MHF 4U0 **Semester 2, 2008 - 2009**

Grade 12 Advanced Functions (University Preparation)
Trigonometry Unit Test

Great work Mr. S.!!

Mr. N. Nolfi

Victim: Mr. Solutions

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14/14	27/27	17/17	14/14

1. Use the three methods indicated below to demonstrate that the equation $\cos(\pi-\theta) = -\cos\theta$ *is* an identity.

Compound Angle Identity **(3 KU)**

$\cos(\pi-\theta)$

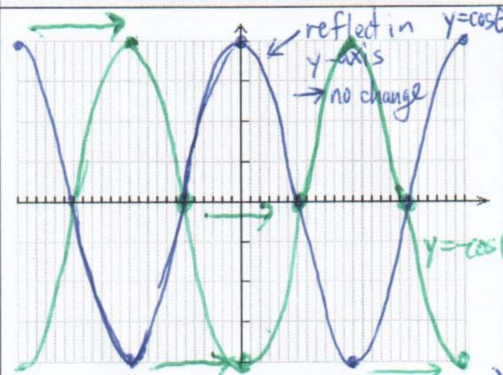
$= \cos\pi\cos\theta + \sin\pi\sin\theta$

$= (-1)\cos\theta + (0)\sin\theta$

$= -\cos\theta$

$\therefore \cos(\pi-\theta) = -\cos\theta$

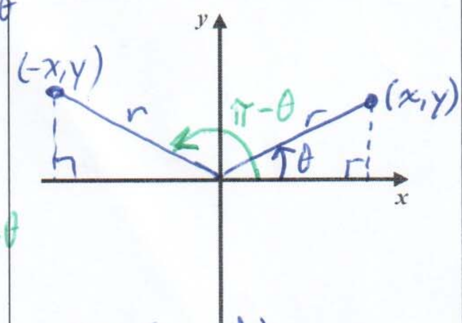
Graphical (Transformations) **(3 APP)**



Let $y = \cos(\pi-\theta) = \cos(-1(\theta-\pi))$

To obtain the graph of $y=\cos(\pi-\theta)$, take the graph of $y=\cos\theta$, reflect it in the y-axis then shift π radians right. The graph of $y = -\cos\theta$ is obtained.

Angles of Rotation **(3 TIPS)**



$\cos(\pi-\theta)$

$= \dfrac{-x}{r}$

$= -\dfrac{x}{r} = -\cos\theta$

$\therefore \cos(\pi-\theta) = -\cos\theta$

2. Evaluate the following trig ratios without using a calculator. Exact values are required!

(a) $\sin 105°$ **(4 APP)**

$\sin 105°$

$= \sin(45° + 60°)$

$= \sin 45°\cos 60° + \cos 45°\sin 60°$

$= \left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{2}\right) + \left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{\sqrt{3}}{2}\right)$

$= \dfrac{1}{2\sqrt{2}} + \dfrac{\sqrt{3}}{2\sqrt{2}}$

$= \dfrac{1+\sqrt{3}}{2\sqrt{2}}$

$$\left.\begin{aligned}&= \frac{(1+\sqrt{3})\sqrt{2}}{(2\sqrt{2})\sqrt{2}}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}\right\}\text{ optional steps}$$

(b) $\cos 22.5°$ **(4 TIPS)**

$\cos 2\theta = 2\cos^2\theta - 1$

Let $\theta = 22.5°$

$\therefore \cos(2(22.5°)) = 2\cos^2 22.5° - 1$

$\therefore \cos 45° = 2\cos^2 22.5° - 1$

$\therefore 2\cos^2 22.5° = \cos 45° + 1$

$\therefore \cos^2 22.5° = \dfrac{\cos 45° + 1}{2}$

$\therefore \cos 22.5° = \pm\sqrt{\dfrac{\cos 45° + 1}{2}}$

$= \pm\sqrt{\dfrac{\frac{1}{\sqrt{2}} + 1}{2}}$

$= \pm\sqrt{\dfrac{1+\sqrt{2}}{2\sqrt{2}}}$

$\because 22.5°$ is in quadrant I,

$\cos 22.5° = \sqrt{\dfrac{1+\sqrt{2}}{2\sqrt{2}}}$

optional $\rightarrow = \sqrt{\dfrac{\sqrt{2}+2}{4}} = \dfrac{\sqrt{\sqrt{2}+2}}{2}$

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3. Using the *two* methods listed below, demonstrate that the equation $\cos\left(x+\frac{\pi}{2}\right)=\cos x+\cos\frac{\pi}{2}$ *is not* an identity. → $\cos\frac{\pi}{2}=0$

