Unit 0 – Review of Essential Mathematical Concepts

MHF4UO – Review # 1 – Mechanical Skills

1. ***Simplify*** each of the following ***expressions***.

(a) $3\left(x^2-2xy+2y^2\right)-5\left(2x^2-2xy-y^2\right)$

(b) $6\left(x-y\right)-2\left(2x+7y\right)-\left(3x-2y\right)$

(c) $\dfrac{\sin^2 x}{1-\cos x}$

(d) $\cot x\csc x(\sec x-1)$

(e) $\dfrac{2x}{4x^2+6x}-\dfrac{3}{2x+3}$

(f) $\dfrac{a^{-3}b^2}{a^4b^{-6}}$

(g) $\dfrac{\left(m^{-3}n^{-2}\right)^{-1}}{\left(m^{-4}n^{-6}\right)^3}$

(h) $10a^2b+5ab-15a$

2. ***Factor*** each of the following ***expressions*** fully.

(a) $10a^2b+5ab-15a$

(b) $d^2+3d-10$

(c) $y^2-10y+25$

(d) $2x^2+7x+3$

(e) $4x^2+12x+9$

(f) $6q^3s^7-7q^3s^6-3q^3s^5$

(g) $9a^2-16$

(h) $9a^2b-6ab+b$

(i) $\sin^2 x-\cos^2 x$

(j) $\cot^2 x-4\cot x+4$

3. ***Solve*** each of the following ***equations***.

(a) $\dfrac{x}{3}+\dfrac{1}{2}=0$

(b) $\dfrac{m+2}{2}=\dfrac{m-1}{3}$

(c) $1.2\left(10x-5\right)-\left(4x+7\right)=8$

(d) $2y^2+7y+3=0$

(e) $\left(q+4\right)\left(q-4\right)=-9\left(q+1\right)\left(q-1\right)$

(f) $\left(z+5\right)\left(2z-3\right)=\left(z+3\right)\left(z+4\right)$

(g) $\sin^2 x+2\sin x+1=0$

(h) $\cos^2\theta-4\cos\theta=-4$

4. ***Evaluate*** each of the following without using a calculator.

(a) $\sec 135°$

(b) $\tan 120°$

(c) $\sin 180°$

(d) $\tan 90°$

(e) $64^{\frac{2}{3}}$

(f) $32^{\frac{11}{5}}$

(g) $h\left(-5\right)$, if $h\left(x\right)=4^{-x}$

(h) $f\left(135°\right)$, if $f\left(\theta\right)=5\cos^2\theta-2\sin\theta+\csc\theta$

(i) $g\left(0\right)$, if $g\left(x\right)=-1.3\left(2^{5x}\right)+27$

(j) t_5, if $t_n=-6\left(2^n\right)$

5. ***Graph*** each of the following ***functions*** without using a calculator.

(a) $f\left(x\right)=-3x-5$

(b) $g\left(x\right)=-3x^2-5x+1$

(c) $h(\theta)=1.5\sin\left(2\left(\theta-45°\right)\right)+1$

(d) $p\left(z\right)=-3\left(2^{5(z-1)}\right)+7$

MHF4U0 – Review # 2 – Going beyond the Mechanical

1. Several equations are given below.

(a) $x^2-3x-22=4(x-1)$ **(b)** $f(x)=ax^2+bx+c$ **(c)** $t_n=4\left(2^{n-1}\right)$ **(d)** $x^2+y^2=1$

(e) $\sin x=\frac{1}{2}$ **(f)** $y=2x-1$ **(g)** $f(n)=a+(n-1)d$ **(h)** $\cos^2 x+\sin^2 x=1$

(i) $(x+y)^2=x^2+2xy+y^2$ **(j)** $c^2=a^2+b^2$ **(k)** $1+\tan^2 x=\sec^2 x$ **(l)** $x^2+x=1$

Circle the letters in the table that correspond to the equations that match the type given in the heading of each column.

Equations that are Solved to find the Value(s) of the Unknown	***Equations that Describe a Relationship between two or more Quantities***	***Identities***
(a) **(b)** **(c)** **(d)**	**(a)** **(b)** **(c)** **(d)**	**(a)** **(b)** **(c)** **(d)**
(e) **(f)** **(g)** **(h)**	**(e)** **(f)** **(g)** **(h)**	**(e)** **(f)** **(g)** **(h)**
(i) **(j)** **(k)** **(l)**	**(i)** **(j)** **(k)** **(l)**	**(i)** **(j)** **(k)** **(l)**

2. The following table contains a series of mathematical statements, some of which contain terminological and/or notational errors. Give a geometric interpretation of each statement that does not contain any errors. Suggest corrections for the statements that do contain errors.

Statement(s)	***Give a geometric interpretation of the correct statements. Suggest possible corrections of the statements that contain errors.***
$\sin\theta=\frac{1}{2}$	
The ***factored*** form of x^2-5x+6 is $\left(x-\frac{5}{2}\right)^2-\frac{1}{4}$	
$2^x=-1$ $\therefore x$ is ***impossible***	
$\dfrac{\overset{3}{\cancel{6}}\cancel{x^2}+17\overset{1}{\cancel{x}}+\overset{1}{\cancel{12}}}{\underset{1}{\cancel{2}}\cancel{x^2}+11\underset{1}{\cancel{x}}+\underset{1}{\cancel{12}}}=\dfrac{3+17+1}{1+11+1}=\dfrac{21}{13}$	
The graph of $f(x)=\sec x$ is called a ***parabola*** because it consists of "u-shaped" sections.	
When the ***equation*** $x^2y^3-5x^2y^3+3x^3-5a^2-7x^3$ is ***evaluated***, the result is $-4x^2y^3-4x^3-5a^2$ However, $5a^2-7b^2$ cannot be ***evaluated***.	

3. State whether each of the following is true or false. Provide an explanation to support each response.

Statement	*True or False?*	*Explanation*
If $f(x)=2^x$ then $f(x+y)=f(x)+f(y)$		
$t_n=t_{n-1}+t_{n-2}$ $\therefore nt=(n-1)t+(n-2)t$		
$125^{\frac{1}{3}}=(1/3)125=\dfrac{125}{3}$		
For all functions *f*, $f^{-1}(x)=\dfrac{1}{f(x)}$		
The quadratic expression $253x^2-72x-493$ ***can be*** factored over the integers. (DO NOT TRY TO FACTOR! JUST EXPLAIN WHETHER IT IS POSSIBLE TO FACTOR!)		

4. Although we often think of radioactive substances as dangerous and harmful, they also have many useful applications. Consider the following examples.

- The radioactive substance carbon-14 can be used to estimate the age of fossils that are up to about 40000 years old.
- Sodium-24, another radioactive substance, is used as a tracer in medical diagnostics. For example, it can be used to measure the rate of flow of blood in an artery or a vein.

Time (hours)	*Mass (kg)*
0	0.1
2	0.0911
4	0.0830
6	0.0756
8	0.0689
10	0.0628
12	0.0572
14	0.0521
16	0.0475
18	0.0433
20	0.03944
22	0.03594
24	0.03274

Since radioactive substances are unstable, they decay (break down) over time. For instance, if a 1 kg sample of sodium-24 were left undisturbed, after 14.9 hours only about 500 g of it would be left. This process of decreasing mass over time due to the emission of radiation is called ***radioactive decay***.

The table at the right lists measurements made to determine the rate of decay of sodium-24.

(a) Use regression to find a function that fits the data well. (**Hint:** We have only used quadratic, sinusoidal and exponential regressions. Think of which of these three best fits this situation. If you are unsure, experiment with each type of regression until you find the most logical fit.)

(b) Now use your equation to predict how much of the sample would be left after 14 ***days***.

MHF4U0 – Review #3 – Loads of Mechanical Practice

Functions, Simplifying, Factoring

It is not necessary to complete every question in this review. Complete enough questions to ensure that you have a very solid understanding of the review material.

Omit 6, 7, 11de

1. For each function, find:
   i) $f(-2), f(3), f(1.2)$ ii) the domain and the range iii) the inverse
   a) $f(x) = 3x - 2$ b) $f(x) = 2x^2 - 3x + 1$ c) $f(x) = \dfrac{2-x}{x}$

2. If $f(x) = x^2 - 3x + 1$, find:
   a) $f(2k)$ b) $f(-2k)$ c) $-f(z)$
   d) $f(x + 2)$ e) $f(2x - 1)$ f) $f\left(\frac{1}{x}\right)$.

3. If $f(x) = 2x^2 - 5$, show that $f(x + 3) \neq f(x) + f(3)$.

4. If $f(x) = 2x^2 - 5x + 1$, solve each equation.
   a) $f(x) = -1$ b) $f(x) = 13$ c) $f(2a) = 13$

5. Draw a mapping diagram to represent each set of ordered pairs. Which sets represent functions?
   a) $\{(1,9), (2,7), (3,5), (4,3), (5,1)\}$
   b) $\{(-3,2), (-1,4), (1,2), (-3,7), (5,-1), (6,5)\}$
   c) $\{(-4,13), (-3,7), (-2,3), (-1,1), (0,3), (1,7)\}$

6. If $f(x) = 2x + 5$ and $g(x) = x^2 - 3x + 2$, find:
   a) $g(f(x))$ b) $f(f(x))$ c) $f^{-1}(f^{-1}(x))$ d) $f(g(x))$.

7. Given $f(x) = 3x - 1$ and $g(x) = 2\sqrt{x+1}$
   a) Find.
      i) $f\left(\dfrac{2x-1}{2}\right)$ ii) $g\left(\dfrac{5}{x-2}\right)$ iii) $f(g(x))$ iv) $g(f(x))$
   b) State the domain and the range of each function.
      i) $f(x)$ ii) $g(x)$ iii) $f(g(x))$ iv) $g(f(x))$

8. If $f(x) = 2x - 5$ and $g(x) = (x + 2)^2 - 3$, sketch these graphs.
   a) $f(x)$ and $g(x)$ on the same axes. b) $f(x) + g(x)$ c) $f(x) - g(x)$

9. The perimeter of a rectangle is 10 m. Express the length of its diagonal as a function of its width.

10. Simplify.
    a) $2(3x + 2y) - 5(x - 4y)$
    b) $3m(5m - 7n) - 2m(12m + 5n)$
    c) $x(4x - 3y + 7) - 3x(2x + 5y - 8)$
    d) $(5a - 3b)(2a + 7b)$
    e) $(3x + 2)^2 - 2(4x - 1)(x + 3)$
    f) $3(2m - 1)(-4m + 7n - 1) - (2m + n - 3)^2$

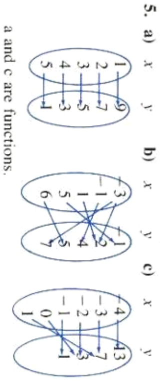
11. Factor.
    a) $15x^2 - 6xy + 21x$ b) $x^2 - 17x + 42$
    c) $10x^2 + 7xy - 12y^2$ d) $m^2(3m - 2) - 5m(3m - 2) + 9m - 6$
    e) $15x^3 + 21x^2y - 18xy^2$ f) $100p^2 - 36q^2$

Answers

1. a) i) $-8, 7, 1.6$ **ii)** R; R
iii) $y = \dfrac{x+2}{3}$ **b) i)** 15, 10, 0.28
ii) R; $\left\{y \mid y \geq -\frac{1}{8}, y \in R\right\}$
iii) $y = \dfrac{\pm\sqrt{8x+1}+3}{4}$ **c) i)** $-2, -\frac{1}{3}, \frac{2}{3}$
ii) $\{x \mid x \neq 0, x \in R\}, \{y \mid y \neq -1, y \in R\}$
iii) $y = \dfrac{2}{x+1}$

2. a) $4k^2 - 6k + 1$ **b)** $4k^2 + 6k + 1$
c) $-z^2 + 3z - 1$ **d)** $x^2 + x - 1$
e) $4x^2 - 10x + 5$ **f)** $\dfrac{1}{x^2} - \dfrac{3}{x} + 1$

3. $2x^2 + 12x + 13 \neq 2x^2 + 8$
4. a) 0.5, 2 **b)** $-1.5, 4$ **c)** $-0.75, 2$
5. a) **b)** **c)**



a and c are functions.

