Investigating Rational Functions

What is a Rational Function?

Just as a ***rational number*** is the ***ratio of two integers***, a ***rational function*** is the ***ratio of two polynomial functions***.

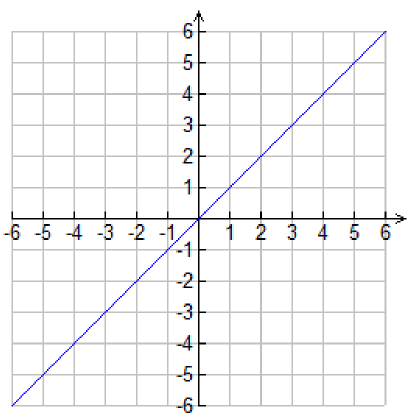
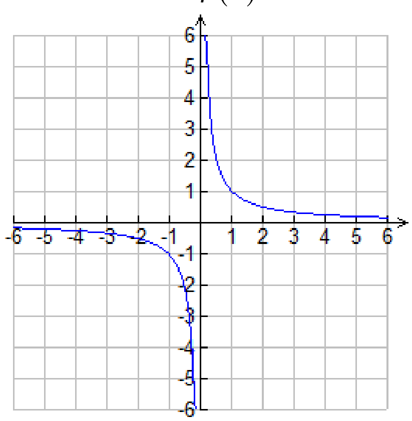
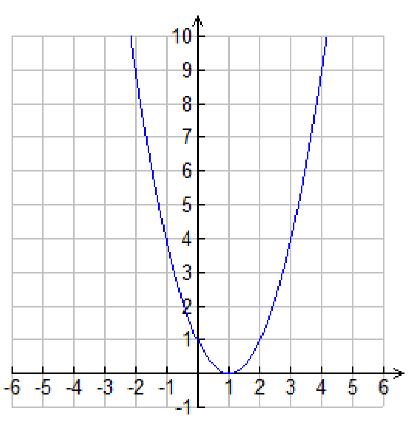
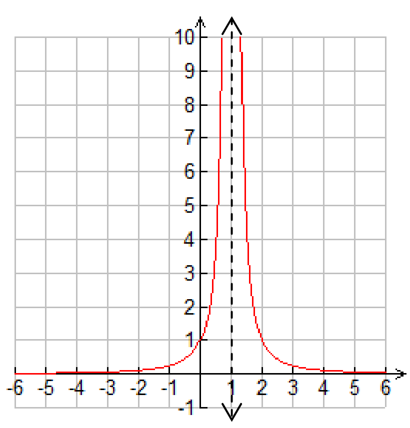
- ***Rational Numbers*** have the form $r = \frac{a}{b}$ where $a \in \mathbb{Z},\ b \in \mathbb{Z},\ \mathrm{b} \neq 0$ (*a* and *b* are integers, *b* must be nonzero)
- ***Rational Functions*** have the form $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions such that $q(x) \neq 0$.

