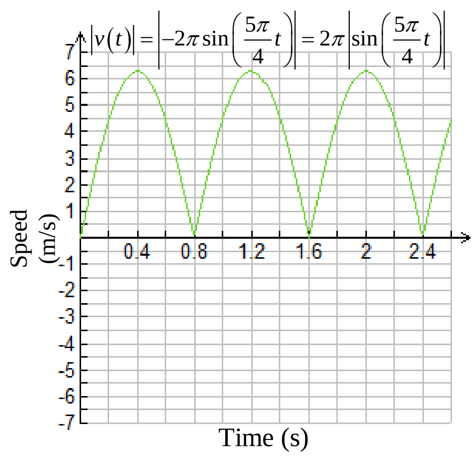


At $t = 2$ seconds, Anshul's instantaneous velocity is approximately –6.28 m/s, which means that he is moving to the left (toward the motion sensor) with a speed of about 6.28 m/s. Notice that the answer –6.28 m/s agrees with the graph shown above. The tangent line at $t = 2$ goes downward to the right, which means that its slope should be negative. In addition, a quick, rough calculation of $\frac{\text{rise}}{\text{run}}$ yields $\frac{4-0}{1.7-2.35} \doteq -6.15$, which is in close agreement with the answer obtained above.

Now use the following graphs to confirm the answers given above for the instantaneous quantities.







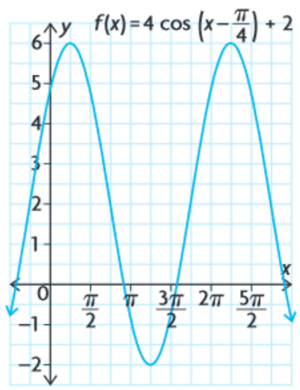
Homework

2. For this graph of a function, state two points where the function has an instantaneous rate of change in $f(x)$ that is

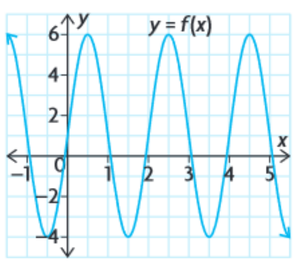
a) zero

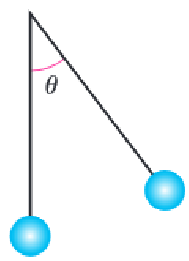
b) a negative value

c) a positive value



3. Use the graph to calculate the average rate of change in $f(x)$ on the interval $2 \leq x \leq 5$.





4. Determine the average rate of change of the function $y = 2\cos\left(x - \frac{\pi}{3}\right) + 1$ for each interval.

a) $0 \leq x \leq \frac{\pi}{2}$

b) $\frac{\pi}{6} \leq x \leq \frac{\pi}{2}$

c) $\frac{\pi}{3} \leq x \leq \frac{\pi}{2}$

d) $\frac{\pi}{2} \leq x \leq \frac{5\pi}{4}$

6. State two points where the function $y = -2\sin(2\pi x) + 7$ has an instantaneous rate of change that is

a) zero

b) a negative value

c) a positive value

8. T The height of the tip of an airplane propeller above the ground once the airplane reaches full speed can be modelled by a sine function. At full speed, the propeller makes 200 revolutions per second. At $t = 0$, the tip of the propeller is at its minimum height above the ground. Determine whether the instantaneous rate of change in height at $t = \frac{1}{300}$ is a negative value, a positive value, or zero.

12. A A ship that is docked in a harbour rises and falls with the waves. The function $h(t) = \sin\left(\frac{\pi}{5}t\right)$ models the vertical movement of the ship, h in metres, at t seconds.

a) Determine the average rate of change in the height of the ship over the first 5 s.

b) Estimate the instantaneous rate of change in the height of the ship at $t = 6$.

13. For a certain pendulum, the angle θ shown is given by the equation $\theta = \frac{1}{5}\sin\left(\frac{1}{2}\pi t\right)$ where t is in seconds and θ is in radians.

a) Sketch a graph of the function given by the equation.

b) Calculate the average rate of change in the angle the pendulum swings through in the interval $t \in [0, 1]$.

c) Estimate the instantaneous rate of change in the angle the pendulum swings through at $t = 1.5$ s.

d) On the interval $t \in [0, 8]$, estimate the times when the pendulum's speed is greatest.

15. In calculus, the derivative of a function is a function that yields the instantaneous rate of change of a function at any given point.

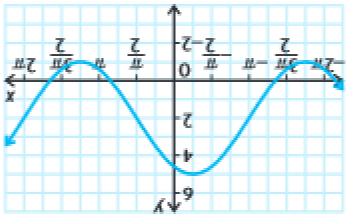
a) Estimate the instantaneous rate of change of the function $f(x) = \sin x$ for the following values of x: $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}$, and π.

b) Plot the points that represent the instantaneous rate of change, and draw a sinusoidal curve through them. What function have you graphed? Based on this information, what is the derivative of $f(x) = \sin x$?

Answers

2. 2, 0.5, $y \doteq 0.973\,94$

3.

