UNIT 4 – POLYNOMIAL AND RATIONAL FUNCTIONS

INTERVAL NOTATION

Introduction

You have already made use of ***set-builder notation*** to describe the domain and range of functions. While set-builder notation is extremely precise, it can also be rather cumbersome, especially in situations in which it is necessary to use it frequently. To remedy this problem, ***interval notation*** was created.

Interval Notation Symbols and their Meanings

Interval Definition

A ***real interval*** is a set of real numbers that has the property that any number that lies *between two numbers in the set* is also included in the set. ***Informally***, we can think of an interval as any ***continuous*** range of real numbers, that is, a set of real numbers that does not contain any "gaps." ***Formally***, a real interval can be defined as follows:

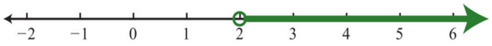
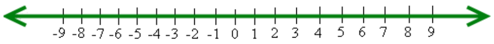
Let $a\in\mathbb{R}$ and $b\in\mathbb{R}$ such that $a\le b$. In addition, let *I* represent any subset of $\mathbb{R}$ that contains both *a* and *b*. Then *I* is called a ***real interval*** if $x\in I$ for every $x\in\mathbb{R}$ satisfying $a\le x\le b$.

Symbol	***Name***	***Meaning***
(	Parenthesis	The ***left*** endpoint ***IS NOT*** included in the interval. (***left-open***)
)	Parenthesis	The ***right*** endpoint ***IS NOT*** included in the interval. (***right-open***)
[	Bracket	The ***left*** endpoint ***IS*** included in the interval. (***left-closed***)
]	Bracket	The ***right*** endpoint ***IS*** included in the interval. (***right-closed***)
∞	Infinity	The interval does not have an ***upper bound***. (***right-unbounded***).
$-\infty$	Negative Infinity	The interval does not have a ***lower bound***. (***left-unbounded***).

The plural of the word *parenthesis* is *parentheses*.

Note that *parenthesis* is the formally correct term that refers to what we informally call a "round bracket."

Examples

Words	***Set-Builder Notation***	***Interval Notation***	***Classification of Interval***	***Graphical Representation***
"*x* is greater than or equal to 2 and less than or equal to 6"	$\{x\in\mathbb{R}:2\le x\le 6\}$	$[2,6]$	Closed	
"*x* is greater than 2 and less than 6"	$\{x\in\mathbb{R}:2< x< 6\}$	$(2,6)$	Open	
"*x* is greater than or equal to 2 and less than 6"	$\{x\in\mathbb{R}:2\le x< 6\}$	$[2,6)$	Left-closed, Right-open	
"*x* is greater than 2 and less than or equal to 6"	$\{x\in\mathbb{R}:2< x\le 6\}$	$(2,6]$	Left-open, Right-closed	
"*x* is greater than or equal to 2"	$\{x\in\mathbb{R}:x\ge 2\}$	$[2,\infty)$	Left-closed, Right-unbounded	
"*x* is greater than 2"	$\{x\in\mathbb{R}:x> 2\}$	$(2,\infty)$	Left-open, Right-unbounded	
"*x* is less than or equal to 6"	$\{x\in\mathbb{R}:x\le 6\}$	$(-\infty,6]$	Left-unbounded, Right-closed	
"*x* is less than 6"	$\{x\in\mathbb{R}:x< 6\}$	$(-\infty,6)$	Left-unbounded, Right-open	
"*x* is any real number"	$\{x\in\mathbb{R}\}=\mathbb{R}$	$(-\infty,\infty)$	Unbounded	

Homework

1. Complete the following table.

Words	***Set-Builder Notation***	***Interval Notation***	***Classification of Interval***	***Graphical Representation***
"x is greater than or equal to –5 and less than or equal to 1"				
"x is greater than –5 and less than 7"				
"x is greater than or equal to –2 and less than –1"				
"x is greater than –2 and less than or equal to 6"				
"x is greater than or equal to –8"				
"x is greater than –3"				
"x is less than or equal to 100"				
"x is negative"				
"x is positive"				
"x is nonnegative"				
"x is non-positive"				

2. Would the set of even numbers be considered a real interval? Explain.

3. Explain why it is appropriate to call the interval $[2,6)$ ***half-open*** or ***half-closed***?

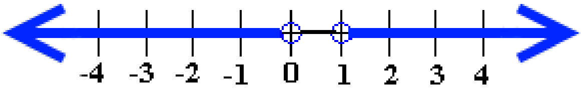
4. The ***empty set*** or ***null set***, denoted $\{\ \}$ or $\varnothing$, is simply the ***unique*** set that contains no elements. (At first glance, the concept of the empty set might seem puzzlingly purposeless. Likening the empty set to an empty container usually helps to relieve this feeling of bewilderment.)

Which of the following intervals is equivalent to the empty set?

Interval	***Equivalent to the Empty Set?***	***If equivalent, explain why. If not, state the set to which the interval is equivalent.***
$(3,3)$		
$(3,3]$		
$[3,3]$		
$\left[1,\dfrac{1001}{1000}\right)$		

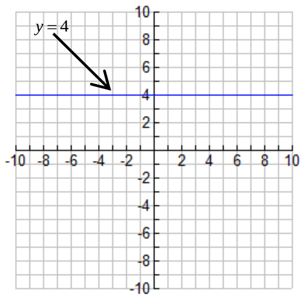
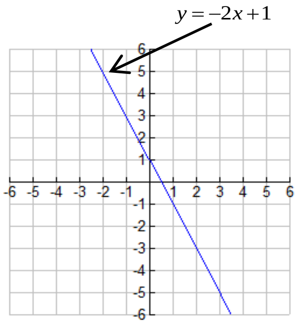
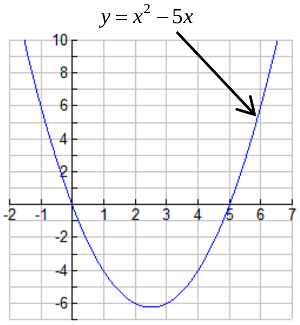
5. Does it make sense to write intervals such as $[-\infty,3]$ or $(3,\infty]$? Explain.

6. The symbols $\bigcup$ and $\bigcap$ are set operations that are called the ***union*** and ***intersection*** operators respectively. If *A* and *B* are sets, then $A\bigcup B=\{x : x\in A \text{ or } x\in B\}$ and $A\bigcap B=\{x : x\in A \text{ and } x\in B\}$. Informally, $A\bigcup B$ (**the union of *A* and *B***) is the set that is formed by combining the elements of *A* and *B* while $A\bigcap B$ (**the intersection of *A* and *B***) is the set that is formed by including only those elements that are common to both *A* and *B*. Use the above information to complete the following table. The first row is done for you.

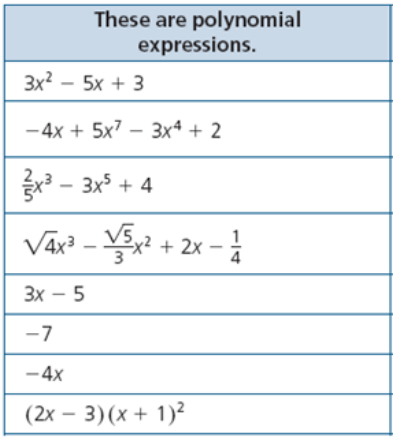
Subset of $\mathbb{R}$ Described using Set-Builder Notation	***Subset of $\mathbb{R}$ Described using Interval Notation***	***Graphical Representation***
$\{x\in\mathbb{R} : x<0 \text{ or } x>1\}$	$(-\infty,0)\bigcup(1,\infty)$	
$\{x\in\mathbb{R} : x\le 0 \text{ or } x\ge 1\}$		
$\{x\in\mathbb{R} : x<-5 \text{ or } 5<x\le 10\}$		
$\{x\in\mathbb{R} : -3\le x<2 \text{ or } x>5\}$		
$\{x\in\mathbb{R} : -3\le x<20 \text{ and } 15\le x<35\}$		

INTRODUCTION TO POLYNOMIAL FUNCTIONS

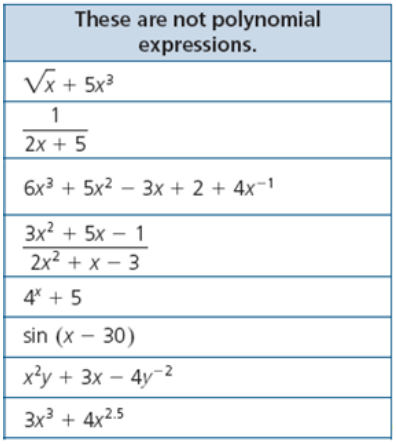
Polynomials that you Already Know and Love

Class of Polynomial Function	***General Equation***	***Example Graph***	***Features***
Constant Polynomial Functions	$f(x)=C,\ \ C\in\mathbb{R}$		• slope = rate of change of *y* with respect to *x* = 0
Linear Polynomial Functions	$f(x)=mx+b$		• *m* = slope = steepness of line = vertical stretch factor = rate of change of *y* with respect to *x* • *b* = *y*-intercept • $-\dfrac{b}{m}$ = *x*-intercept
Quadratic Polynomial Functions	$f(x)=ax^2+bx+c$ (Standard Form) $f(x)=a(x-h)^2+k$ (Vertex Form)		• *a* = vertical stretch factor • $a>0\rightarrow$parabola opens upward • $a<0\rightarrow$parabola opens downward • $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ = *x*-intercept(s) = zeros = roots If $b^2-4ac>0$, there are ***two*** *x*-intercepts If $b^2-4ac=0$, there is ***one*** *x*-intercept If $b^2-4ac<0$, there are ***no*** *x*-intercepts • *c* = *y*-intercept • co-ordinates of vertex: $\left(-\dfrac{b}{2a},\dfrac{4ac-b^2}{4a}\right)$

Examples



These are polynomial expressions.
$3x^2-5x+3$
$-4x+5x^7-3x^4+2$
$\frac{2}{5}x^3-3x^5+4$
$\sqrt{4}x^3-\frac{\sqrt{5}}{3}x^2+2x-\frac{1}{4}$
$3x-5$
-7
$-4x$
$(2x-3)(x+1)^2$



These are not polynomial expressions.
$\sqrt{x}+5x^3$
$\frac{1}{2x+5}$
$6x^3+5x^2-3x+2+4x^{-1}$
$\frac{3x^2+5x-1}{2x^2+x-3}$
4^x+5
$\sin(x-30)$
$x^2y+3x-4y^{-2}$
$3x^3+4x^{2.5}$

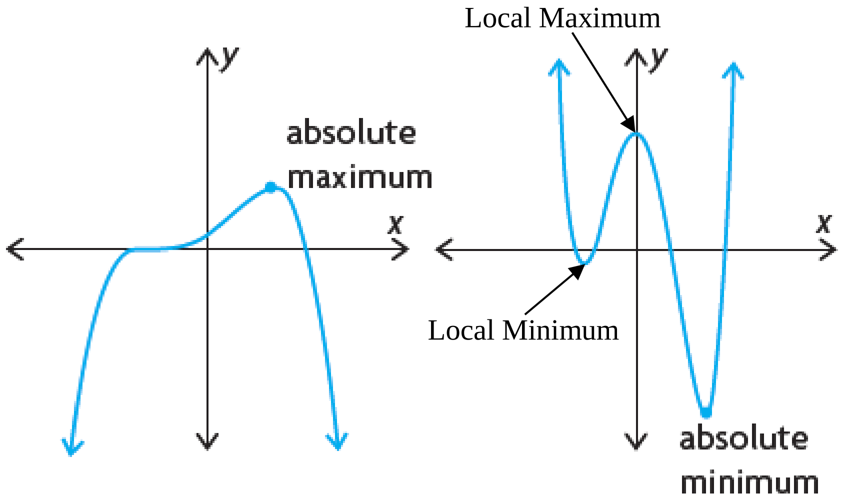
General Form of a Polynomial Function

- Let n be any whole number and $a_0,\ a_1,\ a_2, \ldots a_{n-1},\ a_n$ be real numbers such that $a_n \neq 0$. Then, the function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is called ***a polynomial of degree n***.
- The ***degree n*** of the polynomial is equal to the ***exponent*** of the highest power x^n.
- The numbers $a_0,\ a_1,\ a_2, \ldots, a_{n-1},\ a_n$ are called the ***numerical coefficients*** or the ***coefficients*** of the polynomial.
- The coefficient a_n of the highest power x^n is called the ***leading coefficient*** of the polynomial.

Extreme (Turning) Points

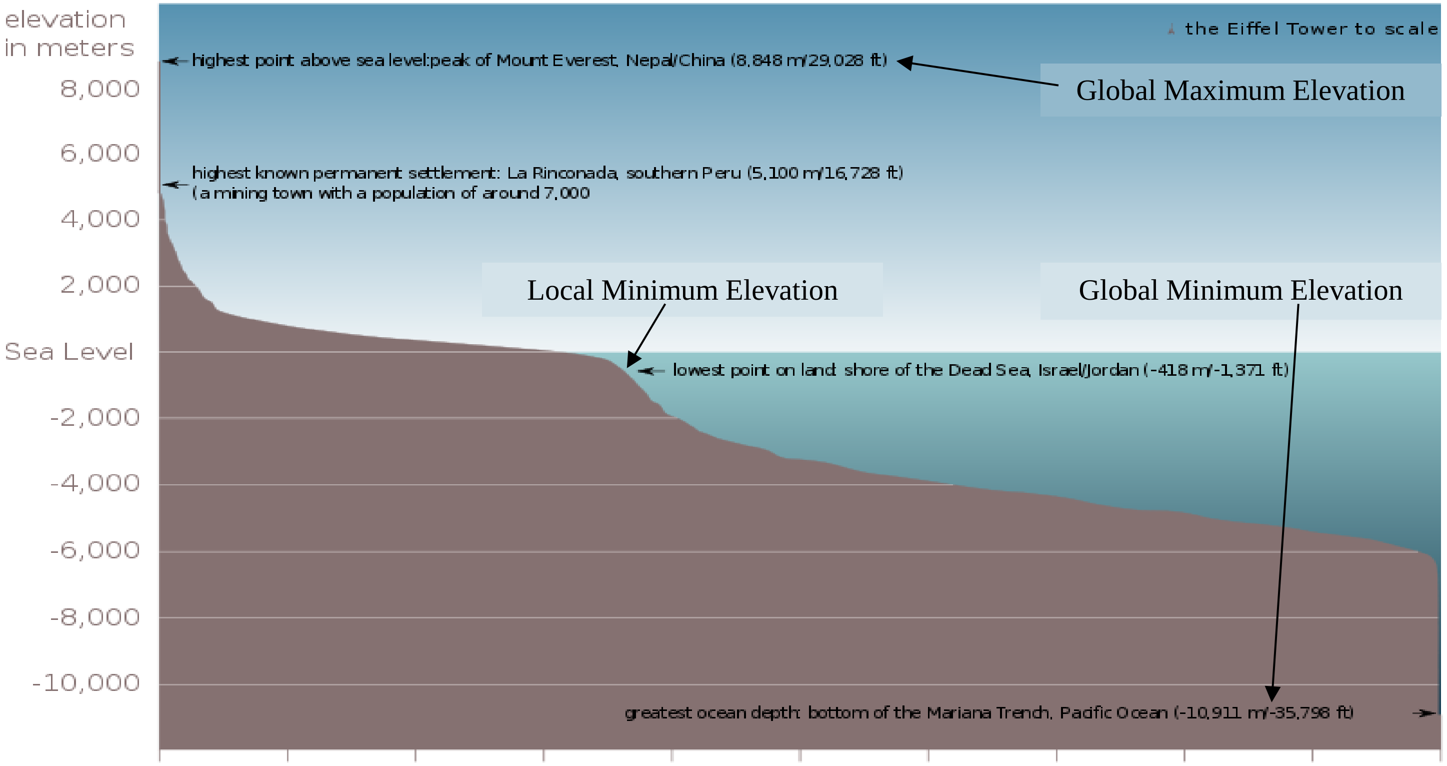
In general, an ***extreme point*** or a ***turning point*** of a function is any point at which the function ***changes direction***.

- If a point on a graph has a y-co-ordinate that is greater than or equal to the y-co-ordinate of any other point on the graph, it is called an ***absolute*** or ***global maximum point***.
- If a point on a graph has a y-co-ordinate that is less than or equal to the y-co-ordinate of any other point on the graph, it is called an ***absolute*** or ***global minimum point***.
- Collectively, maximum and minimum points are called ***extreme points***.



- By contrast, ***local extreme*** points are maximum or minimum points in a restricted region of a function.
- The global maximum and minimum points of the Earth's crust are, respectively, the peak of Mount Everest (approximately 8848 m above sea level) and the deepest part of the Mariana Trench in the Pacific Ocean (approximately 11033 m below sea level).

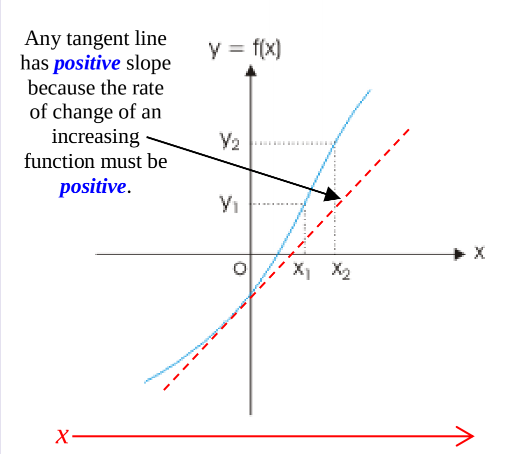
Elevation Histogram of the Earth's Crust



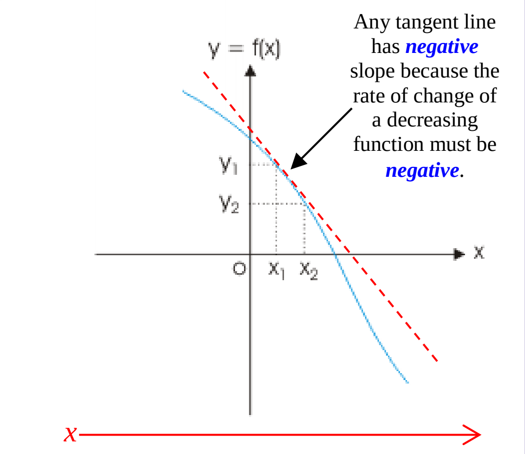
(each tick-mark represents 10% of the surface of the Earth)

Increasing and Decreasing Functions

- The function *f* is said to be ***(strictly) increasing*** if for all choices of x_1 and x_2 such that $x_1 < x_2$, $f\left(x_1\right) < f\left(x_2\right)$.
  - ➢ That is, *f* is said to be ***(strictly) increasing*** if $y = f\left(x\right)$ increases as *x* increases.
  - ➢ If you imagine walking along the graph of an increasing function, you would always be walking ***uphill*** as you move from left to right.
- The function *f* is said to be ***(strictly) decreasing*** if for all choices of x_1 and x_2 such that $x_1 < x_2$, $f\left(x_1\right) > f\left(x_2\right)$.
  - ➢ That is, *f* is said to be ***(strictly) decreasing*** if $y = f\left(x\right)$ decreases as *x* increases.
  - ➢ If you imagine walking along the graph of a decreasing function, you would always be walking ***downhill*** as you move from left to right.


