MHF 4U0 — Semester 2, 2009 - 2010

Grade 12 Advanced Functions (University Preparation)

Unit 2 – Mid-unit Quest (Radian Measure, Angular Frequency, Transformations)

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Victim: Mr. Solutions   Well done Mr. S.!!

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15/15	20/20	13/13	16/16

- Unless otherwise noted, ***radian measure must be used***. COM marks will be deducted for the use of degree measure.
- Up to 10 COM marks may be deducted for poor mathematical form, inappropriate use of terminology, etc.

1. Perform the following conversions. Show your work!

(a) Convert 542° to radians. **(2 KU)**

$180^\circ = \pi$ rad

$\therefore 1^\circ = \frac{\pi}{180}$ rad

$\therefore 542^\circ = \frac{542\pi}{180}$

$= \frac{271\pi}{90} \doteq 9.46$ rad

(b) Convert −57.3 radians to degrees. **(2 KU)**

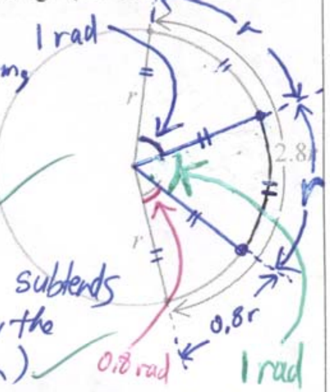
π rad $= 180^\circ$

$\therefore 1$ rad $= \frac{180}{\pi}^\circ$

$\therefore -57.3$ rad $= \frac{-57.3(180)}{\pi}^\circ$

$\doteq -3283.05^\circ$

2. ***Use the definition of the radian*** to evaluate θ (in radians, of course). Explain your reasoning. (***Do not*** use the formula $l = r\theta$.) **(3 COM)**



As shown in the diagram, the arc of length 2.8r is divided into 3 parts:

$2.8r = r + r + 0.8r$

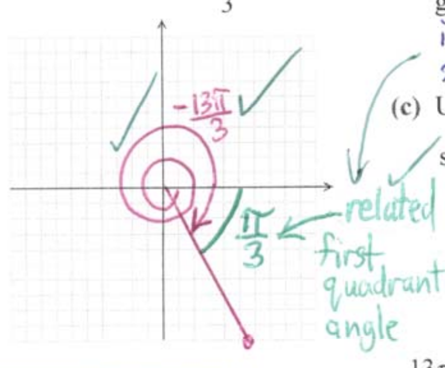
Each arc of length r subtends an angle of 1 rad (by the definition of the radian)

The arc of length 0.8r must subtend an angle of 0.8 rad

$\therefore \theta = 1 + 1 + 0.8 = 2.8$

3. Answer each of the following questions.

(a) Use the grid below to sketch the angle of rotation $\frac{-13\pi}{3}$. **(2 KU)**



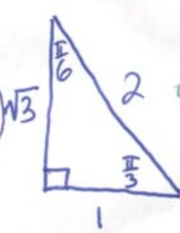
(b) Find the ***first quadrant (acute) angle related*** to $\frac{-13\pi}{3}$. Mark it on the grid given for part (a). **(2 KU)** The related first quadrant angle is the angle between the terminal arm and the x-axis, which is $\frac{\pi}{3}$.

(c) Use the ***related first quadrant (acute) angle*** and a ***special triangle*** to evaluate $\sec\frac{-13\pi}{3}$. **(3 KU)**

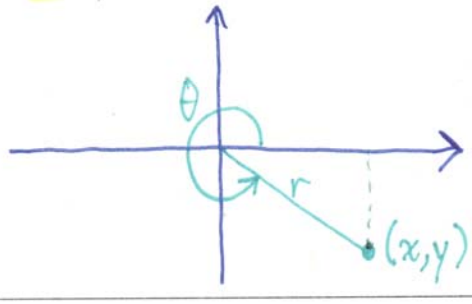
$\sec\frac{-13\pi}{3}$

$= +\sec\frac{\pi}{3}$ (cos and sec are positive in quad IV)

$= 2$



(d) For negative two marks, evaluate $\sin\frac{-13\pi}{3}$. For three communication marks, explain why both the tangent and cotangent of any angle in quadrant IV must be negative. (Use a diagram to illustrate your answer.) For *one* bonus mark , write *one* sentence that describes the purpose of mathematical modelling. **(3 COM)**



Consider the angle θ shown in the diagram. In quadrant IV, the x-co-ordinate must be positive and the y-co-ordinate must be negative. Now $\tan\theta = \frac{y}{x}$ and $\cot\theta = \frac{x}{y}$. Since x and y are of opposite sign, both ratios must be negative.

