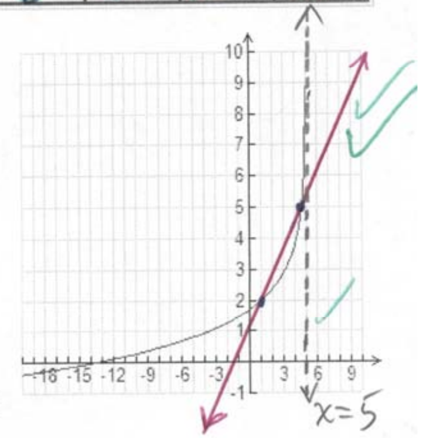
MHF 4U0 **Semester 2, 2009 - 2010**

Grade 12 Advanced Functions (University Preparation)
Unit 1 – Exponential and Logarithmic Functions – Major Test

Mr. N. Nolfi
Victim: Mr. Solutions   Brilliant work Mr. S.!!

KU	APP	TIPS	COM
6 /6	23/22	15/15	15/15

1. The graph of the function $f(x) = -3\log\left(-\frac{1}{4}(x-5)\right)+2$ is shown at the right.



(a) The function *f* defined above has one asymptote. Sketch the asymptote and state its equation below. **(2 KU)**

KU $x=5$

(b) Sketch a line whose slope equals the average rate of change of $f(x)$ with respect to x when x changes from 1 to $\frac{23}{5}$. **(2 KU)**   (see graph)

KU

(c) Calculate the average rate of change of $f(x)$ with respect to x when x changes from 1 to $\frac{23}{5}$. Write your answer as an expression that gives the exact rate of change. **(3 APP)**

- For negative one mark evaluate your expression with a calculator.
- For one bonus mark, underline or highlight this sentence and do not evaluate your expression with a calculator. +1A

$$\frac{\Delta y}{\Delta x} = \frac{f\left(\frac{23}{5}\right)-f(1)}{\frac{23}{5}-1}$$

APP
$$= \frac{-3\log\left(-\frac{1}{4}\left(\frac{23}{5}-5\right)\right)+2-\left(-3\log\left(-\frac{1}{4}(1-5)\right)+2\right)}{\left(\frac{18}{5}\right)}$$

$$= \frac{5\left(-3\log\left(\frac{1}{10}\right)+2+3\log 1-2\right)}{18} = \frac{5(-3(-1))}{18} = \frac{15}{18} = \frac{5}{6}$$

(d) As $x\to-\infty$, $f(x)\to$ $-\infty$.   KU   As $x\to-3$, $f(x)\to$ 1 (approx.). **(2 KU)**

2. Suppose that $g(x)=1.5\log_2\left(0.5(x+6)\right)+2$. **(10 APP)**   (APP)

(a) State the transformations required to obtain *g* from the base function $f(x)=\log_2 x$.

Horizontal	*Vertical*
1. Stretch by a factor of $\frac{1}{0.5}=2$	1. Stretch by a factor of 1.5
2. Translate 6 units to the left	2. Translate 2 units upward

(b) Express the transformation in *mapping notation*.

$(x,y)\to(2x-6,\ 1.5y+2)$

(c) Now apply the transformation to a few key points on the graph of the base function $f(x)=\log_2 x$.

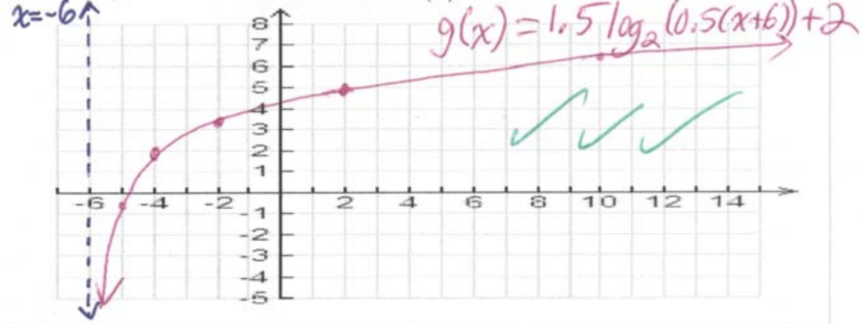
Pre-image Point on $y=f(x)$	*Image Point on* $y=g(x)$
(1,0)	(-4,2)
(2,1)	(-2,3.5)
(4,2)	(2,5)
(8,3)	(10,6.5)
$(\frac{1}{2},-1)$	(-5,0.5)

