MHF 4U0 **Semester 2, 2008 - 2009**

Grade 12 Advanced Functions (University Preparation)
Unit 3 – Major Test on Polynomial Functions

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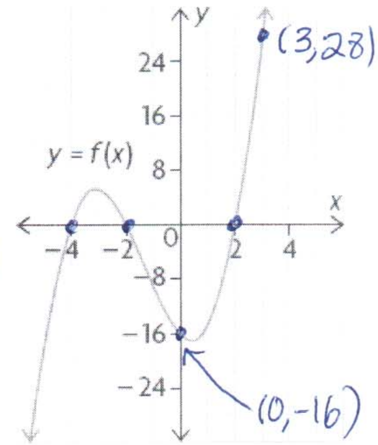
Victim: Mr. Solutions. You work is thorough and clean Mr. S...!!

KU	APP	TIPS	COM
7 /7	25 /25	18 /18	10 /10

1. Determine an equation of the polynomial function $y = f(x)$ whose graph is shown at the right. Express the polynomial expression in your equation in *factored form*. **(4 KU)**



Assume that the given polynomial function has degree 3 (opposite end behaviours → degree must be odd)

$\therefore$ the equation of f has the form

$f(x) = a(x+4)(x+2)(x-2)$ ($\because$ zeros are $-4, -2, 2$)

Since the point $(3,28)$ lies on the curve, $f(3) = 28$

$\therefore a(3+4)(3+2)(3-2) = 28$

$\therefore a = \frac{28}{35} = \frac{4}{5}$

$\therefore f(x) = \frac{4}{5}(x+4)(x+2)(x-2)$

Can also use the point $(0,-16)$, in which case the equation is found to be $f(x) = (x+4)(x+2)(x-2)$

2. Consider the quintic function $p(x) = 4x^5 + 32x^2$.

(a) Fully factor the polynomial. **(3 KU)**

$4x^5 + 32x^2$

$= 4x^2(x^3+8)$

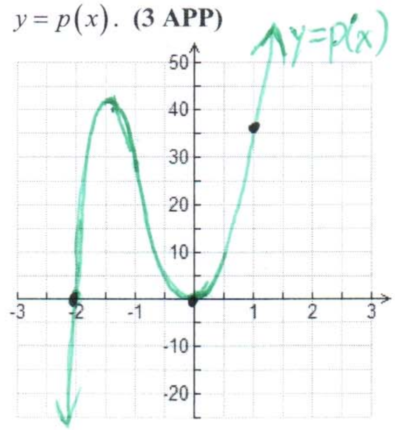
$= 4x^2(x+2)(x^2-2x+4)$

x^2 → order 2 → tangent

zeros are $0, -2$

x^2-2x+4 does not factor because $b^2-4ac<0$

(b) Use the factored form of the polynomial to sketch the graph of $y = p(x)$. **(3 APP)**



(c) Lisa took the graph of $y = p(x)$ and applied the following transformations to produce the function g.

1. Vertical stretch by a factor of 2.
2. Vertical translation one unit up.

How many zeros does g have? **(2 TIPS)**

The function g should have only one zero. The vertical stretch has no effect on the zeros because any point on the x-axis has a y-co-ordinate of zero. The vertical shift, however, causes the point $(0,0)$ to move up, which means that the function g only intersects the x-axis at one point.

3. The following transformations are applied to the function $y = x^4$

(i) Vertical stretch by a factor of 3, reflection in the x-axis, vertical translation 2 units down.

(ii) Horizontal compression by a factor of 4, reflection in the y-axis, horizontal translation 5 units left.

$4 \to \frac{1}{4}$

(a) Write an equation of the transformed function. **(4 APP)**

$y = -3(-4(x+5))^4 - 2$

$\therefore y = -768(x+5)^4 - 2$

(b) State the image of the point $(1,1)$ under the transformation. **(3 APP)**

$(x,y) \to (-\frac{1}{4}x - 5, -3y-2)$

$\therefore (1,1) \to (-\frac{1}{4}(1) - 5, -3(1)-2)$

$\therefore (1,1) \to (-\frac{21}{4}, -5)$

4. Solve the equation $3x^3 - 3x^2 - 7x + 5 = x^3 - 2x^2 - 1$. **(6 APP)**

Let $f(x) = 2x^3 - x^2 - 7x + 6$

$\because f(1) = 0$, $x-1$ must be a factor of $f(x)$ (factor theorem)

$$\begin{array}{r} 2x^2 + x - 6 \\ x-1 \,\overline{)\, 2x^3 - x^2 - 7x + 6} \\ \underline{2x^3 - 2x^2} \\ x^2 - 7x \\ \underline{x^2 - x} \\ -6x + 6 \\ \underline{-6x + 6} \\ 0 \end{array}$$

$\therefore 2x^3 - x^2 - 7x + 6 = 0$

$\therefore (x-1)(2x^2 + x - 6) = 0$

$\therefore (x-1)(2x-3)(x+2) = 0$

$\therefore x-1=0$ or $2x-3=0$ or $x+2=0$

$\therefore x=1$ or $x=\frac{3}{2}$ or $x=-2$

5. Solve the following inequalities. In both cases, express the solution both as a set and with the use of a number line.