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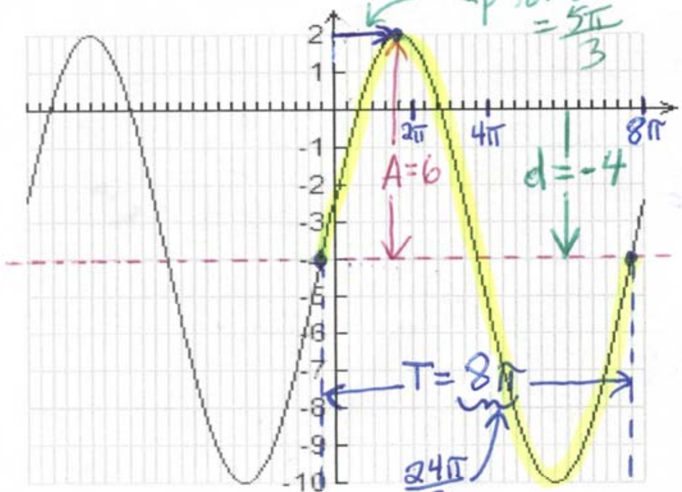
$T = 2\pi\left(\frac{1}{\omega}\right) = \frac{2\pi}{\omega} \quad \therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{8\pi} = \frac{1}{4}$

4. Each tick mark on the horizontal axis represents $\pi/3$ radians. State *two* equations of the graph, one using $y = \sin x$ as the mother function and the other using $y = \cos x$ as the mother function. **(4 APP)**



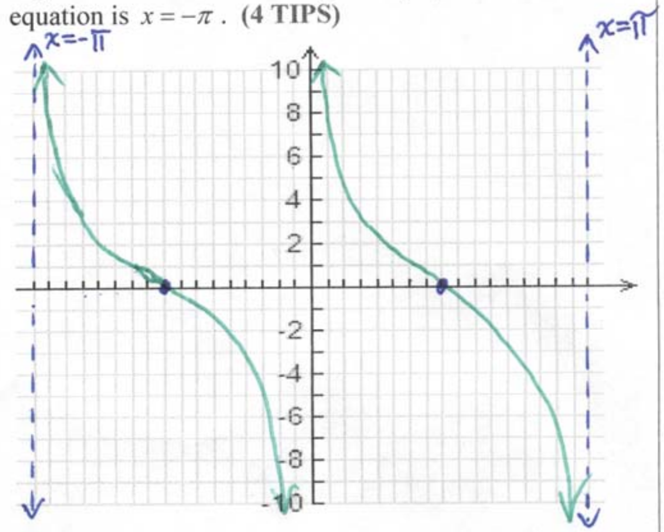
$A =$ 6, $d =$ -4, $T =$ 8π, $\omega =$ $\frac{1}{4}$, $p =$ $-\frac{\pi}{3}$

for base function $y = \sin x$

$y = 6 \sin\left(\frac{1}{4}\left(x + \frac{\pi}{3}\right)\right) + (-4)$

$y = 6 \cos\left(\frac{1}{4}\left(x - \frac{5\pi}{3}\right)\right) + (-4)$

5. State an ***equation*** and sketch ***two cycles*** of the graph of a trigonometric function having an asymptote whose equation is $x = -\pi$. **(4 TIPS)**



Equation

$y = \cot x$

6. Suppose that $g(x) = -2\cot\left(0.25\left(x + \pi/4\right)\right)$. **(12 APP)**

(a) State the transformations required to obtain g from the base/parent/mother function $f(x) = \cot x$. 2A

Horizontal	*Vertical*
1. Stretch horizontally by a factor of 4.	1. Stretch vertically by a factor of 2
2. Translate $\frac{\pi}{4}$ units to the left.	2. Reflect in x-axis.
The stretch MUST be performed before the translation.	The order is unimportant in this case.