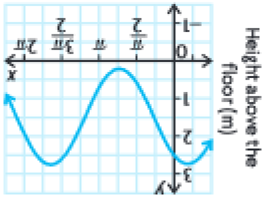


$x = 1.3$

4. amplitude and equation of the axis

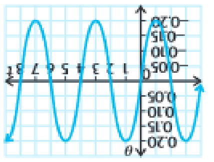
6. $y = 90\sin\left(\frac{\pi}{12}x\right) + 30$

8. $y = -1.25\sin\left(\frac{4}{5}x\right) + 1.5$



12. The student should graph the height of the nail above the ground as a function of the total distance travelled by the nail, because the nail would not be travelling at a constant speed. If the student graphed the height of the nail above the ground as a function of time, the graph would not be sinusoidal.

13. a)



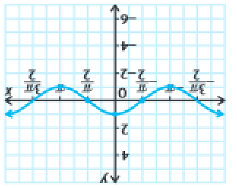
b) 0.2 radians/s

c) Answers may vary. For example, about $-\frac{2}{3}$ radians/s.

d) $t = 0, 2, 4, 6$, and 8

15. a) $-1, 0, 1, 0$, and -1

b)



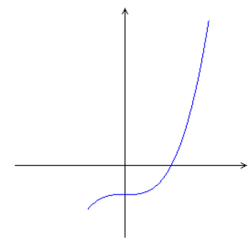
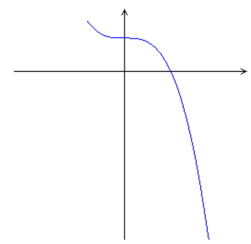
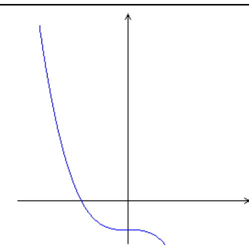
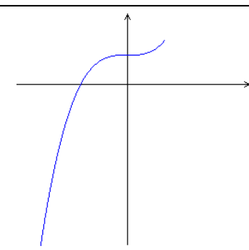
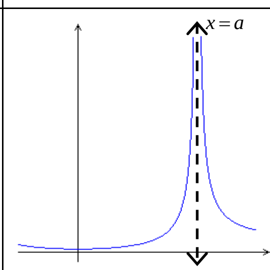
The function is $f(x) = \cos x$. Based on this information, the derivative of $f(x) = \sin x$ is $\cos x$.

End Behaviours and other Tendencies of Trigonometric Functions

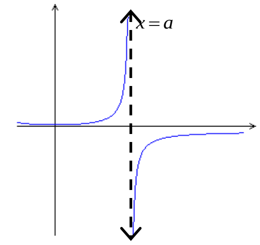
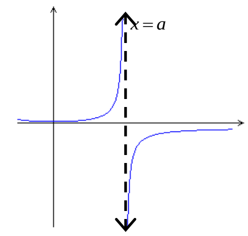
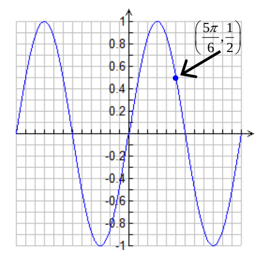
Notation

Examples are given in the following table of notation that is used to describe the behaviour of some function f as x undergoes some change such as tending toward a value or getting larger and larger without bound.

- Note that $x \to \infty$ can also be written $x \to +\infty$.
- "Arbitrarily far from" means "as far as desired from."
- "Arbitrarily close to" means "as close as desired to."

Notation used in this Course	***Calculus Notation****	***Meaning***	***What it Looks Like***
As $x \to \infty$, $f(x) \to \infty$	$\lim_{x\to\infty} f(x) = \infty$	**Read:** As x approaches (positive) infinity, $f(x)$ approaches (positive) infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***positive direction*** by making x far enough from the origin in the ***positive direction***.	
As $x \to \infty$, $f(x) \to -\infty$	$\lim_{x\to\infty} f(x) = -\infty$	**Read:** As x approaches (positive) infinity, $f(x)$ approaches negative infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***negative direction*** by making x far enough from the origin in the ***positive direction***.	
As $x \to -\infty$, $f(x) \to \infty$	$\lim_{x\to-\infty} f(x) = \infty$	**Read:** As x approaches negative infinity, $f(x)$ approaches (positive) infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***positive direction*** by making x far enough from the origin in the ***negative direction***.	
As $x \to -\infty$ $f(x) \to -\infty$	$\lim_{x\to-\infty} f(x) = -\infty$	**Read:** As x approaches negative infinity, $f(x)$ approaches negative infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***negative direction*** by making x far enough from the origin in the ***negative direction***.	
As $x \to a$ $f(x) \to \infty$	$\lim_{x\to a} f(x) = \infty$	**Read:** As x approaches a, $f(x)$ approaches (positive) infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***positive direction*** by making x close enough but not equal to a (from both the left and right sides).	

* This is just a preview of calculus. You are not required to use calculus notation in this course.