Any tangent line has ***positive*** slope because the rate of change of an increasing function must be ***positive***.

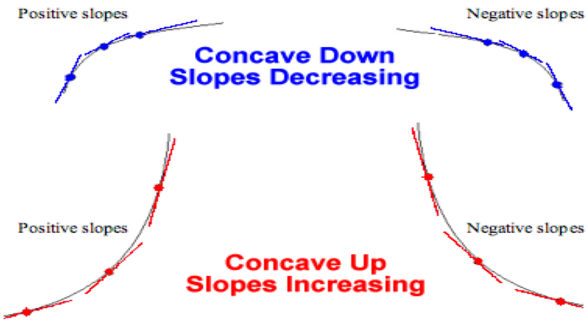
A Strictly Increasing Function
As *x* increases, so does $y = f\left(x\right)$.

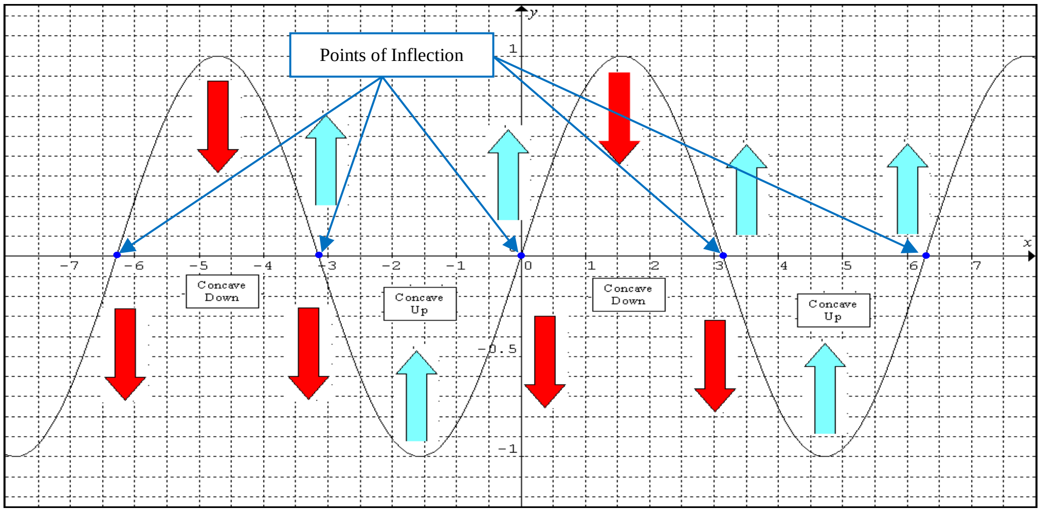

Any tangent line has ***negative*** slope because the rate of change of a decreasing function must be ***negative***.

A Strictly Decreasing Function
As *x* increases, $y = f\left(x\right)$ decreases.

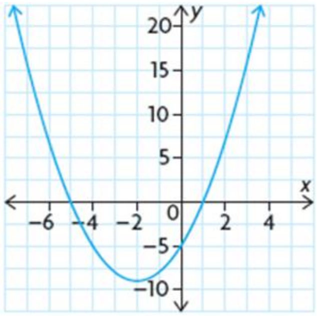
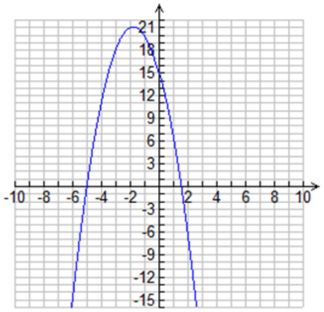
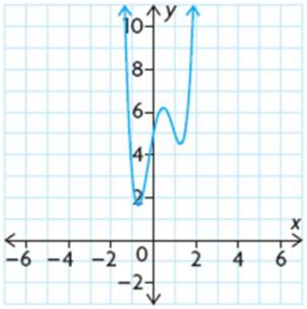
Concavity and Points of Inflection (Optional Topic)

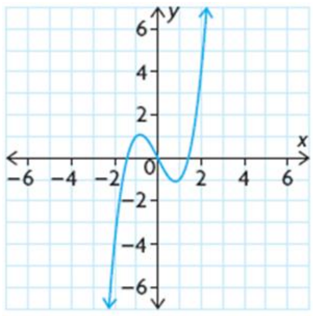
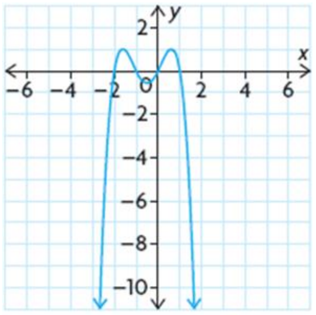
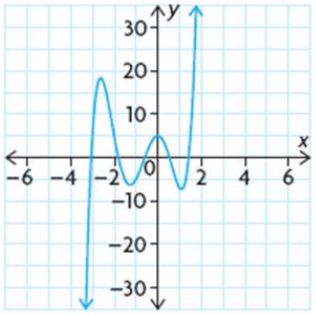
- ⌒ *Concave Down* shape (“frowny”)
  $\rightarrow$ slope decreases
  $\rightarrow$ rate of change decreases
- ◡ *Concave Up* shape (“smiley”)
  $\rightarrow$ slope increases
  $\rightarrow$ rate of change increases
- A ***point of inflection*** is a point at which concavity changes.

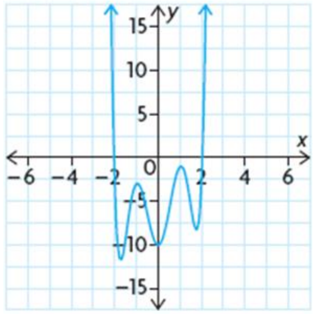
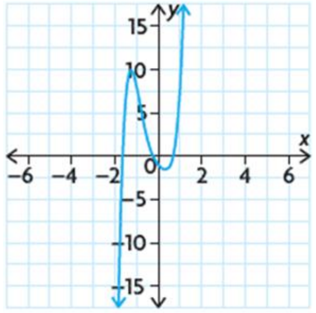
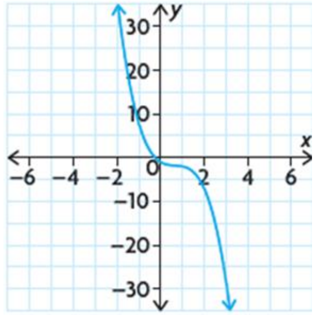


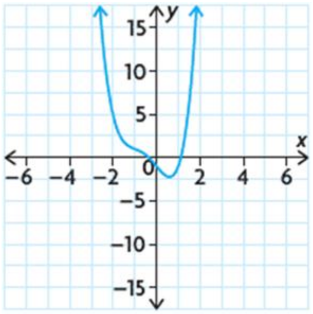


BEHAVIOUR OF POLYNOMIAL FUNCTIONS

Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviours		# of Turning Points	# of Points of Inflection	Intervals of Decrease	Intervals of Increase	Intervals where f is Concave Down	Intervals where f is Concave Up
				$x\to-\infty$	$x\to+\infty$						
 $f(x) = x^2 + 4x - 5$	2	even	+1	$y\to+\infty$	$y\to+\infty$	1	0	$(-\infty,-2)$	$(-2,\infty)$	none	$(-\infty,\infty)$
 $f\left(x\right)=-2x^2-7x+15$	2	even	–2	$y\to-\infty$	$y\to-\infty$	1	0	$\left(-\frac{7}{4},\infty\right)$	$\left(-\infty,-\frac{7}{4}\right)$	$(-\infty,\infty)$	none
 $f(x) = 3x^4 - 4x^3 - 4x^2 + 5x + 5$	4	even	+3	$y\to+\infty$	$y\to+\infty$	3	2	approximate values given here $(-\infty,-0.9)$ $(0.4,1.2)$	approximate values given here $(-0.9,0.4)$ $(1.2,\infty)$	approximate values given here $(0,0.9)$	approximate values given here $(-\infty,0)$ $(0.9,\infty)$

Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviours		# of Turning Points	# of Points of Inflection	Intervals of Decrease	Intervals of Increase	Intervals where *f* is Concave Down	Intervals where *f* is Concave Up
				$x \to -\infty$	$x \to +\infty$						
 $f(x) = x^3 - 2x$											
 $f(x) = -x^4 - 2x^3 + x^2 + 2x$											
 $f(x) = 2x^5 + 7x^4 - 3x^3 - 18x^2 + 5$											

Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviours		# of Turning Points	# of Points of Inflection	Intervals of Decrease	Intervals of Increase	Intervals where *f* is Concave Down	Intervals where *f* is Concave Up
				$x \to -\infty$	$x \to +\infty$						
 $f(x) = 2x^6 - 12x^4 + 18x^2 + x - 10$											
 $f(x) = 5x^5 + 5x^4 - 2x^3 + 4x^2 - 3x$											
 $f(x) = -2x^3 + 4x^2 - 3x - 1$											

Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviours		# of Turning Points	# of Points of Inflection	Intervals of Decrease	Intervals of Increase	Intervals where f is Concave Down	Intervals where f is Concave Up
				$x \to -\infty$	$x \to +\infty$						
 $f(x) = x^4 + 2x^3 - 3x - 1$											

In Summary

Key Idea

- A polynomial in one variable is an expression of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number. The expression contains only one variable, with the powers arranged in descending order. For example, $2x + 5$, $3x^2 + 2x - 1$, and $5x^4 + 3x^3 - 6x^2 + 5x - 8$.

Need to Know

- In any polynomial expression, the exponents on the variable must be whole numbers.
- A polynomial function is any function that contains a polynomial expression in one variable. The degree of the function is the highest exponent in the expression. For example, $f(x) = 6x^3 - 3x^2 + 4x - 9$ has a degree of 3.
- The nth finite differences of a polynomial function of degree n are constant.
- The domain of a polynomial function is the set of real numbers, $\{x \in \mathbf{R}\}$.
- The range of a polynomial function may be all real numbers, or it may have a lower bound or an upper bound (but not both).
- The graphs of polynomial functions do not have horizontal or vertical asymptotes.
- The graphs of polynomial functions of degree zero are horizontal lines. The shape of other graphs depends on the degree of the function. Five typical shapes are shown for various degrees:

Linear ($n = 1$) Quadratic ($n = 2$) Cubic ($n = 3$) Quartic ($n = 4$) Quintic ($n = 5$)

Determining the Behaviour of Polynomial Functions

In Summary

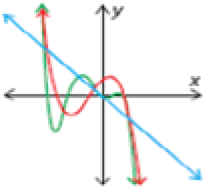
Key Ideas

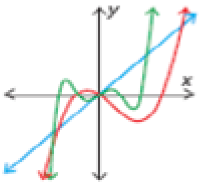
- Polynomial functions of the same degree have similar characteristics.
- The degree and the leading coefficient in the equation of a polynomial function indicate the end behaviours of the graph.
- The degree of a polynomial function provides information about the shape, turning points, and zeros of the graph.

Need to Know

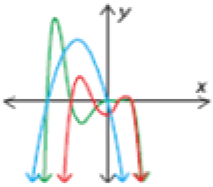
End Behaviours

- An odd-degree polynomial function has opposite end behaviours.
  - If the leading coefficient is negative, then the function extends from the second quadrant to the fourth quadrant; that is, as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$.

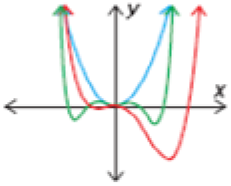




  - If the leading coefficient is positive, then the function extends from the third quadrant to the first quadrant; that is, as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$.



- An even-degree polynomial function has the same end behaviours.
  - If the leading coefficient is negative, then the function extends from the third quadrant to the fourth quadrant; that is, as $x \to \pm\infty$, $y \to -\infty$.



  - If the leading coefficient is positive, then the function extends from the second quadrant to the first quadrant; that is, as $x \to \pm\infty$, $y \to \infty$.

Turning Points

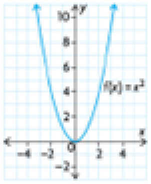
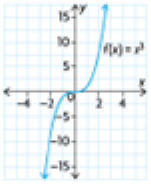
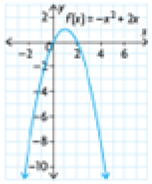
- A polynomial function of degree n has at most $n - 1$ turning points.
- An even degree polynomial has an odd number of turning points. An odd degree polynomial has an even number of turning points.

Number of Zeros

- A polynomial function of degree n may have up to n distinct zeros.
- A polynomial function of odd degree must have at least one zero.
- A polynomial function of even degree may have no zeros.

Symmetry

- Some polynomial functions are symmetrical in the y-axis. These are even functions, where $f(-x) = f(x)$.
- Some polynomial functions have rotational symmetry about the origin. These are odd functions, where $f(-x) = -f(x)$.
- Most polynomial functions have no symmetrical properties. These are functions that are neither even nor odd, with no relationship between $f(-x)$ and $f(x)$.

Even Functions	Odd Functions	Neither Even nor Odd Functions
(symmetry in the y-axis)	(rotational symmetry around the origin)	(neither of these symmetries)
		

Number of Points of Inflection

- A polynomial function of degree n has at most $n - 2$ points of inflection.
- A polynomial of odd degree ≥ 3 must have at least one point of inflection.
- The number of points of inflection of a polynomial function can exceed the number of turning points. (e.g. $f(x) = x^3$ has no turning points and one point of inflection)
- If the degree of the polynomial is odd, it must have an odd number of points of inflection.
- If the degree of the polynomial is even, it must have an even number of points of inflection.

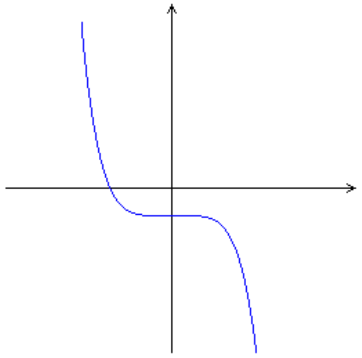
Example

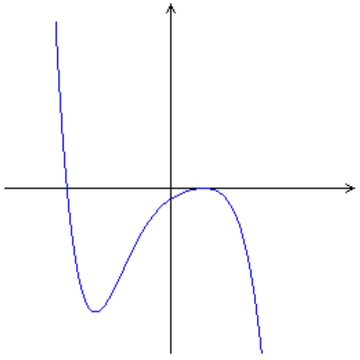
Equation $f(x) = -3x^5 + 4x^3 - 8x^2 + 7x - 5$		General Comments	End Behaviours		# of Zeros Possible	# of Turning Points Possible	# of Points of Inflection Possible	Absolute Max, Min or Neither?
			$x \to -\infty$	$x \to +\infty$				
Degree	5	Since the degree of this polynomial is odd, it has ***opposite*** end behaviours. The leading coefficient is negative, which means that the graph must extend from quadrant II to quadrant IV.	$y \to +\infty$ When x is a very large ***negative*** number such as -1000, $-3x^5$ has an extremely large ***positive*** value and has a greater effect on the value of the function than the other terms. Therefore, as $x \to -\infty$, $y \to +\infty$.	$y \to -\infty$ When x is a very large ***positive*** number such as 1000, $-3x^5$ has an extremely large ***negative*** value and has a greater effect on the value of the function than the other terms. Therefore, as $x \to +\infty$, $y \to -\infty$.	The degree is odd, which means that the polynomial function must have at least one zero. Since the degree is 5, there can be no more than 5 zeros.	The degree of *f* is odd, which means that it has opposite end behaviours. Hence, it must have an ***even*** number of turning points. As a result, *f* can have 0, 2 or 4 turning points.	For a polynomial function, the number of points of inflection is at most two less than the degree of the polynomial. In addition, the number of points of inflection is odd if the degree is odd and even if the degree is even. Hence, there are either 1 or 3 points of inflection.	Since the degree of *f* is odd, it cannot have any absolute extreme points.
Even or Odd Degree?	Odd							
Leading Coefficient	−3							

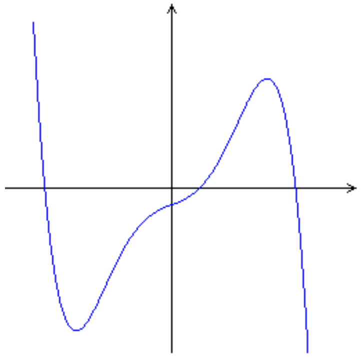
Summary

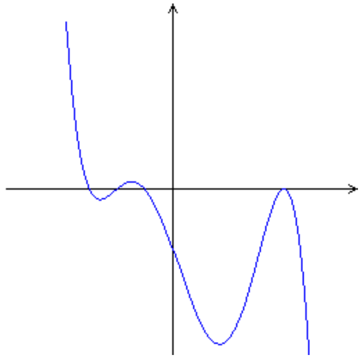
- *f* has opposite end behaviours: As $x \to -\infty$, $f(x) \to \infty$ ($\lim_{x \to -\infty} f(x) = \infty$). As $x \to \infty$, $f(x) \to -\infty$ ($\lim_{x \to \infty} f(x) = -\infty$).
- Since $f(x) = -3x^5 + 4x^3 - 8x^2 + 7x - 5$, then $f(0) = -5$, which means that the *y*-intercept must be –5.
- *f* has 1 to 5 zeroes
- *f* has 0, 2 or 4 turning points but cannot have any absolute (global) turning points
- *f* has 1 or 3 points of inflection (the number of points of inflection must be odd if the degree is odd and even if the degree is even)