6. a) $4x^2 + 14x + 12$ **b)** $4x + 15$ **c)** $\dfrac{x-15}{4}$
d) $2x^2 - 6x + 9$

7. a) i) $\dfrac{6x-5}{2}$ **ii)** $2\sqrt{\dfrac{x+3}{x-2}}$
iii) $6\sqrt{x+1} - 1$ **iv)** $2\sqrt{3x}$
b) i) R; R
ii) $\{x \mid x \geq -1, x \in R\}; \{y \mid y \geq 0, y \in R\}$
iii) $\{x \mid x \geq -1, x \in R\}, \{y \mid y \geq -1, y \in R\}$
iv) $\{x \mid x \geq 0, x \in R\}, \{y \mid y \geq 0, y \in R\}$

9. $d = \sqrt{2w^2 - 10w + 25}$
10. a) $x + 24y$ **b)** $-9m^2 - 31mn$
c) $-2x^2 - 18xy + 31x$
d) $10a^2 + 29ab - 21b^2$ **e)** $x^2 - 10x + 10$
f) $-28m^2 + 38mn + 18m - 15n - n^2 - 6$
11. a) $3x(5x - 2y + 7)$ **b)** $(x - 14)(x - 3)$
c) $(2x + 3y)(5x - 4y)$
d) $(3m - 2)(m^2 - 5m + 3)$
e) $3x(x + 2y)(5x - 3y)$
f) $4(5p - 3q)(5p + 3q)$

Trigonometry

Omit 1efgh, 4ii, 6, 9b, 10, 11

1. Draw each angle in standard position, then find two angles which are coterminal with it.
   a) $65°$ b) $135°$ c) $200°$ d) $-450°$
   e) $\frac{\pi}{3}$ f) $\frac{5\pi}{4}$ g) $-\frac{\pi}{6}$ h) $\frac{8\pi}{3}$

2. Determine the sine, the cosine, and the tangent to 3 decimal places of each angle in *Exercise 1*.

3. Each point P is on the terminal arm of angle θ. Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ to 3 decimal places.
   a) P(4,9) b) P(8, −15) c) P(−4,7) d) P(−6, −5)

4. Find each value of θ in *Exercise 3*:
   i) in degrees to 1 decimal place ii) in radians to 3 decimal places.

5. Solve for θ to the nearest degree, $0° \leqslant \theta \leqslant 360°$.
   a) $\sin\theta = 0.7295$ b) $\cos\theta = -0.3862$ c) $\tan\theta = -5.1730$

6. Solve for θ in radians to 2 decimal places, $0 \leqslant \theta \leqslant 2\pi$.
   a) $\cos\theta = 0.2681$ b) $\tan\theta = 1.0744$ c) $\sin\theta = -0.4683$

7. Solve for θ to the nearest degree, $0° \leqslant \theta \leqslant 360°$.
   a) $3\sin\theta + 2 = 0$ b) $2\tan\theta - 5 = 2$
   c) $12\sin^2\theta - 11\sin\theta + 2 = 0$ d) $3\cos^2\theta + 4\cos\theta - 2 = 0$
   e) $3\cos^2\theta + 2 = 4$ f) $2\sin^2\theta + 5\sin\theta + 1 = 0$

8. Draw graphs of $y = \sin\theta$ and $y = \cos\theta$ for $-360° \leqslant \theta \leqslant 360°$. For each graph
   a) State the maximum value of y, and the values of θ for which it occurs.
   b) State the minimum value of y, and the values of θ for which it occurs.
   c) State the θ- and y-intercepts.

9. Find the amplitude, the period, the phase shift, and the vertical displacement for each function.
   a) $y = 3\sin 2(\theta - 45°) - 4$ b) $y = -2\cos 5\left(\theta + \frac{\pi}{3}\right) + 1$

10. Sketch the graphs of each set of functions on the same grid for $-2\pi \leqslant \theta \leqslant 2\pi$.
    a) $y = \sin\theta$ $y = 3\sin\theta$ $y = 3\sin\theta + 2$
    b) $y = \frac{1}{2}\cos\theta$ $y = \frac{1}{2}\cos\left(\theta + \frac{\pi}{3}\right)$ $y = \frac{1}{2}\cos\left(\theta + \frac{\pi}{3}\right) + 2$

11. Sketch the graph of each function.
    a) $y = \frac{1}{2}\sin 2\pi\frac{(t+1)}{2} - 3$ b) $y = -3\sin 2\pi\frac{(t-2)}{5} + 2$

12. A Ferris wheel of radius 16 m rotates once every 48 s. The passengers get on at a point 1 m above ground level.
    a) Write an equation to express the height h metres of a passenger above the ground at any time t seconds.
    b) How high is a passenger after: i) 10 s ii) 25 s?

Answers

1. Typical answers: **a)** $425°, -295°$ **b)** $495°, -225°$ **c)** $560°, -160°$ **d)** $270°, -90°$ **e)** $\frac{7\pi}{3}, -\frac{5\pi}{3}$ **f)** $\frac{13\pi}{4}, -\frac{3\pi}{4}$ **g)** $\frac{11\pi}{6}, -\frac{13\pi}{6}$ **h)** $\frac{2\pi}{3}, -\frac{4\pi}{3}$
2. a) 0.906, 0.423, 2.145
b) 0.707, −0.707, −1.000
c) −0.342, −0.940, 0.364
d) −1.000, 0, undefined
e) 0.866, 0.500, 1.732
f) −0.707, −0.707, 1.000
g) −0.500, 0.866, −0.577
h) 0.866, −0.500, −1.732
3. a) 0.914, 0.406, 2.250
b) −0.882, 0.471, −1.875
c) 0.868, −0.496, −1.75
d) −0.640, −0.768, 0.833
4. a) 66.0°, 1.153 **b)** 298.1°, 5.202
c) 119.7°, 2.090 **d)** 219.8°, 3.836
5. a) 47°, 133° **b)** 113°, 293° **c)** 101°, 281°
6. a) 1.30, 4.98 **b)** 0.82, 3.96 **c)** 5.80, 3.63
7. a) 222°, 318° **b)** 74°, 254°
c) 14°, 42°, 138°, 166° **d)** 67°, 93°
e) 48°, 132°, 228°, 312° **f)** 193°, 347°
8. a) $\sin\theta$: 1 at −270°, 90°; $\cos\theta$: 1 at 0°, ± 360°
b) $\sin\theta$: −1 at −90°, 270°; $\cos\theta$: −1 at ± 180°
c) $\sin\theta$: y-intercept 0, θ-intercepts ± 360°, ± 180°, 0°; $\cos\theta$: y-intercept 1, θ-intercepts ± 270°, ± 90°, 0°
9. a) 3; π; 45°; 4 **b)** 2; $\frac{2\pi}{5}$; $-\frac{\pi}{3}$; 1
c) 4; $\frac{2\pi}{3}$; $-\frac{\pi}{6}$; −5
12. a) $h = 16\cos\frac{2\pi(t-24)}{48} + 17$
b) 12.9 m, 32.9 m

More Trigonometry

1. Evaluate each trigonometric ratio to 3 decimal places.
   a) sin 55° b) tan 27° c) sec 81° d) cot 37° e) cos 65° f) csc 22°
   g) tan 138° h) sin 102° i) csc 95° j) cot 107° k) cos 141° l) sec 170°

2. Find each value of θ to the nearest degree if $0° < \theta < 180°$.
   a) $\cos\theta = 0.274$ b) $\cot\theta = 1.912$ c) $\sec\theta = 1.125$
   d) $\sin\theta = 0.469$ e) $\csc\theta = 3.150$ f) $\tan\theta = 2.247$

3. State the exact values of the six trigonometric ratios of each angle.
   a) 0° b) 30° c) 45° d) 60° e) 90°
   f) 120° g) 135° h) 150° i) 180°

4. Given θ is an acute angle, find the values of the other five trigonometric ratios.
   a) $\cos\theta = \dfrac{8}{17}$ b) $\tan\theta = \dfrac{12}{5}$ c) $\sec\theta = \dfrac{21}{11}$ d) $\sin\theta = \dfrac{5}{9}$

5. Given θ is an acute angle, find expressions for the other five trigonometric ratios.
   a) $\sin\theta = \dfrac{a}{b}$ b) $\tan\theta = \dfrac{p}{p+q}$ c) $\sec\theta = \dfrac{2m-1}{m+3}$

6. Solve $\triangle$ABC, if $\angle B = 90°$, and:
   a) AB = 15, BC = 27 b) AC = 18, BC = 10
   c) AB = 42, $\angle C = 72°$ d) AC = 12, $\angle A = 35°$.
   Give the answers to 1 decimal place where necessary.

7. Solve each $\triangle$PQR. Give the answers to 1 decimal place.
   a) $\angle Q = 75°$, $r = 8$, $p = 11$ b) $\angle R = 52°$, $r = 28$, $q = 25$
   c) $\angle P = 38°$, $\angle Q = 105°$, $p = 32$ d) $r = 17$, $p = 14$, $q = 26$
   e) $\angle Q = 57°$, $q = 42$, $r = 45$ f) $\angle R = 33°$, $p = 14$, $q = 24$

8. A wheelchair ramp 8.2 m long rises 94 cm. Find its angle of inclination to 1 decimal place.

9. The angle of elevation of the sun is 68° when a tree casts a shadow 14.3 m long. How tall is the tree?

10. A cable car rises 762 m as it moves a horizontal distance of 628 m.
    a) How long is the ride?
    b) What is the angle of inclination of the cable to the nearest degree?

11. Two identical apartment buildings are 41.3 m apart. From her balcony, Kudo notices that the angle of elevation to the top of the adjacent building is 57°. The angle of depression to the base of the building is 28°. Find the height of the buildings.

12. Rectangle PQRS has sides whose lengths are in the ratio of 3 : 2. Points A and B are the midpoints of PQ and PS respectively. Find the measure of $\angle$BAR to 1 decimal place.

13. When watching a rocket launch, Nema is 0.8 km closer to the launching pad than Joel is. When the rocket disappears from view, its angle of elevation for Nema is 36.5° and for Joel is 31.9°. How high is the rocket at this point?

Answers

1. a) 0.819 **b)** 0.510 **c)** 6.392 **d)** 1.327 **e)** 0.423 **f)** 2.669 **g)** −0.900 **h)** 0.978 **i)** 1.004 **j)** −0.306 **k)** −0.777 **l)** −1.015 **2. a)** 74° **b)** 28° **c)** 27° **d)** 28°, 152° **e)** 19°, 161° **f)** 66°

3. sin: **a)** 0 **b)** $\frac{1}{2}$ **c)** $\frac{1}{\sqrt{2}}$ **d)** $\frac{\sqrt{3}}{2}$ **e)** 1 **f)** $\frac{\sqrt{3}}{2}$ **g)** $\frac{1}{\sqrt{2}}$ **h)** $\frac{1}{2}$ **i)** 0;

csc: **a), i)** Undefined **b)** 2 **c)** $\sqrt{2}$ **d)** $\frac{2}{\sqrt{3}}$ **e)** 1 **f)** $\frac{2}{\sqrt{3}}$ **g)** $\sqrt{2}$ **h)** 2;

cos: **a)** 1 **b)** $\frac{\sqrt{3}}{2}$ **c)** $\frac{1}{\sqrt{2}}$ **d)** $\frac{1}{2}$ **e)** 0 **f)** $-\frac{1}{2}$ **g)** $-\frac{1}{\sqrt{2}}$ **h)** $-\frac{\sqrt{3}}{2}$ **i)** −1;

sec: **a)** 1 **b)** $\frac{2}{\sqrt{3}}$ **c)** $\sqrt{2}$ **d)** 2 **e)** Undefined **f)** −2 **g)** $-\sqrt{2}$ **h)** $-\frac{2}{\sqrt{3}}$ **i)** −1;

tan: **a)** 0 **b)** $\frac{1}{\sqrt{3}}$ **c)** 1 **d)** $\sqrt{3}$ **e)** Undefined **f)** $-\sqrt{3}$ **g)** −1 **h)** $-\frac{1}{\sqrt{3}}$ **i)** 0;

cot: **a), i)** Undefined **b)** $\sqrt{3}$ **c)** 1 **d)** $\frac{1}{\sqrt{3}}$ **e)** 0 **f)** $-\frac{1}{\sqrt{3}}$ **g)** −1 **h)** $-\sqrt{3}$

4. a) $\sin\theta = \frac{15}{17}$; $\csc\theta = \frac{17}{15}$; $\sec\theta = \frac{17}{8}$; $\tan\theta = \frac{15}{8}$; $\cot\theta = \frac{8}{15}$
b) $\sin\theta = \frac{12}{13}$; $\csc\theta = \frac{13}{12}$; $\cos\theta = \frac{12}{13}$; $\sec\theta = \frac{13}{12}$; $\cot\theta = \frac{5}{12}$
c) $\sin\theta = \frac{8\sqrt{5}}{21}$; $\csc\theta = \frac{21}{8\sqrt{5}}$; $\cos\theta = \frac{11}{21}$; $\tan\theta = \frac{8\sqrt{5}}{11}$; $\cot\theta = \frac{11}{8\sqrt{5}}$
d) $\csc\theta = \frac{9}{5}$; $\cos\theta = \frac{2\sqrt{14}}{9}$; $\sec\theta = \frac{9}{2\sqrt{14}}$; $\tan\theta = \frac{5}{2\sqrt{14}}$; $\cot\theta = \frac{2\sqrt{14}}{5}$

