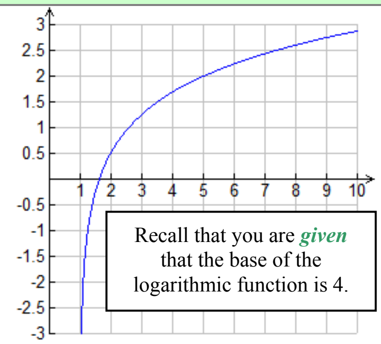
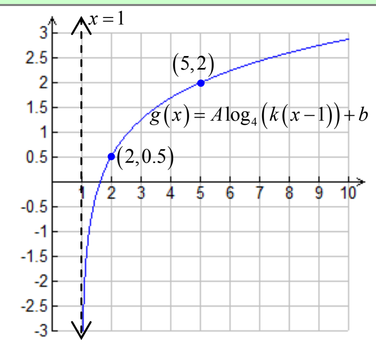
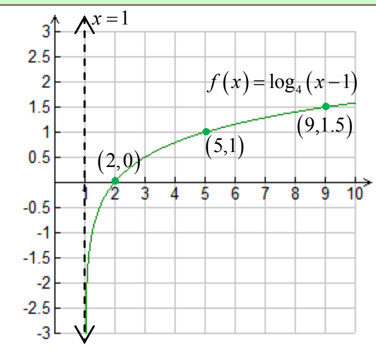
Determining the Equation of a Function Given an Accurate Graph

The Analysis

The given Graph	*A Detailed Look at the given Graph*	*A Detailed Look at its Pre-Image*
		

A close look at the given graph shows us that *g* must be a transformation of $f(x)=\log_4(x-1)$. The critical clue in recognizing this is the vertical asymptote with equation $x=1$. Thus, the equation of *g* must be of the form $g(x)=A\log_4(k(x-1))+b$.

Since the points $(2,0.5)$ and $(5,2)$ lie on the graph of *g*, they must satisfy the equation of *g*. Therefore we can we can write the following two equations:

$$A\log_4 k + b = 0.5 \qquad (1)$$
$$A\log_4 4k + b = 2 \qquad (2)$$

Subtracting equation (1) from equation (2) yields equation (3):

$$A\log_4 4k - A\log_4 k = 1.5 \qquad (3)$$

By manipulating equation (3), we can solve for *A*. Once we know the value of *A*, we shall be able to calculate the values of the other parameters (*k* and *b*).

$$A\log_4 4k - A\log_4 k = 1.5$$
$$\therefore A(\log_4 4k - \log_4 k) = 1.5$$
$$\therefore A\log_4 \frac{4k}{k} = 1.5$$
$$\therefore A\log_4 4 = 1.5$$
$$\therefore A(1) = 1.5$$
$$\therefore A = 1.5$$
$$\therefore g(x) = 1.5\log_4(k(x-1)) + b$$

Now we can determine the value(s) of *k* by substituting $A=1.5$ into equation (3):

$$1.5\log_4 4k - 1.5\log_4 k = 1.5 \qquad (4)$$
$$\therefore 1.5(\log_4 4k - \log_4 k) = 1.5$$
$$\therefore \log_4 4k - \log_4 k = 1$$

A close examination of this final equation (i.e. $\log_4 4k - \log_4 k = 1$) reveals that it is an ***identity***! This is easily shown as follows: L.S.$=\log_4 4k - \log_4 k = \log_4 \frac{4k}{k} = \log_4 4 = 1=$R.S. This means that we can choose any value of *k* whatsoever, provided that for such a choice, the logarithmic function is defined. Once the value of *k* is chosen, the value of *b* can be calculated.

e.g. 1: If *k* is chosen to have a value of ¼ = 0.25, then $b = 2 - 1.5\log_4(4(0.25)) = 2 - 1.5\log_4 1 = 2 - 0 = 2$

e.g. 2: If *k* is chosen to have a value of 1, then $b = 2 - 1.5\log_4(4(1)) = 2 - 1.5\log_4 4 = 2 - 1.5 = 0.5$