(b) Express the transformation in *mapping notation*. 2A

$(x, y) \to \left(4x - \frac{\pi}{4}, -2y\right)$

(c) Apply the transformation to a few key points on the graph of the base function $f(x) = \cot x$ 2A

Pre-image Point on $y = f(x)$	*Image Point on $y = g(x)$*
$\left(\frac{\pi}{6}, \sqrt{3}\right)$	$\left(\frac{5\pi}{12}, -2\sqrt{3}\right)$
$\left(\frac{\pi}{4}, 1\right)$	$\left(\frac{3\pi}{4}, -2\right)$
$\left(\frac{\pi}{3}, \frac{1}{\sqrt{3}}\right)$	$\left(\frac{13\pi}{12}, -\frac{2}{\sqrt{3}}\right)$
$\left(\frac{\pi}{2}, 0\right)$	$\left(\frac{7\pi}{4}, 0\right)$

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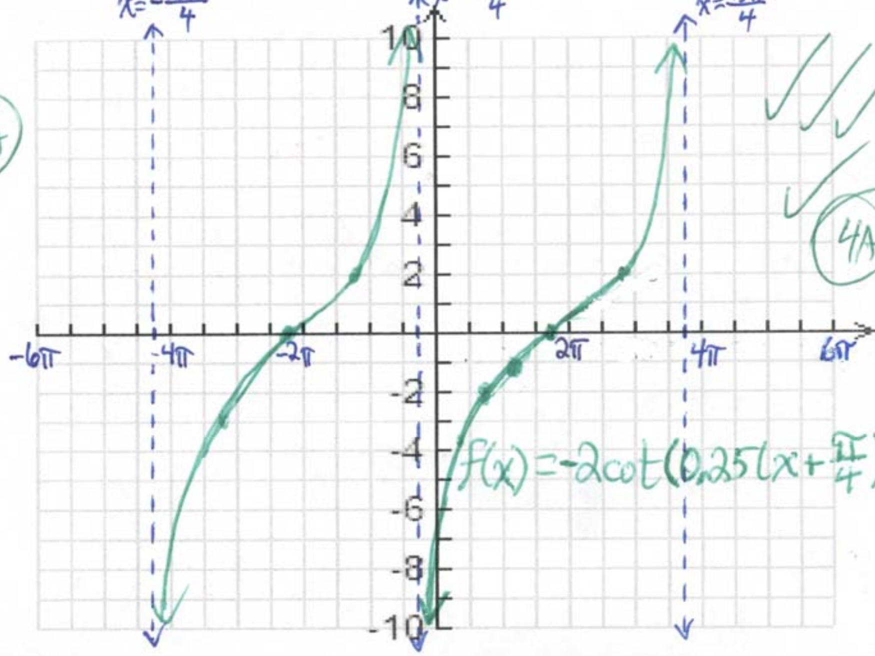
(d) Apply the transformation to a few of the asymptotes of $f(x) = \cot x$ ✓✓

Pre-image Asymptote of $y = f(x)$	*Image Asymptote of $y = g(x)$*
$x = -\pi$	$x = 4(-\pi) - \frac{\pi}{4}$ $\therefore x = \frac{-16\pi}{4} - \frac{\pi}{4}$ $\therefore x = -\frac{17\pi}{4}$
$x = 0$	$x = 4(0) - \frac{\pi}{4}$ $\therefore x = -\frac{\pi}{4}$
$x = \pi$	$x = 4(\pi) - \frac{\pi}{4}$ $\therefore x = \frac{16\pi}{4} - \frac{\pi}{4}$ $\therefore x = \frac{15\pi}{4}$

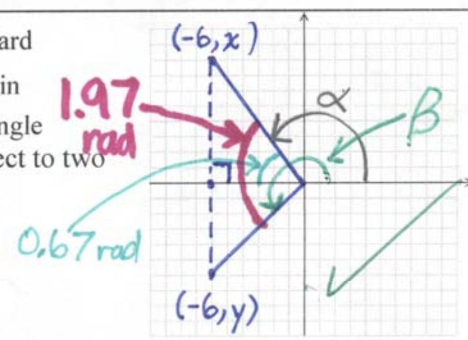
2A

(e) Finally, sketch the graph of $y = g(x)$. $T = \pi\left(\frac{1}{0.25}\right) = 4\pi$



4A

7. The point $(-6, x)$ lies on the terminal arm of a second quadrant angle α in standard position. The point $(-6, y)$ lies on the terminal arm of a third quadrant angle β in standard position. The first quadrant angle related to α is 0.67 radians and the angle between the two terminal arms is 1.97 radians. Find the values of x and y correct to two decimal places. **(5 TIPS)**