5. a) $\csc\theta = \frac{b}{a}$; $\cos\theta = \frac{\sqrt{b^2 - a^2}}{b}$; $\sec\theta = \frac{b}{\sqrt{b^2 - a^2}}$; $\tan\theta = \frac{a}{\sqrt{b^2 - a^2}}$; $\cot\theta = \frac{\sqrt{b^2 - a^2}}{a}$
b) $\sin\theta = \frac{p}{\sqrt{2p^2 + 2pq + q^2}}$; $\csc\theta = \frac{\sqrt{2p^2 + 2pq + q^2}}{p}$; $\cos\theta = \frac{p+q}{\sqrt{2p^2 + 2pq + q^2}}$; $\sec\theta = \frac{\sqrt{2p^2 + 2pq + q^2}}{p+q}$; $\cot\theta = \frac{p+q}{p}$
c) $\sin\theta = \frac{\sqrt{3m^2 - 10m - 8}}{2m-1}$; $\csc\theta = \frac{2m-1}{\sqrt{3m^2 - 10m - 8}}$; $\cos\theta = \frac{m+3}{2m-1}$; $\tan\theta = \frac{\sqrt{3m^2 - 10m - 8}}{m+3}$; $\cot\theta = \frac{m+3}{\sqrt{3m^2 - 10m - 8}}$

6. a) AC = 31.0, $\angle A = 61°$, $\angle C = 29°$ **b)** AB = 15, $\angle C = 56°$, $\angle A = 34°$ **c)** BC = 13.6, AC = 44.2, $\angle A = 18°$ **d)** $\angle C = 55°$, AB = 9.8, BC = 6.9

7. a) $q = 11.8$, $\angle P = 64.0°$, $\angle R = 41.0°$ **b)** $\angle Q = 44.7°$, $\angle P = 83.3°$, $p = 35.3$ **c)** $\angle R = 37.0°$, $q = 50.3$, $r = 31.3$ **d)** $\angle Q = 113.7°$, $\angle R = 36.8°$, $\angle P = 29.6°$ **e)** $\angle R = 64.0°$, $\angle P = 59.0°$, $p = 42.9$ or $\angle R = 116.0°$, $\angle P = 7.0°$, $p = 6.1$ **f)** $r = 14.4$, $\angle P = 32.0°$, $\angle Q = 115.0°$

8. 6.6° **9.** 35.4 m **10. a)** 987 m **b)** 51° **11.** 85.6 m **12.** 52.1° **13.** 4.2 km

Quadratics, Equations, Factoring, Roots

Omit 13, 14, 16, 18bcde, 21b, 23, 24

12. Factor.
    a) $x^2 + 8xy + 16y^2 - 9$ b) $16x^2 - 4y^2 + 12y - 9$
    c) $4(x + 2y + z)^2 - 9(x - 2y + z)^2$ d) $a^2 - 2a + 1 - x^2 - 2xy - y^2$
    e) $2x^2(3x - 1) - 5x(3x - 1) + 6x - 2$ f) $(3x + 2)^2 + 6xy + 4y + y^2$

13. One factor of $6m^3 - 6 - 29m - 31m^2$ is $1 + 3m$. Find the other factors.

14. Divide $x^4 - 4x^3 - 6x^5 + x + 5x^2 + 15$ by $2x^2 - x + 3$.

15. If $p(x) = 2x^2 + 5x - 4$, evaluate:
    a) $p(-2)$ b) $p(3a)$ c) $p(8x - 5)$

16. Factor completely.
    a) $x^3 + 4x^2 + 5x + 2$ b) $x^3 + 2x^2 - 3$
    c) $27x^3 + 125y^3$ d) $4x^3 - x + 8x^2 - 2$

17. Solve and check.
    a) $6(x + 1) - 12x = 3 - 4(2x - 1)$ b) $\frac{2x - 4}{5} - \frac{x - 3}{4} = \frac{5 - 3x}{8}$
    c) $3x^2 - x - 14 = 0$ d) $2x^2 + 4x - 7 = 0$
    e) $5x^2 + 8x + 3 = 0$ f) $\frac{4}{x - 1} - \frac{5}{x + 2} = \frac{3}{x}$

18. Solve.
    a) $2x^2 - 5x + 4 = 0$ b) $4x^3 - 3x^2 + 2x = 0$
    c) $x^3 - 7x - 6 = 0$ d) $2x^3 - 3x^2 - 8x + 12 = 0$
    e) $(x^2 - 4x + 5)(x^2 - 4x + 2) = -2$ f) $\frac{4}{x + 1} - \frac{12}{x + 3} = \frac{-5}{x + 2}$

19. Solve for the variable indicated.
    a) $s = 3u + \frac{1}{2}at^2$, t b) $m = 2\sqrt{3n + 5} - 1$, n

20. Determine the nature of the roots of each equation.
    a) $4x^2 + 20x + 25 = 0$ b) $2x^2 - 5x + 2 = 0$ c) $3x^2 - 4x + 8 = 0$

21. Write a quadratic equation given:
    a) the roots $\frac{3}{4}$ and -2 b) the root $-2 + 3i$.

22. For what values of k does:
    a) $3x^2 - kx + 2 = 0$ have equal roots
    b) $2x^2 - 5x - k = 0$ have complex roots?

23. Solve.
    a) $|2x - 1| = 7$ b) $|x + 5| - x + 2 = 3$
    c) $x + 2|x + 1| = 8$ d) $|3x + 2| = 2x - 4$

24. Solve.
    a) $\sqrt{5x + 4} = 7$ b) $\sqrt{2x - 3} + 3 = x$
    c) $\sqrt{x + 5} + x = 7$ d) $2x + 2\sqrt{x} = 5$

Answers

12. **a)** $(x + 4y + 3)(x + 4y - 3)$
b) $(4x - 2y + 3)(4x + 2y - 3)$
c) $(5x - 2y + 5z)(-x + 10y - z)$
d) $(a - 1 - x - y)(a - 1 + x + y)$
e) $(3x - 1)(2x - 1)(x - 2)$
f) $(3x + y + 2)^2$

13. $2m + 1$, $m - 6$ **14.** $-3x^3 - x^2 + 2x + 5$

15. **a)** -6 **b)** $18a^2 + 15a - 4$
c) $128x^2 - 120x + 21$

16. **a)** $(x + 1)^2(x + 2)$ **b)** $(x - 1)(x^2 + 3x + 3)$
c) $(3x + 5y)(9x^2 - 15xy + 25y^2)$
d) $(2x + 1)(2x - 1)(x + 2)$

17. **a)** $\frac{1}{2}$ **b)** $\frac{9}{7}$ **c)** $\frac{7}{3}, -2$ **d)** $\frac{-2 \pm 3\sqrt{2}}{2}$
e) $-\frac{3}{5}, -1$ **f)** $-\frac{1}{2}, 3$ **18.** **a)** $\frac{5 \pm \sqrt{7}i}{4}$
b) $0, \frac{3 \pm \sqrt{23}i}{8}$ **c)** $-1, -2, 3$ **d)** $\frac{3}{2}, \pm 2$
e) 1, 3, 2, 2 **f)** $-\frac{5}{3}, 3$

19. **a)** $t = \pm\sqrt{\frac{2(s - 3u)}{a}}$ **b)** $n = \frac{(m + 1)^2 - 20}{12}$

20. **a)** 2 equal real roots **b)** 2 different real roots
c) 2 complex roots

21. **a)** $4x^2 + 5x - 6 = 0$ **b)** $x^2 + 4x + 13 = 0$

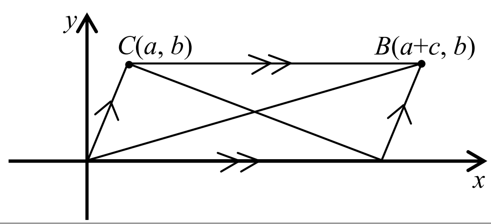
22. **a)** $k = \pm 2\sqrt{6}$ **b)** $k < -\frac{25}{8}$

23. **a)** $4, -3$ **b)** -3 **c)** $2, -10$ **d)** No solution

24. **a)** 9 **b)** 6 **c)** 4 **d)** $\frac{6 - \sqrt{11}}{2}$

MHF4U0 – Review # 4 – The Language of Mathematics

1. Complete the tables given below. This kind of exercise will help you to develop the ability to associate mathematical symbols, expressions and equations with concrete ideas.

Expression, Equation or Inequality	***Diagram***	***Conclusion, Interpretation or Explanation***
$3x+4y=5$		
$x^2+y^2=25$		
		
$b^2-4ac<0$		
$b^2-4ac>0$		
$b^2-4ac=0$		
$\sqrt{x+5}=x^2$		

Expression, Equation or Inequality	***Diagram***	***Conclusion, Interpretation or Explanation***
$f(2)=6$		
$f(a)=b$		
$\dfrac{f(10)-f(2)}{10-2}=-4$		
$\dfrac{f(b)-f(a)}{b-a}=c$		
$\dfrac{f(1+h)-f(1)}{h}=3$		
$\dfrac{f(a+h)-f(a)}{h}=b$		
$\sin x=\dfrac{1}{2}x-\pi$		

2. State whether each of the following is true or false. Provide an explanation to support each response. Keep the following points in mind:

- If a mathematical statement is said to be ***true***, it must be true in ***all possible cases***.
- A ***general proof*** is required to demonstrate that a statement is ***true***. The proof must demonstrate the truth of the statement in all possible cases! Clearly, any number of examples cannot accomplish this goal.
- To demonstrate that a statement is ***false***, it is only necessary to produce a ***single example*** that contradicts the statement. Such an example is called a ***counterexample***.

Statement	***True or False?***	***Proof, Counterexample or Explanation***
$(a+b)^2 = a^2 + b^2$		
$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$		
For all functions f and all real numbers u and c, $f(u+c) = f(u) + f(c)$		
The slope of the line $3x + 4y - 6 = 0$ is 3.		
The equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$ describes a function.		
For the function $g(x) = \sqrt{x+3} - 5$, $D = \{x \in \mathbb{R} : x \geq -3\}$ and $R = \{y \in \mathbb{R} : y \geq -5\}$. (Here D and R represent domain and range respectively.)		
Suppose that $g(x) = -3f(2x-8) + 6$. To obtain the graph of g, the following transformations must be performed to f: • Vertical stretch by a factor of -3 followed by a shift up by 6 units • Horizontal compression by a factor of $1/2$ followed by a shift 8 units right.		

Appendix – Notes on Prerequisite Material

Operator Precedence (Order of Operations)

Standard Order (no parentheses)	*Notes*	*Example*	*Example with Parentheses*
1. Exponents		$12 \div 4 \times 3 - 2 \times 4^3 \div 32$ $= 3 \times 3 - 2 \times 64 \div 32$ $= 9 - 128 \div 32$ $= 9 - 4$ $= 5$	$12 \div 4 \times (3-2) \times 64 \div 32$ $= 3 \times 1 \times 64 \div 32$ $= 3 \times 64 \div 32$ $= 192 \div 32$ $= 6$
2. Multiplication and Division	Performed in order of occurrence from left to right when both of these operations occur in a given term.		
3. Addition and Subtraction	Performed in order of occurrence from left to right when both of these operations occur in a given expression.		

Parentheses are used to ***override*** the standard order of precedence. When a departure from the standard order is required (for example when subtraction needs to be performed before multiplication) parentheses must be used.