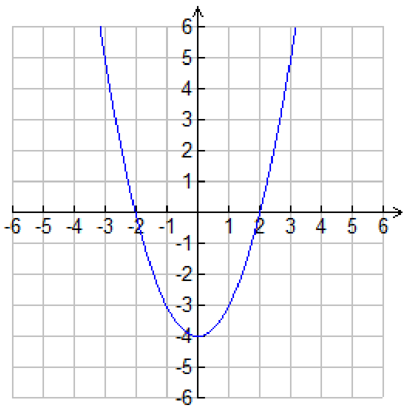
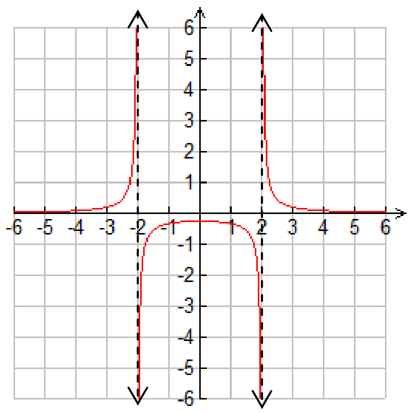
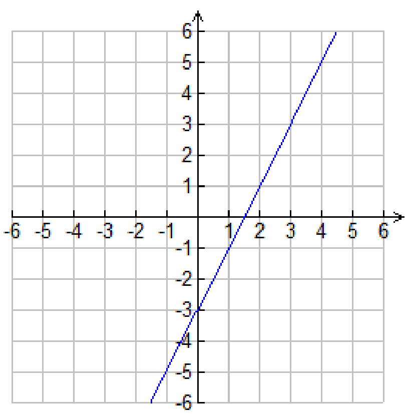
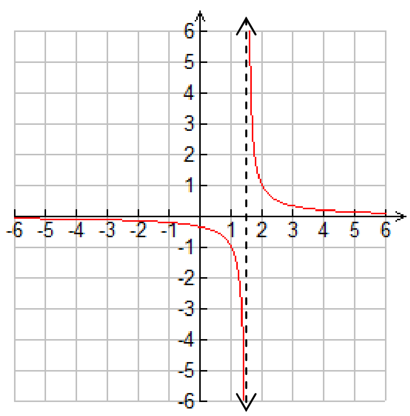
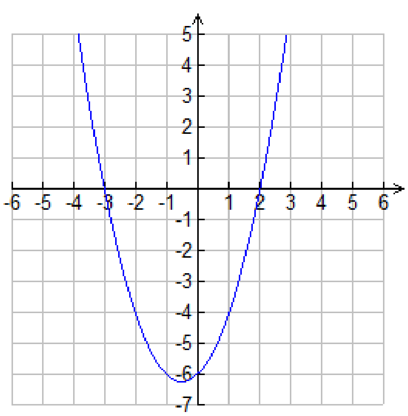
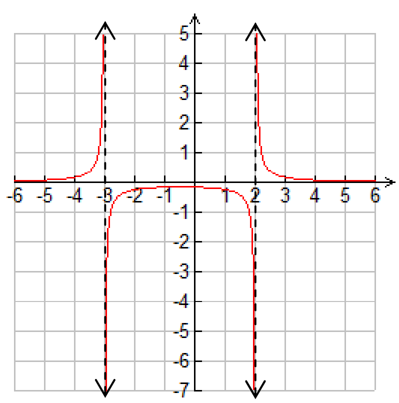
Graphs of the Simplest Rational Functions – Reciprocals of Polynomial Functions

Use a graphing calculator or graphing software to complete the following table.

Legend

VA→Vertical Asymptote(s) HA→Horizontal Asymptote(s) IP→Intervals on which the Function is Positive
IN→Intervals on which the Function is Negative II→Intervals on which the Function is Increasing
ID→Intervals on which the Function is Decreasing ICU→Intervals on which the Function is Concave Up
ICD→Intervals on which the Function is Concave Down PON→Points at which the Function is 1 or –1

Graph of Function	*Graph of its Reciprocal*	*Characteristics of Function*	*Characteristics of the Reciprocal of the Function*
$f(x) = x$ 	$g(x) = \frac{1}{f(x)} = \frac{1}{x}$ 	Zeros: 1 (at $x = 0$) VA: none HA: none IP: $(0, \infty)$ IN: $(-\infty, 0)$ II: $(-\infty, \infty)$ ID: none ICU: none ICD: none PON: $(-1,-1)$, $(1,1)$	Zeros: none VA: $x = 0$ HA: $y = 0$ IP: $(0, \infty)$ IN: $(-\infty, 0)$ II: none ID: $(-\infty, 0), (0, \infty)$ ICU: $(0, \infty)$ ICD: $(-\infty, 0)$ PON: $(-1,-1)$, $(1,1)$
$f(x) = (x-1)^2$ 	$g(x) = \frac{1}{f(x)} = \frac{1}{(x-1)^2}$ 	Zeros: 1 (at $x = 1$) VA: none HA: none IP: $(-\infty, 1)$, $(1, \infty)$ IN: none II: $(1, \infty)$ ID: $(-\infty, 1)$ ICU: $(-\infty, \infty)$ ICD: none PON: $(0,1)$, $(2,1)$	Zeros: none VA: $x = 1$ HA: $y = 0$ IP: $(-\infty, 1)$, $(1, \infty)$ IN: none II: $(-\infty, 1)$ ID: $(1, \infty)$ ICU: $(-\infty, 1)$, $(1, \infty)$ ICD: none PON: $(0,1)$, $(2,1)$