KU	APP	TIPS	COM

(d) Apply the transformation to the asymptote of $y = f(x)$.

Pre-image Asymptote on $y = f(x)$	Image Asymptote on $y = g(x)$
$x = 0$	$x = 2(0) - 6$ $\therefore x = -6$

(e) Finally, sketch the graph of $y = g(x)$.



3. *Explain* the following.

(a) There are no solutions to the equation $y = \log_5(-10)$. **(3 COM)**

Writing in exponential form,

$5^y = -10$.

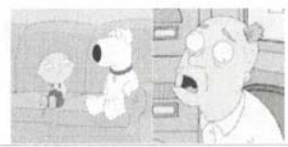
Since the base of the power is positive, there is no exponent y such that $5^y \leq 0$. Therefore, the given equation has no solutions.

(b) If $g(x) = (-10)^x$, $g\left(\frac{1}{2}\right)$ is undefined. **(3 COM)**

$g(\frac{1}{2}) = (-10)^{\frac{1}{2}}$

$= \sqrt{-10}$, which is undefined since the square root function is only defined for non-negtive real numbers

4. Opie lent Brian and Stewie an amount of money at a rate of 8.40% p.a. (per year), compounded *monthly*. When Brian and Stewie finally repaid Opie, they had to pay back (in one lump sum) an amount equal to 1.5 times what they had originally borrowed. How much time passed before Brian and Stewie repaid Opie?



(a) Assuming that P represents the original amount borrowed, complete the following: **(3 TIPS)**

annual rate = 0.084

monthly rate = 0.007

Time (months)	Amount ($)
0	P
1	$P(1.007)$
2	$P(1.007)^2$
3	$P(1.007)^3$
⋮	⋮
t	$P(1.007)^t$

(b) Now solve the problem! **(4 APP)**

Let $V(t)$ represent the value of the investment after t months. If P represents the amount borrowed, then

$V(t) = P(1.007)^t$

When the amount owed is 1.5 times what was originally borrowed, $V(t) = 1.5P$.

$\therefore P(1.007)^t = 1.5P$

$\therefore (1.007)^t = 1.5$

$\therefore t(\log 1.007) = \log 1.5$

$\therefore t = \frac{\log 1.5}{\log 1.007} \doteq 58$

Brian and Stewie repaid Opie after about 58 months (4 years, 10 months)

KU	APP	TIPS	COM
–	–	–	–

5. Complete the following table. **(8 TIPS)**

Statement	*True or False?*	*Proof, Counterexample or Explanation*
$\log_{-5} 25 = 2$	F	Logarithmic functions are only defined for positive bases (excluding one)
$\dfrac{\log_a x}{\log_a y} = \log_a\left(\dfrac{x}{y}\right)$	F	Let $a=2$, $x=32$, $y=16$ $L.S. = \frac{\log_2 32}{\log_2 16} = \frac{5}{4}$ $R.S. = \log_a\left(\frac{32}{16}\right) = \log_2 2 = 1$ $\therefore$ L.S. $\neq$ R.S.
$\log_a\left(5\sqrt[3]{z}\right) = \frac{1}{3}\log_a(5z)$	F	$R.S. = \log_a(5z)^{\frac{1}{3}} = \log_a(5^{\frac{1}{3}}z^{\frac{1}{3}}) = \log_a(\sqrt[3]{5}\sqrt[3]{z})$ $L.S. = \log_a(5\sqrt[3]{z}) \neq \log_a(\sqrt[3]{5}\sqrt[3]{z})$ $\therefore L.S. \neq R.S.$
When light passes through a certain material, it loses 2.5% of its intensity per millimetre of thickness. This process can be modelled using the equation $I(d) = I_0(1.025)^d$, where d represents the distance travelled by the light through the material, I_0 represents the initial intensity of the light and $I(d)$ represents the intensity of the light after having passed through d millimetres of the material.	F	The given equation CANNOT be correct because the base is greater than 1, making it an equation describing exponential growth. Clearly, the given situation involves exponential decay.