(a) *Counterexample* **(3 APP)**

Let $x=\frac{\pi}{2}$. Then,

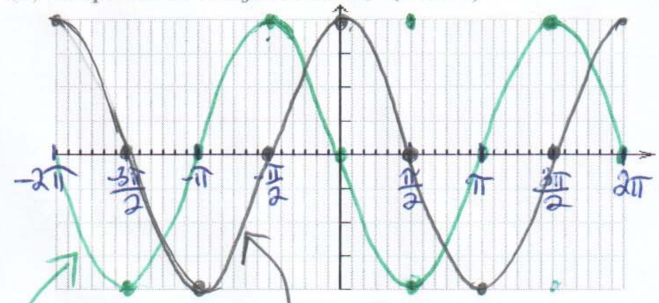
$L.S. = \cos\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\cos\pi=-1$

$R.S. = \cos\frac{\pi}{2}+\cos\frac{\pi}{2}=0+0=0$

$\because -1\neq 0$

$\therefore L.S. \neq R.S.$

(b) *Graphical (Transformations)* **(3 APP)**



$y=\cos\left(x+\frac{\pi}{2}\right)$

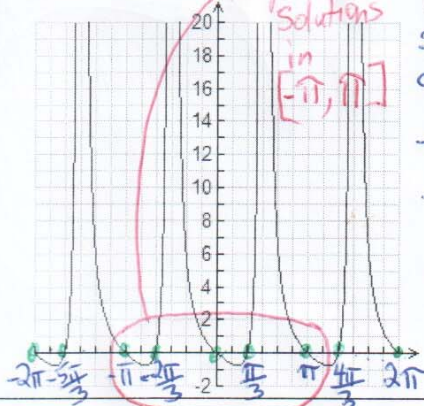
$y=\cos x+\cos\frac{\pi}{2}=\cos x+0=\cos x$

Since the graphs are different, the equation cannot be an identity.

4. The following question deals with solving certain trigonometric equations.

(a) Shown below is the graph of $y=\tan^2 x-\sqrt{3}\tan x$. Given that each tick mark on the x-axis represents $\frac{\pi}{6}$ radians, state *approximate* solutions to the equation $\tan^2 x-\sqrt{3}\tan x=0$. (Mark the solutions on the graph.) **(3 KU)**

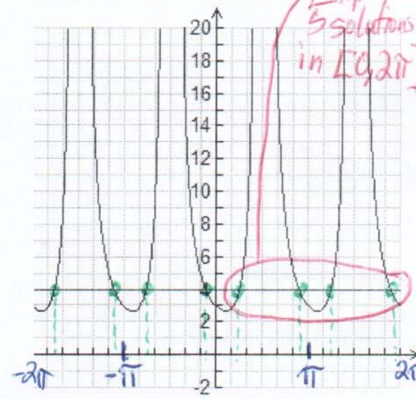
Expect 5 solutions in $[-\pi,\pi]$



Approximate solutions appear to be -2π, $-\frac{5\pi}{3}$, $-\pi$, $-\frac{2\pi}{3}$, 0, $\frac{\pi}{3}$, π, $\frac{4\pi}{3}$, 2π

(b) Shown below are the graphs of $y=3\sec^2\theta-2\tan\theta$ and $y=4$. Given that each tick mark on the x-axis represents $\frac{\pi}{6}$ radians, state *approximate* solutions to the equation $3\sec^2\theta-2\tan\theta=4$. (Mark the solutions on the graph.) **(3 KU)**

Expect 5 solutions in $[0,2\pi]$



Approximate solutions are $\frac{\pi}{4}$, 3, $\pi+\frac{\pi}{4}=\frac{5\pi}{4}$, 6, -0.3, $-\frac{3\pi}{4}$, -3.3, $-\frac{7\pi}{4}$

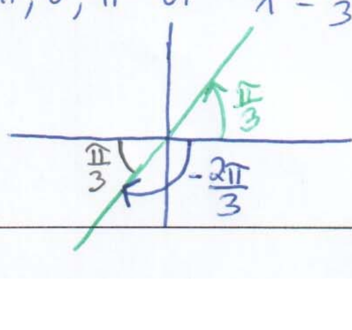
(c) Use an algebraic method to solve the equation $\tan^2 x-\sqrt{3}\tan x=0$, where $x\in[-\pi,\pi]$. **(4 APP)**

$\therefore \tan x(\tan x-\sqrt{3})=0$

$\therefore \tan x=0$ or $\tan x-\sqrt{3}=0$

$\therefore \tan x=0$ or $\tan x=\sqrt{3}$

$\therefore x=-\pi, 0, \pi$ or $x=\frac{\pi}{3}, -\frac{2\pi}{3}$



(d) Use an algebraic method to solve the equation $3\sec^2\theta-2\tan\theta=4$, where $\theta\in[0,2\pi]$. **(5 APP)**