(a) $-4 \le -3x - 3 \le 5$ **(4 APP)**

$\therefore -4+3 \le -3x-3+3 \le 5+3$

$\therefore -1 \le -3x \le 8$

$\therefore \frac{-1}{-3} \geq \frac{-3x}{-3} \geq \frac{8}{-3}$

$\therefore \frac{1}{3} \geq x \geq -\frac{8}{3}$

Solution set

$\{x \in \mathbb{R} \mid -\frac{8}{3} \le x \le \frac{1}{3}\}$

(interval notation: $[-\frac{8}{3}, \frac{1}{3}]$)

(b) $(2x-4)^2(x+3) \le 0$ **(5 APP)**

only $x=2$ satisfies this

$\therefore (2x-4)^2 \geq 0$ and $x+3 \le 0$ or $(2x-4)^2 \le 0$ and $x+3 \geq 0$

$\because (2x-4)^2 \geq 0$ for all $x \in \mathbb{R}$,

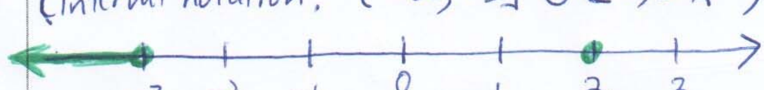
$\therefore x+3 \le 0$ or $(2x-4)^2 = 0$ and $x+3 \geq 0$

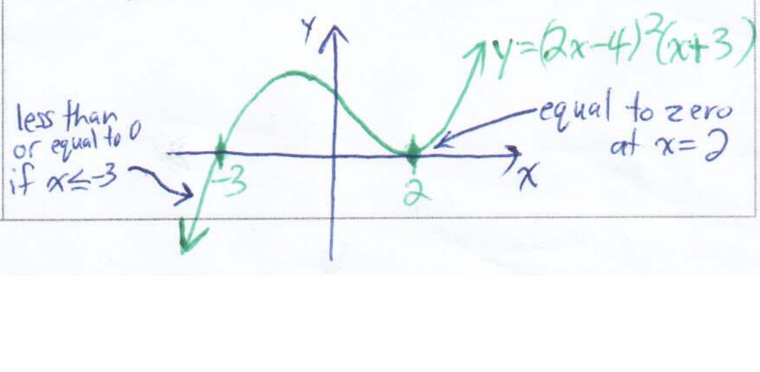
$\therefore x \le -3$ or $x=2$ and $x \geq -3$

$\therefore x \le -3$ or $x = 2$

Solution Set: $\{x \in \mathbb{R} \mid x \le -3 \text{ or } x = 2\}$

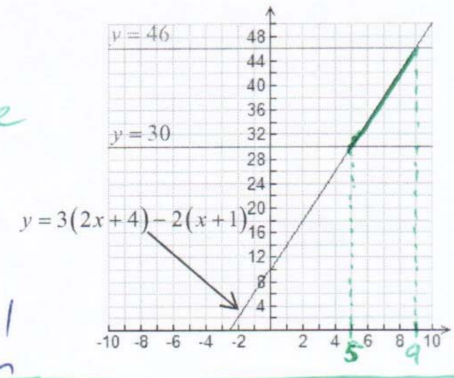
(interval notation: $(-\infty, -3] \cup [2, 2]$)





6. Write an inequality that corresponds to the diagram given at the right. In addition, state the solution set. (Do not solve the inequality. You should be able to see the solution just by looking at the graphs.) **(3 TIPS)**

May also use $<$

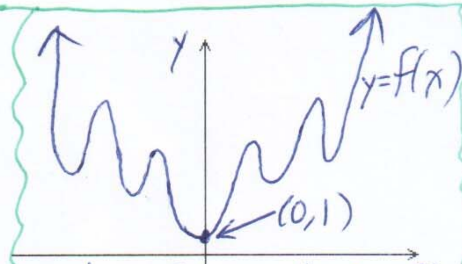


$$30 \le 3(2x+4)-2(x+1) \le 46$$

Solution Set

$\{x \in \mathbb{R} \mid 5 \le x \le 9\}$ or $[5, 9]$ in interval notation

7. Explain why the polynomial equation $2x^{10}+13x^8+5x^6+x^4+19x^2+1=0$ has *no real roots*. In addition, sketch a possible graph of the polynomial function $f(x)=2x^{10}+13x^8+5x^6+x^4+19x^2+1$. **(4 TIPS)**



A possible graph of $y=f(x)$

The graph must lie entirely above the x-axis, have same end behaviours and up to 9 turning points.