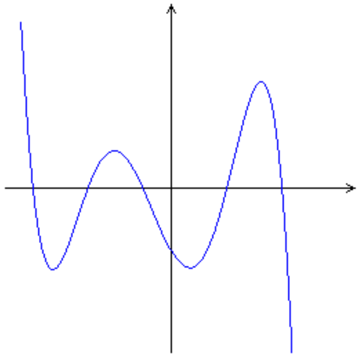
Possible graphs of $f(x)$


One Zero
Zero Turning Points
One Point of Inflection


Two Zeros
Two Turning Points
One Point of Inflection

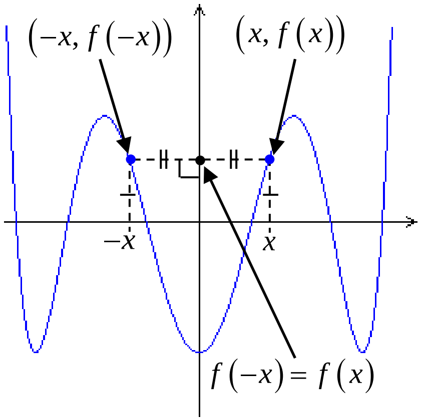
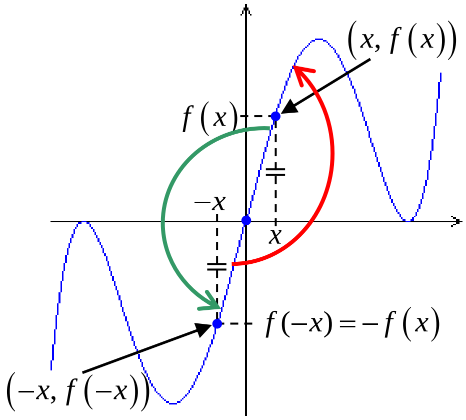

Three Zeros
Two Turning Points
Three Points of Inflection

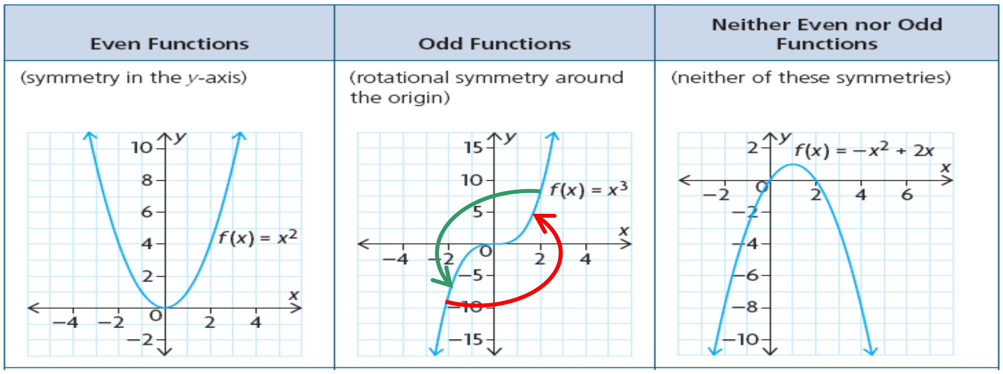

Four Zeros
Four Turning Points
Three Points of Inflection


Five Zeros
Four Turning Points
Three Points of Inflection

Symmetry – Even and Odd Functions

Certain functions can be classified according to symmetry that they exhibit. Two important categories of symmetries are shown in the diagrams below.

Even Functions	***Odd Functions***
 $f(x)=x^2(x-2)^2(x+2.5)^2-5$ ***An EVEN Polynomial Function*** For all ***even*** functions *f*, $f(-x)=f(x)$ and *f* is ***symmetric in the y-axis***.	 $f(x)=x(x-2)^2(x+2.5)^2$ ***An ODD Polynomial Function*** For all ***odd*** functions *f*, $f(-x)=-f(x)$ and *f* has ***rotational symmetry about the origin***.



Important Questions

1. Complete the following table. (Don't forget to investigate the graph of each function!)

Function	***Odd or Even?***	***Explanation***	***Function***	***Odd or Even?***	***Explanation***
$f(x)=2^x$	Neither	$f(-x)=2^{-x}=\frac{1}{2^x}\neq f(x)$ $f(-x)=\frac{1}{2^x}\neq -2^x=-f(x)$	$f(x)=\tan x$		
$f(x)=3x^{100}-10$			$f(x)=\csc x$		
$f(x)=\sin x$			$f(x)=\sec x$		
$f(x)=\cos x$			$f(x)=\cot x$		

2. Is it possible for a function to have symmetry in the *x*-axis? Explain.

3. Besides symmetry in the *y*-axis and rotational symmetry about the origin, are there any other symmetries that a function can have?

4. Complete the following tables.

Equation $g(x)=2x^4+x^2+2$		General Comments (Including any Symmetry)	End Behaviours		# of Zeros Possible	# of Turning Points Possible	# of Points of Inflection Possible	Absolute Max, Min or Neither?
			$x\to-\infty$	$x\to+\infty$				
Degree								
Even or Odd Degree?								
Leading Coefficient								

Possible graphs of $g(x)$

THE ADVANTAGES OF WRITING POLYNOMIAL EXPRESSIONS IN FACTORED FORM

Factoring – The "F" Word of Math

To the disappointment of many students, a great deal of time is spent developing factoring skills in high school mathematics. While factoring in and of itself is often tedious and sometimes may even appear to be pointless, its importance in understanding polynomial functions cannot be underestimated. For instance, consider the polynomial function $f(x)=x^4-4x^3-7x^2+22x+24$. The table given below illustrates how much more convenient and informative the factored form of the equation can be.

Information that can be obtained easily when the Polynomial Equation is written in Standard Form	***Information that can be obtained easily when the Polynomial Equation is written in Factored Form***
$f(x)=x^4-4x^3-7x^2+22x+24$ From this form of the equation, we can only determine the following. • The highest power is x^4. Therefore, the end behaviours are the same (as $x\to\pm\infty$, $y\to\infty$). • The *y*-intercept is $f(0)=24$.	$f(x)=(x+1)(x+2)(x-3)(x-4)$ From this form of the equation, we can easily determine much more. • The highest power is $x(x)(x)(x)=x^4$. Therefore, the end behaviours are the same (as $x\to\pm\infty$, $y\to\infty$). • The *y*-intercept is $f(0)=1(2)(-3)(-4)=24$. • The zeros of the function are –1, –2, 3, 4 • There must be 3 turning points

Example 1

Sketch a possible graph of $f(x)=-(x+2)(x-1)(x-3)^2$

Solution

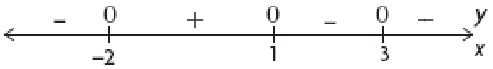
$$
\begin{aligned}
f(0) &= -(0+2)(0-1)(0-3)^2\\
&= -(2)(-1)(-3)^2\\
&= 18
\end{aligned}
$$

Calculate the *y*-intercept.

$$0=-(x+2)(x-1)(x-3)^2$$
$$x=-2,\ x=1,\text{ or } x=3$$

Determine the *x*-intercepts by letting $f(x)=0$. Use the factors to solve the resulting equation for *x*.

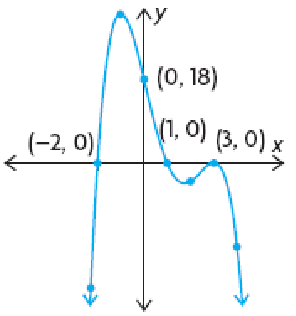
Use values of *x* that fall between the *x*-intercepts as test values to determine the location of the function above or below the *x*-axis.



$f(-3)=-144$ $f(-1)=32$ $f(2)=-4$ $f(4)=-18$

Since the function lies below the *x*-axis on both sides of $x=3$, the graph must just touch the *x*-axis and not cross over at this point. The order of 2 on the factor $(x-3)^2$ confirms the parabolic shape near $x=3$.

Determine the end behaviours of the function.



$f(x)=-(x+2)(x-1)(x-3)^2$

Because the degree is even and the leading coefficient is negative, the graph extends from third quadrant to the fourth quadrant; that is, as $x\to\pm\infty, y\to-\infty$.

This is a possible graph of $f(x)$ estimating the locations of the turning points.

Order (Multiplicity) of Zeros

Let *r* represent a zero of a polynomial function $f(x)$. The ***order*** or ***multiplicity*** of *r* is equal to the "number of times" that *r* appears as a root of the polynomial equation $f(x)=0$. This can be stated more precisely as follows:

Let $f(x)$ represent a polynomial function and let *r* represent one of its zeros. We say that the zero *r* has ***order*** or ***multiplicity*** *k* if *k* is the largest possible value such that $(x-r)^k$ ***is*** a factor of $f(x)$. (i.e. $(x-r)^k$ ***is*** a factor of $f(x)$ but $(x-r)^m$ ***is not*** a factor of $f(x)$ for $m>k$.)

Example

$$f(x)=x^2(x-5)^4(x-1)^3(x+4)^5(x+2)$$

Zero	Order (Multiplicity)
–4	5
–2	1
0	2
1	3
5	4

Example 2

Write the equation of a cubic function with x-intercepts –2, 3 and $\frac{2}{5}$, and y-intercept 6.

Solution

Let f represent the cubic polynomial function. Note that the x-intercepts of f are the same as the zeros of f. Therefore, the equation of f must take the form $f(x) = a(x+2)(x-3)(5x-2)$ where $a \in \mathbb{R}$. Then,

$$f(x) = a(x+2)(x-3)(5x-2)$$

Use the zeros of the function to create factors for the correct family of polynomials. Since this function has three zeros and it is cubic, the order of each factor must be 1.

$$6 = a(0+2)(0-3)(5(0)-2)$$
$$6 = a(2)(-3)(-2)$$
$$6 = 12a$$
$$a = \frac{1}{2}$$

Use the y-intercept to calculate the value of a.

Substitute $x = 0$ and $y = 6$ into the equation, and solve for a.

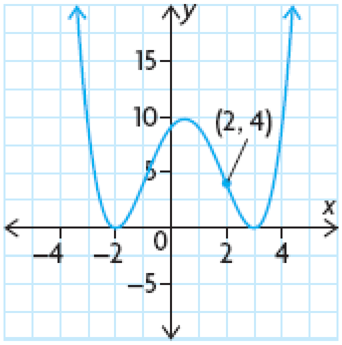
$$f(x) = \frac{1}{2}(x+2)(x-3)(5x-2)$$

Write the equation in factored form.

Therefore, the equation of f in factored form is $f(x) = \frac{1}{2}(x+2)(x-3)(5x-2)$.

Exercise 1

Write an equation of the graph shown at the right. In addition, state its domain and range.



Solution

Exercise 2

Sketch the graph of $f(x) = x^4 + 2x^3$.

Solution

In Summary

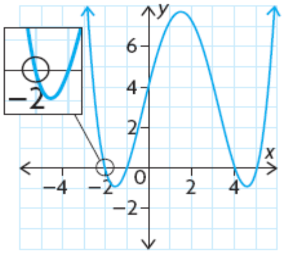
Key Idea

- The zeros of the polynomial function $y = f(x)$ are the same as the roots of the related polynomial equation, $f(x) = 0$.

Need to Know

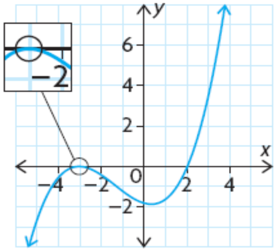
- To determine the equation of a polynomial function in factored form, follow these steps:
  - Substitute the zeros $(x_1, x_2, \ldots, x_n)$ into the general equation of the appropriate family of polynomial functions of the form $y = a(x - x_1)(x - x_2)\ldots(x - x_n)$.
  - Substitute the coordinates of an additional point for x and y, and solve for a to determine the equation.
- If any of the factors of a polynomial function are linear, then the corresponding x-intercept is a point where the curve passes through the x-axis. The graph has a linear shape near this x-intercept.

$y = \frac{1}{10}(x + 2)(x + 1)(x - 4)(x - 5)$



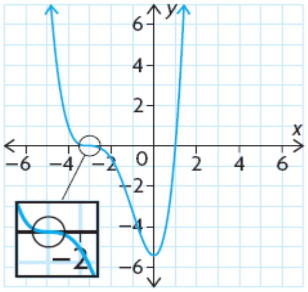
- If any of the factors of a polynomial function are squared, then the corresponding x-intercepts are turning points of the curve and the x-axis is tangent to the curve at these points. The graph has a parabolic shape near these x-intercepts.

$y = \frac{1}{10}(x + 3)^2(x - 2)$



- If any of the factors of a polynomial function are cubed, then the corresponding x-intercepts are points where the x-axis is tangent to the curve and also passes through the x-axis. The graph has a cubic shape near these x-intercepts.

$y = \frac{1}{5}(x + 3)^3(x - 1)$



Repeated Zeros

A factor $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.

1. When k is odd, the graph *crosses* the x-axis at $x = a$.
2. When k is even, the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

Homework

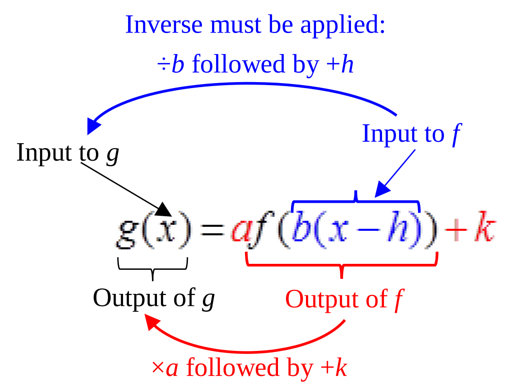
Precalculus (Ron Larson)

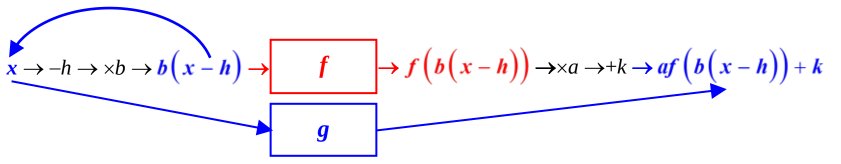
pp. 133 – 137: #1-7, 35-50, 55-74, 89-92, 99, 100, 105-111, 116,

USING TRANSFORMATIONS TO SKETCH THE GRAPHS OF CUBIC AND QUARTIC FUNCTIONS

Review

The graph of $g(x)=af\left(b(x-h)\right)+k$ can be obtained by applying stretches/compressions and translations to the graph of *f*.





Pre-Image Point on *f* → Image Point on *g*	
$\left(b(x-h),\ f\left(b(x-h)\right)\right)\rightarrow\left(x, af\left(b(x-h)\right)+k\right)$	$(x,y)\rightarrow\left(\frac{1}{b}x+h, ay+k\right)$

Horizontal

1. Stretch/compress by a factor of $1/b=b^{-1}$. If *b* is negative, there is also a reflection in the *y*-axis.

2. Translate *h* units right if $h>0$ or *h* units left if $h<0$.

Vertical

1. Stretch/compress by a factor of *a*. If *a* is negative, there is also a reflection in the *x*-axis.

2. Translate *k* units up/down depending on whether *k* is positive or negative respectively.

Application to Polynomial Functions of Degree n

Base Function for Polynomials of Degree n	$f(x)=x^n$
General Equation of g	$g(x)=af\left(b(x-h)\right)+k=a\left(b(x-h)\right)^n+k=ab^n(x-h)^n+k$

Example 1

Sketch the graph of $g(x)=-4\left(\frac{1}{3}x+1\right)^3+2$ by applying transformations to the graph of $f(x)=x^3$.

Solution

Method 1

First, write the equation of *g* to conform with the general form $g(x)=af\left(b(x-h)\right)+k$:

$$g(x)=-4\left(\frac{1}{3}x+1\right)^3+2=-4\left(\frac{1}{3}(x+3)\right)^3+2$$

Then decide what the transformations should be. By now, you should be able to glance at the equation and immediately write the transformation using mapping notation:

$$(x,y)\rightarrow(3x-3,-4y+2)$$

Method 2

First, simplify the equation fully:

$$g(x)=-4\left(\frac{1}{3}x+1\right)^3+2$$

$$=-4\left(\frac{1}{3}(x+3)\right)^3+2$$

$$=-4\left(\frac{1}{3}\right)^3(x+3)^3+2$$

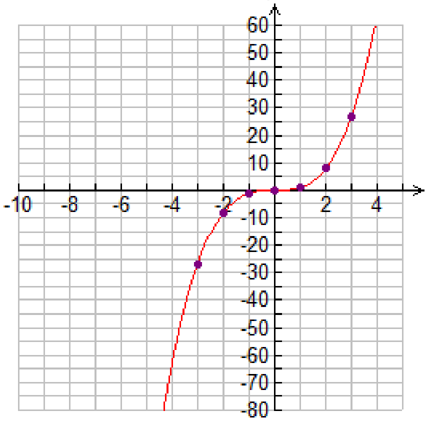
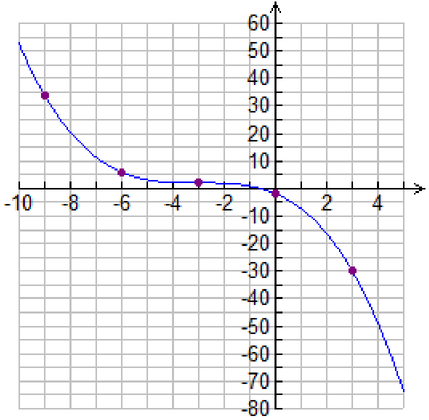
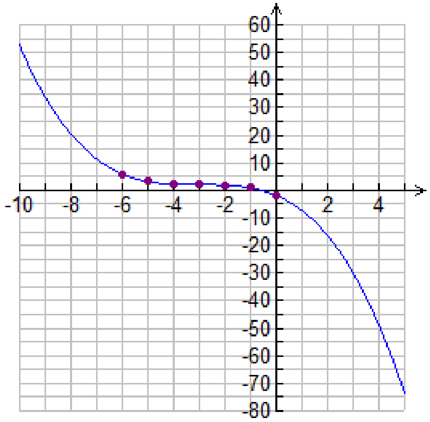
$$=-\frac{4}{27}(x+3)^3+2$$

Then write the transformation using mapping notation:

$$(x,y)\rightarrow\left(x-3,-\frac{4}{27}y+2\right)$$

Although this answer is not the same as that obtained using method 1, it is equivalent to it when applied to the base function $f(x)=x^3$

Pre-image Points on Graph of $f(x)=x^3$	Image Points on Graph of $g(x)=-4((1/3)x+1)^3+2$	
(x,y)	Method 1: $(3x-3,-4y+2)$	Method 2: $(x-3,(-4/27)y+2)$
$(0,0)$	$(-3,2)$	$(-3,2)$
$(1,1)$	$(0,-2)$	$(-2,\frac{50}{27})$
$(-1,-1)$	$(-6,6)$	$(-4,\frac{58}{27})$
$(2,8)$	$(3,-30)$	$(-1,\frac{22}{27})$
$(-2,-8)$	$(-9,34)$	$(-5,\frac{86}{27})$
$(3,27)$	$(6,-106)$	$(0,-2)$
$(-3,-27)$	$(-12,110)$	$(-6,6)$

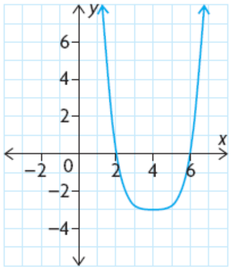
Base Graph: $y=x^3$	***Image under the Transformation given Above***	
Note the Pre-image Points	Image Points obtained using Method 1	Image Points obtained using Method 2
		

Example 2

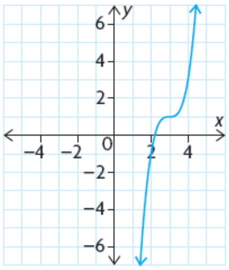
Match each function with the most suitable graph. Explain your reasoning.

a) $y=2(x-3)^3+1$ b) $y=-\frac{1}{3}(x+1)^3-1$ c) $y=0.2(x-4)^4-3$ d) $y=-1.5(x+3)^4+4$

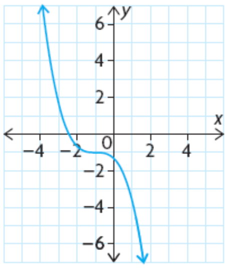
A



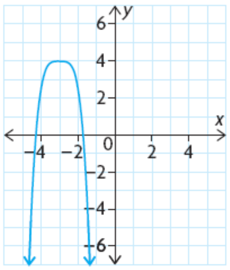
B



C



D



Solution

a) → B

b) → C

c) → A

d) → D

- a) and b) → odd degree → opposite end behaviour → can only match with graphs B and C
- a) must match graph B because that graph has the correct end behaviour (as $x\to\infty$, $y\to\infty$ & as $x\to-\infty$, $y\to-\infty$). Therefore, equation b) must match graph C.
- Graph A has same end behaviours and as $x\to\pm\infty$, $y\to\infty$, which means it can only match equation c). By a process of elimination, graph D must match with equation d).