$\therefore 3(1+\tan^2\theta)-2\tan\theta-4=0$

$\therefore 3\tan^2\theta-2\tan\theta-1=0$

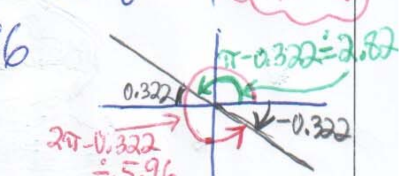
$\therefore (3\tan\theta+1)(\tan\theta-1)=0$

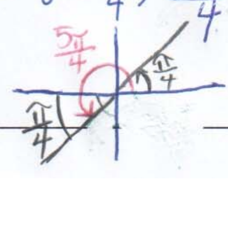
$\therefore \tan\theta=-\frac{1}{3}$ or $\tan\theta=1$

$\therefore \theta \doteq 2.82, 5.96$

or $\theta=\frac{\pi}{4}, \frac{5\pi}{4}$

Note: The calculator solution $\theta\doteq-0.322$ is outside $[0,2\pi]$





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5. Prove that the following equations *are* identities.

since $2\cos^2 x - 1 = \cos 2x = \cos^2 x - \sin^2 x$

(a) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ **(5 KU)**

$$
\begin{aligned}
L.S. &= \cos 3\theta \\
&= \cos(2\theta + \theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta) \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta \\
&= R.S.
\end{aligned}
$$

$\therefore L.S. = R.S.$

(b) $\dfrac{2\cos^2 x - 1}{\sin x + \cos x} + 2\sin\dfrac{x}{2}\cos\dfrac{x}{2} = \sin x \cot x$ **(5 TIPS)**

$$
\begin{aligned}
L.S. &= \frac{2\cos^2 x - 1}{\sin x + \cos x} + 2\sin\frac{x}{2}\cos\frac{x}{2} \\
&= \frac{\cos^2 x - \sin^2 x}{\sin x + \cos x} + \sin\left(2\left(\frac{x}{2}\right)\right) \\
&= \frac{(\cos x - \sin x)\cancel{(\cos x + \sin x)}^{\,1}}{\cancel{\cos x + \sin x}_{\,1}} + \sin x \\
&= \cos x - \sin x + \sin x \\
&= \cos x
\end{aligned}
$$

$$
\begin{aligned}
R.S. &= \sin x \cot x \\
&= \left(\frac{\cancel{\sin x}^{\,1}}{1}\right)\left(\frac{\cos x}{\cancel{\sin x}_{\,1}}\right) \\
&= \cos x
\end{aligned}
$$

$\therefore L.S. = R.S.$

6. The students in Mr. Meaner's math class were asked to prove that the equation $\dfrac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \tan x$ is an identity.

The most troublesome student in the class, Payne Indabutt, was very busy texting so he quickly and carelessly scribbled the following "proof" on a crumpled piece of paper and hurriedly handed it in. Upon reading the solution, Mr. Meaner became very angry, assigned Payne a mark of zero and then proceeded to tear up the piece of paper. Explain what is wrong with Payne's work and describe how you would correct it to Mr. Meaner's liking. **(4 COM)**

$$
\begin{aligned}
\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} &= \tan x \\
\frac{\sin x + 2\sin x\cos x}{1 + \cos x + \cos 2x} &= \tan x \\
\frac{\sin x(1 + 2\cos x)}{1 + \cos x + \cos 2x} &= \tan x \\
\frac{\sin x(1 + 2\cos x)}{\cos x + (1 + \cos 2x)} &= \tan x \\
\frac{\sin x(1 + 2\cos x)}{\cos x + 2\cos^2 x} &= \tan x \\
\frac{\sin x(1 + 2\cos x)}{\cos x(1 + 2\cos x)} &= \tan x \\
\frac{\sin x}{\cos x} &= \tan x \\
\tan x &= \tan x
\end{aligned}
$$

Mr. Meaner is quite rightly justified in assigning a mark of zero to this response. The purpose of a proof is to ESTABLISH the "truth" of a statement. In the first line of Payne Indabutt's work, he/she already assumed that the given equation is an identity. To correct this solution, the left and right sides need to be treated separately.