Operating with Integers

Adding and Subtracting Integers

- Movements on a number line
- Moving from one floor to another using an elevator
- Loss/gain of yards in football
- Loss/gain of money in bank account or stock market

Add a Positive Value or Subtract a Negative Value
$+(+\ )$ ***or*** $-(-\ ) \rightarrow$ ***GAIN*** (move ***up*** or ***right***)

Add a Negative Value or Subtract a Positive Value
$+(-\ )$ ***or*** $-(+\ ) \rightarrow$ ***LOSS*** (move ***down*** or ***left***)

Multiplying and Dividing Integers

- Multiplication is ***repeated addition***
  e.g. $5(-2) = 5$ groups of -2
  $= (-2)+(-2)+(-2)+(-2)+(-2) = -10$
- Division is the ***opposite of*** multiplication
  e.g. $-10 \div (-2) =$ How many groups of -2 in -10?
  $= 5$

Multiply or Divide Two Numbers of Like Sign
$(+\ )(+\ )$ ***or*** $(-\ )(-\ ) \rightarrow$ ***POSITIVE RESULT***

Multiply or Divide Two Numbers of Unlike Sign
$(+\ )(-\ )$ ***or*** $(-\ )(+\ ) \rightarrow$ ***NEGATIVE RESULT***

Operating with Fractions

Adding and Subtracting Rational Numbers (Fractions)

- Express each fraction with a ***common denominator***
- Use rules for operating with integers
- $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad}{bd} + \dfrac{bc}{bd} = \dfrac{ad+bc}{bd}$

Example

$$\frac{-3}{6} + \left(\frac{-5}{9}\right) = \frac{-9}{18} + \left(\frac{-10}{18}\right) = \frac{-9+(-10)}{18} = -\frac{19}{18}$$

Multiplying and Dividing Rational Numbers (Fractions)

- ***NO*** common denominator
- $\dfrac{a}{b}\left(\dfrac{c}{d}\right) = \dfrac{ac}{bd}$ and $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}$

Example

$$\frac{-3}{6} \div \left(\frac{-5}{9}\right) = \frac{-3}{6} \times \left(\frac{9}{-5}\right) = \frac{-3}{2} \times \left(\frac{3}{-5}\right) = \frac{9}{10}$$

Simplifying Algebraic Expressions

Adding and Subtracting TERMS

- Collect ***like terms***
- Use rules for adding/subtracting integers

Example

$-3x^2y + 5xy - 6x^2y - 13xy = -9x^2y - 8xy$

Multiplying FACTORS

- Use rules for multiplying integers, laws of exponents and the distributive law

Examples

1. $-3a^2(5a^6)(-6b^7) = (-3)(5)(-6)a^2a^6b^7 = 90a^8b^7$

2. $(2x-7)(3x-8) = 2x(3x) - 2x(8) - 7(3x) - 7(-8) = 6x^2 - 37x + 56$

Dividing Algebraic Expressions

- Use rules for dividing integers
- Use laws of exponents

Examples

1. $\dfrac{-6a^5b^3}{-18a^3b^5} = \dfrac{a^2}{3b^2}$, **2.** $\dfrac{7m^3n - 14m^6n^3}{-2m^2n^7} = \dfrac{7m^3n}{-2m^2n^7} - \dfrac{14m^6n^3}{-2m^2n^7} = -\dfrac{7m}{2n^6} + \dfrac{7m^4}{n^4}$

Factoring

An ***expression*** is ***factored*** if it is written as a ***product***.

Common Factoring	***Factor Simple Trinomial***	***Factor Complex Trinomial***	***Difference of Squares***
Example $-42m^3n^2+13mn^2p-39m^4n^3q$ $=-13mn^2(4m^2-p+3m^3nq)$	***Example*** $n^2-20n+91$ $=(n-7)(n-13)$ ***Rough Work*** $(-7)(-13)=91$ $-7+(-13)=-20$	***Example*** $10x^2-x-21$ $=(10x^2-15x)+(14x-21)$ $=5x(2x-3)+7(2x-3)$ $=(2x-3)(5x+7)$ ***Rough Work*** $(10)(-21)=-210$, $(-15)(14)=-210$ $-15+14=-1$	***Example*** $98x^2-50y^2$ $=2(49x^2-25y^2)$ $=2((7x)^2-(5y)^2)$ $=2(7x-5y)(7x+5y)$

Solving Equations

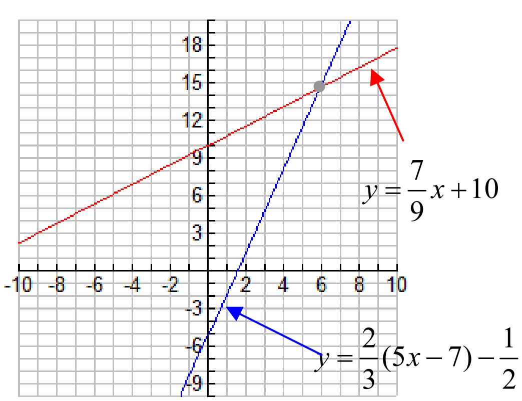
- An ***equation*** has an ***expression*** on the left-hand side, an ***expression*** on the right-hand side and an ***equals sign*** between the left and right sides.
- If an equation needs to be ***solved***, then we must find a value of the unknown that ***satisfies*** the equation. That is, when a solution is substituted into the equation, the left side ***must*** equal the right side.
- It is also very important to understand ***graphical solutions*** of equations.
- While using the balancing method, it is helpful to remember the "dressing/undressing" analogy. Remember that as long as you perform the ***same operation to both sides of any equation***, you will obtain an ***equivalent equation*** (i.e. an equation that has the same solutions as the original).

Solving Linear Equations

- Use the "balancing" method.
- Eliminate fractions by multiplying both sides of the equation by the ***least common multiple of all denominators***.

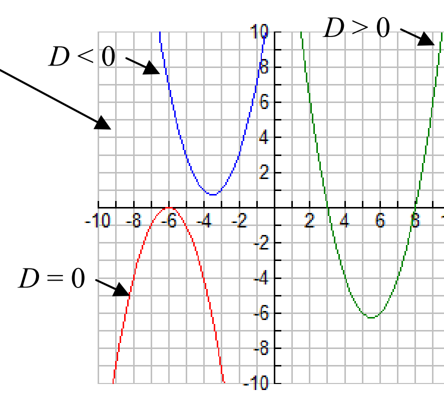
Example

$$\frac{2}{3}(5x-7)-\frac{1}{2}=\frac{7}{9}x+10$$
$$\therefore 18\left[\frac{2}{3}(5x-7)-\frac{1}{2}\right]=18\left[\frac{7}{9}x+10\right]$$
$$\therefore 12(5x-7)-9=14x+180$$
$$\therefore 60x-84-9=14x+180$$
$$\therefore 60x-93=14x+180$$
$$\therefore 60x-93-14x+93=14x+180-14x+93$$
$$\therefore 46x=273$$
$$\therefore x=\frac{273}{46}$$



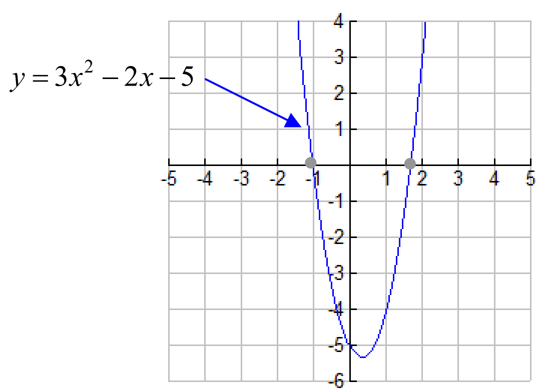
Solving Quadratic Equations

- First write the quadratic equation in the form $ax^2+bx+c=0$.
- Try ***factoring first***.
- If the quadratic does not factor, use the ***quadratic formula***.
- Only use the method of ***completing the square*** or ***partial factoring*** if you are asked to!
- The ***nature of the roots*** can be determined by calculating the ***discriminant*** $D=b^2-4ac$.



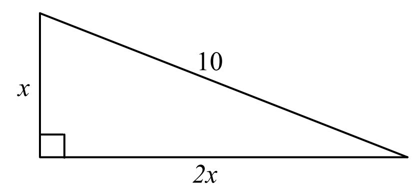
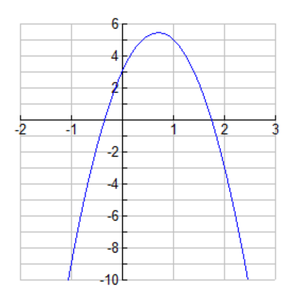
Examples

$$3n^2-2n-5=0$$
$$\therefore 3n^2-5n+3n-5=0$$
$$\therefore (3n^2-5n)+(3n-5)=0$$
$$\therefore n(3n-5)+1(3n-5)=0$$
$$\therefore (3n-5)(n+1)=0$$
$$\therefore 3n-5=0 \text{ or } n+1=0$$
$$\therefore n=\frac{5}{3} \text{ or } n=-1$$



Mathematical Relationships

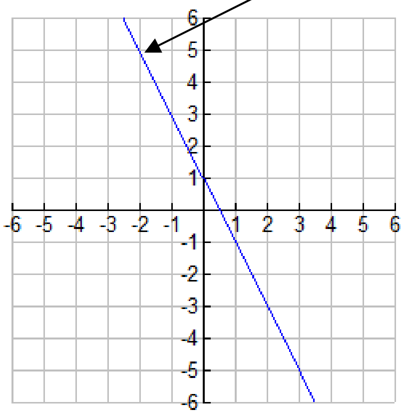
A ***formula*** is an ***equation*** that expresses a ***mathematical relationship*** between/among ***two or more unknowns***. ***All formulas are equations but not all equations are formulas!***

Example Formulas	*Relationship Expressed*	*Example*
$c^2 = a^2 + b^2$ This is the famous Pythagorean Theorem	This formula expresses the relationship among the sides of a right triangle.	**The hypotenuse of a right triangle has a length of 10 m and the lengths of other sides are in the ratio 2:1. Find the lengths of the other sides.** $x^2 + (2x)^2 = 10^2$ $\therefore x^2 + 4x^2 = 100$ $\therefore 5x^2 = 100$ $\therefore x^2 = 20$ $\therefore x = \sqrt{20} = 2\sqrt{5}$  The lengths of the other two sides are $2\sqrt{5}$ m and $4\sqrt{5}$ m.
$Ax + By + C = 0$ This is the so-called ***standard form*** of a linear equation.	This formula expresses the relationship between the x-co-ordinate and the y-co-ordinate of any point lying on a straight line with slope equal to $-\frac{A}{B}$ and y-intercept equal to $-\frac{C}{B}$.	**Find the slope and y-intercept of a straight line with equation $Ax + By + C = 0$.** First solve for y. Then compare to the $y = mx + b$ form. $Ax + By + C = 0$ $\therefore By = -Ax - C$ $\therefore y = -\frac{A}{B}x - \frac{C}{B}$ By comparing to the $y = mx + b$ form, we see that $m = -\frac{A}{B}$ and $b = -\frac{C}{B}$.
$y = ax^2 + bx + c$ This is the so-called ***standard form*** of a quadratic equation.	This formula expresses the relationship between the x-co-ordinate and the y-co-ordinate of any point lying on a parabola with vertex $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ and vertical stretch factor a. If $a > 0$ then the parabola opens ***upward***. Otherwise, if $a < 0$ then the parabola opens ***downward***.	**Find the vertex, vertical stretch factor and direction of opening of a parabola with equation $y = -5x^2 + 7x + 3$.** $y = -5x^2 + 7x + 3$ $\therefore y = -5\left(x^2 - \frac{7}{5}x\right) + 3$ $\therefore y = -5\left(x^2 - \frac{7}{5}x + \left(\frac{7}{10}\right)^2 - \left(\frac{7}{10}\right)^2\right) + 3$ $\therefore y = -5\left(x - \frac{7}{10}\right)^2 + 5\left(\frac{7}{10}\right)^2 + 3$ $\therefore y = -5\left(x - \frac{7}{10}\right)^2 + \frac{109}{20}$  Therefore, the vertex is $\left(\frac{7}{10}, \frac{109}{20}\right)$, the direction of opening is downward and the vertical stretch factor is 5.

Rate of Change (How fast a Quantity Changes relative to another Quantity)

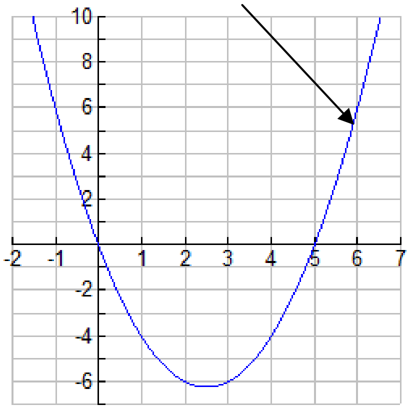
- If y is ***linearly related*** to x, then the ***rate of change of y is constant relative to x***. That is, for a given Δx, Δy is always the same. Another way of expressing this is that the ***first differences are constant***.
- If y is quadratically related to x, then the ***rate of change of y is NOT constant relative to x***. That is, for a given Δx, Δy is ***NOT*** always the same. For quadratic relationships, the ***first differences change linearly*** and ***the second differences are constant***.