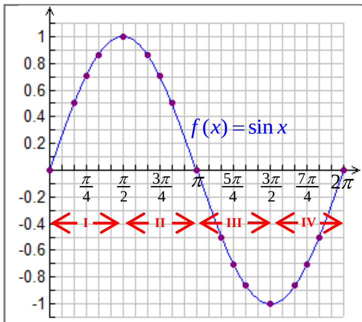
Notation used in this Course	*Calculus Notation**	*Meaning*	*What it Looks Like*
As $x \to a^+$, $f(x) \to -\infty$	$\lim\limits_{x \to a^+} f(x) = -\infty$	**Read:** As x approaches a ***from the right***, $f(x)$ approaches negative infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***negative direction*** by making x close enough to a from the right side but not equal to a.	
As $x \to a^-$, $f(x) \to \infty$	$\lim\limits_{x \to a^-} f(x) = \infty$	**Read:** As x approaches a ***from the left***, $f(x)$ approaches (positive) infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***positive direction*** by making x close enough to a from the left side but not equal to a.	
As $x \to a$, $f(x) \to L$	$\lim\limits_{x \to a} f(x) = L$	**Read:** As x approaches a, $f(x)$ approaches L. **Meaning:** We can make $f(x)$ arbitrarily close to L by making x close enough but not equal to a.	

In the example given in the previous row, as $x \to \frac{5\pi}{6}$, $f(x) \to \frac{1}{2}$. This means that we can make $f(x) = \sin x$ as close as we desire to $\frac{1}{2}$ by making x close enough but not equal to $\frac{5\pi}{6}$. In the language of calculus, this is written $\lim\limits_{x \to \frac{5\pi}{6}} f(x) = \frac{1}{2}$.

* This is just a preview of calculus. You are not required to use calculus notation in this course.

Exercises

Determine the tendency of $f(x)$.



As $x \to 0^+$, $f(x) \to$ ______

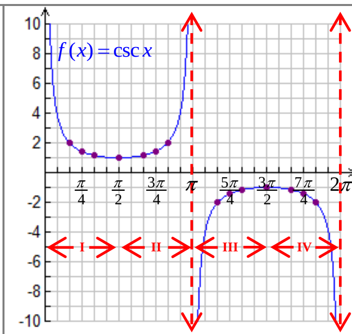
As $x \to 0^-$, $f(x) \to$ ______

As $x \to 0$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^+$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}$, $f(x) \to$ ______



As $x \to 0^+$, $f(x) \to$ ______

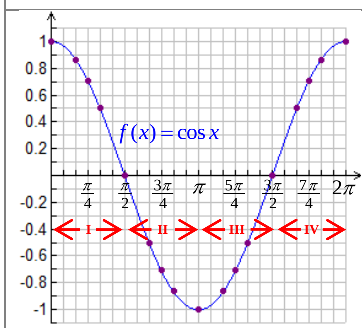
As $x \to 0^-$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}$, $f(x) \to$ ______

As $x \to \frac{3\pi}{2}$, $f(x) \to$ ______

As $x \to \pi^-$, $f(x) \to$ ______

As $x \to \pi^+$, $f(x) \to$ ______



As $x \to 0^+$, $f(x) \to$ ______

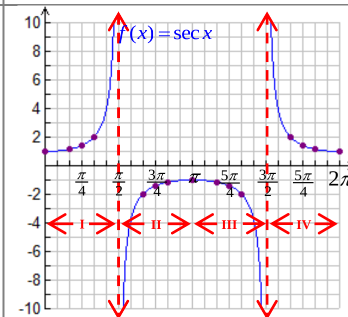
As $x \to 0^-$, $f(x) \to$ ______

As $x \to 0$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^+$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}$, $f(x) \to$ ______



As $x \to 0$, $f(x) \to$ ______

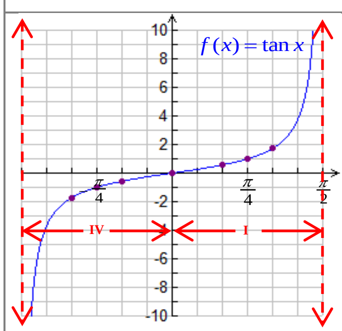
As $x \to \pi$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^+$, $f(x) \to$ ______

As $x \to \frac{3\pi}{2}^-$, $f(x) \to$ ______

As $x \to \frac{3\pi}{2}^+$, $f(x) \to$ ______



As $x \to 0$, $f(x) \to$ ______

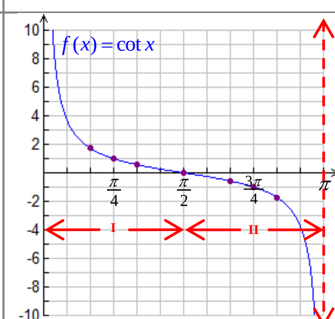
As $x \to \frac{\pi}{4}$, $f(x) \to$ ______

As $x \to \frac{\pi}{6}$, $f(x) \to$ ______

As $x \to -\frac{\pi}{2}^+$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ ______

