Solutions – Problems on Page 6 of Unit 1

1. Given $f(x)=\left(\frac{1}{2}\right)^{x}$, sketch the graph of $g(x)=3f(-2(x+1))-4$.

Equation of g

$$g(x)=3\left(\frac{1}{2}\right)^{-2(x+1)}-4$$

Transformations of f expressed in Words

Horizontal

1. Compress by a factor of –1/2. (Reflect in the *y*-axis and compress by a factor of 1/2).
2. Shift 1 unit to the left.

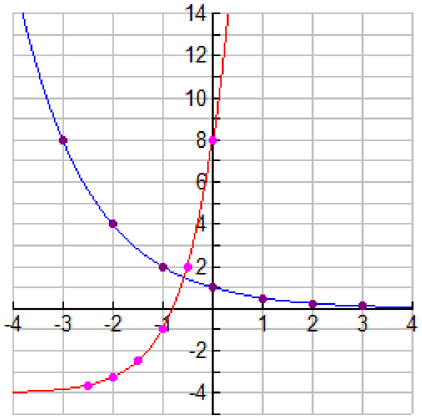
Vertical

1. Stretch by a factor of 3.
2. Shift down 4 units.

Transformation in Mapping Notation

$$(x,y)\rightarrow\left(-\frac{1}{2}x-1,3y-4\right)$$

Pre-image	*Image*
$(-3,8)$	$(1/2,20)$
$(-2,4)$	$(0,8)$
$(-1,2)$	$(-1/2,2)$
$(0,1)$	$(-1,-1)$
$(1,1/2)$	$(-3/2,-5/2)$
$(2,1/4)$	$(-2,-13/4)$
$(3,1/8)$	$(-5/2,-29/8)$



2. Lemmings are small rodents usually found in or near the Arctic. Contrary to popular belief, lemmings ***do not commit mass suicide*** when they migrate. Driven by strong biological urges, they will migrate in large groups when population density becomes too great. During such migrations, lemmings may choose to swim across bodies of water in search of a new habitat. Many lemmings drown during such treks, which may in part explain the myth of mass suicide.

What is true about lemmings is that they reproduce at a very fast rate, causing populations to increase dramatically over a very short time. Possibly because of limited resources and the life cycles of their predators, lemming populations tend to plummet every four years. These periodic "boom-and-bust" cycles may also contribute to the mass suicide myth.

Using the data in the table at the right, model the lemming population for a four year cycle.

Time (Years)	Population Per Hectare
0	5
0.5	7.2
1	10.4
1.5	15
2	21.6
2.5	31.2
3	45
3.5	64.9
4	93.6

Solution

Let *t* represent the number of years and *P*(*t*) represent the population per hectare.

Notice that the population ***triples*** every 1.5 years. Then, $P(t)=5\left(3^{t/1.5}\right)=5\left(3^{2t/3}\right)$

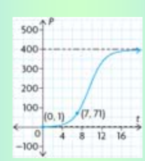
Alternative Solution – Samina's Approach

$\frac{P(1)}{P(0)}=\frac{10.4}{5}=2.08$, $\frac{P(2)}{P(1)}=\frac{21.6}{10.4}=2.08$, $\frac{P(3)}{P(2)}=\frac{45}{21.6}=2.08$, $\frac{P(4)}{P(3)}=\frac{93.6}{45}=2.08$

Then, $P(t)=5\left(2.08^{t}\right)$.

3. Since exponential growth is so fast, it usually cannot be sustained for very long. The rate of growth of any system is constrained by the availability of resources. Once the growth rate outstrips the rate of growth of resources, the system's growth is necessarily curbed. In such cases, a ***logistic function*** is likely a better mathematical model than an exponential function.

The general equation of a ***logistic function*** is $f(x)=\frac{c}{1+ab^{x}}$, where c represents the ***carrying capacity*** (upper limit) of the function. The following is an example of a graph of a logistic function.



Construct both an ***exponential model*** and a ***logistic model*** for the following.
(a) According to each model, how long would it take to reach the maximum rate of infection?
(b) Which model describes the given situation more realistically?
A town has a population of 5000 people. During a March Break trip, one of the town's residents contracted a virus. One week after her return to the town, 70 additional people had become infected with the same virus. Detailed scientific studies of the transmission of this virus determined that it infects approximately 8% of a given population.