6. Through detailed studies, scientists have determined that in *living* carbonaceous material, the ratio of ^{14}C atoms to ^{12}C atoms is $1:10^{12}$. In a wooden artifact found in an archaeological excavation, the ratio of ^{14}C atoms to ^{12}C atoms is measured to be $1:2.7\times10^{12}$. Estimate the age of the wood used to make the artifact. (Recall that the half-life of ^{14}C is 5730 years.) **(5 APP)**

t (years)	$^{14}C : ^{12}C$
0	10^{-12}
5730	$10^{-12}\left(\frac{1}{2}\right)$
11460	$10^{-12}\left(\frac{1}{2}\right)^2$
17190	$10^{-12}\left(\frac{1}{2}\right)^3$
⋮	⋮
t	$10^{-12}\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

Note: $1:10^{12} = \frac{1}{10^{12}} = 10^{-12}$

Let $R(t)$ represent the ratio of ^{14}C atoms to ^{12}C atoms, t years after the death of the organism.

Then $R(t) = 10^{-12}\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

For the given artifact, $R(t) = \frac{1}{2.7\times10^{12}}$

$\therefore 10^{-12}\left(\frac{1}{2}\right)^{\frac{t}{5730}} = \frac{1}{2.7\times10^{12}}$

$\therefore \left(\frac{1}{2}\right)^{\frac{t}{5730}} = \frac{10^{12}}{2.7\times10^{12}} = \frac{1}{2.7}$

$\therefore \frac{t}{5730}\left(\log\frac{1}{2}\right) = \log\left(\frac{1}{2.7}\right)$

$\therefore t = \dfrac{5730\log\frac{1}{2.7}}{\log\frac{1}{2}} \doteq 8210$

The artifact is approximately 8200 years old.

KU	APP	TIPS	COM
–	–	–	–

7. Equal volumes of two solutions are mixed. If the pH of one solution is 1.8 and the pH of the other solution is 3.8, what will be the pH of the mixture? (Assume that neither solution contains buffers, which would resist changes in pH.) **(4 TIPS)**

TIPS

Solution 1

$pH = -\log[H^+]$

$\therefore 1.8 = -\log[H^+]$

$\therefore \log[H^+] = -1.8$

$\therefore [H^+] = 10^{-1.8}$ mol/L

Solution 2

$pH = -\log[H^+]$

$\therefore 3.8 = -\log[H^+]$

$\therefore \log[H^+] =$

$\therefore [H^+] = 10^{-3.8}$ mol/L

Therefore, the hydrogen ion concentration of the mixture is

$$\frac{10^{-1.8} + 10^{-3.8}}{2} \text{ mol/L}$$

(volumes are equal, which means that the hydrogen ion concentration of the mixture is the average of the two)

The pH of the mixture is found as follows:

$$pH = -\log\left(\frac{10^{-1.8} + 10^{-3.8}}{2}\right) \doteq 2.1$$

The pH of the mixture is about 2.1. //

8. When is it appropriate to use a logarithmic scale? Give two examples to support your answer. **(3 COM)**

Presentation of data on a logarithmic scale is very useful when the data cover a large range of values. The logarithm reduces the range to a more manageable range

e.g. pH, Richter scale

COM

0 to 14 as opposed to 10^{-14} to 10^{0}

KU	APP	TIPS	COM