As $x \to \frac{3\pi}{2}^+$, $f(x) \to$ ______



As $x \to 0^+$, $f(x) \to$ ______

As $x \to \frac{\pi}{4}$, $f(x) \to$ ______

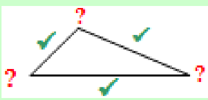
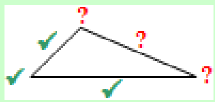
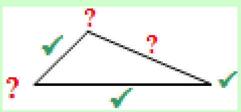
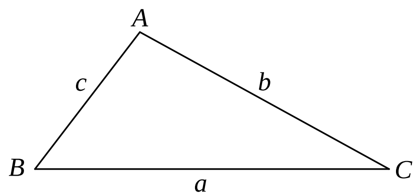
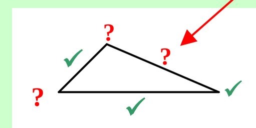
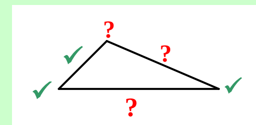
As $x \to \frac{\pi}{6}$, $f(x) \to$ ______

As $x \to \frac{\pi}{2}$, $f(x) \to$ ______

As $x \to \pi^-$, $f(x) \to$ ______

As $x \to \pi^+$, $f(x) \to$ ______

Law of Sines and Law of Cosines

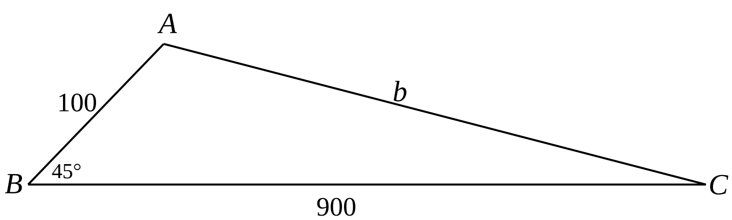
Law of Cosines		*Law of Sines*
$c^2 = a^2 + b^2 - 2ab\cos C$ **The law of cosines is a *generalization* of the Pythagorean Theorem.** The law of cosines is most useful in the following two cases.   The law of cosines also can be used in the following case. However, a quadratic equation needs to be solved. 	While the Pythagorean Theorem ***holds only for right triangles***, the Sine Law and the Cosine Law hold for ***all triangles***!  However, you ***should not*** use these laws when working with right triangles. It is much easier to use the basic trigonometric ratios and the Pythagorean Theorem.	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ The law of sines is used in the following two cases. **However, you must beware of the ambiguous case (SSA).**  

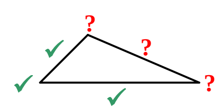
Important Exercise

Rearrange the equation for the law of cosines in such a way that you solve for $\cos C$. In what situation might you want to solve for $\cos C$?

Examples

1. Solve the following triangle.





Solution

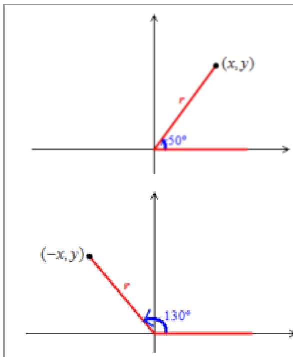
Given: $a = 900$, $c = 100$, $\angle B = 45°$

Required to Find (RTF): $b = ?$, $\angle A = ?$, $\angle C = ?$ (Note that "RTF" should not be confused with "WTF.")

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$= 900^2 + 100^2 - 2(900)(100)\cos 45°$$
$$= 810000 + 10000 - 180000\left(\frac{1}{\sqrt{2}}\right)$$
$$= 820000 - \frac{180000}{\sqrt{2}}$$
$$= \frac{820000\sqrt{2} - 180000}{\sqrt{2}}$$
$$\therefore b = \sqrt{\frac{820000\sqrt{2} - 180000}{\sqrt{2}}} \doteq 832.2985$$

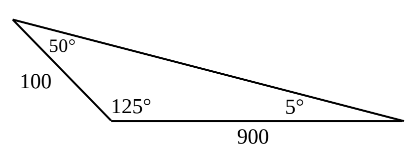
$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\therefore \frac{\sin A}{900} = \frac{\sin 45°}{\sqrt{\frac{820000\sqrt{2} - 180000}{\sqrt{2}}}}$$
$$\therefore \sin A \doteq \frac{900\left(\frac{1}{\sqrt{2}}\right)}{832.2985} \doteq 0.764624834$$
$$\therefore \angle A \doteq \sin^{-1}(0.764624834) \doteq 50° \text{ or } 130°$$

This is an example of how the *ambiguous* case of the sine law can arise.



The problem here is that $\sin 50° = \sin 130°$. Which answer is correct, 50º or 130º?

Because the given angle (45º) is enclosed by the two given sides (SAS), we must conclude that $A \doteq 130°$. If we allow $\angle A$ to have a measure of 50º, then the only triangle that can be constructed is the one shown at the right. Clearly, this triangle contradicts the given information because it lacks a 45 º angle.