From the diagram, it's clear that

$$\alpha = \pi - 0.67$$
$$\beta = \alpha + 1.97 = \pi - 0.67 + 1.97 = \pi + 1.3$$

Now,

$\tan\alpha = \frac{x}{-6}$

$\therefore x = -6\tan\alpha$

$= -6\tan(\pi - 0.67)$

$\doteq 4.75$

$\tan\beta = \frac{y}{-6}$

$\therefore y = -6\tan\beta$

$\doteq -6\tan(\pi + 1.3)$

$\doteq -21.61$

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8. The planet Venus, which has a diameter of about 12100 km, takes 243.01 days to spin *once* about its axis

(a) Calculate the angular frequency, in radians per day, of Venus' rotation about its axis. **(2 KU)**

2π radians in 243.01 days

$\therefore$ Venus rotates through $\frac{2\pi}{243.01}$ radians in one day ✓

$\therefore \omega = \frac{2\pi}{243.01} \doteq 0.0259$ rad/day ✓

(b) Suppose that you were standing at a point on Venus' equator. How long would it take you to travel a distance of 100000 km? **(4 APP)**

$r = 6050$, $\ell = 100000$, $\theta = ?$ (angle through which Venus must rotate for arc length = 100000)

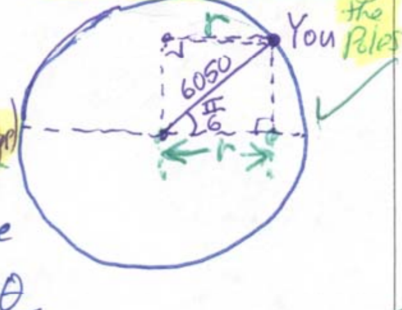
$\ell = r\theta$ ✓

$\therefore \theta = \frac{\ell}{r} = \frac{100000}{6050} \doteq 16.53$ rad ✓

$\therefore t = \frac{\theta}{\omega} = \frac{\left(\frac{1000000}{6050}\right)}{\left(\frac{2\pi}{243.01}\right)} \doteq 639.3$ days ✓

(c) Now suppose that you were standing at a point on a line of latitude $\frac{\pi}{6}$ radians north of Venus' equator. How long would it take you to travel a distance of 100000 km? **(4 TIPS)**

As Venus rotates about its axis, you would travel along a circle of radius r as shown in the diagram (think of spinning a coin standing on edge)



$\frac{r}{6050} = \cos\frac{\pi}{6}$

$\therefore r = 6050\cos\frac{\pi}{6}$

$= \cancel{6050}^{3025}\left(\frac{\sqrt{3}}{\cancel{2}}\right)$

$= 3025\sqrt{3}$ ✓

$r = 3025\sqrt{3}$, $\ell = 100000$, $\theta = ?$

(θ = angle through which Venus must rotate for $\ell = 100000$)

$\theta = \frac{\ell}{r}$

$= \frac{100000}{3025\sqrt{3}}$

$\doteq 19.1$ rad ✓

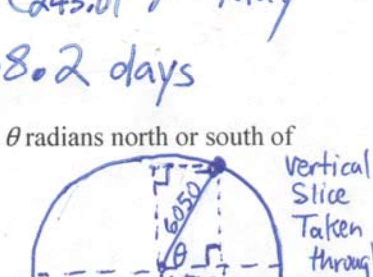
$t = \frac{\theta}{\omega}$

$= \frac{\left(\frac{1000000}{3025\sqrt{3}}\right)\text{ rad}}{\left(\frac{2\pi}{243.01}\right)\text{ rad/day}}$ ✓

$\doteq 738.2$ days

(d) ***Bonus – Attempt only if you have answered all other questions.***

Write an equation that describes the ~~angular frequency~~ velocity (v) of a point on a line of latitude θ radians north or south of Venus' equator. (Your equation should relate ~~ω~~ v to θ.)



$r = 6050\cos\theta$

Distance travelled in one rotation

$= 2\pi r$

$= 2\pi(6050\cos\theta)$

$= 12100\pi\cos\theta$

$v = \frac{d}{t}$

$= \frac{12100\pi\cos\theta\text{ km}}{243.01\text{ days}}$

$= \frac{12100\pi}{243.01}\cos\theta$ km/day

$= \frac{12100\pi}{24(243.01)}\cos\theta$ km/h

$\therefore v = \frac{3025\pi}{1458.06}\cos\theta$ km/h

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