Graph of Function	*Graph of its Reciprocal*	*Characteristics of Function*	*Characteristics of the Reciprocal of the Function*
$f(x)=x^2-4$ 	$g(x)=\frac{1}{f(x)}=\frac{1}{x^2-4}$ 	Zeros: 2 (at $x=-2,2$) VA: none HA: none IP: $(-\infty,-2)$, $(2,\infty)$ IN: $(-2,2)$ II: $(0,\infty)$ ID: $(-\infty,0)$ ICU: $(-\infty,\infty)$ ICD: none PON: $(\sqrt{5},1)$, $(-\sqrt{5},1)$ $(\sqrt{3},-1)$, $(-\sqrt{3},-1)$	Zeros: none VA: $x=-2$, $x=2$ HA: $y=0$ IP: $(-\infty,-2)$, $(2,\infty)$ IN: none II: $(-\infty,-2)$, $(-2,0)$ ID: $(0,2)$, $(2,\infty)$ ICU: $(-\infty,-2)$, $(2,\infty)$ ICD: $(-2,2)$ PON: $(\sqrt{5},1)$, $(-\sqrt{5},1)$ $(\sqrt{3},-1)$, $(-\sqrt{3},-1)$
$f(x)=2x-3$ 	$g(x)=\frac{1}{f(x)}=\frac{1}{2x-3}$ 	Zeros: 1 (at $x=1.5$) VA: none HA: none IP: $(1.5,\infty)$ IN: $(-\infty,1.5)$ II: $(-\infty,\infty)$ ID: none ICU: none ICD: none PON: $(1,-1)$, $(2,1)$	Zeros: none VA: $x=1.5$ HA: $y=0$ IP: $(1.5,\infty)$ IN: $(-\infty,1.5)$ II: none ID: $(-\infty,1.5)$, $(1.5,\infty)$ ICU: , $(1.5,\infty)$ ICD: $(-\infty,1.5)$ PON: $(1,-1)$, $(2,1)$
$f(x)=(x-2)(x+3)$ 	$g(x)=\frac{1}{f(x)}=\frac{1}{(x-2)(x+3)}$ 	Zeros: 2 (at $x=-3,2$) VA: none HA: none IP: $(-\infty,-3)$, $(2,\infty)$ IN: $(-3,2)$ II: $(-0.5,\infty)$ ID: $(-\infty,-0.5)$ ICU: $(-\infty,\infty)$ ICD: none PON: $\left(\frac{1+\sqrt{29}}{2},1\right)$, $\left(\frac{1-\sqrt{29}}{2},1\right)$, $\left(\frac{1+\sqrt{21}}{2},-1\right)$ $\left(\frac{1-\sqrt{21}}{2},-1\right)$	Zeros: none VA: $x=1$ HA: $y=0$ IP: $(-\infty,-3)$, $(2,\infty)$ IN: $(-3,2)$ II: $(-\infty,1)$ ID: $(1,\infty)$ ICU: $(-\infty,-3)$, $(2,\infty)$ ICD: $(-3,2)$ PON: $\left(\frac{1+\sqrt{29}}{2},1\right)$, $\left(\frac{1-\sqrt{29}}{2},1\right)$, $\left(\frac{1+\sqrt{21}}{2},-1\right)$ $\left(\frac{1-\sqrt{21}}{2},-1\right)$

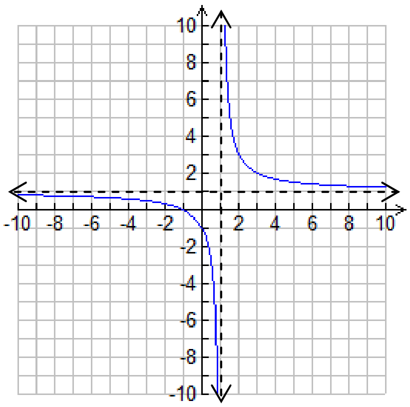
Graphs of Rational Functions of the Form $f(x)=\dfrac{ax+b}{cx+d}$ *(Quotients of Linear Polynomials)*

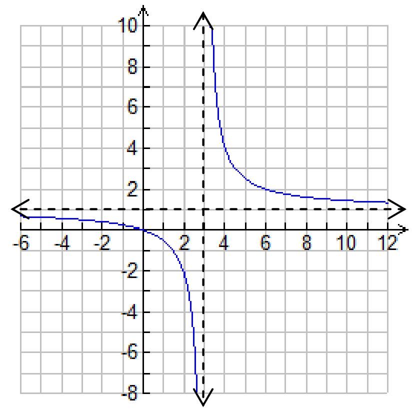
Complete the following table.