Final Answer: $b \doteq 832.2985$, $\angle A \doteq 130°$, $\angle C \doteq 5°$

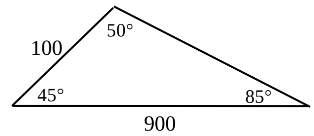
2. Answer the following questions.

(a) Explain the meaning of the word "ambiguous."

(b) Explain why the law of sines ***has*** an ambiguous case.

(c) Explain why the law of cosines ***does not have*** an ambiguous case.

(d) Refer to example 1. Explain why it is not possible for $\angle A$ to have a measure of 50º. That is, explain why it is not possible to construct the triangle shown at the right.



(e) Suppose that a solution of the equation $\sin\theta = k$ is the first quadrant angle α. What would be a second quadrant solution?

(f) In ΔABC, $\angle A = 37°$, a = 3 cm and c = 4 cm. How many different triangles are possible?

Solutions

(a) **Ambiguous:** Open to two or more interpretations; or of uncertain nature or significance; or intended to mislead e.g. "The polling had a complex and ***ambiguous*** message for potential female candidates."

(b) Since $\sin\theta > 0$ in quadrants I and II, solving an equation such as $\sin\theta = 0.5$ will result in two answers for θ, one that is in quadrant I and one that is in quadrant II. Since angles in a triangle can have any measure between 0º and 180º, then ***both*** answers are possible!

(c) Since $\cos\theta > 0$ in quadrant I and $\cos\theta < 0$ in quadrant II, there is no ambiguity. A positive cosine implies a first quadrant angle while a negative cosine implies a second quadrant angle.

(d) In any triangle, the largest angle must be opposite the longest side. In the triangle shown above, the largest angle is opposite the shortest side, which is impossible. To confirm that this is the case, we can calculate $\angle A$ by using the law of cosines. Beginning with the equation $a^2 = b^2 + c^2 - 2bc\cos A$, we solve for $\cos A$:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore \cos A \doteq \frac{832.2985^2 + 100^2 - 900^2}{2(832.2985)(100)}$$

$$\therefore \cos A \doteq -0.64447555$$

$$\therefore \angle A \doteq \cos^{-1}(-0.64447555)$$

$$\therefore \angle A \doteq 130°$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\therefore 2bc\cos A = b^2 + c^2 - a^2$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

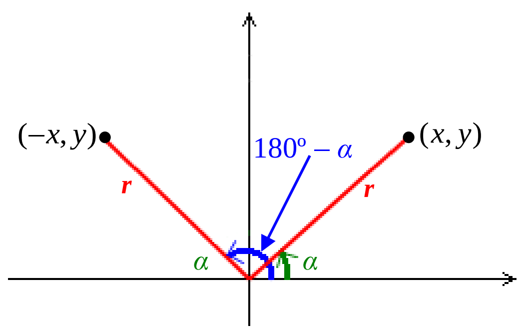
(e) Consider the diagram at the right.

$$\sin\alpha = \frac{y}{r}$$

$$\sin(180° - \alpha) = \frac{y}{r}$$

$$\therefore \sin\alpha = \sin(180° - \alpha)$$

Therefore, if α is a first quadrant solution to the given equation, then 180º – α is a second quadrant solution.



(f) As shown at the right, there are two possible triangles that meet the given criteria. To confirm this algebraically, the law of cosines can be used.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\therefore 3^2 = b^2 + 4^2 - 2(b)(4)\cos 37°$$

$$\therefore 9 = b^2 + 16 - (8\cos 37°)b$$

$$\therefore b^2 - (8\cos 37°)b + 7 = 0$$

$$\therefore b = \frac{8\cos 37° \pm \sqrt{(8\cos 37°)^2 - 4(1)(7)}}{2(1)}$$

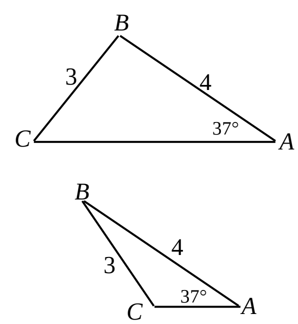
$$\therefore b \doteq 4.99 \text{ or } b \doteq 1.40$$

Since we obtain two answers for b, there are two possible triangles.

The number of roots of this quadratic equation can be determined very quickly by calculating the discriminant:

$$D = (8\cos 37°)^2 - 4(1)(7) \doteq 12.8$$

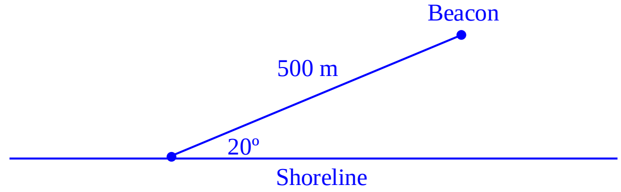
Since D > 0, there are two real roots.



3. The light from a rotating offshore beacon can illuminate effectively up to a distance of 250 m. A point on the shore is 500 m from the beacon. From this point, the sight line to the beacon makes an angle of 20º with the shoreline.