Homework

Precalculus (Ron Larson)
pp. 133 – 137: #9-18, 51-54, 75-88, 97, 98, 101-104, 112-114, 115abcdef

SOLVING POLYNOMIAL EQUATIONS OF DEGREE 3 OR HIGHER

Introduction

Polynomial Function	*Degree*	*Corresponding Polynomial Equation*	*# Zeros of Function (=# of Roots of Equation)*	*General Solution of Polynomial Equation in Terms of Coefficients*
$f(x)=a_1x+a_0$	1	$a_1x+a_0=0$	exactly 1	$x=-\dfrac{a_0}{a_1}$
$f(x)=a_2x^2+a_1x+a_0$	2	$a_2x^2+a_1x+a_0=0$	0, 1 or 2	$x=\dfrac{-a_1\pm\sqrt{a_1^2-4a_2a_0}}{2a_2}$
$f(x)=a_3x^3+a_2x^2+a_1x+a_0$	3	$a_3x^3+a_2x^2+a_1x+a_0=0$	1, 2 or 3	The roots of $a_3x^3+a_2x^2+a_1x+a_0=0$ can be expressed in terms of the coefficients a_3, a_2, a_1 and a_0. However, doing so involves complicated algebraic manipulations. See Cubic Function for more information.
$f(x)=a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$	4	$a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$	0, 1, 2, 3 or 4	The roots of $a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$ can be expressed in terms of the coefficients a_4, a_3, a_2, a_1 and a_0. However, doing so involves complicated algebraic manipulations. See Quartic Equation for more information.
$f(x)=a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$	5	$a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$	1, 2, 3, 4 or 5	For polynomial equations of degree 5 or higher, it is ***impossible in general*** to find the roots using a finite number of additions, subtractions, multiplications divisions and root extractions (i.e. radicals). Of course, ***certain*** quintic and higher-degree polynomial equations ***are*** solvable using these operations (e.g. $x^5-1=0$) but there is no way to do this for ***arbitrary*** polynomial equations of degree five or higher. For more information, see Abel-Ruffini Theorem.

Long Division and the Remainder Theorem – A Limited Approach to Solving Polynomial Equations

As summarized above, linear and quadratic polynomial equations are the ***only ones*** for which it is always easy to find exact solutions! Any cubic or quartic polynomial equation can also be solved exactly but the process is somewhat long and tedious. Exact solutions for polynomial equations of degree five or higher can be found only in certain special cases. In general, it is ***impossible*** to solve polynomial equations of degree five or higher using a finite number of additions, subtractions, multiplications divisions and root extractions (i.e. radicals).

- Only a small class of equations can be solved by using algebraic methods. Therefore, methods of approximation are used. Such methods must be executed by a computer because they involve copious calculations.
- For our purposes, we shall solve certain polynomial equations of ***degree three*** or higher by ***guessing*** one of the roots. This method is based on a theorem known as the ***Remainder Theorem***.

The Remainder Theorem

Suppose that *p* is a polynomial function of degree *n* and that $a \in \mathbb{R}$. Then, *p* can be expressed uniquely in the form

$$p(x) = (x-a)q(x) + R,$$

where *q* is a polynomial of degree *n* – 1 and $R \in \mathbb{R}$. The value *R* is called the ***remainder***.

It follows immediately that $p(a) = R$, that is, the remainder can be calculated simply by substituting *a* into the polynomial. That is, when $p(x)$ is divided by $x-a$, the remainder is $p(a)$.

The remainder theorem gives us a quick way to calculate the remainder obtained when a polynomial is divided by a linear binomial. It states that a polynomial can be divided by a linear binomial to obtain a quotient $q(x)$, which is a polynomial of degree *n* – 1, and a remainder *R*, which is a real number.

Examples

$$\begin{array}{r}
x^2+2x+5 \\
x-2\overline{\smash{\big)}\,x^3+0x^2+x+1} \\
\underline{x^3-2x^2} \\
2x^2+x \\
\underline{2x^2-4x} \\
5x+1 \\
\underline{5x-10} \\
11
\end{array}$$

$$\therefore p(x) = x^3+x+1 = (x-2)\left(x^2+2x+5\right)+11$$

Notice that

$$\begin{aligned} p(2) &= (2-2)\left(2^2+2(2)+5\right)+11 \\ &= 0\left(2^2+2(2)+5\right)+11 \\ &= 11 \end{aligned}$$

$$\begin{array}{r}
x^2+2x+5 \\
x-2\overline{\smash{\big)}\,x^3+0x^2+x-10} \\
\underline{x^3-2x^2} \\
2x^2+x \\
\underline{2x^2-4x} \\
5x-10 \\
\underline{5x-10} \\
0
\end{array}$$

$$\therefore p(x) = x^3+x-10 = (x-2)\left(x^2+2x+5\right)$$

Notice that

$$\begin{aligned} p(2) &= (2-2)\left(2^2+2(2)+5\right) \\ &= 0\left(2^2+2(2)+5\right) \\ &= 0 \end{aligned}$$

The remainder theorem gives us a method to solve certain polynomial equations of degree 3 or higher. As we can see from the examples at the left, $x-a$ divides into a polynomial $p(x)$ whenever the remainder *R* is zero. By the remainder theorem, $p(a)=R$. Therefore, $x-a$ divides into a polynomial $p(x)$ whenever $p(a)=0$. Another way of stating this is that $x-a$ divides into $p(x)$ if *a* is a root of the polynomial equation $p(x)=0$. The example below shows how we can exploit this to solve a cubic equation.

How to find a Value "a" such that $p(a)=0$

To apply the remainder theorem, we need to find a value *a* such that $p(a)=0$. How do we go about finding such a value? A simple observation can help us narrow down the possibilities. Let $p(x)=bx^3+cx^2+dx+e$.

If $p(a)=0$, then by the remainder theorem, $p(x)=(x-a)\left(mx^2+nx+s\right)=mx^3+(n-am)x^2-anx-as$.

Comparing the two different forms of $p(x)$, we can conclude that $b=m$, $c=n-am$, $d=-an$ and $e=-as$

Since $e=-as$, we can conclude that *a* must divide into *e*.

If $p(x)=bx^3+cx^2+dx+e$ and $p(a)=0$, then *a* must divide into *e*.

A Corollary of the Remainder Theorem – The Factor Theorem

The Factor Theorem

Let *p* represent any polynomial function. Then, $x-a$ is a factor of $p(x)$ ***if and only if*** $p(a)=0$.

Note

The phrase "if and only if" in the above statement expresses the ***logical equivalence*** of the statements "$x-a$ is a factor of $p(x)$" and "$p(a)=0$." That is, the use of "if and only if" means that ***both*** of the following statements are true.

- If $x-a$ is a factor of $p(x)$, then $p(a)=0$. (This statement is true.)
- If $p(a)=0$, then $x-a$ is a factor of $p(x)$. (The ***converse*** of the above statement is ***also*** true.)

In general, consider the following statements:

- If *P* is true then *Q* is true. (This is called a ***conditional statement***.)
- If *Q* is true then *P* is true. (This statement is called the ***converse*** of the above ***conditional statement***.)

If ***both*** of the above statements are true, then we can write

- *P* is true ***if and only*** if *Q* is true. (This statement is called a ***biconditional statement*** or a ***logical equivalence***.)

Note that ***not all statements*** are biconditional. Consider the following:

1. If I do all my homework each and every day, then Mr. Nolfi is extremely happy with my effort. (The statement)
2. If Mr. Nolfi is extremely happy with my effort, then I do all my homework each and every day. (The converse.)

3. If it is raining, then the roads are wet. (The statement.)
4. If the roads are wet, then it is raining. (The converse.)

Statements 1 and 3 are true in all possible cases and hence, we call them "true." However, the converses of the statements are false in some cases, and so, we call them false. It is possible that Mr. Nolfi is extremely happy with a student's efforts even if the student did not complete all of his/her homework. Similarly, the roads can be wet even if it is not raining.

Example of using the Factor Theorem to Solve a Cubic Equation

Solve $2r^3+45r^2-2052=0$

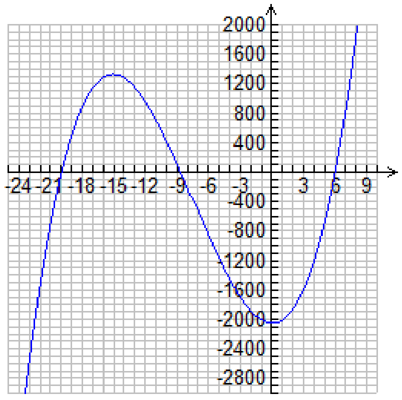
Solution 1

Let $f(r)=2r^3+45r^2-2052$.

Using the result on the previous page, we know that if $f(a)=0$, then *a* must divide into 2052. Therefore, it is only necessary to try factors of 2052 when searching for values of *a* such that $f(a)=0$.

a (Factors of 2052)	$f(a)$	***Conclusion***
±1	$f(-1)=-2009$ $f(1)=-2005$	$(r-(-1))=(r+1)$ is not a factor of $f(r)$ $(r-1)$ is not a factor of $f(r)$
±2	$f(-2)=-1888$ $f(2)=-1856$	$(r-(-2))=(r+2)$ is not a factor of $f(r)$ $(r-2)$ is not a factor of $f(r)$
±3	$f(-3)=-1701$ $f(3)=-1593$	$(r-(-3))=(r+3)$ is not a factor of $f(r)$ $(r-3)$ is not a factor of $f(r)$
±4	$f(-4)=-1460$ $f(4)=-1204$	$(r-(-4))=(r+4)$ is not a factor of $f(r)$ $(r-4)$ is not a factor of $f(r)$
±6	$f(-6)=-864$ $f(6)=0$	$(r-(-6))=(r+6)$ is not a factor of $f(r)$ $(r-6)$ ***is*** a factor of $f(r)$

The Graph of $f(r)=2r^3+45r^2-2052$



Since the degree of *f* is odd, *f* must have opposite end behaviours.

As shown above, by trial and error we find that $f(6)=0$. Therefore, $r-6$ divides into $f(r)$ with remainder zero. This means that $r-6$ is a factor of $f(r)$.

By ***long division***, we find that $f(r)=(r-6)\left(2r^2+57r+342\right)$.

$$\begin{array}{r|l} & \quad 2r^2+57r+342 \\ \cline{2-2} r-6 & 2r^3+45r^2+0r-2052 \\ & \underline{2r^3-12r^2} \\ & \qquad 57r^2+0r \\ & \qquad \underline{57r^2-342r} \\ & \qquad\qquad 342r-2052 \\ & \qquad\qquad \underline{342r-2052} \\ & \qquad\qquad\qquad\qquad 0 \end{array}$$

$$2r^3+45r^2-2052=0$$
$$\therefore (r-6)\left(2r^2+57r+342\right)=0$$
$$\therefore r-6=0 \text{ or } 2r^2+57r+342=0$$
$$\therefore r=6 \text{ or } r=\frac{-57\pm\sqrt{513}}{4}$$

Solution 2

We know that $r-6$ is a factor of $f(r)$. Therefore, there exist real numbers *a*, *b* and *c* such that

$$(r-6)\left(ar^2+br+c\right)=2r^3+45r^2+0r-2052$$
$$\therefore ar^3+br^2+cr-6ar^2-6br-6c=2r^3+45r^2+0r-2052$$
$$\therefore ar^3+(b-6a)r^2+(c-6b)r-6c=2r^3+45r^2+0r-2052$$
$$\therefore a=2,\ b-6a=45,\ c-6b=0,\ -6c=-2052$$
$$\therefore a=2,\ b-12=45,\ c-6b=0,\ c=342$$
$$\therefore a=2,\ b=57,\ c=342$$
$$\therefore 2r^3+45r^2-2052=(r-6)\left(2r^2+57r+342\right)$$

Homework

Precalculus (Ron Larson)

Ignore any references to synthetic division; use long division instead. Students interested in learning synthetic division may read about it on page 141 of the textbook. For more information, consult Synthetic Division.

pp. 144 – 146: #5-10, 11-25 (odd-numbered questions), 47-65 (odd-numbered questions), 81-92, 95, 96

FACTORING POLYNOMIALS

Review – Common Factoring and Factoring Quadratic Polynomials

An ***expression*** is ***factored*** if it is written as a ***product***.

Common Factoring	***Factor Simple Trinomial***	***Factor Complex Trinomial***	***Difference of Squares***
Example $-42m^3n^2+13mn^2p-39m^4n^3q$ $=-13mn^2(4m^2-p+3m^3nq)$	***Example*** $n^2-20n+91$ $=(n-7)(n-13)$ ***Rough Work*** $(-7)(-13)=91$ $-7+(-13)=-20$	***Example*** $10x^2-x-21$ $=(10x^2-15x)+(14x-21)$ $=5x(2x-3)+7(2x-3)$ $=(2x-3)(5x+7)$ ***Rough Work*** $(10)(-21)=-210$, $(-15)(14)=-210$ $-15+14=-1$	***Example*** $98x^2-50y^2$ $=2(49x^2-25y^2)$ $=2((7x)^2-(5y)^2)$ $=2(7x-5y)(7x+5y)$

Factoring Polynomials of Degree Three or Higher

Factor each of the following polynomials.

$$\begin{aligned}&x^4-6x^3+2x^2-12x\\&=\left(x^4-6x^3\right)+\left(2x^2-12x\right)\\&=x^3(x-6)+2x(x-6)\\&=(x-6)\left(x^3+2x\right)\\&=x(x-6)\left(x^2+2\right)\end{aligned}$$

This method is called ***factoring by grouping***. It can be used in certain special cases but does not work very well in general.

In general, the factor theorem is more useful in factoring polynomials of degree three or higher.

$$\begin{aligned}&4x^4+6x^3-6x^2-4x\\&=x\left(4x^3+6x^2-6x-4\right)\\&=x(x-1)\left(4x^2+10x+4\right)\\&=2x(x-1)\left(2x^2+5x+2\right)\\&=2x(x-1)(2x+1)(x+2)\end{aligned}$$

Let $f(x)=4x^3+6x^2-6x-4$.

Then,

$$f(1)=4\left(1^3\right)+6\left(1^2\right)-6(1)-4=0$$

Therefore, by the factor theorem, $x-1$ is a factor of $f(x)$.

$$\begin{array}{r}4x^2+10x+4\\ x-1\overline{)4x^3+6x^2-6x-4}\\ \underline{4x^3-4x^2}\qquad\quad\\ 10x^2-6x\quad\\ \underline{10x^2-10x}\quad\\ 4x-4\\ \underline{4x-4}\\ 0\end{array}$$

$$\begin{aligned}&x^3-64\\&=(x-4)\left(x^2+4x+16\right)\end{aligned}$$

This is an example of how a ***difference of cubes*** is factored.
Notice that the quadratic factor cannot be factored further.

$$\begin{aligned}&b^2-4ac\\&=4^2-4(1)(16)\\&=16-64\\&=-48\\&<0\end{aligned}$$

Let $f(x)=x^3-64$.

Then,

$$f(4)=4^3-64=0$$

Therefore, by the factor theorem, $x-4$ is a factor of $f(x)$.

$$\begin{array}{r}x^2+4x+16\\ x-4\overline{)x^3+0x^2+0x-64}\\ \underline{x^3-4x^2}\qquad\quad\\ 4x^2+0x\quad\\ \underline{4x^2-16x}\quad\\ 16x-64\\ \underline{16x-64}\\ 0\end{array}$$

$$\begin{aligned}&x^3+27\\&=(x+3)\left(x^2-3x+9\right)\end{aligned}$$

This is an example of how a ***sum of cubes*** is factored.
Notice that the quadratic factor cannot be factored further.

$$\begin{aligned}&b^2-4ac\\&=(-3)^2-4(1)(9)\\&=9-36\\&=-27\\&<0\end{aligned}$$

Let $f(x)=x^3+27$. Then,

$$f(-3)=(-3)^3+27=0$$

Therefore, by the factor theorem, $x+3$ is a factor of $f(x)$.

$$\begin{array}{r}x^2-3x+9\\ x+3\overline{)x^3+0x^2+0x+27}\\ \underline{x^3+3x^2}\qquad\quad\\ -3x^2+0x\quad\\ \underline{-3x^2-9x}\quad\\ 9x+27\\ \underline{9x+27}\\ 0\end{array}$$

Factoring Sums and Differences of Cubes

The last two examples on the previous page suggest a general method for factoring sums and differences of cubes.