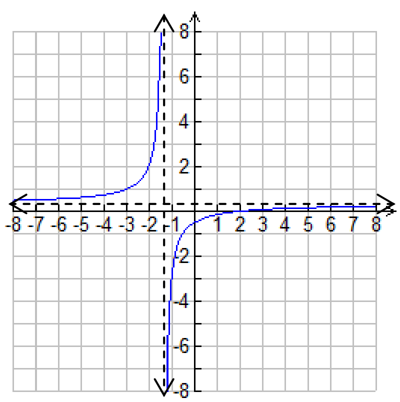
Legend

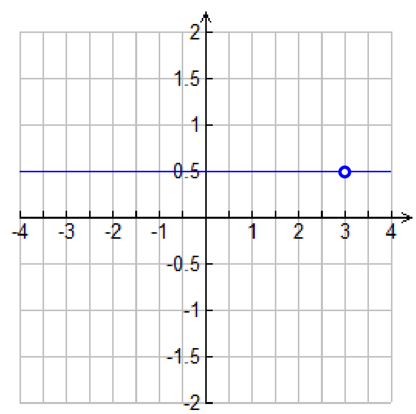
The following symbols are used in addition to the abbreviations given above.

D→Domain R→Range X→*x*-intercept(s) Y→*y*-intercept

<table>
<tr><th colspan="2">Graph and Characteristics of Rational Function</th><th colspan="2">Graph and Characteristics of Rational Function</th></tr>
<tr>
<td>$f(x)=\dfrac{x+1}{x-1}$


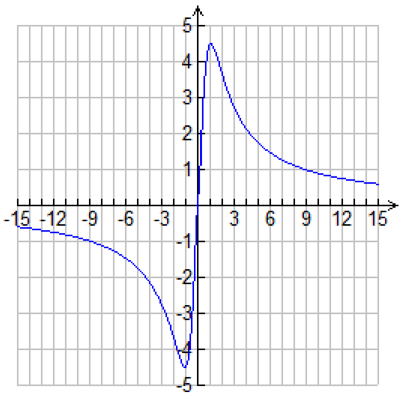
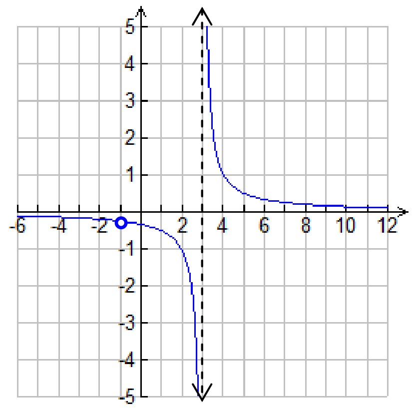
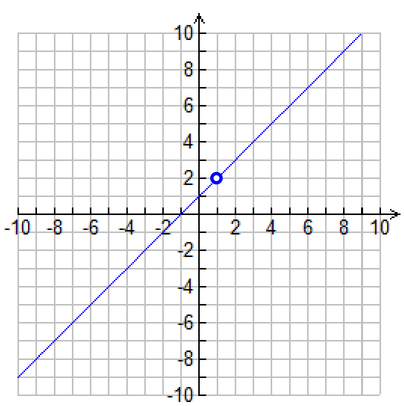
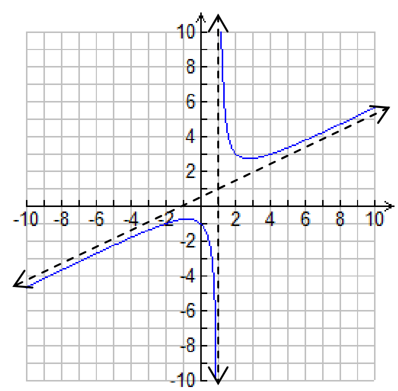
As $x\to\infty$, $f(x)\to 1$
As $x\to-\infty$, $f(x)\to 1$
As $x\to 1^-$, $f(x)\to-\infty$
As $x\to 1^+$, $f(x)\to\infty$</td>
<td>Zeros: 1 (at $x=-1$)
VA: $x=1$
HA: $y=1$
IP: $(-\infty,-1)$, $(1,\infty)$
IN: $(-1,1)$
II: none
ID: $(-\infty,1)$, $(1,\infty)$
ICU: $(1,\infty)$
ICD: $(-\infty,1)$
$D=\{x\in\mathbb{R}:x\neq 1\}$
$R=\{y\in\mathbb{R}:y\neq 1\}$
X: -1
Y: -1</td>
<td>$f(x)=\dfrac{x}{x-3}$