Solution

- Let t represent time in days
- Let $N(t)$ represent the number of people infected after t days

Time (Days)	*Number of People Infected*
0	1
7	71

Exponential Model

$$\frac{N(7)}{N(0)} = \frac{71}{1} = 71$$

Therefore, $N(t) = 1\left(71^{t/7}\right) = 71^{t/7}$.

Logistic Model

$N(t) = \dfrac{c}{1+ab^t}$, where c represents the upper limit (carrying capacity). Since 8% of the population will get infected, the upper limit = carrying capacity = max # of people infected= $c = 0.08(5000) = 400$.

Therefore, $N(t) = \dfrac{400}{1+ab^t}$.

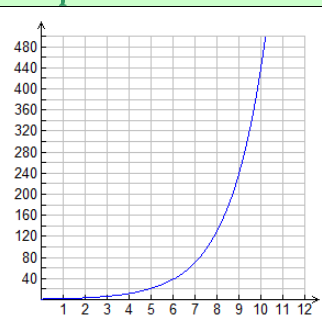
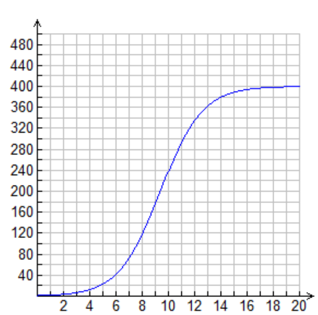
To calculate a and b, we can use the data in the table shown above.

Since $N(0) = 1$, it follows that $\dfrac{400}{1+a} = 1$. Solving this equation for a, we obtain a = 399.

Since $N(7) = 71$, it follows that $\dfrac{400}{1+ab^7} = \dfrac{400}{1+399b^7} = 71$. (In the previous step we determined that a = 399.)
Solving for b, we obtain $b \doteq 0.5291$.

Therefore, the equation for the logistic model is $N(t) = \dfrac{400}{1+399(0.5291)^t}$

Comparing the Two Models

Model	*Equation*	*Graph*	*Explanation*
Exponential	$N(t) = 1\left(71^{t/7}\right) = 71^{t/7}$		The exponential model predicts slow initial growth followed by much faster growth. In addition, it predicts that the virus will infect the maximum number of people in about 10 days. As can be seen easily from the graph, however, an exponential model is unrealistic because it predicts that the number of infections will continue to rise rapidly beyond the maximum number of expected infections.
Logistic	$N(t) = \dfrac{400}{1+399(0.5291)^t}$		The logistic model predicts slow growth followed by rapid growth, and then a slowing of the growth rate again as the maximum number of infected people approaches 400. In addition, it predicts that the virus will infect the maximum number of people in about 16 days. The logistic model is much more realistic than the exponential model because it correctly predicts that the number of infections will approach an upper limit of about 400.

4. How long would it take for an investment of \$5000.00 to double if it is invested at a rate of 2.4% per annum (per year) compounded monthly?

Solutions

annual interest rate = r = 0.024 periodic (monthly) interest rate = i = 0.024 ÷ 12 = 0.002

Time (Months)	***Value of Investment (\$)***
0	5000
1	$5000(1.002)$ $=5000(1.002)^1$
2	$5000(1.002)(1.002)$ $=5000(1.002)^2$
3	$5000(1.002)^2(1.002)$ $=5000(1.002)^3$
t	$5000(1.002)^t$

Let t represent the time in months and let $A(t)$ represent the value of the investment, in dollars, after t months.

Using the table shown at the left, it's clear that $A(t)=5000(1.002)^t$.

Double the initial value of the investment is \$10000. To determine the time required for the value of the investment to grow to \$10000, we simply need to solve the equation $5000(1.002)^t=10000$.

$$5000(1.002)^t=10000$$
$$\therefore (1.002)^t=2$$

Without logarithms, there is no way of solving this equation other than using trial and error. Since $(1.002)^{347}\doteq 2$, we can conclude that $t\doteq 347$. Therefore, it would take about 347 months (28.9 years) for the value of the investment to double.