Difference of Cubes	***Sum of Cubes***
Let $f(x)=x^3-y^3$, where *y* represents some constant. Since $f(y)=y^3-y^3=0$, by the factor theorem, $x-y$ must be a factor of $f(x)$. $\begin{array}{r} x^2+yx+y^2 \\ x-y\overline{\smash{)}x^3+0x^2+0x-y^3} \\ \underline{x^3-yx^2} \\ yx^2+0x \\ \underline{yx^2-y^2x} \\ y^2x-y^3 \\ \underline{y^2x-y^3} \\ 0 \end{array}$ Therefore, $f(x)=x^3-y^3=(x-y)(x^2+yx+y^2)$	Let, $f(x)=x^3+y^3$, where *y* represents some constant. Since $f(-y)=(-y)^3+y^3=0$, by the factor theorem, $x-(-y)=x+y$ must be a factor of $f(x)$. $\begin{array}{r} x^2-yx+y^2 \\ x+y\overline{\smash{)}x^3+0x^2+0x+y^3} \\ \underline{x^3+yx^2} \\ -yx^2+0x \\ \underline{-yx^2-y^2x} \\ y^2x+y^3 \\ \underline{y^2x+y^3} \\ 0 \end{array}$ Therefore, $f(x)=x^3+y^3=(x+y)(x^2-yx+y^2)$

Summary

cubes

$$x^3-y^3=(x-y)(x^2+xy+y^2)$$

difference

cubes

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$

sum

Examples

Factor each of the following:

(a) $8x^3+27$

$=(2x)^3+3^3$

$=(2x+3)\left((2x)^2-3(2x)+3^2\right)$

$=(2x+3)(4x^2-6x+9)$

(b) $64a^3-8b^3$

$=8(8a^3-b^3)$

$=8\left((2a)^3-b^3\right)$

$=8(2a-b)\left((2a)^2+(2a)(b)+b^2\right)$

$=8(2a-b)(4a^2+2ab+b^2)$

(c) $7m^4-448m$

$=7m(m^3-64)$

$=7m(m^3-4^3)$

$=7m(m-4)(m^2+4m+16)$

(d) $125x^9+512$

$=(5x^3)^3+8^3$

$=(5x^3+8)\left((5x^3)^2-8(5x^3)+8^2\right)$

$=(5x^3+8)(125x^6-40x^3+64)$

Important Question

Calculate the discriminant (b^2-4ac) of the quadratic polynomials in *x* obtained in the factorizations of x^3-y^3 and x^3+y^3. What do you notice? What conclusions can you draw?

Homework

4. State the remainder when $x + 2$ is divided into each polynomial.
K
a) $x^2 + 7x + 9$ **d)** $x^4 - 2x^3 - 11x^2 + 10x - 2$
b) $6x^3 + 19x^2 + 11x - 11$ **e)** $x^3 + 3x^2 - 10x + 6$
c) $x^4 - 5x^2 + 4$ **f)** $4x^4 + 12x^3 - 13x^2 - 33x + 18$

5. Determine whether $2x - 5$ is a factor of each polynomial.
a) $2x^3 - 5x^2 - 2x + 5$ **c)** $2x^4 - 7x^3 - 13x^2 + 63x - 45$
b) $3x^3 + 2x^2 - 3x - 2$ **d)** $6x^4 + x^3 - 7x^2 - x + 1$

6. Factor each polynomial using the factor theorem.
a) $x^3 - 3x^2 - 10x + 24$ **d)** $4x^4 + 7x^3 - 80x^2 - 21x + 270$
b) $4x^3 + 12x^2 - x - 15$ **e)** $x^5 - 5x^4 - 7x^3 + 29x^2 + 30x$
c) $x^4 + 8x^3 + 4x^2 - 48x$ **f)** $x^4 + 2x^3 - 23x^2 - 24x + 144$

7. Factor fully.
a) $f(x) = x^3 + 9x^2 + 8x - 60$ **d)** $f(x) = x^4 + 3x^3 - 38x^2 + 24x + 64$
b) $f(x) = x^3 - 7x - 6$ **e)** $f(x) = x^3 - x^2 + x - 1$
c) $f(x) = x^4 - 5x^2 + 4$ **f)** $f(x) = x^5 - x^4 + 2x^3 - 2x^2 + x - 1$

8. Use the factored form of $f(x)$ to sketch the graph of each function in question 7.

9. The polynomial $12x^3 + kx^2 - x - 6$ has $2x - 1$ as one of its factors. Determine the value of k.

10. When $ax^3 - x^2 + 2x + b$ is divided by $x - 1$, the remainder is 10. When it is divided by $x - 2$, the remainder is 51. Find a and b.
A

11. Determine a general rule to help decide whether $x - a$ and $x + a$ are factors of $x^n - a^n$ and $x^n + a^n$.
T

12. The function $f(x) = ax^3 - x^2 + bx - 24$ has three factors. Two of these factors are $x - 2$ and $x + 4$. Determine the values of a and b, and then determine the other factor.

13. Consider the function $f(x) = x^3 + 4x^2 + kx - 4$. The remainder from $f(x) \div (x + 2)$ is twice the remainder from $f(x) \div (x - 2)$. Determine the value of k.

14. Show that $x - a$ is a factor of $x^4 - a^4$.

15. Explain why the factor theorem works.
C

Extending

16. Use the factor theorem to prove that $x^2 - x - 2$ is a factor of $x^3 - 6x^2 + 3x + 10$.

17. Prove that $x + a$ is a factor of $(x + a)^5 + (x + c)^5 + (a - c)^5$.

Answers

4. **a)** -1 **c)** 0 **e)** 30
b) -5 **d)** -34 **f)** 0
5. **a)** yes **c)** yes
b) no **d)** no
6. **a)** $(x - 2)(x - 4)(x + 3)$
b) $(x - 1)(2x + 3)(2x + 5)$
c) $x(x - 2)(x + 4)(x + 6)$
d) $(x + 2)(x + 5)(4x - 9)(x - 3)$
e) $x(x + 2)(x + 1)(x - 3)(x - 5)$
f) $(x - 3)(x - 3)(x + 4)(x + 4)$
7. **a)** $(x - 2)(x + 5)(x + 6)$
b) $(x + 1)(x - 3)(x + 2)$
c) $(x + 1)(x - 1)(x - 2)(x + 2)$
d) $(x - 2)(x + 1)(x + 8)(x - 4)$
e) $(x - 1)(x^2 + 1)$
f) $(x - 1)(x^2 + 1)(x^2 + 1)$

9. 20
10. $a = 6, b = 3$
11. For $x^n - a^n$, if n is even, they're both factors. If n is odd, only $(x - a)$ is a factor. For $x^n + a^n$, if n is even, neither is a factor. If n is odd, only $(x + a)$ is a factor.
12. $a = -2, b = 22$;
The other factor is $-2x + 3$.
13. -6

14. $x^4 - a^4$
$= (x^2)^2 - (a^2)^2$
$= (x^2 + a^2)(x^2 - a^2)$
$= (x^2 + a^2)(x + a)(x - a)$
15. Answers may vary. For example: if $f(x) = k(x - a)$, then $f(a) = k(a - a)$ $= k(0) = 0$.
16. $x^2 - x - 2 = (x - 2)(x + 1)$;
If $f(x) = x^3 - 6x^2 + 3x + 10$, then $f(2) = 0$ and $f(-1) = 0$.
17. If $f(x) = (x + a)^5 + (x + c)^5 + (a - c)^5$, then $f(-a) = 0$

4. Factor.

K

a) $x^3 - 343$	**d)** $125x^3 - 512$	**g)** $512x^3 + 1$
b) $216x^3 - 1$	**e)** $64x^3 - 1331$	**h)** $1331x^3 + 1728$
c) $x^3 + 1000$	**f)** $343x^3 + 27$	**i)** $512 - 1331x^3$

5. Factor each expression.

a) $\frac{1}{27}x^3 - \frac{8}{125}$ **c)** $(x - 3)^3 + (3x - 2)^3$

b) $-432x^5 - 128x^2$ **d)** $\frac{1}{512}x^9 - 512$

6. Jarred claims that the expression

A $\frac{(a + b)(a^2 - ab + b^2) + (a - b)(a^2 + ab + b^2)}{2a^3}$ is equivalent to 1.

Do you agree or disagree with Jarred? Justify your decision.

7. 1729 is a very interesting number. It is the smallest whole number that can be expressed as a sum of two cubes in two ways: $1^3 + 12^3$ and $9^3 + 10^3$. Use the factorization for the sum of cubes to verify that these sums are equal.

8. Prove that $(x^2 + y^2)(x^4 - x^2y^2 + y^4)(x^{12} - x^6y^6 + y^{12}) + 2x^9y^9$

T equals $(x^9 + y^9)^2$ using the factorization for the sum of cubes.

9. Some students might argue that if you know how to factor a sum

C of cubes, then you do not need to know how to factor a difference of cubes. Explain why you agree or disagree.

Extending

10. The number 1729, in question 7, is called a taxicab number.

a) Use the Internet to find out why 1729 is called a taxicab number.

b) Are there other taxicab numbers? If so, what are they?

Answers

4. **a)** $(x - 7)(x^2 + 7x + 49)$

b) $(6x - 1)(36x^2 + 6x + 1)$

c) $(x + 10)(x^2 - 10x + 100)$

d) $(5x - 8)(25x^2 + 40x + 64)$

e) $(4x - 11)(16x^2 + 44x + 121)$

f) $(7x + 3)(49x^2 - 21x + 9)$

g) $(8x + 1)(64x^2 - 8x + 1)$

h) $(11x + 12)(121x^2 - 132x + 144)$

i) $(8 - 11x)(64 + 88x + 121x^2)$

5. **a)** $\left(\frac{1}{3}x - \frac{2}{5}\right)\left(\frac{1}{9}x^2 + \frac{2}{15}x + \frac{4}{25}\right)$

b) $-16x^2(3x + 2)(9x^2 - 6x + 4)$

c) $7(4x - 5)(x^2 - x + 1)$

d) $\left(\frac{1}{2}x - 2\right)\left(\frac{1}{4}x^2 + x + 4\right)$
$\left(\frac{1}{64}x^6 + x^3 + 64\right)$

6. Agree; by the formulas for factoring the sum and difference of cubes, the numerator of the fraction is equivalent to $(a^3 + b^3) + (a^3 - b^3)$. Since $(a^3 + b^3) + (a^3 - b^3) = 2a^3$, the entire fraction is equal to 1.

7. **a)**
$$\begin{aligned}1^3 + 12^3 &= (1 + 12)(1^2 - (1)(12) + 12^2)\\ &= (13)(133) = 1729\end{aligned}$$

b)
$$\begin{aligned}9^3 + 10^3 &= (9 + 10)(9^2 - (9)(10) + 10^2)\\ &= (19)(91) = 1729\end{aligned}$$

8.
$$\begin{aligned}&x^9 + y^9\\ &= x^{18} + 2x^9y^9 + y^{18}\\ &= (x^{18} + y^{18}) + 2x^9y^9\\ &= (x^6 + y^6)(x^{12} - x^6y^6 + y^{12}) + 2x^9y^9\\ &= (x^2 + y^2)(x^4 - x^2y^2 + y^4)\\ &\quad(x^{12} - x^6y^6 + y^{12}) + 2x^9y^9\end{aligned}$$

9. Answers may vary. For example, this statement is true because $a^3 - b^3$ is the same as $a^3 + (-b)^3$.

10. **a)** 1729 was the number of the taxicab that G. H. Hardy rode in when going to visit the mathematician Ramanujan. When Hardy told Ramanujan that the number of the taxicab he rode in was uninteresting, Ramanujan replied that the number was interesting because it was the smallest number that could be expressed as the sum of two cubes in two different ways. This is how such numbers came to be known as taxicab numbers.

b) Yes;

TN(1) = 2
TN(2) = 1729
TN(3) = 87 539 319
TN(4) = 6 963 472 309 248
TN(5) = 48 988 659 276 962 496
TN(6) = 24 153 319 581 254 312 065 344

SOLVING POLYNOMIAL INEQUALITIES

Introductory Problem

Preetika has a monthly budget of \$1750. Each month she must pay \$950 for rent, \$375 for transportation, \$250 for food and \$100 in miscellaneous expenses. Assuming that there are no other expenses, determine the number of times per month that Preetika can go to the movies. (Assume an average price of admission of \$10.00 per movie.)

Solution

Although you might be tempted to solve this problem by using an equation, technically it would be incorrect. To see this, consider the following restatement of the problem:

total monthly expenses ***must be less than or equal to*** \$1750.00

$$\therefore \$950 + \$375 + \$250 + \$100 + \text{cost of movies} \le \$1750$$

$$\therefore \$1675 + \text{cost of movies} \le \$1750$$

Now if we let *x* represent the number of times that Preetika goes to the movies in one month, then we can represent this problem using the following inequality:

$$1675 + 10x \le 1750$$

We can solve this inequality as follows:

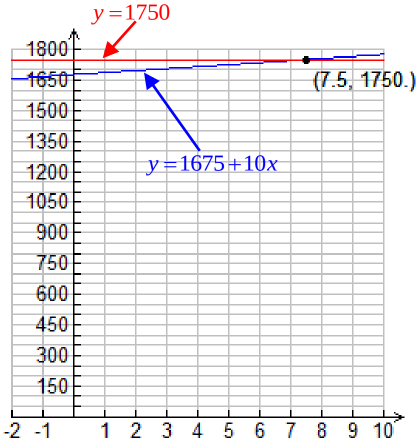
$$1675 + 10x \le 1750$$

$$\therefore 10x \le 1750 - 1675$$

$$\therefore 10x \le 75$$

$$\therefore \frac{10x}{10} \le \frac{75}{10}$$

$$\therefore x \le 7.5$$



The solution set of the inequality is $\{x \in \mathbb{R} : x \le 7.5\}$, or in interval notation, $(-\infty, 7.5]$.

The solution informs us that Preetika can stay within her monthly budget if she goes to the movies 7.5 or fewer times per month. Obviously, the number of times that Preetika goes to the movies must be a whole number. Therefore, Preetika can afford to go to the movies no more than 7 times per month.

Represent the solution on a number line.
A solid dot is placed on 7.5 since this number is included in the solution set.

Investigation

Consider the following series of inequalities. In which cases is the reasoning invalid? Can you draw any conclusions?

$\because 2 < 5$	$\because -3 < -1$	$\because 2 > 1$	$\because 2 > 1$	$-3 < -1$
$\therefore 2 + 6 < 5 + 6$	$\therefore -3 + 4 < -1 + 4$	$\therefore 2 - 6 > 1 - 6$	$\therefore 2 + 3 > 1 + 3$	$\therefore -3 + 2 < -1 + 2$
$\therefore 8 < 11$	$\therefore 1 < 3$	$\therefore -4 > -5$	$\therefore 5 > 4$	$\therefore -1 < 1$
$\therefore 8(2) < 11(2)$	$\therefore \frac{1}{5} < \frac{3}{5}$	$\therefore 3(-4) > 3(-5)$	$\therefore -3(5) > -3(4)$	$\therefore \frac{-1}{-5} < \frac{1}{-5}$
$\therefore 16 < 22$		$\therefore -12 > -15$	$\therefore -15 > -12$	$\therefore \frac{1}{5} < -\frac{1}{5}$
?	?	?	?	?

Working with Equations and Inequalities – Similarities and Differences

1. Whether solving an equation or an inequality, whatever operation is performed to one side must also be performed to the other side. Equivalently, ***whatever function is applied to one side must also be applied to the other side***.
2. Whether solving an equation or an inequality, the ***general approach is to apply inverse functions (operations) to both sides in the order OPPOSITE the standard order of operations***.
3. When the same function is applied to both sides of an ***equation, equality is always preserved***.
4. When the same function is applied to both sides of an ***inequality, the inequality is NOT always preserved***.
    - Whenever a strictly increasing function is applied to both sides, the inequality is preserved.
    - However, when other functions are applied to both sides, the inequality may not be preserved.
    - In particular, if both sides of an inequality are ***multiplied or divided by a negative number***, the inequality ***must be reversed***. This happens because multiplying or dividing an increasing or decreasing sequence of numbers by a negative value produces a new sequence in which the order of the numbers is inverted.

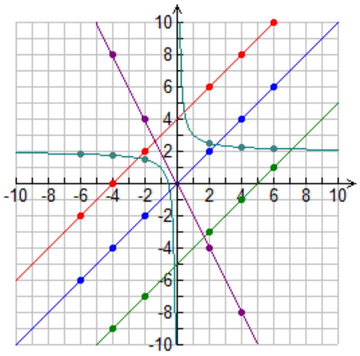
Summary

- If $x=y$, then $f(x)=f(y)$ ***for all*** functions *f* and ***all*** real numbers *x* and *y*.
- If $x\neq y$, then it ***is not necessarily the case*** that $f(x)\neq f(y)$. For some functions *f* and some real numbers *x* and *y*, it ***is*** the case that $f(x)=f(y)$. This kind of inequality is only preserved if *f* is strictly increasing or decreasing.
- If $x<y$, $x\leq y$, $x>y$ or $x\geq y$, then it ***is not necessarily the case*** that $f(x)<f(y)$, $f(x)\leq f(y)$, $f(x)>f(y)$ or $f(x)\geq f(y)$ ***respectively***. These kinds of inequalities are only preserved if *f* is strictly increasing.

Understanding why an Inequality sometimes needs to be Reversed

Consider the values in the following table. Note that in the first three columns, the values are in ***ascending*** order. Another way of interpreting this is that the operations of adding and subtracting ***preserved the order of the values***. In the fourth column, however, the values are in ***descending*** order. The operation of multiplying by −2 caused the ***order of the values to be reversed***. A slightly more extreme example is given in the fifth column. When the function $y=\frac{1}{x}+2$ is applied, the numbers are in descending order up to a point, then briefly ascend only to descend once again.

x	$y=x+4$	$y=x-5$	$y=-2x$	$y=\frac{1}{x}+2$
−6	−2	−11	12	11/6
−4	0	−9	8	7/4
−2	2	−7	4	3/2
2	6	−3	−4	5/2
4	8	−1	−8	9/4
6	10	1	−12	13/6



To the left is a graphical view of the table. Certain operations (functions) preserve order, some reverse order and others produce mixed results.