As $x\to\infty$, $f(x)\to 1$
As $x\to-\infty$, $f(x)\to 1$
As $x\to 3^-$, $f(x)\to-\infty$
As $x\to 3^+$, $f(x)\to\infty$</td>
<td>Zeros: 1 (at $x=0$)
VA: $x=3$
HA: $y=1$
IP: $(-\infty,0)$, $(3,\infty)$
IN: $(0,3)$
II: none
ID: $(-\infty,3)$, $(3,\infty)$
ICU: $(3,\infty)$
ICD: $(-\infty,3)$
$D=\{x\in\mathbb{R}:x\neq 3\}$
$R=\{y\in\mathbb{R}:y\neq 1\}$
X: 0
Y: 0</td>
</tr>
<tr>
<td>$f(x)=\dfrac{x-2}{3x+4}$


As $x\to\infty$, $f(x)\to\dfrac{1}{3}$
As $x\to-\infty$, $f(x)\to\dfrac{1}{3}$
As $x\to-\frac{4}{3}^-$, $f(x)\to\infty$
As $x\to-\frac{4}{3}^+$, $f(x)\to-\infty$</td>
<td>Zeros: 1 (at $x=-1$)
VA: $x=-4/3$
HA: $y=1/3$
IP: $(-\infty,-\frac{4}{3})$, $(2,\infty)$
IN: $(-4/3,2)$
II: $(-\infty,-\frac{4}{3})$, $(-\frac{4}{3},\infty)$
ID: none
ICU: $(-\infty,-4/3)$
ICD: $(-4/3,\infty)$
$D=\{x\in\mathbb{R}:x\neq 1\}$
$R=\{y\in\mathbb{R}:y\neq 1\}$
X: 2
Y: $-1/2$</td>
<td>$f(x)=\dfrac{x-3}{2x-6}$


As $x\to\infty$, $f(x)\to 0.5$
As $x\to-\infty$, $f(x)\to 0.5$
As $x\to 3^-$, $f(x)\to 0.5$
As $x\to 3^+$, $f(x)\to 0.5$</td>
<td>Zeros: none
VA: none
HA: none
IP: $(-\infty,3)$, $(3,\infty)$
IN: none
II: none
ID: none
ICU: none
ICD: none
$D=\{x\in\mathbb{R}:x\neq 3\}$
$R=\{0.5\}$
X: none
Y: 0.5</td>
</tr>
</table>

Graphs of other Rational Functions

Graph and Characteristics of Rational Function	Graph and Characteristics of Rational Function
$f(x)=\dfrac{9x}{1+x^2}$  As $x\to\infty$, $f(x)\to 0$ As $x\to-\infty$, $f(x)\to 0$ As $x\to 1^-$, $f(x)\to\dfrac{9}{2}$ As $x\to 1^+$, $f(x)\to\dfrac{9}{2}$ Zeros: 1 (at $x=0$) VA: none HA: $y=0$ IP: $(0,\infty)$ IN: $(-\infty,0)$ II: $(-1,1)$ ID: $(-\infty,-1)$, $(1,\infty)$ ICU: $(-2,0)$, $(2,\infty)$ ICD: $(-\infty,-2)$, $(0,2)$ $D=\mathbb{R}$ $R=\left\{y\in\mathbb{R}:-\dfrac{9}{2}\le y\le\dfrac{9}{2}\right\}$ X: 0 Y: 0	$f(x)=\dfrac{x+1}{x^2-2x-3}$  As $x\to\infty$, $f(x)\to 0$ As $x\to-\infty$, $f(x)\to 0$ As $x\to 3^-$, $f(x)\to-\infty$ As $x\to 3^+$, $f(x)\to\infty$ Zeros: none VA: $x=3$ HA: $y=0$ IP: $(2,\infty)$ IN: $(-\infty,2)$ II: none ID: $(-\infty,3)$, $(3,\infty)$ ICU: $(3,\infty)$ ICD: $(-\infty,3)$ $D=\{x\in\mathbb{R}:x\neq-1,x\neq 3\}$ $R=\{y\in\mathbb{R}:y\neq 0\}$ X: none Y: none
$f(x)=\dfrac{x^2-1}{x-1}$  As $x\to\infty$, $f(x)\to\infty$ As $x\to-\infty$, $f(x)\to-\infty$ As $x\to 1^-$, $f(x)\to 2$ As $x\to 1^+$, $f(x)\to 2$ Zeros: 1 (at $x=-1$) VA: none HA: none IP: $(-1,\infty)$ IN: $(-\infty,-1)$ II: $(-\infty,1)$, $(1,\infty)$ ID: none ICU: none ICD: none $D=\{x\in\mathbb{R}:x\neq 1\}$ $R=\{y\in\mathbb{R}:y\neq 2\}$ X: -1 Y: 1	$f(x)=\dfrac{0.5x^2+1}{x-1}$  As $x\to\infty$, $f(x)\to\infty$ As $x\to-\infty$, $f(x)\to-\infty$ As $x\to 1^-$, $f(x)\to-\infty$ As $x\to 1^+$, $f(x)\to\infty$ Zeros: none VA: $x=1$ HA: none IP: $(1,\infty)$ IN: $(-\infty,1)$ II: $\left(-\infty,1-\sqrt{3}\right)$, $\left(1+\sqrt{3},\infty\right)$ ID: none ICU: $(1,\infty)$ ICD: $(-\infty,1)$ $D=\{x\in\mathbb{R}:x\neq 1\}$ R (see below) X: none Y: -1