$y = -2x + 1$



x	Δx	$y = -2x + 1$	First Differences Δy	Second Differences $\Delta(\Delta y)$
-1		3		
0	1	1	-2	
1	1	-1	-2	0
2	1	-3	-2	0
3	1	-5	-2	0
4	1	-7	-2	0
5	1	-9	-2	0
6	1	-11	-2	0

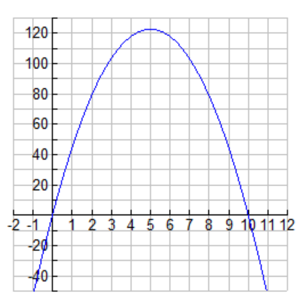
$y = x^2 - 5x$



x	Δx	$y = x^2 - 5x$	First Differences Δy	Second Differences $\Delta(\Delta y)$
-1		6		
0	1	0	-6	
1	1	-4	-4	2
2	1	-6	-2	2
3	1	-6	0	2
4	1	-4	2	2
5	1	0	4	2
6	1	6	6	2

What is Mathematics?

- In a nutshell, ***mathematics*** is the ***investigation*** of ***axiomatically defined abstract structures*** using ***logic*** and ***mathematical notation***. Accordingly, mathematics can be seen as an ***extension*** of ***spoken*** and ***written natural languages***, with an extremely precisely defined vocabulary and grammar, for the purpose of ***describing and exploring physical and conceptual RELATIONSHIPS.*** (Math is like a ***dating service***. It's all about ***relationships***! The lonely and very simple "Mr. x" is looking for a lovely but perhaps somewhat complex "Miss y.")
- Mathematical relationships can be viewed from a variety of different perspectives as shown below:

Algebraic	***Geometric***	***Physical***	***Verbal***	***Numerical***
$h = 49t - 4.9t^2$		A cannonball is fired vertically into the air with an initial speed of 49 m/s. Its height above the ground at a given time t is quadratically related to t.	The value of h is equal to the product of 49 and t reduced by the product of 4.9 and the square of t.	(see table below)

x	y1 (x) 49x-4.9x^2
-2	-117.6
-1	-53.9
0	0
1	44.1
2	78.4
3	102.9
4	117.6
5	122.5
6	117.6
7	102.9
8	78.4
9	44.1
10	-5.68E-14
11	-53.9

Each of these perspectives has an important role to play in the understanding of mathematical relationships.

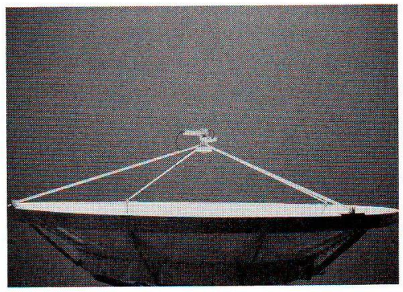
Linear Relationships

- These are the ***simplest*** of all mathematical relationships. They are very easy to analyze mathematically and are completely understood (i.e. there are no unresolved problems regarding linear relationships).
- The graphs of linear relationships are ***straight lines***.
- If y is linearly related to x, then the ***rate of change of y is constant***. That is, for a given Δx, Δy is always the same. Another way of expressing this is that the ***first differences are constant***.
- Linear relationships can be used to ***model*** any quantity that ***changes at a constant rate***. For example, for a car that is travelling at a constant speed, distance travelled is linearly related to time elapsed. That is, the graph of distance versus time would be a straight line.
- The general equation of a linear relationship is $Ax + By + C = 0$. This equation can be rewritten in the form $y = -\frac{A}{B}x - \frac{C}{B}$. By comparing this to the slope-intercept form a linear equation, we see that slope $= -\frac{A}{B}$ and y-intercept $= -\frac{C}{B}$. By comparing to the slope-intercept form of a linear relationship we find $m = -\frac{A}{B}$ and $b = -\frac{C}{B}$.
- Any equation involving only linear terms can be solved by using the ***balancing method***. For example, an equation such as $\frac{2}{3}(5x-7) - \frac{1}{2} = \frac{7}{9}x + 10$ can be solved by balancing.
- While using the balancing method, it is helpful to remember the "dressing/undressing" analogy. Remember that as long as you perform the ***same operation to both sides of any equation***, you will obtain an equivalent equation (i.e. having the same solutions).

Quadratic Relationships

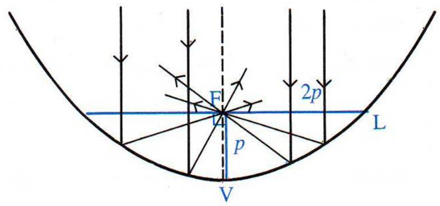
- These are not as simple as linear relationships but are still quite easy to analyze mathematically. Like linear relationships, quadratics are completely understood (i.e. there are no unresolved problems regarding quadratics).
- The graphs of quadratic relationships are ***parabolas***.
- If y is quadratically related to x, then the ***rate of change of y is NOT constant***. That is, for a given Δx, Δy is ***NOT*** always the same. For quadratic relationships, the ***first differences change linearly*** and ***the second differences are constant***.
- Quadratic relationships can be used to ***model*** many different quantities. For example, the position of any object moving solely under the influence of gravity (close to the surface of the Earth) changes quadratically. (See cannonball example in the "What is Mathematics?" section.
- The general equation of a quadratic relationship is $y = ax^2 + bx + c$. By ***completing the square***, this equation can be rewritten in the more convenient form $y = a(x-h)^2 + k$, where (h, k) is the vertex of the parabola.
- An equation involving quadratic terms ***cannot*** be solved entirely by using the ***balancing method***. Quadratic equations can be solved by ***factoring***, ***the quadratic formula***, ***partial factoring*** and ***completing the square***.
- The roots of a quadratic equation are completely characterized by the discriminant, which equals $b^2 - 4ac$. This expression appears under the square root sign in the quadratic formula, which is what allows us to decide the nature of the roots.
- Once we know the nature of the roots of a quadratic, we can deduce whether its associated parabola lies entirely above the x-axis, entirely below the x-axis, crosses the x-axis at two points or just "touches" it at one point.
- Contrary to Jonathan Coulson's and others claims, quadratics have a wide variety of applications. An interesting one is given below. Satellite dishes are really just antennas that are designed to receive signals originating from satellites in geostationary orbits around the Earth. Most satellite dishes have a parabolic cross-section.

Reflector Property of the Parabola



We have all seen dish antennas for receiving TV signals from satellites. These antennas have parabolic cross sections. When the antenna is aimed at a satellite, the signals entering the antenna are reflected to the receiver, which is placed at the focus of the antenna.

Every parabola has a *focus*, which is a particular point on the axis of symmetry. The position of the focus can be defined as follows.



For any parabola, the *focus* is the point on the axis of symmetry which is half as far from the vertex as it is from the parabola, measured along a line perpendicular to the axis of symmetry. For example, in the diagram, FV $= p$, and FL $= 2p$. That is, F is half as far from V as from L. Hence, F is the focus of the parabola. Every parabola has one and only one focus.

You can illustrate the reflector property of the parabola by completing the questions below.

QUESTIONS

1. a) Use a table of values to construct an accurate graph of the parabola defined by $y = \frac{1}{8}x^2$ for values of x between -8 and 8.
   b) Mark the point F(0,2) on the graph. Verify that F satisfies the above definition of the focus.
2. a) Mark any point P on the parabola you constructed in *Question 1*. Join PF, and draw a line PM parallel to the axis of symmetry. By estimation, draw a tangent to the parabola at P. Verify that PF and PM form equal angles with the tangent.
   b) Repeat part a) for other points P on the parabola.
3. Use the above definition of the focus to prove that the coordinates of the focus of the parabola defined by $y = ax^2$ are $\left(0, \frac{1}{4a}\right)$.

Terminology

- By this point in your mathematics education, you must understand and use the following terminology correctly: ***equation***, ***expression***, ***term***, ***factor***, ***polynomial***, ***monomial***, ***binomial***, ***trinomial***, ***factor***, ***expand***, ***solve***, ***simplify***, ***evaluate***, ***roots***, ***discriminant***, ***intercept***, ***intersect***, ***vertex***, ***axis of symmetry***, ***(simultaneous) system of equations***, ***relation***, ***relationship***, ***rate of change***, ***distance***, ***length***, ***area***, ***volume***, ***speed***, ***surface area***

Measurement

- Pythagorean Theorem and distance between two points (length of a line segment)
- Midpoint of a line segment
- Area of rectangle, parallelogram, triangle, trapezoid, circle
- Surface area of cylinder, cone, sphere, prism
- Volume of cylinder, cone, sphere, prism

Systems of Linear Equations (Two Linear Equations in Two Unknowns)

- Solve by using the method of substitution
- Solve by using the method of elimination
- Solve graphically

Viewing Relations and Functions from a Variety of Different Perspectives

Set of Ordered Pairs Perspective

- ***Ordered Pair***
  Two numbers written in the form (x, y) form an ***ordered pair***. As the name implies, ***order is important***.
- ***Set***
  A ***set*** is a group or collection of numbers, variables, geometric figures or just about anything else. Sets are written using ***set braces*** "{}." For example, {1, 2, 3} is the set containing the elements 1, 2 and 3. Note that ***order does not matter*** in a set. The sets $\{a, b, c\}$ and $\{c, a, b\}$ are the same set. Repetition does not matter either, so $\{a, b\}$ and $\{a, a, b, b, b\}$ are the same set.

  The main idea of a set is to group objects that have common properties. For example, the symbol "$\mathbb{Q}$" represents the set of ***rational numbers***, the set of all ***fractions***, including negative fractions and zero. Alternatively, we can think of a ***rational number*** as a ***ratio*** of two integers. That is, all rational numbers share the common property that they can be written in the form $\frac{a}{b}$, where a and b are both integers and $b \neq 0$. Formally, this is written

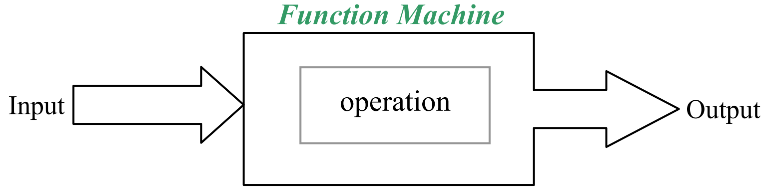
  $\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0 \right\}$. Note that the symbol $\mathbb{Z}$ ***denotes the set of integers*** and the symbol "$\in$" means "is an element of."
- ***Relation***
  A ***relation*** is any set of ***ordered pairs***. For example, the set $\{(2,0),(2,1),(2,2),(2,3),(2,4),(2,5),\ldots\}$ is a relation. In this relation, the number "2" is associated with every non-negative integer. This relation can be expressed more precisely as follows: $\{(x, y) : x = 2, y \in \mathbb{Z}, y \geq 0\}$. This is read as, "the set of all ordered pairs (x, y) such that x is equal to 2 and y is any non-negative integer."
- ***Function***
  A ***function*** is a ***relation*** in which for every "x-value," there is one and only one corresponding "y-value." The relation given above is ***not*** a function because for the "x-value" 2, there are an infinite number of corresponding "y-values." However, the relation $\{(0,2),(1,2),(2,2),(3,2),(4,2),(5,2),\ldots\}$ = $\{(x, y) : x \in \mathbb{Z}, x \geq 0, y = 2\}$ ***is*** a function because there is only one possible "y-value" for each "x-value." Formally, this idea is expressed as follows:

  Suppose that R represents a ***relation***, that is, a set of ordered pairs. Suppose further that $a = b$ whenever $(a,c) \in R$ and $(b,c) \in R$ (i.e. a must equal b if (a,c) and (b,c) are both in R). Then R is called a ***function***.

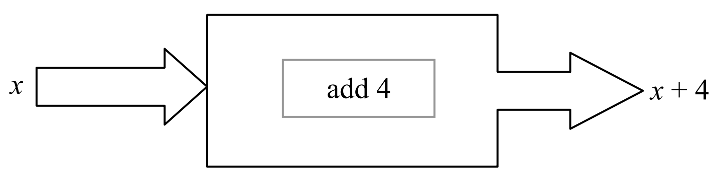
Machine Perspective

Sets of ordered pairs allow us to give very precise mathematical definitions of relations and functions. However, being a quite ***abstract*** concept, the notion of a set may be somewhat difficult to understand. You can also think of a function as a machine that accepts an ***input*** and then produces a ***single output***. ***There is one and only one "output" for every "input."***

Function Machine



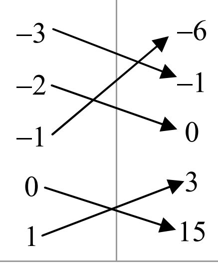
For example, consider the function machine that adds 4 to the input to produce the output.



Mapping Diagram Perspective

In addition to function machines, functions can be represented using ***mapping diagrams***, ***tables of values***, ***graphs*** and ***equations***. You may not know it but you already have a great deal of experience with functions from previous math courses.

A mapping diagram uses arrows to map each element of the input to its corresponding output value. Remember that a function has only one output for each input.