Valid Deductions involving Inequalities			*Invalid Deductions involving Inequalities*	
$\because -6<-4$ $\therefore -6+4<-4+4$ $\therefore -2<0$	$\because -6<-4$ $\therefore -6-5<-4-5$ $\therefore -11<-9$	$\because -2<2$ $\therefore -2-5<2-5$ $\therefore -7<-3$	$\because -6<6$ $\therefore -2(-6)<-2(6)$ $\therefore 12<-12$	$\because 4>2$ $\therefore 1/4>1/2$
$\because -2<4$ $\therefore -2(-2)>2(-2)$ $\therefore 4>-4$	$\because -6<6$ $\therefore -2(-6)>-2(6)$ $\therefore 12>-12$	$\because 6>-4$ $\therefore -2(6)<-2(-4)$ $\therefore -12<8$	$\because 4>2$ $\therefore 1/4>1/2$ $\therefore 1/4+2>1/2+2$ $\therefore 9/4>5/2$	$\because 4>2$ $\therefore 4^2>2^2$ $\therefore 1/4^2>1/2^2$ $\therefore 1/16>1/4$

Examples

Solve each of the following inequalities. Note that a good way to visualize the solution is to use a number line.

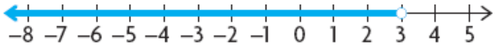
1. $-3x+1>-8$

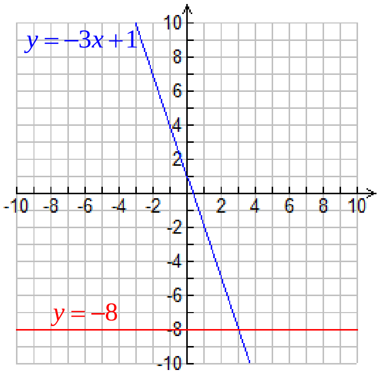
Solution

$$-3x+1>-8$$
$$\therefore -3x+1-1>-8-1$$
$$\therefore -3x>-9$$
$$\therefore \frac{-3x}{-3}<\frac{-9}{-3}$$
$$\therefore x<3$$

Remember to reverse the inequality when multiplying or dividing by a negative number.

The solution set of the inequality is $\{x\in\mathbb{R}:x<3\}$, or in interval notation, $(-\infty,3)$.



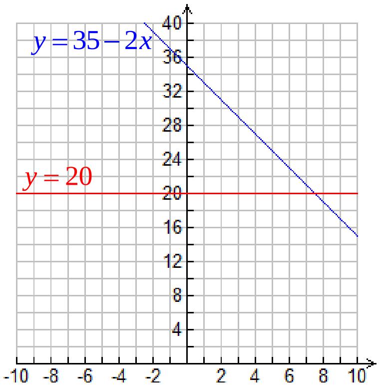


2. $35-2x\geq 20$

Solution

$$35-2x\geq 20$$
$$\therefore 35-2x-35\geq 20-35$$
$$\therefore -2x\geq -15$$
$$\therefore \frac{-2x}{-2}\leq\frac{-15}{-2}$$
$$\therefore x\leq\frac{15}{2}$$

The solution set of the inequality is $\left\{x\in\mathbb{R}:x\leq\frac{15}{2}\right\}$, or in interval notation, $\left(-\infty,\frac{15}{2}\right]$.

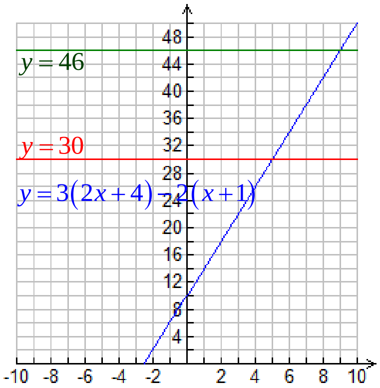


3. $30\leq 3(2x+4)-2(x+1)\leq 46$

Solution

$$30\leq 3(2x+4)-2(x+1)\leq 46$$
$$\therefore 30\leq 6x+12-2x-2\leq 46$$
$$\therefore 30\leq 4x+10\leq 46$$
$$\therefore 30-10\leq 4x+10-10\leq 46-10$$
$$\therefore 20\leq 4x\leq 36$$
$$\therefore \frac{20}{4}\leq\frac{4x}{4}\leq\frac{36}{4}$$
$$\therefore 5\leq x\leq 9$$

The solution set of the inequality is $\{x\in\mathbb{R}:5\leq x\leq 9\}$, or in interval notation, $[5,9]$.



4. $x^2-5x+6>0$

Solution

$$x^2-5x+6>0$$
$$\therefore (x-2)(x-3)>0$$

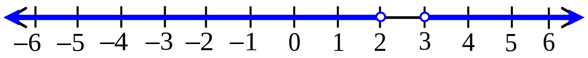
If the product $(x-2)(x-3)$ is positive, then either both factors are positive OR both factors are negative.

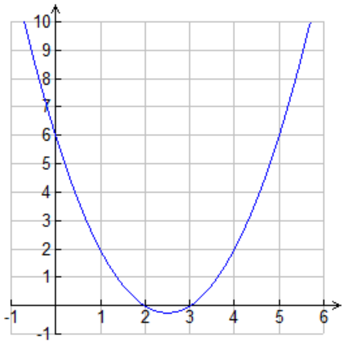
$\therefore x-2>0$ and $x-3>0$ ***or*** $x-2<0$ and $x-3<0$

$\therefore x>2$ and $x>3$ ***or*** $x<2$ and $x<3$

$\therefore x>3$ ***or*** $x<2$

The solution set of the inequality is $\{x\in\mathbb{R}:x<2 \text{ or } x>3\}$, or in interval notation, $(-\infty,2)\cup(3,\infty)$.





Shown above is the graph of $y=x^2-5x+6$. Notice that the parabola dips below the *x*-axis between *x* = 2 and *x* = 3 and that it is above the *x*-axis wherever *x* < 2 or *x* > 3. Remember that wherever a graph is ***above*** the *x*-axis, all *y*-values of points on that part of the graph are ***positive***. Similarly, wherever a graph is ***below*** the *x*-axis, all *y*-values of points on that part of the graph are ***negative***.

5. $2x^3+3x^2-17x+12>0$

Solution

$2x^3+3x^2-17x+12>0$

$\therefore (x-1)(2x^2+5x-12)>0$

$\therefore (x-1)(2x-3)(x+4)>0$

$\therefore (x-1)(x+4)(2x-3)>0$

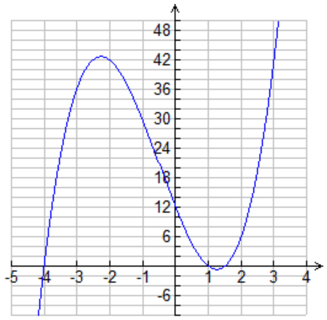
As can be seen from the factorization of $f(x)=2x^3+3x^2-17x+12$, its zeros are –4, 1 and $\frac{3}{2}$.

From the graph of $f(x)$ shown at the right, it's obvious that $f(x)>0$ (i.e. the graph is ***above*** the *x*-axis) if *x* is between –4 and 1 or if *x* is greater than $\frac{3}{2}$. Thus, the solution set of the inequality is $\left\{x\in\mathbb{R}:-4<x<1 \text{ or } x>\frac{3}{2}\right\}$, which can also be written as $(4,1)\cup\left(\frac{3}{2},\infty\right)$ using interval notation.

To verify this solution set, consider the table shown below.

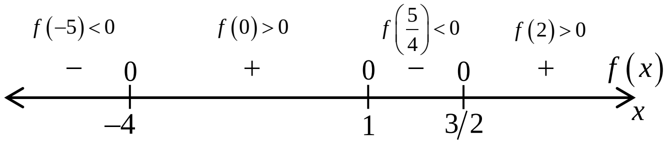
Let $f(x)=2x^3+3x^2-17x+12$. Since, $f(1)=0$ by the factor theorem, $x-1$ must be a factor of $f(x)$.

$$\begin{array}{r} 2x^2+5x-12 \\ x-1\overline{)2x^3+3x^2-17x+12} \\ \underline{2x^3-2x^2} \\ 5x^2-17x \\ \underline{5x^2-5x} \\ -12x+12 \\ \underline{-12x+12} \\ 0 \end{array}$$



	$x<-4$	$-4<x<1$	$1<x<\frac{3}{2}$	$x>\frac{3}{2}$
$(x-1)$	–	–	+	+
$(x+4)$	–	+	+	+
$\left(x-\frac{3}{2}\right)$	–	–	–	+
Their Product	–	+	–	+

We can also use a number line to determine the sign of $f(x)$ over each interval.

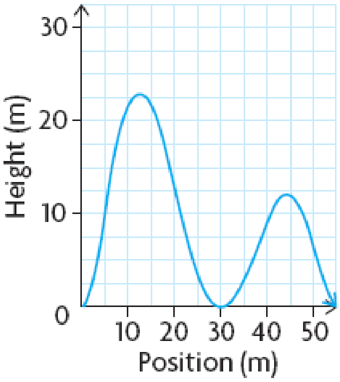


The zeros of a polynomial divide the real number line into a group of intervals that are sometimes called ***test intervals***.

Formally, a test interval of a polynomial function *p* is any interval of the form (a,b), such that $p(a)=0$, $p(b)=0$ and for all $x\in(a,b)$, $p(x)\neq 0$.

Over a test interval, a polynomial must either be entirely positive or entirely negative.

6. The height of one section of a rollercoaster can be modelled by the polynomial function $h(x)=\dfrac{1}{40000000}x^2(x-30)^2(x-55)^2$, where $h(x)$ is the height above the ground in metres, measured at the position x metres along the ground from the start. At what points will the rollercoaster car be more than 9 metres above the ground?

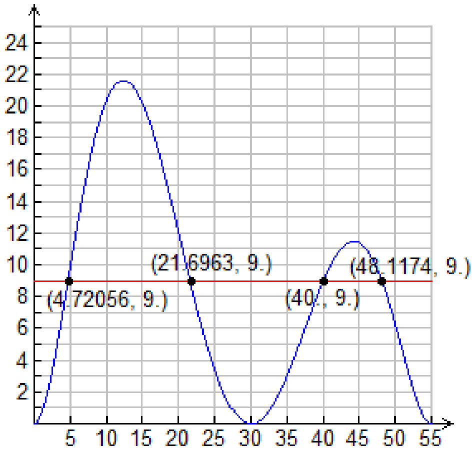


Solution

This problem is equivalent to solving the inequality $\dfrac{1}{40000000}x^2(x-30)^2(x-55)^2>9$.

Expanding and simplifying produces a sixth-degree polynomial inequality. Thanks to Abel, Galois and Ruffini, we know that polynomial equations of degree five or higher cannot in general be solved by a finite number of additions, subtractions, multiplications, divisions and root extractions. Thus, it is best to use a graphical approach in this case.

By using graphing software or a graphing calculator, sketch the graphs of $y=\dfrac{1}{40000000}x^2(x-30)^2(x-55)^2$ and $y=9$. Then find the points of intersection. Once the points of intersection are found, it's easy to see the approximate solution set of the inequality. The roller coaster is more than 9 m above the ground wherever the graph of $y=\dfrac{1}{40000000}x^2(x-30)^2(x-55)^2$ lies above the graph of $y=9$.



Therefore, the rollercoaster will be more than 9 m above the ground approximately between 4.7 m and 21.7 m from the starting point and approximately between 40 m and 48.1 m from the starting point.

Homework

Precalculus (Ron Larson)

REMINDER

The "back of the book" is ***NOT*** the only resource available to you for verifying your solutions. You can also use Desmos (or any other graphing software) as well as reminding yourself ***always*** to ask the time-tested question "Is my answer reasonable?"

pp. 187 – 189: 1, 13-37 (odd-numbered questions), 53-56, 67-70, 73-75, 77-80, 83-90

INVESTIGATING RATIONAL FUNCTIONS

What is a Rational Function?

Just as a ***rational number*** is the ***ratio of two integers***, a ***rational function*** is the ***ratio of two polynomial functions***.

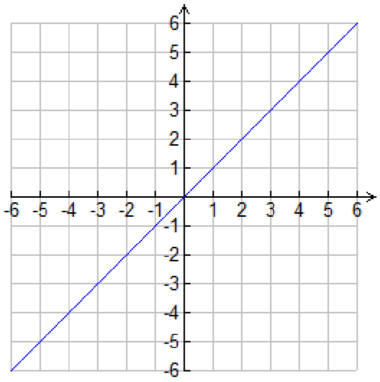
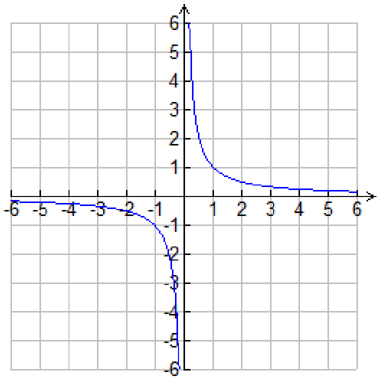
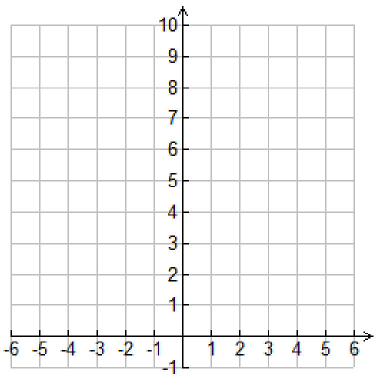
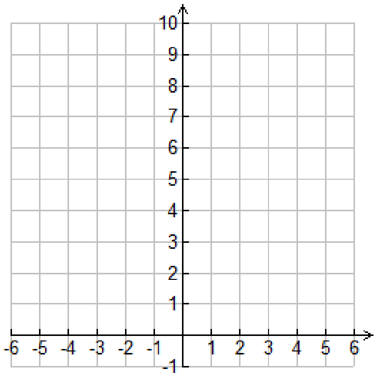
- ***Rational Numbers*** have the form $r=\frac{a}{b}$ where $a\in\mathbb{Z},\ b\in\mathbb{Z},\ \mathrm{b}\neq 0$ (*a* and *b* are integers, *b* must be nonzero)
- ***Rational Functions*** have the form $q(x)=\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions such that $g(x)\neq 0$.

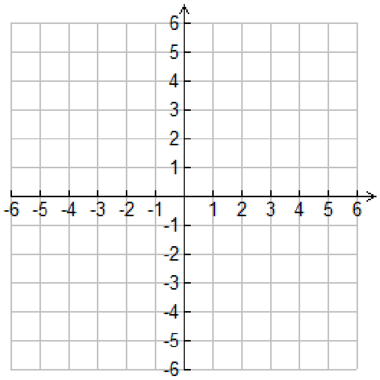
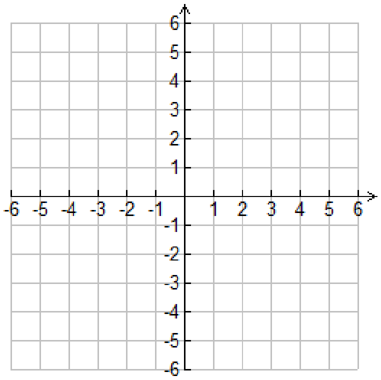
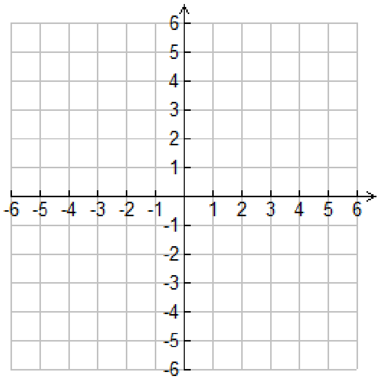
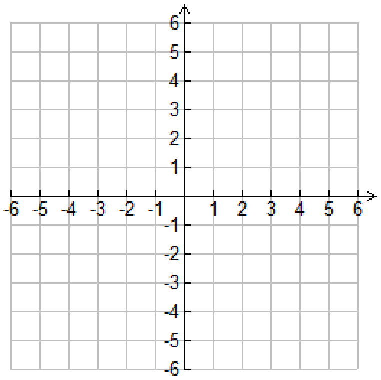
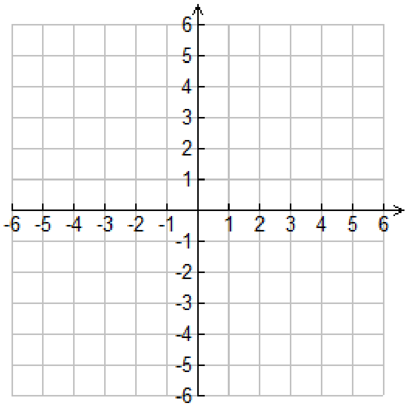
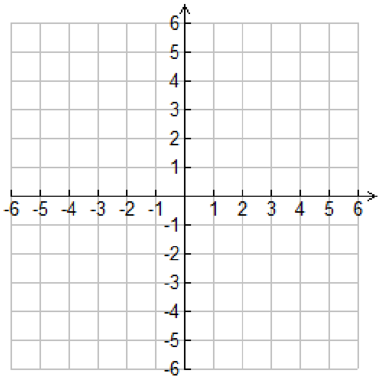
Graphs of the Simplest Rational Functions – Reciprocals of Polynomial Functions

Use a graphing calculator or graphing software to complete the following table.