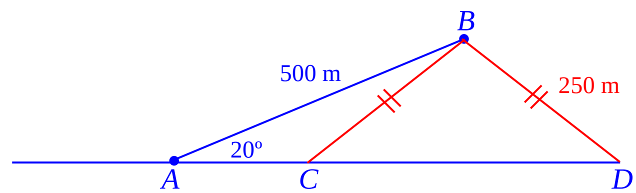
(a) What length of shoreline is illuminated effectively by the beacon?

(b) What area of the shore is illuminated effectively by the beacon?



Solution

(a) Using the diagram at the right, it is clear that CD is the portion of the shoreline that is illuminated effectively by the beacon.



Using ΔABD and the law of sines, we obtain

$$\frac{\sin D}{AB} = \frac{\sin A}{BD}$$

$$\therefore \frac{\sin D}{500} = \frac{\sin 20°}{250}$$

$$\therefore \sin D = 500\left(\frac{\sin 20°}{250}\right)$$

$$\therefore \sin D = 2\sin 20°$$

$$\therefore \angle D = \sin^{-1}(2\sin 20°)$$

$$\therefore \angle D \doteq 43.2°$$

Since ΔBCD is isosceles,

$$\therefore \angle BCD \doteq 43.2°$$

$$\therefore \angle CBD \doteq 180° - 43.2° - 43.2° = 93.6°$$

Using ΔBCD and the law of sines, we obtain

$$\frac{CD}{\sin\angle CBD} = \frac{BC}{\sin D}$$

$$\therefore \frac{CD}{\sin 93.6°} = \frac{250}{\sin 43.2°}$$

$$\therefore CD = \sin 93.6°\left(\frac{250}{\sin 43.2°}\right)$$

$$\therefore CD \doteq 364$$

Therefore, the length of the shoreline effectively illuminated by the beacon is approximately 364 m.

(b) Let S represent the required area. The portion of the shore illuminated by the beacon is shaded in the diagram below. The area of this portion of the shore can be found by subtracting the area of ΔBCD from the area of sector BCD (shape of a slice of pie). That is,

S = (area of sector BCD) – (area of ΔBCD)

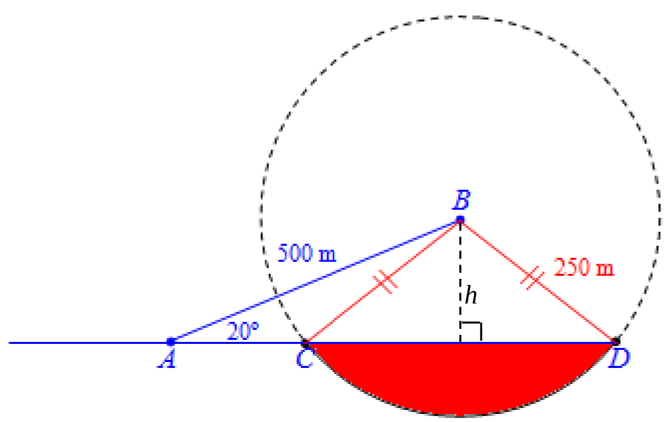
Since $\angle CBD \doteq 93.6°$, the area of sector BCD is about $\frac{93.6}{360}$ the area of the entire circle. Thus,

$$S \doteq \frac{\theta}{360}\left(\pi r^2\right) - \frac{1}{2}bh$$

$$\therefore S \doteq \frac{93.6}{360}\left(\pi(250)^2\right) - \frac{1}{2}(364)\sqrt{250^2 - 182^2}$$

$$\therefore S \doteq 19857$$

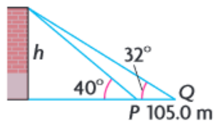
The area of the shore effectively illuminated by the beacon is approximately 19587 m^2.



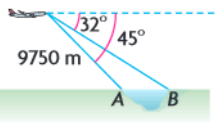
Homework

Law of Sines

7. A building of height h is observed from two points, P and Q, that are 105.0 m apart as shown. The angles of elevation at P and Q are 40° and 32°, respectively. Calculate the height, h, to the nearest tenth of a metre.



8. A surveyor in an airplane observes that the angle of depression to two points on the opposite shores of a lake are 32° and 45°, respectively, as shown. What is the width of the lake, to the nearest metre, at those two points?



9. The Pont du Gard near Nîmes, France, is a Roman aqueduct. An observer in a hot-air balloon some distance away from the aqueduct determines that the angle of depression to each end is 54° and 71°, respectively. The closest end of the aqueduct is 270.0 m from the balloon. Calculate the length of the aqueduct to the nearest tenth of a metre.

10. A wind tower at the top of a hill casts a shadow 30 m long along the side of the hill. An observer at the farthest edge of the shadow from the tower estimates the angle of elevation to the top of the tower to be 34°. If the slope of the hill is 13° from the horizontal, how high is the tower to the nearest metre?