Relation as a Mapping Diagram		***Is it a function? Why or why not?***	***Set of ordered pairs***
Domain	***Range***		
		Since every element of the domain has only one corresponding element in the range, this relation is a function. This function has a ***one-to-one mapping***.	{(–3, –1), (–2,0), (–1, –6), (0, 15), (1, 3)}

Domain and Range

The set of all possible input values of a function is referred to as the ***domain***. That is, if D represents the domain of a function f, then $D = \{x : (x, y) \in f\}$.

The set of all possible output values of a function is referred to as the ***range***. That is, if R represents the range of a function f, then $R = \{y : (x, y) \in f\}$.

Numerical Perspective – Tables of Values

Consider the following relations expressed in table form.

(a) Which relations are functions? Justify your answers.
(b) Draw a mapping diagram for each relation.
(c) Write the set of ordered pairs for each relation.

<table>
<tr><th>Relation as a Table of Values</th><th>Is it a function? Why or why not?</th><th>Mapping Diagram</th><th>Set of Ordered Pairs</th></tr>
<tr><td><table><tr><th>x</th><th>y</th></tr><tr><td>–3</td><td>9</td></tr><tr><td>–1</td><td>1</td></tr><tr><td>1</td><td>1</td></tr><tr><td>3</td><td>9</td></tr></table></td><td></td><td></td><td></td></tr>
</table>

Geometric Perspective – Graphs of Functions

Discrete Relations

A ***discrete relation*** either has a ***finite number*** of ordered pairs ***OR*** the ordered pairs can be ***numbered*** using integers. Graphs of discrete relations consist of either a ***finite*** or an ***infinite*** number of "disconnected" points, much like a "connect-the-dots" picture ***before*** the dots are connected.

Continuous Relations

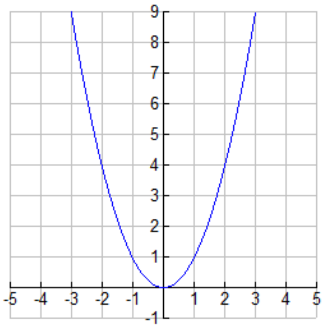
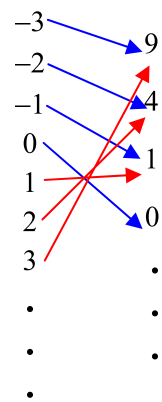
Unlike the graphs on the previous page, the following graphs ***do not represent discrete relations***. They are called ***continuous relations*** because their graphs do not consist of disconnected points. (You need to study calculus to learn a more precise definition of continuity. For now, it suffices to think of continuous relations as those whose graphs are "unbroken.")

Algebraic Perspective – Equations of Relations

The perspectives given above help us to understand the properties of relations and functions. However, when it comes time to computing with relations and functions, then equations become an indispensable tool!

For each of the following (the first is done for you)

(a) determine whether it is a function.
(b) state an equation that describes the relation.
(c) complete a table of values and a mapping diagram for each relation. Note that the given relations are all ***continuous***, which means that it is impossible to create a complete table of values or mapping diagram. (Why?)
(d) state the domain and range and the type of mapping.

Relation in Graphical Form	*Is it a function? Explain.*	*Equation*	*Table of Values*	*Mapping Diagram*	*Domain and Range (D & R)*	*Type of Mapping*
	This is a function because any vertical line will intersect at only a single point.	$f(x)=x^2$	x $f(x)$ -2 4 -1 1 0 0 1 1 2 4 • • • • • •		The domain is the set of all real numbers, that is, $D=\mathbb{R}$. The range is the set of all real numbers greater than or equal to zero, that is, $R=\{y\in\mathbb{R}: y\geq 0\}$ s	many-to-one

Transformations of Functions – Combining Stretches, Reflections and Translations

To obtain the graph of $y=g(x)=af(b(x-h))+k$ from the graph of $y=f(x)$, perform the following transformations. (Note that ***f*** is called the ***base function***, ***parent function*** or ***mother function***.)

Vertical (Follow the order of operations)

1. First ***stretch*** or ***compress vertically*** by the factor a. $(x,y)\to(x,ay)$

If $|a|>1$, ***stretch*** by a factor of a. ($|a|>1$ is a short form for "$a>1$ or $a<-1$")

If $0<|a|<1$, ***compress*** by a factor of a.

If $a<0$ (i.e. a is negative), the stretch or compression is combined with a ***reflection in the x-axis***.

2. Then ***translate*** k units ***up*** if $k>0$ or k units ***down*** if $k<0$. $(x,y)\to(x,y+k)$

Horizontal (Reverse the operations in the order opposite the order of operations)

1. First ***stretch*** or ***compress horizontally*** by the factor $\frac{1}{b}=b^{-1}$. $(x,y)\to(b^{-1}x,y)$

If $|b|>1$, ***compress*** by the factor b^{-1}.

If $0<|b|<1$, ***stretch*** by the factor b^{-1}.

If $b<0$ (i.e. b is negative), the stretch or compression is combined with a ***reflection in the y-axis***.

2. Then ***translate*** h units ***right*** if $h>0$ or h units ***left*** if $h<0$. $(x,y)\to(x+h,y)$

Summary using Mapping Notation

The following shows how an ordered pair belonging to f (pre-image) is mapped to an ordered pair belonging to g (image) under the transformation defined by $g(x)=af(b(x-h))+k$.

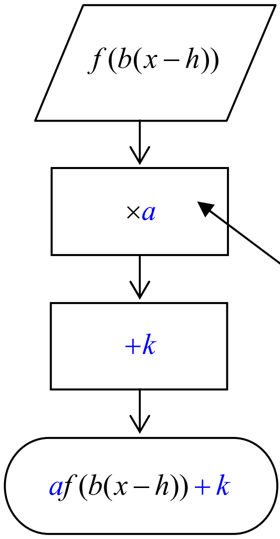
pre-image → image $(x,y)\to(b^{-1}x+h,ay+k)$

Explanation

Vertical	*Horizontal*
$af(b(x-h))+k$	$b(x-h)$

Vertical

$$af(b(x-h))+k$$



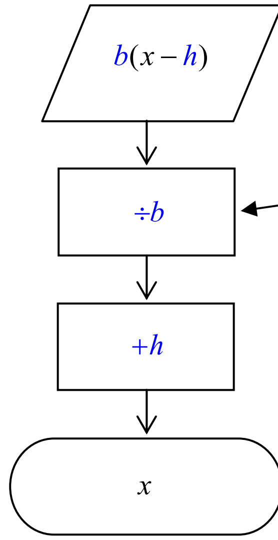
Because of the order of operations, $f(b(x-h))$ must be multiplied by a ***before*** k is added.

Therefore, the vertical stretch by a factor of a must be performed ***before*** the vertical translation by k units.

$$(x,y)\rightarrow(x,ay+k)$$

Horizontal

$$b(x-h)$$



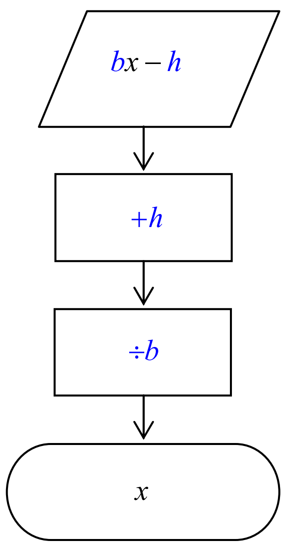
To find the output of g when the input x is given, we look at the output of f when the input $b(x-h)$ is given.

Therefore, the transformations must be performed in the reverse order, ***opposite*** the order of operations. ***First*** f is ***stretched or compressed horizontally by a factor of b^{-1}*** and ***then*** the resulting function is translated ***h units right if $h>0$*** or ***h units left if $h<0$***.

$$(x,y)\rightarrow(b^{-1}x+h,y)$$

Important Note

Notice that in $g(x)=af(b(x-h))+k$, the ***input*** to the function f (that is, $b(x-h)$) is given in ***factored*** form. When this is the case, the flowchart on the upper right applies. If the input is ***not*** given in factored form, then the following flowchart would apply. Remember that you must ***reverse the operations in the order opposite the order of operations.***



This time, the transformations must be done in the opposite order.

First f is translated ***h units right if $h>0$*** or ***h units left if $h<0$***. Then, it is ***stretched or compressed horizontally by a factor of b^{-1}***

Extremely Important Terminology

Term	*Meaning*	*Example(s)*
Expression	A combination of constants, operators, and variables representing numbers or quantities	**1.** $3(5)^2-4(5)(-1)+(-1)^2$ **2.** $3x^2-4xy+y^2$
Equation	A mathematical statement asserting that two expressions have the same value	$2(6z-1)(z-1)=-6(z-1)(2z-5)+3z+25$
Evaluate	Ascertain the numerical value of an expression	If $x=5$ and $y=-1$, ***evaluate*** $3x^2-4xy+y^2$. $3x^2-4xy+y^2$ $=3(5)^2-4(5)(-1)+(-1)^2$ $=3(25)+20+1$ $=96$
Simplify	Convert a mathematical expression to a simpler form	***Simplify*** $5(x-1)(x-2)-(5x-7)(x+10)$ $5(x-1)(x-2)-(5x-7)(x+10)$ $=5(x^2-3x+2)-(5x^2+43x-70)$ $=5x^2-15x+10-5x^2-43x+70$ $=5x^2-5x^2-15x-43x+10+70$ $=-48x+80$
Term	A mathematical expression that is associated to another through the operation of addition.	The algebraic expression $3x^2-4xy+y^2$ can be rewritten as $3x^2+(-4xy)+y^2$. Therefore, its ***terms*** are $3x^2$, $-4xy$ and y^2.
Factor	***Noun:*** One of two or more numbers or quantities that can be multiplied together to give a particular number or quantity. **e.g.** The factors of 15 are 1, 3, 5 and 15. ***Verb:*** To determine the factors of a number or expression.	***Factor*** $3n^2-2n-5$ $3n^2-2n-5$ $=3n^2-5n+3n-5$ $=(3n^2-5n)+(3n-5)$ $=n(3n-5)+1(3n-5)$ $=(3n-5)(n+1)$
Solve	Work out the solution to an equation	***Solve*** $3n^2-2n-5=0$ $3n^2-2n-5=0$ $\therefore 3n^2-5n+3n-5=0$ $\therefore (3n^2-5n)+(3n-5)=0$ $\therefore n(3n-5)+1(3n-5)=0$ $\therefore (3n-5)(n+1)=0$ $\therefore 3n-5=0$ or $n+1=0$ $\therefore n=\frac{5}{3}$ or $n=-1$
Relationship or Relation	A property of association shared by ordered pairs of terms or objects	The equation $y=x^2$ expresses a ***relationship*** between the x-co-ordinate and the y-co-ordinate of any point that lies on the upward opening parabola with vertex (0, 0) and vertical stretch factor 1.

Simultaneous Systems of Linear Equations

Solving Simultaneous Systems of Linear Equations

- Use ***substitution*** or ***elimination***.
- The solutions ***must satisfy both equations***.

Example

$x-2y=-5$ (1)

$5x+6y=7$ (2)

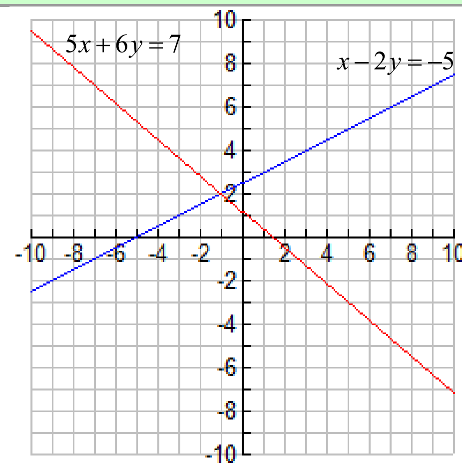
$(1)\times 3,\quad 3x-6y=-15$ (3)

$(2)+(3),\ 8x=-8$

$\therefore x=-1$

$\therefore y=2$

Graphical Solution



George Polya's Four Steps of Problem Solving

1. Understand the Problem
2. Choose a Strategy
3. Carry out the Strategy
4. Check the Solution

STEP 1 IS OF CRITICAL IMPORTANCE!