Legend

VA: Vertical Asymptote(s)
HA: Horizontal Asymptote(s)
IP: Intervals on which the Function is Positive
IN: Intervals on which the Function is Negative
II: Intervals on which the Function is Increasing
ID: Intervals on which the Function is Decreasing
ICU: Intervals on which the Function is Concave Up
ICD: Intervals on which the Function is Concave Down
PON: Points at which the Function is 1 or –1

Graph of Function	***Characteristics of Function***	***Graph of the Reciprocal of the Function***	***Characteristics of the Reciprocal of the Function***
$f(x)=x$ 	Zeros: 1 (at $x=0$) VA: none HA: none IP: $(0,\infty)$ IN: $(-\infty,0)$ II: $(-\infty,\infty)$ ID: none ICU: none ICD: none PON: $(-1,-1)$, $(1,1)$	$g(x)=\frac{1}{f(x)}=\frac{1}{x}$ 	Zeros: none VA: $x=0$ HA: $y=0$ IP: $(0,\infty)$ IN: $(-\infty,0)$ II: none ID: $(-\infty,0),(0,\infty)$ ICU: $(0,\infty)$ ICD: $(-\infty,0)$ PON: $(-1,-1)$, $(1,1)$
$f(x)=(x-1)^2$ 	Zeros: ______________ VA: ______________ HA: ______________ IP: ______________ IN: ______________ II: ______________ ID: ______________ ICU: ______________ ICD: ______________ PON: ______________	$g(x)=\frac{1}{f(x)}=\frac{1}{(x-1)^2}$ 	Zeros: ______________ VA: ______________ HA: ______________ IP: ______________ IN: ______________ II: ______________ ID: ______________ ICU: ______________ ICD: ______________ PON: ______________

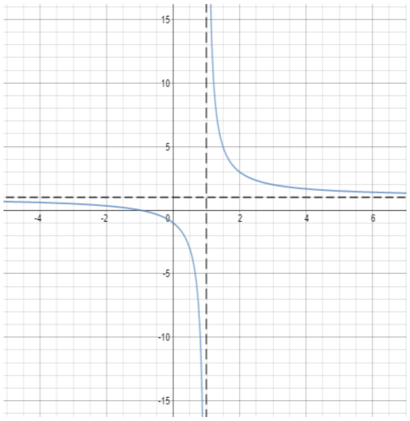
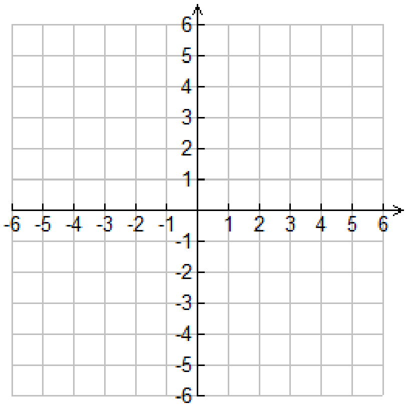
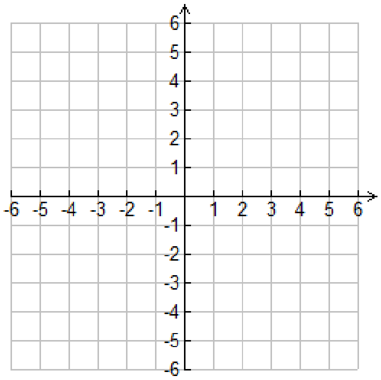
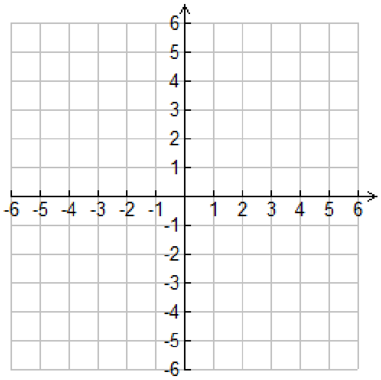
Graph of Function	***Characteristics of Function***	***Graph of its Reciprocal***	***Characteristics of the Reciprocal of the Function***
$f(x)=x^2-4$ 	Zeros: ____________ VA: ____________ HA: ____________ IP: ____________ IN: ____________ II: ____________ ID: ____________ ICU: ____________ ICD: ____________ PON: ____________	$g(x)=\frac{1}{f(x)}=\frac{1}{x^2-4}$ 	Zeros: ____________ VA: ____________ HA: ____________ IP: ____________ IN: ____________ II: ____________ ID: ____________ ICU: ____________ ICD: ____________ PON:____________
$f(x)=2x-3$ 	Zeros: ____________ VA: ____________ HA: ____________ IP: ____________ IN: ____________ II: ____________ ID: ____________ ICU: ____________ ICD: ____________ PON: ____________	$g(x)=\frac{1}{f(x)}=\frac{1}{2x-3}$ 	Zeros: ____________ VA: ____________ HA: ____________ IP: ____________ IN: ____________ II: ____________ ID: ____________ ICU: ____________ ICD: ____________ PON:____________
$f(x)=(x-2)(x+3)$ 	Zeros: ____________ VA: ____________ HA: ____________ IP: ____________ IN: ____________ II: ____________ ID: ____________ ICU: ____________ ICD: ____________ PON: ____________	$g(x)=\frac{1}{f(x)}=\frac{1}{(x-2)(x+3)}$ 	Zeros: ____________ VA: ____________ HA: ____________ IP: ____________ IN: ____________ II: ____________ ID: ____________ ICU: ____________ ICD: ____________ PON: ____________

Graphs of Rational Functions of the Form $f(x)=\frac{ax+b}{cx+d}$ *(Quotients of Linear Polynomials)*

Legend

The following abbreviations are used in addition those used on page 35: **D**: Domain **R**: Range **X**: *x*-intercept(s) **Y**:*y*-intercept “ $x\to 1^-$ ” ***means*** “as *x* approaches 1 from the left” “ $x\to 1^+$ ” ***means*** “as *x* approaches 1 from the right”

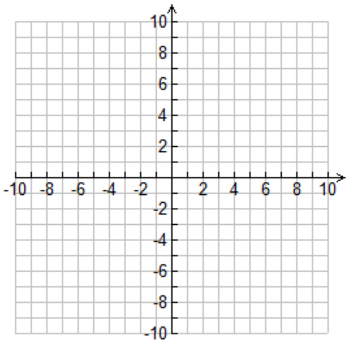
Complete the following table.

Graph and Characteristics of Rational Function	*Graph and Characteristics of Rational Function*
$f(x)=\frac{x+1}{x-1}$  As $x\to\infty$, $f(x)\to 1$ As $x\to-\infty$, $f(x)\to 1$ As $x\to 1^-$, $f(x)\to-\infty$ As $x\to 1^+$, $f(x)\to\infty$ Zeros: 1 (at $x=-1$) VA: $x=1$ HA: $y=1$ IP: $(-\infty,-1)$, $(1,\infty)$ IN: $(-1,1)$ II: none ID: $(-\infty,1)$, $(1,\infty)$ ICU: $(1,\infty)$ ICD: $(-\infty,1)$ D: $\{x\in\mathbb{R}:x\neq 1\}$ R: $\{y\in\mathbb{R}:y\neq 1\}$ X: -1 Y: -1	$f(x)=\frac{x}{x-3}$  As $x\to\infty$, $f(x)\to$ ________ As $x\to-\infty$, $f(x)\to$ ________ As $x\to 3^-$, $f(x)\to$ ________ As $x\to 3^+$, $f(x)\to$ ________ Zeros: ____________ VA: ____________ HA: ____________ IP: ____________ IN: ____________ II: ____________ ID: ____________ ICU: ____________ ICD: ____________ D: ____________ R: ____________ X: ____________ Y: ____________
$f(x)=\frac{x-2}{3x+4}$  As $x\to\infty$, $f(x)\to$ ________ As $x\to-\infty$, $f(x)\to$ ________ As $x\to-\frac{4}{3}^-$, $f(x)\to$ ________ As $x\to-\frac{4}{3}^+$, $f(x)\to$ ________ Zeros: ____________ VA: ____________ HA: ____________ IP: ____________ IN: ____________ II: ____________ ID: ____________ ICU: ____________ ICD: ____________ D: ____________ R: ____________ X: ____________ Y: ____________	$f(x)=\frac{x-3}{2x-6}$  As $x\to\infty$, $f(x)\to$ ________ As $x\to-\infty$, $f(x)\to$ ________ As $x\to 3^-$, $f(x)\to$ ________ As $x\to 3^+$, $f(x)\to$ ________ Zeros: ____________ VA: ____________ HA: ____________ IP: ____________ IN: ____________ II: ____________ ID: ____________ ICU: ____________ ICD: ____________ D: ____________ R: ____________ X: ____________ Y: ____________

Graphs of other Rational Functions

Graph and Characteristics of Rational Function

$$f(x)=\frac{9x}{1+x^2}$$



Zeros: ______

VA: ________

HA: ________

IP: ________

IN: ________

II: ________

ID: ________

ICU: ________

ICD: ________

D: ________

R: ________

X: ________

Y: ________

As $x \to \infty$, $f(x) \to$ ____

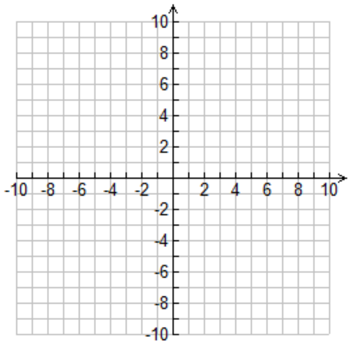
As $x \to -\infty$, $f(x) \to$ ____

As $x \to 1^-$, $f(x) \to$ ____

As $x \to 1^+$, $f(x) \to$ ____

Graph and Characteristics of Rational Function

$$f(x)=\frac{x+1}{x^2-2x-3}$$



Zeros: ____________________

VA: ____________________

HA: ____________________

IP: ____________________

IN: ____________________

II: ____________________

ID: ____________________

ICU: ____________________

ICD: ____________________

D: ____________________

R: ____________________

X: ____________________

Y: ____________________

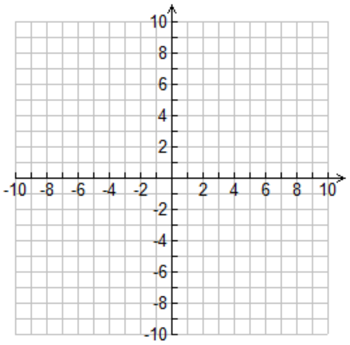
As $x \to \infty$, $f(x) \to$ _____

As $x \to -\infty$, $f(x) \to$ _____

As $x \to 3^-$, $f(x) \to$ _____

As $x \to 3^+$, $f(x) \to$ _____

$$f(x)=\frac{x^2-1}{x-1}$$



Zeros: ______

VA: ________

HA: ________

IP: ________

IN: ________

II: ________

ID: ________

ICU: ________

ICD: ________

D: ________

R: ________

X: ________

Y: ________

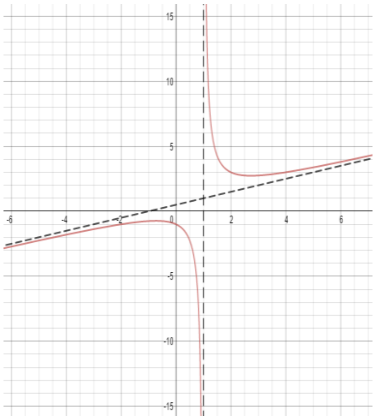
As $x \to \infty$, $f(x) \to$ ____

As $x \to -\infty$, $f(x) \to$ ____

As $x \to 1^-$, $f(x) \to$ ____

As $x \to 1^+$, $f(x) \to$ ____

$$f(x)=\frac{0.5x^2+1}{x-1}$$



Zeros: none

VA: $x=1$

HA: none

Oblique (Slant) Asymptote (OA): $y=0.5x+0.5$

IP: $(1,\infty)$

IN: $(-\infty,-1)$

II: $(-\infty,-0.73)$, $(2.73,\infty)$

ID: $(-0.73,1)$, $(1,2.73)$

ICU: $(1,\infty)$

ICD: $(-\infty,1)$

D: $\{x\in\mathbb{R}:x\neq 1\}$

R: $\{y\in\mathbb{R}:y\leq -0.73 \text{ or } y\geq 2.73\}$

X: None

Y: -1

As $x \to \infty$, $f(x) \to 0.5x+0.5$

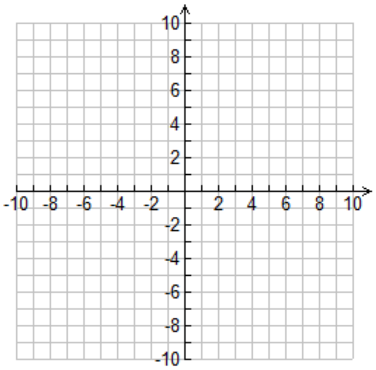
As $x \to -\infty$, $f(x) \to 0.5x+0.5$

As $x \to 1^-$, $f(x) \to -\infty$

As $x \to 1^+$, $f(x) \to \infty$

Graph and Characteristics of Rational Function

$$f(x)=\frac{2x^3}{x^2-1}$$



Zeros: ________

VA: ________

OA: ________

IP: ________

IN: ________

II: ________

ID: ________

ICU: ________

ICD: ________

D: ________

R: ________

X: ________

Y: ________

As $x\to\infty$, $f(x)\to$ ________

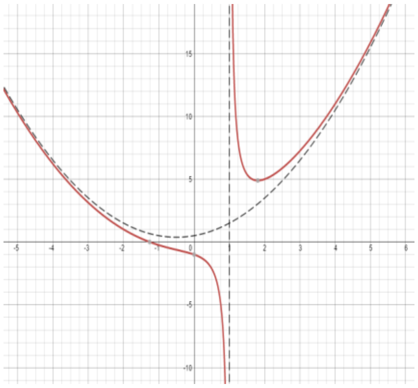
As $x\to-\infty$, $f(x)\to$ ________

As $x\to 1^-$, $f(x)\to$ ________

As $x\to 1^+$, $f(x)\to$ ________

Graph and Characteristics of Rational Function

$$f(x)=\frac{0.5x^3+1}{x-1}$$



Zeros: 1 ($x\doteq-1.26$)

VA: $x=1$

HA: none

Curvilinear Asymptote

(CA): $y=0.5x^2+0.5x+0.5$

IP: $(-\infty,-1.26)$, $(1.806,\infty)$

IN: $(-1.26,1)$

II: $(1.806,\infty)$

ID: $(-\infty,1)$, $(1,1.806)$

ICU: $(-\infty,-0.44)$, $(0,\infty)$

ICD: $(-0.44,0)$

D: $\{x\in\mathbb{R}: x\neq 1\}$

R: $\mathbb{R}$

X: -1.26

Y: -1

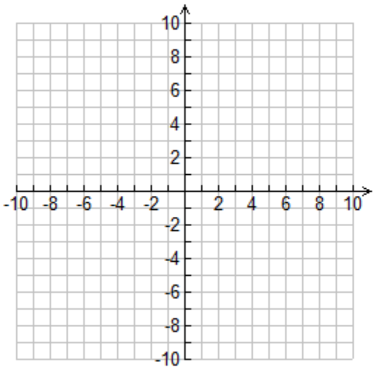
As $x\to\infty$, $f(x)\to\infty$

As $x\to-\infty$, $f(x)\to\infty$

As $x\to 1^-$, $f(x)\to-\infty$

As $x\to 1^+$, $f(x)\to\infty$

$$f(x)=\frac{x^3+1}{x+1}$$



Zeros: ________

VA: ________

HA: ________

IP: ________

IN: ________

II: ________

ID: ________

ICU: ________

ICD: ________

D: ________

R: ________

X: ________

Y: ________

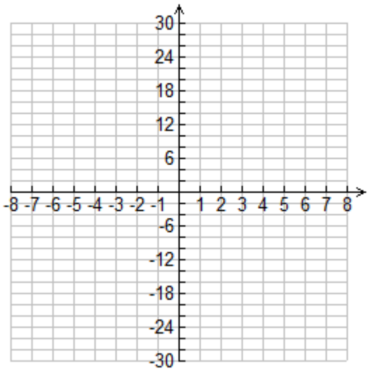
As $x\to\infty$, $f(x)\to$ ______

As $x\to-\infty$, $f(x)\to$ ______

As $x\to 1^-$, $f(x)\to$ ______

As $x\to 1^+$, $f(x)\to$ ______

$$f(x)=\frac{x^3-2x-3}{x+1}$$



Zeros: ________________

VA: ________________

CA: ________________

IP: ________________

IN: ________________

II: ________________

ID: ________________

ICU: ________________

ICD: ________________

D: ________________

R: ________________

X: ________________

Y: ________________

As $x\to\infty$, $f(x)\to$ ________

As $x\to-\infty$, $f(x)\to$ ________

As $x\to -1^-$, $f(x)\to$ ________

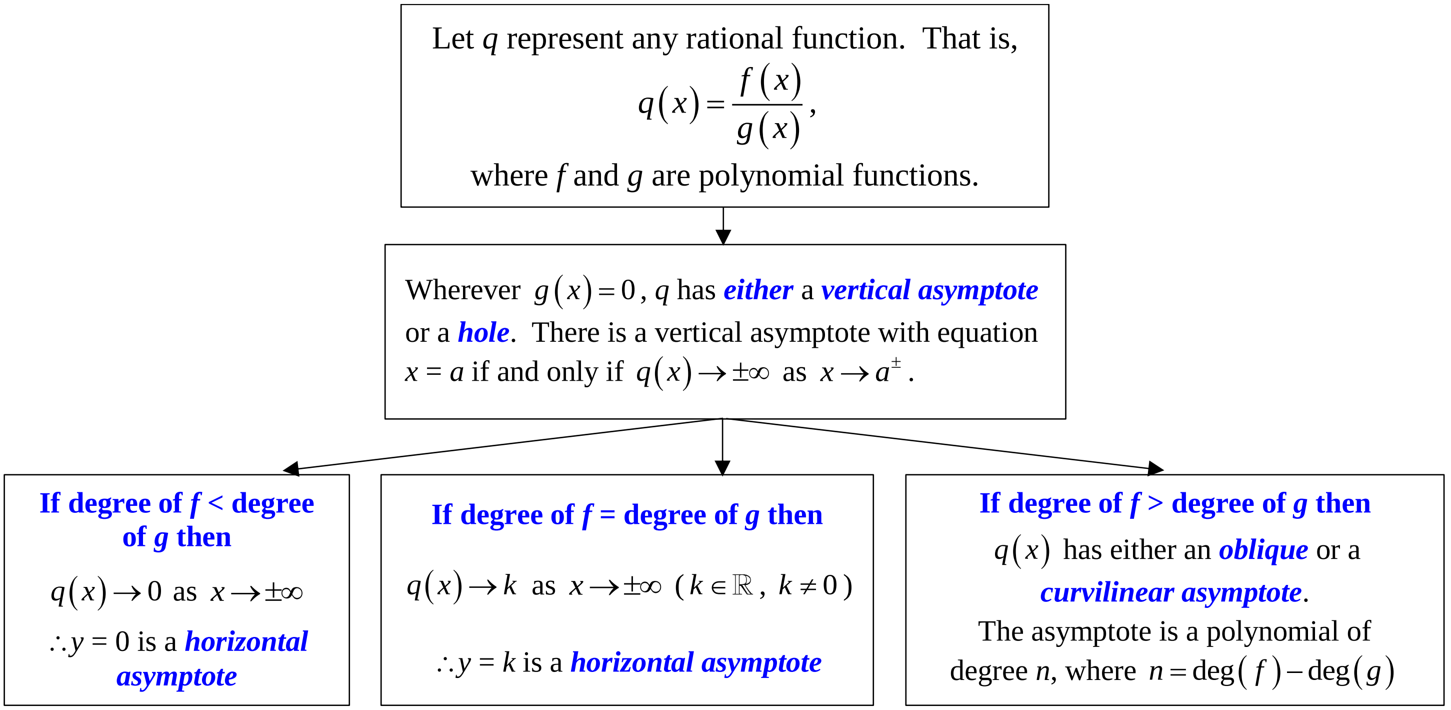
As $x\to -1^+$, $f(x)\to$ ________

Legend

VA: Vertical Asymptote(s) **HA**: Horizontal Asymptote(s) **IP**: Intervals on which the Function is Positive
IN: Intervals on which the Function is Negative **II**: Intervals on which the Function is Increasing
ID: Intervals on which the Function is Decreasing **ICU**: Intervals on which the Function is Concave Up
ICD: Intervals on which the Function is Concave Down **PON**: Points at which the Function is 1 or –1
D: Domain **R**: Range **X**: *x*-intercept(s) **Y**: *y*-intercept **OA**: Oblique Asymptote
CA: Curvilinear Asymptote