15. A sailor out in a lake sees two lighthouses 11 km apart along the shore and gets bearings of 285° from his present position for lighthouse A and 237° for lighthouse B. From lighthouse B, lighthouse A has a bearing of 45°.

a) How far, to the nearest kilometre, is the sailor from both lighthouses?

b) What is the shortest distance, to the nearest kilometre, from the sailor to the shore?

Law of Cosines

5. The posts of a hockey goal are 2.0 m apart. A player attempts to score by shooting the puck along the ice from a point 6.5 m from one post and 8.0 m from the other. Within what angle θ must the shot be made? Round your answer to the nearest degree.

6. While golfing, Sahar hits a tee shot from T toward a hole at H, but the ball veers 23° and lands at B. The scorecard says that H is 270 m from T. If Sahar walks 160 m to the ball (B), how far, to the nearest metre, is the ball from the hole?

9. Two roads intersect at an angle of 15°. Darryl is standing on one of the roads 270 m from the intersection.

a) Create a question that requires using the sine law to solve it. Include a complete solution and a sketch.

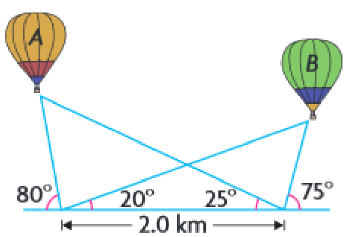
b) Create a question that requires using the cosine law to solve it. Include a complete solution and a sketch.

10. The Leaning Tower of Pisa is 55.9 m tall and leans 5.5° from the vertical. If its shadow is 90.0 m long, what is the distance from the top of the tower to the top edge of its shadow? Assume that the ground around the tower is level. Round your answer to the nearest metre.

14. Two hot-air balloons are moored to level ground below, each at a different location. An observer at each location determines the angle of elevation to the opposite balloon as shown at the right. The observers are 2.0 km apart.

a) What is the distance separating the balloons, to the nearest tenth of a kilometre?

b) Determine the difference in height (above the ground) between the two balloons. Round your answer to the nearest metre.



Three-Dimensional Problems

1. Morana is trolling for salmon in Lake Ontario. She sets the fishing rod so that its tip is 1 m above water and the line forms an angle of 35° with the water's surface. She knows that there are fish at a depth of 45 m. Describe the steps you would use to calculate the length of line she must let out.

2. Josh is building a garden shed that is 4.0 m wide. The two sides of the roof are equal in length and must meet at an angle of 80°. There will be a 0.5 m overhang on each side of the shed. Josh wants to determine the length of each side of the roof.

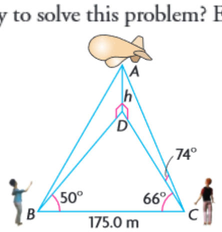
a) Should he use the sine law or the cosine law? Explain.

b) How could Josh use the primary trigonometric ratios to calculate x? Explain.

4. As a project, a group of students was asked to determine the altitude, h, of a promotional blimp. The students' measurements are shown in the sketch at the left.

a) Determine h to the nearest tenth of a metre. Explain each of your steps.

b) Is there another way to solve this problem? Explain.

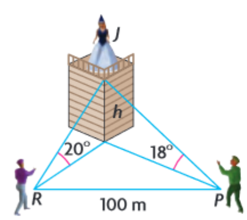


6. The observation deck of the Skylon Tower in Niagara Falls, Ontario, is 166 m above the Niagara River. A tourist in the observation deck notices two boats on the water. From the tourist's position,

- the bearing of boat A is 180° at an angle of depression of 40°
- the bearing of boat B is 250° at an angle of depression of 34°

Calculate the distance between the two boats to the nearest metre.

7. Suppose Romeo is serenading Juliet while she is on her balcony. Romeo is facing north and sees the balcony at an angle of elevation of 20°. Paris, Juliet's other suitor, is observing the situation and is facing west. Paris sees the balcony at an angle of elevation of 18°. Romeo and Paris are 100 m apart as shown. Determine the height of Juliet's balcony above the ground, to the nearest metre.



15. An airport radar operator locates two planes flying toward the airport. The first plane, P, is 120 km from the airport, A, at a bearing of 70° and with an altitude of 2.7 km. The other plane, Q, is 180 km away on a bearing of 125° and with an altitude of 1.8 km. Calculate the distance between the two planes to the nearest tenth of a kilometre.

Proofs of the Pythagorean Theorem, the Law of Cosines and the Law of Sines

Introduction

"A demonstration is an argument that will convince a reasonable man. A proof is an argument that can convince even an unreasonable man." (Mark Kac, 20th century Polish-American mathematician)

Until now, you have simply ***accepted*** the "truth" of much of what you have been taught in mathematics. But how do you know that the claims made by your math teachers really are true? To be sure that you are not being duped, you should always seek ***proof***, or at the very least, a very convincing demonstration.

Nolfi's Intuitive Definition of "Proof"

A proof is a ***series*** or ***"chain" of inferences*** (i.e. "if…then" statements, formally known as ***logical implications*** or ***conditional statements***) that allows us to make ***logical deductions*** that lead from a ***premise***, which is known or assumed to be true, to a desired ***conclusion***.