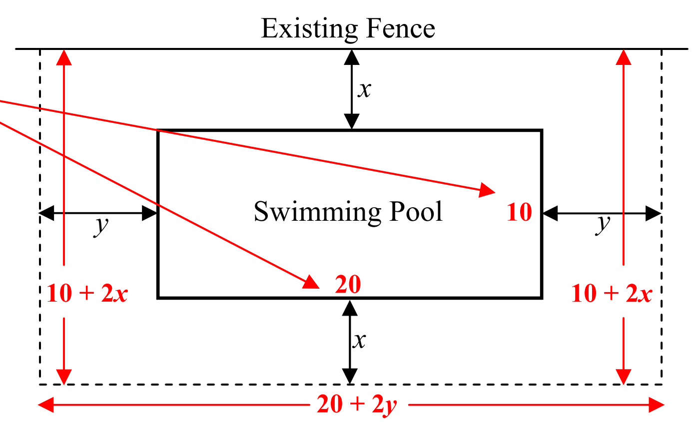
To complete step 1, you must ***understand*** the ***terminology*** used in the problem statement!

- ***equation*** → L.H.S. = R.H.S. → a complete mathematical "***sentence***"
- ***expression*** → ***not*** a complete mathematical "sentence" → more like a ***phrase***

Solving the so-called "word problems" that you are given in school is usually just a matter of ***translating English sentences into mathematical equations***.

Example

To help prevent drowning accidents, a protective fence is to be erected around a **pool** whose dimensions are **20 m by 10 m**. Since there is an existing fence on one side, new fencing is only required around ***three*** sides of the pool (see diagram). In addition, the gap between the fence and the edge of the pool must be uniform on opposite sides of the pool (see diagram). If **100 m of fencing material is available**, what is the **maximum area that can be enclosed by the fencing**?



100 m of fencing material is available

↓

Length of fence is 100 m

$10 + 2x + 20 + 2y + 10 + 2x = 100$

$\therefore 4x + 2y + 40 = 100$

$\therefore 4x + 2y = 60$

$\therefore 2x + y = 30$

$\therefore 2x = 30 - y$

area that can be enclosed by the fencing

↓

area of fence = length × width

$A = (20 + 2y)(10 + 2x)$

$\therefore A = (20 + 2y)(10 + 30 - y)$

$\therefore A = (20 + 2y)(40 - y)$

The only Equation you need to know to find an Equation of a Line

Slope = Slope

Example

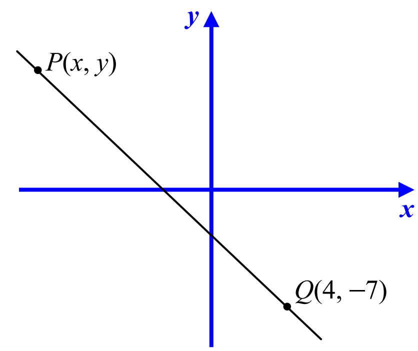
Find an equation of the line with slope $-\frac{2}{3}$ and passing through the point (4, −7).

Solution

Let $P(x, y)$ be any point on the line other than (4, -7).

Now since

$$\begin{aligned} &\text{slope} = \text{slope}, \\ &\therefore \frac{\Delta y}{\Delta x} = -\frac{2}{3} \\ &\therefore \frac{y-(-7)}{x-4} = -\frac{2}{3} \\ &\therefore y+7 = -\frac{2}{3}(x-4) \\ &\therefore y = -\frac{2}{3}x - \frac{13}{3} \end{aligned}$$



Therefore, $y = -\frac{2}{3}x - \frac{13}{3}$ is ***an*** (not "the") equation of the required line. //

Review of finding the Distance between Two Points

The only Equation you need to know to find the Distance between Two Points

The Pythagorean Theorem

Example

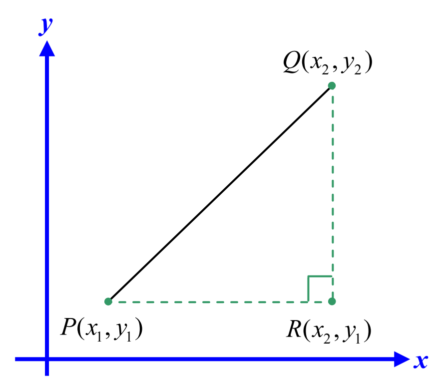
Find the distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Solution

Let P, Q and R be as shown in the diagram.

Clearly, $PR = |x_2 - x_1|$ and $QR = |y_2 - y_1|$ Therefore, by the Pythagorean Theorem,

$$\begin{aligned} &PQ^2 = PR^2 + QR^2 \\ &\therefore PQ^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 \\ &\therefore PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &\therefore PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$



Therefore, finding the distance between two points is a simple application of the Pythagorean Theorem. //

Review of finding the Midpoint of a Line Segment

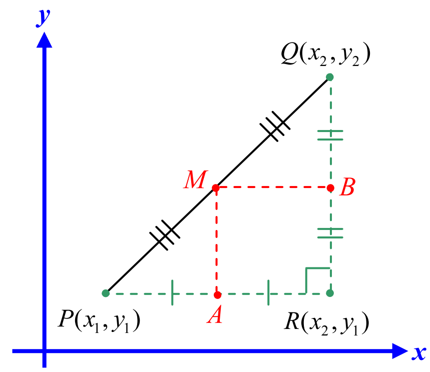
The only Equation you need to know to find the Midpoint of a Line Segment

The average of two numbers a and b is $\dfrac{a+b}{2}$

Example

Find the midpoint of the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Solution



As shown in the diagram, let *M*, *A* and *B* represent the midpoints of the line segments *PQ*, *PR* and *QR* respectively. Since *A* lies on *PR* and *PR* is parallel to the *x*-axis, the *y*-co-ordinate of *A* must be y_1. Also, since *A* lies exactly half way between *P* and *R*, its *x*-co-ordinate must be the average of the *x*-co-ordinates of *P* and *R*. Therefore, the co-ordinates of *A* must be $\left(\dfrac{x_1+x_2}{2}, y_1\right)$. Using similar reasoning, the co-ordinates of *B* must be $\left(x_2, \dfrac{y_1+y_2}{2}\right)$.

Since *MA* is parallel to the *y*-axis, the *x*-co-ordinate of *M* must equal that of *A*. Since *MB* is parallel to the *x*-axis, the *y*-co-ordinate of *M* must equal that of *B*. Therefore, the co-ordinates of *M* must be

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Therefore, the midpoint *M* of *PQ* must have co-ordinates $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$. //

Important Exercises

1. Using a diagram and an argument similar to those given above, explain why parallel lines have equal slope.

2. Using a diagram and an argument similar to those given above, explain why any line perpendicular to a line with slope m must have slope $-\dfrac{1}{m}$.

3. Give a physical interpretation of slope.

4. Given the points $P(-3,5)$ and $Q(11,11)$, find an equation of the line passing through the midpoint of the line segment *PQ* and having slope $-\dfrac{2}{7}$.

5. Given the points $P(-3,5)$ and $Q(11,11)$, find an equation of the line perpendicular to *PQ* and passing through *Q*.

6. Find the distance from the point $P(-3,5)$ to the line $y = 2x + 3$. (If you hope to solve this problem, a diagram is a must!)

What is Trigonometry?

The Canadarm2 robotic manipulator on the International Space Station is operated by controlling the angles of its joints. Calculating the final position of the astronaut at the end of the arm requires repeated use of the trigonometric functions of those angles.

Trigonometry (Greek *trigōnon* "triangle" + *metron* "measure") is a branch of mathematics that deals with the relationships among the interior angles and side lengths of triangles, as well as with the study of trigonometric functions. Although the word "trigonometry" emerged in the mathematical literature only about 500 years ago, the origins of the subject can be traced back more than 4000 years to the ancient civilizations of Egypt, Mesopotamia and the Indus Valley. Trigonometry has evolved into its present form through important contributions made by, among others, the Greek, Chinese, Indian, Sinhalese, Persian and European civilizations. ***Trigonometry is very useful because it is generally easier to measure angles than it is to measure distances.***

Why Triangles?

Triangles are the basic building blocks from which any shape (with straight boundaries) can be constructed. A square, pentagon or any other polygon can be divided into triangles, for instance, using straight lines that radiate from one vertex to all the others.

Examples of Problems that can be solved using Trigonometry

- ☺ How tall is Mount Everest? How tall is the CN Tower?
- ☺ What is the distance from the Earth to the sun? How far is the Alpha Centauri star system from the Earth?
- ☺ What is the diameter of Mars? What is the diameter of the sun?
- ☺ At what times of the day will the tide come in?

General Applications of Trigonometric Functions

Trigonometry is one of the most widely applied branches of mathematics. The following is just a small sample of its myriad uses.

Application	*Examples*
Modelling of ***cyclic*** (periodic) processes	Orbits (of planets, moons, etc.) Hours of Daylight Tides
Measurement	Navigation Engineering Construction Surveying
Electronics	Circuit Analysis (Modelling of Voltage Versus Time in AC Circuits, Fourier Analysis, etc)

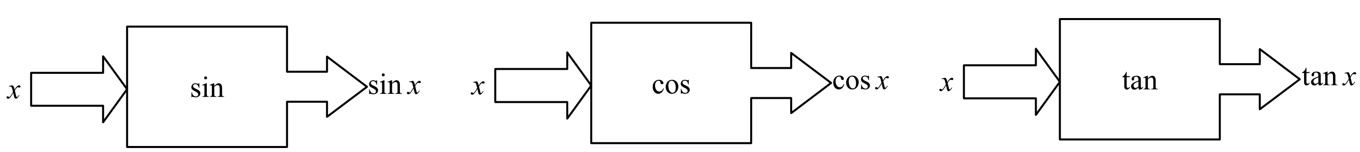
Extremely Important Prerequisite Knowledge

Before tackling the details of trigonometric functions, it is important to review the trigonometry that you learned in grades 10 and 11. However, now that you have a solid foundation in functions, the following should be noted.

Input

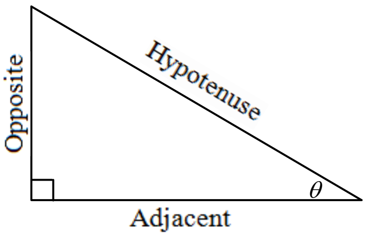
Name of Function → $\sin x = \sin(x)$ ← Parentheses can be used but are usually omitted.

Output



Trigonometry of Right Triangles – Trigonometric Ratios of Acute Angles

Right triangles can be used to define the trigonometric ratios of ***acute*** angles (angles that measure less than 90º).



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

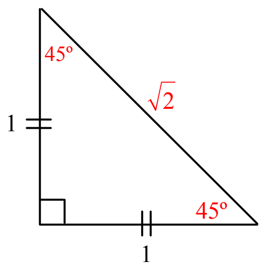
SOH CAH TOA

"**S**hout **O**ut **H**ey" "**C**anadians **A**re **H**ot" "**T**ight **O**iled **A**bs"

Have fun by creating your own mnemonic!

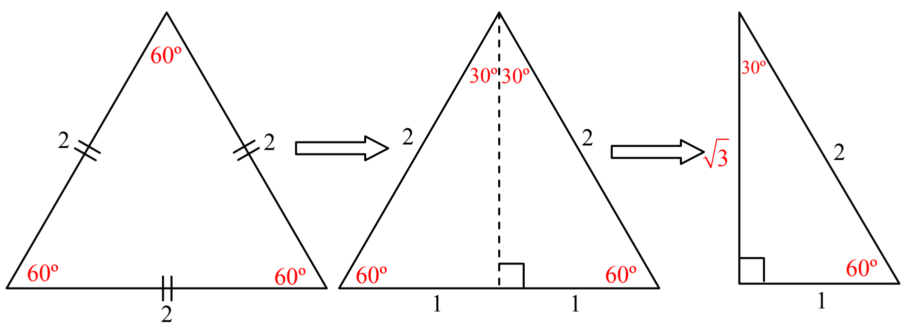
The Special Triangles – Trigonometric Ratios of Special Angles

For certain ***special angles***, it is possible to calculate the ***exact value*** of the trigonometric ratios. As I have mentioned on many occasions, it is not advisable to memorize blindly! Instead, you can deduce the values that you need to calculate the trig ratios by ***understanding*** the following triangles!



- Isosceles Right Triangle
- Let the length of the equal sides be 1 unit.
- By the Pythagorean Theorem, the length of the hypotenuse must be $\sqrt{2}$.