Summary – General Characteristics of Rational Functions

Reciprocals of Polynomials	***Rational Functions of the Form*** $f(x)=\frac{ax+b}{cx+d}$	***General Rational Functions***
Let *f* represent a polynomial function of degree one or greater. • Wherever $y=f(x)$ has a ***zero***, $y=\frac{1}{f(x)}$ has a ***vertical asymptote***. • The *x*-axis (i.e. the line $y=0$) is always a ***horizontal asymptote*** of $y=\frac{1}{f(x)}$. • Wherever $y=f(x)$ ***increases***, $y=\frac{1}{f(x)}$ ***decreases*** (and vice versa) • Since $y=f(x)$ and $y=\frac{1}{f(x)}$ have the same sign, if $y=f(x)$ lies above the *x*-axis so does $y=\frac{1}{f(x)}$ (and vice versa). • A point whose *y*-co-ordinate is ±1 is ***invariant***. That is, such a point lies on both $y=f(x)$ and $y=\frac{1}{f(x)}$.	• $y=f(x)$ has either a ***vertical asymptote*** or a ***hole*** at $x=-\frac{d}{c}$. • A hole will occur if $ax+b$ and $cx+d$ have a common factor that "divides out" in such a way that only a constant remains. More precisely, a "hole" occurs at $x=-\frac{d}{c}$ if $ax+b=k(cx+d)$ for some non-zero real number *k*. In this case, $f(x)=\frac{k(cx+d)}{cx+d}=k$, where $k\in\mathbb{R}$ and $x\neq-\frac{d}{c}$. • As $x\to\pm\infty$, $f(x)\to\frac{a}{c}$. Therefore, $y=f(x)$ has a ***horizontal asymptote*** at $x=\frac{a}{c}$.	Let *f* and *g* represent polynomial functions and define the rational function *q* as $q(x)=\frac{f(x)}{g(x)}$. • Wherever $g(x)=0$, $q(x)$ either has a ***vertical asymptote*** or a ***hole***. • If $q(x)\to k$ as $x\to\pm\infty$ ($k\in\mathbb{R}$), then $q(x)$ has a ***horizontal asymptote*** at $y=k$. Note that this can occur only if the degree of $f(x)$ is less than or equal to the degree of $g(x)$. • If the degree of $f(x)$ is ***exactly one greater*** than the degree of $g(x)$, then $q(x)$ has an ***oblique (slant) asymptote***. • If the degree of $f(x)$ is exactly *n* greater than the degree of $g(x)$, then $q(x)$ has a ***curvilinear asymptote*** that is a ***polynomial of degree n***.



Let *q* represent any rational function. That is,

$$q(x)=\frac{f(x)}{g(x)},$$

where *f* and *g* are polynomial functions.

Wherever $g(x)=0$, *q* has ***either*** a ***vertical asymptote*** or a ***hole***. There is a vertical asymptote with equation $x=a$ if and only if $q(x)\to\pm\infty$ as $x\to a^{\pm}$.

If degree of *f* < degree of *g* then

$q(x)\to 0$ as $x\to\pm\infty$

$\therefore y=0$ is a ***horizontal asymptote***

If degree of *f* = degree of *g* then

$q(x)\to k$ as $x\to\pm\infty$ $(k\in\mathbb{R},\ k\neq 0)$

$\therefore y=k$ is a ***horizontal asymptote***

If degree of *f* > degree of *g* then

$q(x)$ has either an ***oblique*** or a ***curvilinear asymptote***.

The asymptote is a polynomial of degree *n*, where $n=\deg(f)-\deg(g)$

How to find the Equation of an Oblique (Slant) or Curvilinear Asymptote

Examples

(a) $f\left(x\right)=\dfrac{0.5x^2+1}{x-1}$

$$\begin{array}{r}
0.5x+0.5 \\
x-1\overline{\smash{)}0.5x^2+0x+1} \\
\underline{0.5x^2-0.5x} \\
0.5x+1 \\
\underline{0.5x-0.5} \\
1.5
\end{array}$$

$$\therefore 0.5x^2+1=\left(x-1\right)\left(0.5x+0.5\right)+1.5$$

$$\therefore \frac{0.5x^2+1}{x-1}=\frac{\left(x-1\right)\left(0.5x+0.5\right)}{x-1}+\frac{1.5}{x-1}$$

$$\therefore \frac{0.5x^2+1}{x-1}=0.5x+0.5+\frac{1.5}{x-1}$$

Since $\dfrac{1.5}{x-1}\to 0$ as $x\to\infty$, $\dfrac{0.5x^2+1}{x-1}\to 0.5x+0.5$ as $x\to\infty$. This means that $y=0.5x+0.5$ is an ***oblique (slant) asymptote*** of *f*.

(b) $f\left(x\right)=\dfrac{0.5x^3+1}{x-1}$

$$\begin{array}{r}
0.5x^2+0.5x+0.5 \\
x-1\overline{\smash{)}0.5x^3+0x^2+0x+1} \\
\underline{0.5x^3-0.5x^2} \\
0.5x^2+0x \\
\underline{0.5x^2-0.5x} \\
0.5x+1 \\
\underline{0.5x-0.5} \\
1.5
\end{array}$$

$$\therefore 0.5x^3+1=\left(x-1\right)\left(0.5x^2+0.5x+0.5\right)+1.5$$

$$\therefore \frac{0.5x^3+1}{x-1}=\frac{\left(x-1\right)\left(0.5x^2+0.5x+0.5\right)}{x-1}+\frac{1.5}{x-1}$$

$$\therefore \frac{0.5x^3+1}{x-1}=0.5x^2+0.5x+0.5+\frac{1.5}{x-1}$$

Since $\dfrac{1.5}{x-1}\to 0$ as $x\to\infty$, $\dfrac{0.5x^3+1}{x-1}\to 0.5x^2+0.5x+0.5$ as $x\to\infty$. This means that $y=0.5x^2+0.5x+0.5$ is a ***curvilinear asymptote*** of *f*.

Homework

Precalculus (Ron Larson)

REMINDER

The “back of the book” is ***NOT*** the only resource available to you for verifying your solutions. You can also use Desmos (or any other graphing software) as well as reminding yourself ***always*** to ask the time-tested question “Is my answer reasonable?”

pp. 177–179: 1-4, 5, 7, 17-39 (odd-numbered questions), 45-48, 49-61 (odd-numbered questions), 63-66, 71, 73, 75, 77-82

Enrichment Problem

Find an example of a rational function *f* that has ***ALL*** of the following characteristics:

(a) one hole
(b) two vertical asymptotes
(c) one curvilinear cubic polynomial asymptote

Use Desmos or any other graphing software to verify that *f* has all the required properties. In addition, determine the following features of *f*:

- intervals of increase/decrease
- intervals over which *f* is concave up/concave down
- co-ordinates of turning points and points of inflection
- intercepts
- equations of asymptotes

RATIONAL EQUATIONS AND INEQUALITIES

Example 1

When eating together, it takes Peter and Homer ***two hours*** to eat a certain number of hamburgers. By himself, Homer can eat the same number of hamburgers in ***fifteen fewer minutes*** than it takes Peter to do the same. How long does it take Peter to eat the hamburgers by himself?

Solution

Let t represent the amount of time, in minutes, that it takes Peter to eat all the burgers. Then $t – 15$ represents the time that it takes Homer to do the same. In addition,

- $\frac{1}{t}$ represents the ***fraction*** of burgers eaten by Peter in ***one minute*** when eating alone
- $\frac{1}{t-15}$ represents the ***fraction*** of burgers eaten by Homer in ***one minute*** when eating alone
- $\frac{1}{120}$ represents the ***fraction*** of burgers eaten by Peter and Homer in ***one minute*** when eating together

> To understand this, consider an example. Suppose that it takes Homer 180 minutes to eat all the burgers by himself. This means that in one minute, he eats $\frac{1}{180}$ of the burgers.
>
> Also, $t-15>0$ because it takes Homer more than zero minutes to eat all the burgers. Therefore, $t>15$.

(Fraction Peter eats in one minute) + (Fraction Homer eats in one minute) = fraction they eat together in one minute

Therefore, $\frac{1}{t}+\frac{1}{t-15}=\frac{1}{120}$.

$$\therefore 120t(t-15)\left(\frac{1}{t}+\frac{1}{t-15}\right)=120t(t-15)\left(\frac{1}{120}\right)$$

$$\therefore \left(\frac{120\overset{1}{\cancel{t}}(t-15)}{1}\right)\left(\frac{1}{\underset{1}{\cancel{t}}}\right)+\left(\frac{120t\overset{1}{\cancel{(t-15)}}}{1}\right)\left(\frac{1}{\underset{1}{\cancel{t-15}}}\right)=\left(\frac{\overset{1}{\cancel{120}}t(t-15)}{1}\right)\left(\frac{1}{\underset{1}{\cancel{120}}}\right)$$

> Yikes! Mr. Nolfi is stooping to *cancelling*! You may also use this ***shortcut*** if you ***understand*** its mathematical ***justification***; ***both*** numerator and denominator are ***divided*** by the ***same*** value. This only works if ***both*** numerator and denominator are expressed in ***factored form***.

$$\therefore 120(t-15)+120t=t(t-15)$$

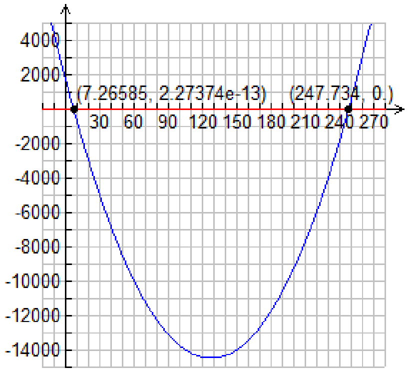
$$\therefore 120t+120t-1800=t^2-15t$$

$$\therefore t^2-255t+1800=0$$

$$\therefore t=\frac{-(-255)\pm\sqrt{(-255)^2-4(1)(1800)}}{2(1)}$$

$$\therefore t=\frac{255\pm\sqrt{57825}}{2}=\frac{255\pm15\sqrt{257}}{2}$$

$$\therefore t=\xcancel{\frac{255-15\sqrt{257}}{2}}\doteq 7.27 \text{ or } t=\frac{255+15\sqrt{257}}{2}\doteq 247.73 \checkmark$$



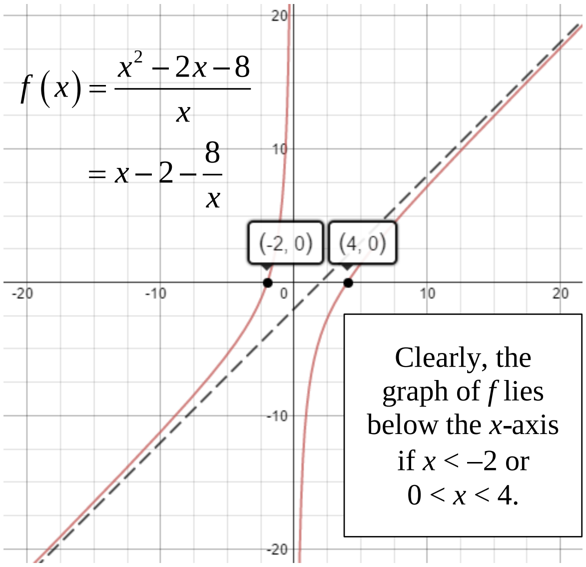
Clearly, 7.27 minutes ***cannot be*** the correct answer. First of all, we observed above that $t>15$. Furthermore, if it takes Homer and Peter ***two hours together*** to eat all the burgers, Peter ***cannot possibly*** eat all of them in only 7.27 minutes. Therefore, it must take Peter about 247.73 minutes (4 hours, 7 minutes, 44 seconds) to eat all the burgers.

Example 2

Solve $x-2<\frac{8}{x}$.

Solution

Since the value of x could be negative, multiplying both sides by x might require that the inequality be reversed. A solution that involves multiplying both sides by x requires two cases, one for $x>0$ and another for $x<0$. Instead of dividing the solution into two cases, it is easier to use the following approach.



$$f(x)=\frac{x^2-2x-8}{x}$$
$$=x-2-\frac{8}{x}$$

Clearly, the graph of f lies below the x-axis if $x<-2$ or $0<x<4$.

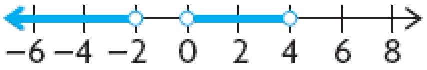
$$x-2<\frac{8}{x},\ x\neq 0$$
$$\therefore x-2-\frac{8}{x}<0$$
$$\therefore \frac{x^2}{x}-\frac{2x}{x}-\frac{8}{x}<0 \longleftarrow$$
$$\therefore \frac{x^2-2x-8}{x}<0$$
$$\therefore \frac{(x-4)(x+2)}{x}<0$$

Write each term with a common denominator

This final inequality states that the expression on the left side must be negative. A chart such as the following is an organized method for determining the intervals on which $\frac{(x-4)(x+2)}{x}$ is negative.

	$x<-2$	$-2<x<0$	$0<x<4$	$x>4$
$x-4$	$-$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$\frac{(x-4)(x+2)}{x}$	$\frac{(-)(-)}{-}=-$	$\frac{(-)(+)}{-}=+$	$\frac{(-)(+)}{+}=-$	$\frac{(+)(+)}{+}=+$

The expression $\frac{(x-4)(x+2)}{x}$ is negative if $x<-2$ or if $0<x<4$. Hence, the solution set of the inequality is $\{x\in\mathbb{R}: x<-2 \text{ or } 0<x<4\}$. In interval notation, this can be written $(-\infty,-2)\cup(0,4)$.



Homework

Precalculus (Ron Larson)

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pp. 187 – 189: 39-51 (odd-numbered questions), 57-60, 71, 72 ***See additional homework questions on the next page.***

Additional Homework Questions

11. Tayla purchased a large box of comic books for \$300. She gave 15 of the comic books to her brother and then sold the rest on an Internet website for \$330, making a profit of \$1.50 on each one. How many comic books were in the box? What was the original price of each comic book?

12. **A** Polluted water flows into a pond. The concentration of pollutant, c, in the pond at time t minutes is modelled by the equation $c(t) = 9 - 90\,000\left(\frac{1}{10\,000 + 3t}\right)$, where c is measured in kilograms per cubic metre.

a) When will the concentration of pollutant in the pond reach 6 kg/m^3?

b) What will happen to the concentration of pollutant over time?

13. **T** Three employees work at a shipping warehouse. Tom can fill an order in s minutes. Paco can fill an order in $s - 2$ minutes. Carl can fill an order in $s + 1$ minutes. When Tom and Paco work together, they take about 1 minute and 20 seconds to fill an order. When Paco and Carl work together, they take about 1 minute and 30 seconds to fill an order.

a) How long does each person take to fill an order?

b) How long would all three of them, working together, take to fill an order?

11. 75; \$4.00

12. a) After 6666.67 s

b) The function appears to approach 9 kg/m^3 as time increases.

13. a) Tom = 4 min; Carl = 5 min; Paco = 2 min

b) 6.4 min

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7. **K** a) Find all the values of x that make the following inequality true:
$$\frac{3x - 8}{2x - 1} > \frac{x - 4}{x + 1}$$

b) Graph the solution set on a number line. Write the solution set using interval notation and set notation.

9. **A** The equation $f(t) = \frac{5t}{t^2 + 3t + 2}$ models the bacteria count, in thousands, for a sample of tap water that is left to sit over time, t, in days. The equation $g(t) = \frac{15t}{t^2 + 9}$ models the bacteria count, in thousands, for a sample of pond water that is also left to sit over several days. In both models, $t > 0$. Will the bacteria count for the tap water sample ever exceed the bacteria count for the pond water? Justify your answer.

11. **T** An economist for a sporting goods company estimates the revenue and cost functions for the production of a new snowboard. These functions are $R(x) = -x^2 + 10x$ and $C(x) = 4x + 5$, respectively, where x is the number of snowboards produced, in thousands. The average profit is defined by the function $AP(x) = \frac{P(x)}{x}$, where $P(x)$ is the profit function. Determine the production levels that make $AP(x) > 0$.

12. **C** a) Explain why the inequalities $\frac{x + 1}{x - 1} < \frac{x + 3}{x + 2}$ and $\frac{x + 5}{(x - 1)(x + 2)} < 0$ are equivalent.

b) Describe how you would use a graphing calculator to solve these inequalities.

c) Explain how you would use a table to solve these inequalities.

7. a) $x < -1$, $-0.2614 < x < 0.5$, $x > 3.065$

b)

c) Interval notation: $(-\infty, -1)$, $(-0.2614, 0.5)$, $(3.065, \infty)$
Set notation: $\{x \in \mathbf{R} \mid x < -1, -0.2614 < x < 0.5, \text{ or } x > 3.065\}$

9. The only values that make the expression greater than 0 are negative. Because the values of t have to be positive, the bacteria count in the tap water will never be greater than that of the pond water.

11. when $x > 5$

12. a) The first inequality can be manipulated algebraically to produce the second inequality.

b) Graph the equation $y = \frac{x + 1}{x - 1} - \frac{x + 3}{x + 2}$ and determine when it is negative.

c) The values that make the factors of the second inequality zero are -5, -2, and 1. Determine the sign of each factor in the intervals corresponding to the zeros. Determine when the entire expression is negative by examining the signs of the factors.