Hopefully, my definition is somewhat easier to understand than that of a former Prime Minister:

Jean Chrétien, a former Prime Minister of Canada, was quoted by CBC News as saying, "A proof is a proof. What kind of a proof? It's a proof. A proof is a proof. And when you have a good proof, it's because it's proven."

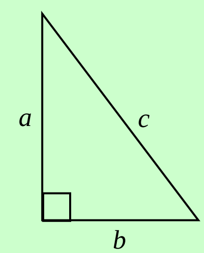
Pythagorean Theorem Example

The ***premise***

The ***conclusion***

In ***any*** right triangle,
the square of the hypotenuse is equal to the sum of the squares of the other two sides.

That is, $c^2 = a^2 + b^2$



Proof:

Begin with ***right*** ΔABC and construct the altitude AD.

Since $\Delta ABC \sim \Delta DBA$ (AA similarity theorem),

$$\therefore \frac{AB}{BC} = \frac{BD}{AB}.$$

Since $\Delta ABC \sim \Delta DAC$ (AA similarity theorem),

$$\therefore \frac{AC}{BC} = \frac{DC}{AC}.$$

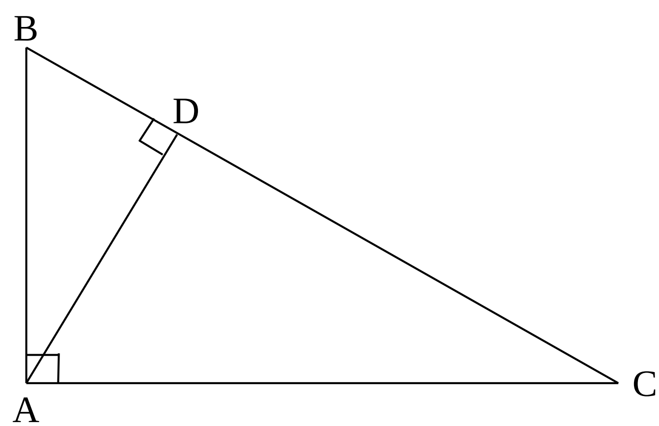
Therefore,

$$(AB)(AB) = (BD)(BC) \text{ and } (AC)(AC) = (DC)(BC)$$

Summing up, we obtain

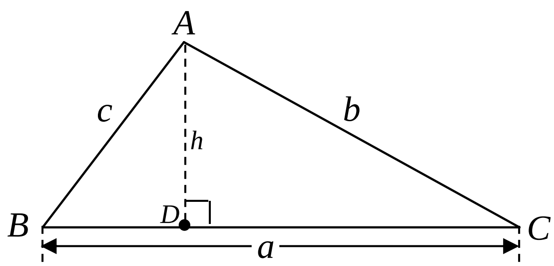
$$\begin{aligned}(AB)(AB) + (AC)(AC) &= (BD)(BC) + (DC)(BC)\\ &= BC(BD + DC)\\ &= (BC)(BC)\\ &= BC^2\end{aligned}$$

$\therefore AB^2 + AC^2 = BC^2$ //



See http://www.cut-the-knot.org/pythagoras/index.shtml for a multitude of proofs of the Pythagorean Theorem.

Now that we have proved the Pythagorean Theorem we can use it to derive more relationships.



For the following proofs, refer to the diagram at the left.

Proof of the Law of Cosines	*Proof of the Law of Sines*
By the Pythagorean Theorem $c^2 = h^2 + BD^2$ and $b^2 = h^2 + DC^2$. Since $a = BD + DC$, $a^2 = (BD + DC)^2 = BD^2 + 2(BD)(DC) + DC^2$. Therefore, $c^2 - a^2 - b^2 = h^2 + BD^2 - (BD^2 + 2(BD)(DC) + DC^2) - (h^2 + DC^2)$ $= h^2 + BD^2 - BD^2 - 2(BD)(DC) - DC^2 - h^2 - DC^2$ $= -2(BD)(DC) - 2DC^2$ $= -2DC(BD + DC)$ $= -2(DC)(a)$ But $\cos C = \dfrac{DC}{b}$, which means that $DC = b\cos C$. Hence, $c^2 - a^2 - b^2 = -2(DC)(a) = -2(b\cos C)(a) = -2ab\cos C$. $\therefore c^2 = a^2 + b^2 - 2ab\cos C$ //	$\dfrac{h}{c} = \sin B$ and $\dfrac{h}{b} = \sin C$ $\therefore h = c\sin B$ and $h = b\sin C$ $\therefore c\sin B = b\sin C$ $\therefore \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ Similarly, it can be shown that $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ Therefore, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ //

Important Questions – Test your Understanding

1. Would it be correct to use the law of cosines to prove the Pythagorean Theorem?

2. Complete the proof of the law of sines. (The word "similarly" was used to indicate that the remaining part of the proof would proceed in the same manner.)

3. In the given proof of the Pythagorean Theorem, it is stated that $\Delta ABC \sim \Delta BDA$ and that $\Delta ABC \sim \Delta ADC$. Explain why each of these statements must be true.