Since all the coefficients are positive and all the powers are even powers of x,

$2x^{10}+13x^8+5x^6+x^4+19x^2 \ge 0$ for all $x \in \mathbb{R}$

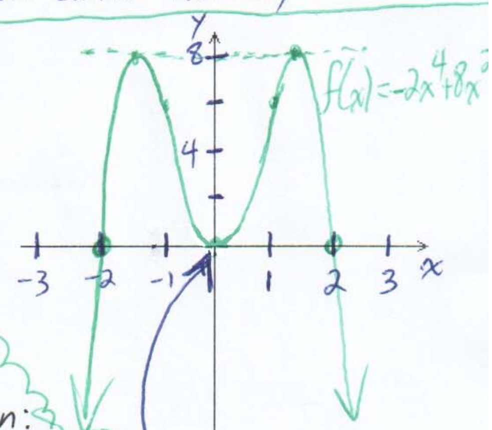
$\therefore 2x^{10}+13x^8+5x^6+x^4+19x^2+1 \ge 1$ for all $x \in \mathbb{R}$

$\therefore f(x) \ge 1$ for all $x \in \mathbb{R}$

That is, no matter what value of x is chosen, the value of $f(x)$ must be at least 1. Therefore, the graph of $y=f(x)$ cannot cross the x-axis, which means that the equation cannot have any real roots

8. The function $f(x)=kx^4+8x^2$ has three turning points, an absolute (global) maximum value of 8 and a zero at $x=2$. Determine the value of k as well as the other zero(s) of f. Then sketch the graph of $y=f(x)$. **(4 TIPS)**

$f(2)=0$



Since f has a zero at $x=2$, $f(2)=0$

$\therefore k(2^4)+8(2^2)=0$

$\therefore 16k+32=0$

$\therefore 16k=-32$

$\therefore k=-2$

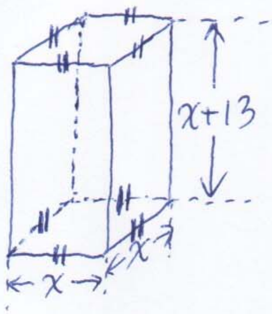
$\therefore f(x)=-2x^4+8x^2$
$= -2x^2(x^2-4)$
$= -2x^2(x-2)(x+2)$

$\therefore$ the other zeros are 0 and −2.

Note that f is an EVEN function:
$f(-x)=-2(-x)^4+8(-x)^2$
$=-2x^4+8x^2=f(x)$
$\therefore f$ is symmetric in the y-axis

Since the order (also called multiplicity) of this zero is 2, the graph looks like a parabola in the vicinity of $x=0$ and is tangent to the x-axis.

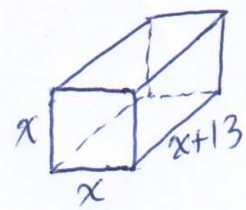
9. A box that holds an expensive pen has square ends and its length is 13 cm greater than its width. If the volume of the box is 60 cm^3, determine its dimensions. **(5 TIPS)**



Let $V(x)$ represent the volume of the box

$\therefore V(x) = x(x)(x+13)$

$\therefore V(x) = x^2(x+13)$



Since the volume of the box is 60 cm^3,

$$V(x) = 60$$

$\therefore x^2(x+13) = 60$

$\therefore x^3 + 13x^2 = 60$

$\therefore x^3 + 13x^2 - 60 = 0$

$\therefore (x-2)(x^2+15x+30) = 0$

$\therefore x-2=0$ or $x^2+15x+30=0$

$\therefore x=2$ or $x = \dfrac{-15 \pm \sqrt{15^2-4(1)(30)}}{2}$

$= \dfrac{-15 \pm \sqrt{105}}{2} < 0$ → inadmissable since a length cannot be negative

Let $f(x) = x^3+13x^2-60$

$\because f(2)=0$

$\therefore x-2$ must be a factor of $f(x)$ (factor theorem)

$$\begin{array}{r|l} & x^2+15x+30 \\ \hline x-2 & x^3+13x^2+0x-60 \\ & x^3-2x^2 \\ \hline & 15x^2+0x \\ & 15x^2-30x \\ \hline & +30x-60 \\ & 30x-60 \\ \hline & 0 \end{array}$$

The dimensions of the box are

2 cm × 2 cm × 15 cm. //