$R=\left\{y\in\mathbb{R}:y<=1-\sqrt{3}\text{ or }y>=1+\sqrt{3}\right\}$ for the range of $f(x)=\dfrac{0.5x^2+1}{x-1}$

Further Analysis of $f(x)=\dfrac{0.5x^2+1}{x-1}$

Note that in the very last example, there is a "slanted" asymptote. Such an asymptote is called an ***oblique asymptote***. In the case of $f(x)=\dfrac{0.5x^2+1}{x-1}$, the equation of the asymptote turns out to be $y=0.5x+0.5$. The following is a detailed explanation of how this equation can be determined.

As shown at the right, when $x-1$ is divided into $0.5x^2+1$, a quotient of $0.5x+0.5$ is obtained, along with a remainder of 1.5. That is,

$$
\begin{array}{r|l}
 & 0.5x+0.5 \\ \hline
x-1 & 0.5x^2+0x+1 \\
 & \underline{0.5x^2-0.5x} \\
 & \quad 0.5x+1 \\
 & \quad \underline{0.5x-0.5} \\
 & \qquad 1.5
\end{array}
$$

$$0.5x^2+1=(x-1)(0.5x+0.5)+1.5$$

$$\therefore \frac{0.5x^2+1}{x-1}=\frac{0.5(x-1)(x+1)+1.5}{x-1}$$

$$\therefore \frac{0.5x^2+1}{x-1}=\frac{0.5(x-1)(x+1)}{x-1}+\frac{1.5}{x-1}$$

$$\therefore \frac{0.5x^2+1}{x-1}=0.5(x+1)+\frac{1.5}{x-1}=0.5x+0.5+\frac{1.5}{x-1}$$

$$\therefore f(x)=\frac{0.5x^2+1}{x-1}=0.5x+0.5+\frac{1.5}{x-1}$$

This means that $f(x)$ can be written as the ***sum*** of the ***linear function*** $0.5x+0.5$ and the ***rational function*** $\dfrac{1.5}{x-1}$. For large enough values of *x*, the rational function $\dfrac{1.5}{x-1}$ takes on increasingly smaller values, which means that it contributes very little to the overall sum. Therefore, the value of $0.5x+0.5+\dfrac{1.5}{x-1}$ gets closer and closer to $0.5x+0.5$ as *x* gets larger because $\dfrac{1.5}{x-1}\to 0$ as $x\to\infty$.

x	y1(x) 0.5x+0.5	y2(x) 1.5/(x-1)	y3(x) 0.5x+0.5+1.5/(x-1)
0	0.5	-1.5	-1
10	5.5	0.166667	5.66667
20	10.5	0.078947	10.5789
30	15.5	0.051724	15.5517
40	20.5	0.038462	20.5385
50	25.5	0.030612	25.5306
60	30.5	0.025424	30.5254
70	35.5	0.021739	35.5217
80	40.5	0.018987	40.519
90	45.5	0.016854	45.5169
100	50.5	0.015152	50.5152
110	55.5	0.013761	55.5138
120	60.5	0.012605	60.5126
130	65.5	0.011628	65.5116
140	70.5	0.010791	70.5108
150	75.5	0.010067	75.5101
160	80.5	0.009434	80.5094
170	85.5	0.008876	85.5089
180	90.5	0.00838	90.5084
190	95.5	0.007937	95.5079
200	100.5	0.007538	100.508
210	105.5	0.007177	105.507
220	110.5	0.006849	110.507