Important Conclusion

The above analysis shows us that there are ***infinitely many equivalent*** equations of the function *g*. The equations have the form

$$g(x)=1.5\log_4\left(k(x-1)\right)-1.5\log_4(4k)+2$$

Using the laws of logarithms, this equation can be simplified considerably.

$$\begin{aligned} & g(x)=1.5\log_4\left(k(x-1)\right)-1.5\log_4(4k)+2 \\ \therefore\ & g(x)=1.5\left(\log_4 k+\log_4(x-1)\right)-1.5\left(\log_4 4+\log_4 k\right)+2 \\ \therefore\ & g(x)=1.5\log_4 k+1.5\log_4(x-1)-1.5(1)-1.5\log_4 k+2 \\ \therefore\ & g(x)=1.5\log_4(x-1)+0.5, \end{aligned}$$

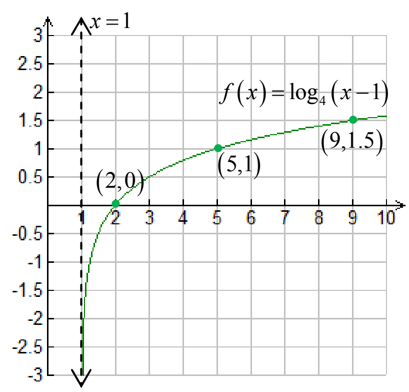
which is the same equation that we obtained by letting $k = 1$.

Thus, we have shown that the equation $1.5\log_4\left(k(x-1)\right)-1.5\log_4(4k)+2=1.5\log_4(x-1)+0.5$ is an identity for all values of *k* such that both $\log_4\left(k(x-1)\right)$ and $\log_4(4k)$ are defined. (If $k>0$, then $x>1$ for $\log_4\left(k(x-1)\right)$ to be defined. If $k<0$, then $x<1$ for $\log_4\left(k(x-1)\right)$ to be defined. For $\log_4(4k)$ to be defined, however, $k>0$. Therefore, the equation $1.5\log_4\left(k(x-1)\right)+2-1.5\log_4(4k)=1.5\log_4(x-1)+0.5$ is an identity for all $k>0$.)

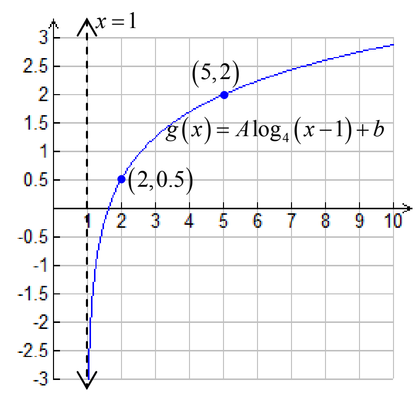
Furthermore, the "change of base" formula, $\log_b x=\dfrac{\log_a x}{\log_a b}$, can be used to write $\log_4(x-1)$ using any base, yielding even more possibilities!

A More Intuitive Approach

Now that we have seen that we can choose the value of *k* arbitrarily as long as $k>0$, let us take the bold step of choosing $k=1$. From the point of view of transformations, this means that there is no horizontal stretch or compression. Therefore, the graph of *g* can be obtained from the graph of *f* by using ***vertical transformations only***.



$$(2,0)\to(2,0.5)$$
$$(5,1)\to(5,2)$$
$$(x,y)\to(x,Ay+b)$$



$$\begin{aligned} \because\ & (x,y)\to(x,Ay+b) \\ \therefore\ & (2,0)\to(2,A(0)+b)=(2,b) \\ \because\ & (2,0)\to(2,0.5)\ \text{(see diagram above)} \\ \therefore\ & b=0.5 \\ \therefore\ & (x,y)\to(x,Ay+0.5) \\ \therefore\ & (5,1)\to(5,A(1)+0.5)=(5,A+0.5) \\ \because\ & (5,1)\to(5,2)\ \text{(see diagram above)} \\ \therefore\ & A+0.5=2 \\ \therefore\ & A=1.5 \end{aligned}$$