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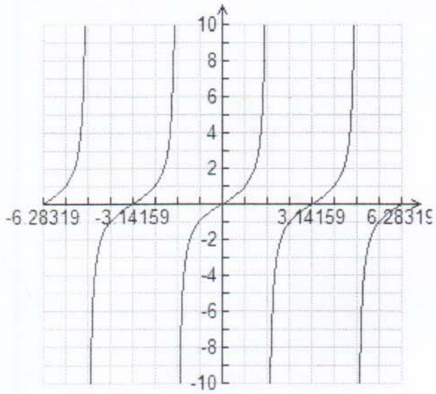
7. When TI-Interactive was used to sketch the graph of $f(x)=\dfrac{2\tan 2x-\sec^2 x\tan 2x}{2}$, the following was obtained. Write an equation for the identity suggested by this graph then prove that the equation is indeed an identity. **(5 TIPS)**



This graph strongly suggests that $\dfrac{2\tan 2x-\sec^2 x\tan 2x}{2}=\tan x$ is an identity.

Proof: $L.S.=\frac{1}{2}\left[\tan 2x\left(2-\sec^2 x\right)\right]$

$$=\frac{1}{\cancel{2}}\left(\frac{\cancel{2}\tan x}{1-\tan^2 x}\right)\left(\frac{2-(1+\tan^2 x)}{1}\right)$$

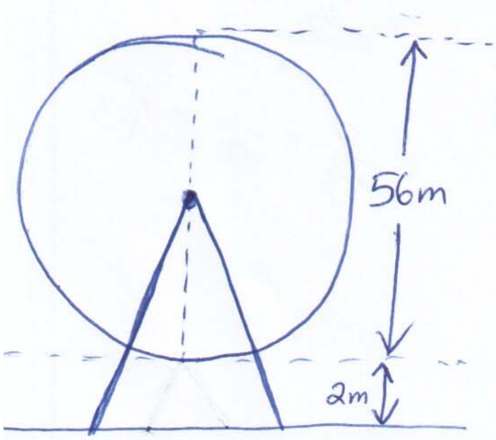
$$=\left(\frac{\tan x}{\cancel{1-\tan^2 x}}\right)\left(\frac{\cancel{1-\tan^2 x}}{1}\right)$$

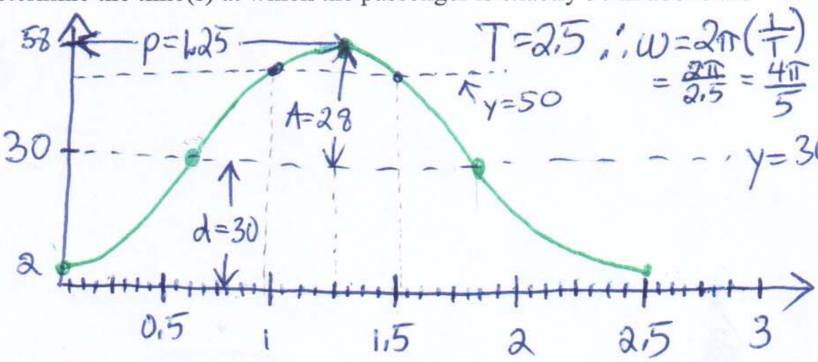
$$=\tan x$$

$$=R.S.$$

$\therefore L.S.=R.S.$

8. A Ferris wheel has a diameter of 56 m and one revolution takes 2.5 minutes to complete. A passenger gets on the Ferris wheel at a point that is 2 m above the ground. Create a mathematical model of the passenger's height above the ground with respect to time. Then use your model to determine the time(s) at which the passenger is exactly 50 m above the ground. **(5 APP)**





$T=2.5 \therefore \omega=2\pi\left(\frac{1}{T}\right)=\frac{2\pi}{2.5}=\frac{4\pi}{5}$

$\therefore$ an equation that models the height of a passenger with respect to time is $h(t)=28\cos\left(\frac{4\pi}{5}(t-1.25)\right)+30$

A passenger's height is exactly 50 m above the ground when $h(t)=50$

$$\therefore 28\cos\left(\frac{4\pi}{5}(t-1.25)\right)+30=50$$

$$\therefore 28\cos\left(\frac{4\pi}{5}(t-1.25)\right)=20$$

$$\therefore \cos\left(\frac{4\pi}{5}(t-1.25)\right)=\frac{20}{28}=\frac{5}{7}$$

$$\therefore \frac{4\pi}{5}(t-1.25)=\cos^{-1}\left(\frac{5}{7}\right)$$

$$\therefore t-1.25=\frac{5}{4\pi}\cos^{-1}\left(\frac{5}{7}\right)$$

$$\therefore t=\frac{5}{4\pi}\cos^{-1}\left(\frac{5}{7}\right)+1.25$$

$$\therefore t \doteq 1.56 \text{ or } t \doteq 1.25-(1.56-1.25)=0.94$$

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A passenger would be about 50 m above the ground at about 0.94 and 1.56 minutes//