$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{1}{\sqrt{2}} \qquad \tan 45° = \frac{1}{1} = 1$$



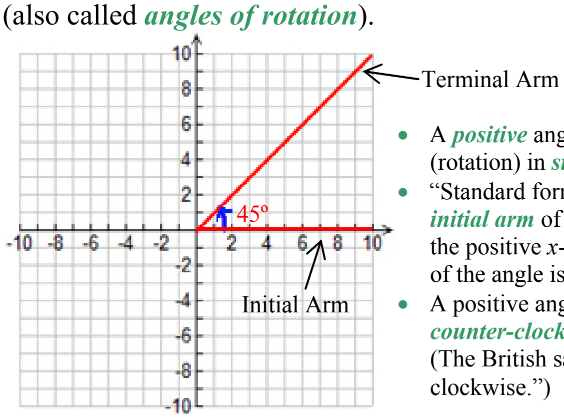
- Begin with an equilateral triangle having sides of length 2 units. Then cut it in half to form a 30º-60º-90º right triangle.
- Use the Pythagorean Theorem to calculate the height of the triangle. ($\sqrt{3}$)

$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}}$$

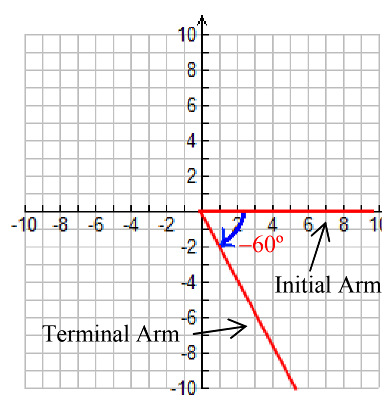
$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \sqrt{3}$$

Trigonometric Ratios of Angles of Revolution (Rotation) – Trigonometric Ratios of Angles of Any Size

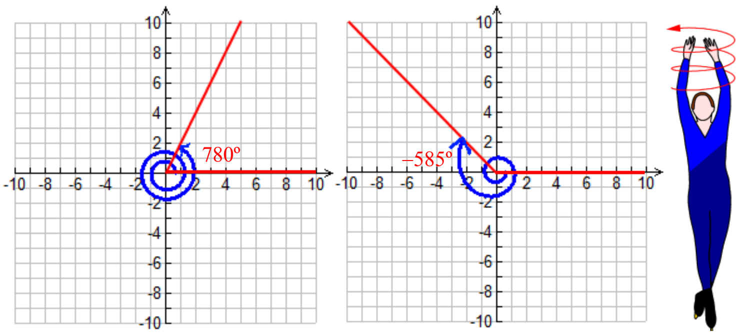
We can extend the idea of trigonometric ratios to angles of any size by introducing the concept of ***angles of revolution*** (also called ***angles of rotation***).



- A ***positive*** angle of revolution (rotation) in ***standard form***.
- "Standard form" means that the ***initial arm*** of the angle lies on the positive x-axis and the ***vertex*** of the angle is at the origin.
- A positive angle results from a ***counter-clockwise*** revolution. (The British say "anti-clockwise.")



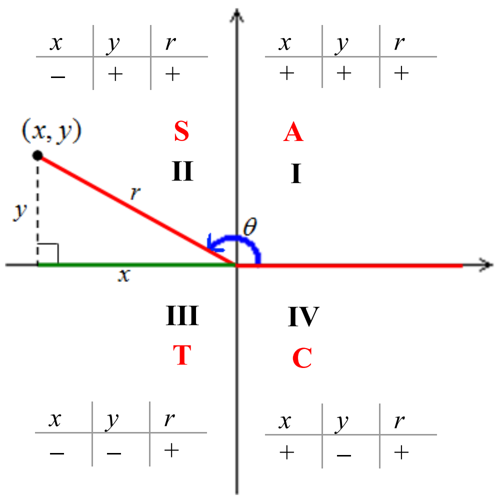
- A ***negative*** angle of revolution (rotation) in ***standard form***.
- "Standard form" means that the ***initial arm*** of the angle lies on the positive x-axis and the ***vertex*** of the angle is at the origin.
- A negative angle results from a ***clockwise*** revolution.



Why Angles of Rotation?

To describe motion that involves moving from one place to another, it makes sense to use units of distance. For instance, it is easy to find your destination if you are told that you need to move 2 km north and 1 km west of your current position.

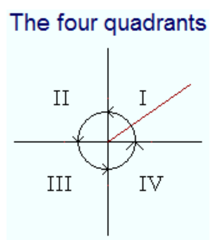
Consider a spinning figure skater. It does not make sense to describe his/her motion using units of distance because he/she is fixed in one spot and rotating. However, it is very easy to describe the motion through angles of rotation.



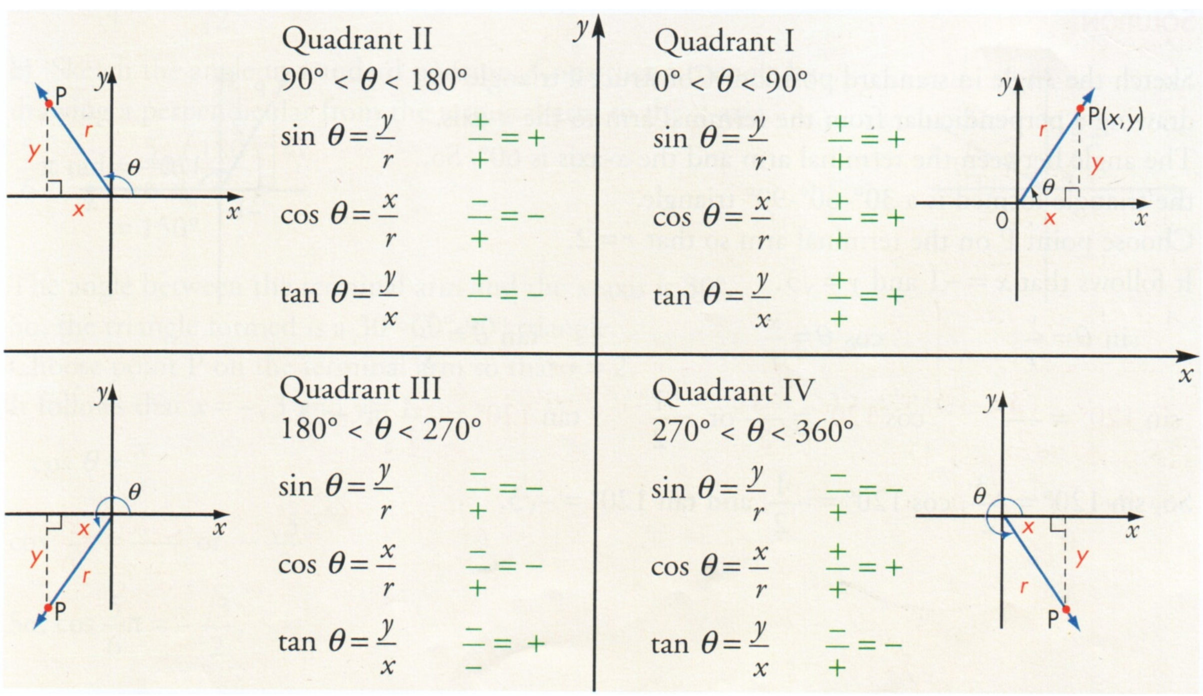
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$



- Since r represents the length of the terminal arm, $r > 0$.
- In quadrant I, $x > 0$ and $y > 0$. Therefore **A**LL the trig ratios are positive.
- In quadrant II, $x < 0$ and $y > 0$. Therefore only **S**INE is positive. The others are negative.
- In quadrant III, $x < 0$ and $y < 0$. Therefore only **T**ANGENT is positive. The others are negative.
- In quadrant IV, $x > 0$ and $y < 0$. Therefore only **C**OSINE is positive. The others are negative.
- Hence the mnemonic, "**A**LL **S**TUDENTS **T**ALK on **C**ELLPHONES"



Coterminal Angles

Angles of revolution are called ***coterminal*** if, when in standard position, they share the same terminal arm. For example, –90º, 270º and 630º are coterminal angles. An angle coterminal to a given angle can be found by adding or subtracting any multiple of 360º.

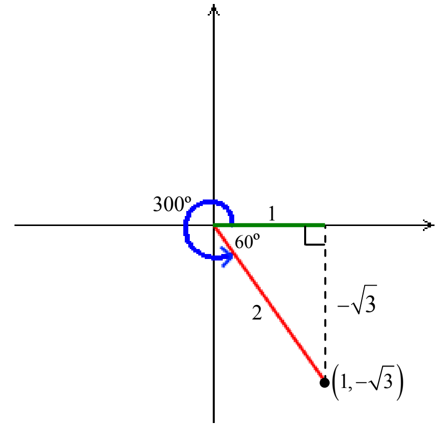
Example

Find the trigonometric ratios of 300º.

> Every angle of rotation has a ***related first quadrant angle***. The related first quadrant angle is found by taking the ***acute between the terminal arm and the x-axis***.
> For 300º, the related first quadrant angle is 60º.

Solution

From the diagram at the right, we can see that for an angle of rotation of 300º, we obtain a 30º-60º-90º right triangle in quadrant IV. In addition, by observing the ***acute angle*** between the ***terminal arm and the x-axis***, we find the ***related first quadrant angle***, 60º.



Therefore,

$$\sin 300^\circ = \frac{y}{r} = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$

$$\cos 300^\circ = \frac{x}{r} = \frac{1}{2}$$

$$\tan 300^\circ = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

> Compare these answers to
> $$\sin 60^\circ = \frac{\sqrt{3}}{2}$$
> $$\cos 60^\circ = \frac{1}{2}$$
> $$\tan 60^\circ = \sqrt{3}$$

Question

How are the trigonometric ratios of 300º related to the trigonometric ratios of 60º?

Answer

- The ***magnitudes*** of the trig ratios of 300º are equal to the ***magnitudes*** of the trig ratios of the related first quadrant angle 60º.
- The ratios may differ only in ***sign***.
- To determine the correct sign, use the ***ASTC*** rule.
- In case you forget how to apply the ASTC rule, just think about the ***signs*** of x and y in each quadrant (see previous page). Don't forget that r is always positive because it represents the length of the terminal arm. Thus, the above ratios could have been calculated as follows:

Angle of Rotation: 300º (quadrant IV) **Related First Quadrant Angle:** 60º

In quadrant IV, $\sin\theta = \frac{y}{r} < 0$ because $\frac{-}{+} = -$, $\cos\theta = \frac{x}{r} > 0$ because $\frac{+}{+} = +$ and $\tan\theta = \frac{y}{x} < 0$ because $\frac{-}{+} = -$.

Hence,

$$\sin 300^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2},$$

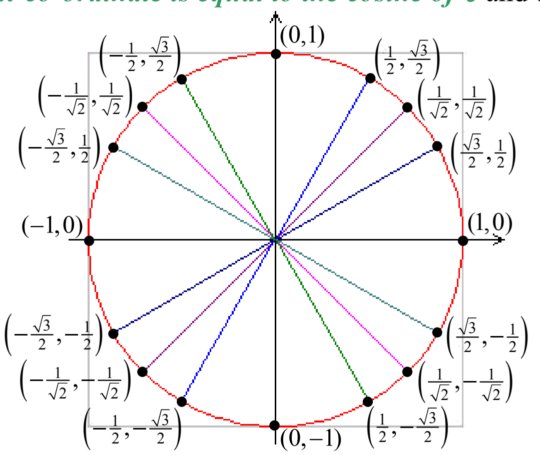
$$\cos 300^\circ = \cos 60^\circ = \frac{1}{2}$$

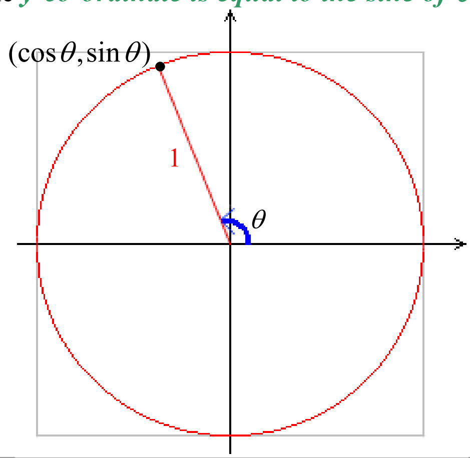
$$\tan 300^\circ = -\tan 60^\circ = -\sqrt{3}.$$

Additional Tools for Determining Trig Ratios of Special Angles

The Unit Circle

A ***unit circle*** is any circle having a radius of ***one unit***. If a unit circle is centred at the origin, it is described by the equation $x^2+y^2=1$, meaning that for any point (x,y) lying on the circle, the value of x^2+y^2 must equal 1. Furthermore, for any point (x,y) lying on the unit circle and for any angle θ, $r=1$. Therefore, $\cos\theta=\frac{x}{r}=\frac{x}{1}=x$ and $\sin\theta=\frac{y}{r}=\frac{y}{1}=y$. In other words, for any point (x,y) lying on the unit circle, the ***x-co-ordinate is equal to the cosine of θ*** and the ***y-co-ordinate is equal to the sine of θ***.





The Rule of Quarters (Beware of Blind Memorization!)

The rule of quarters makes it easy to remember the sine of special angles. ***Be aware, however, that this rule invites blind memorization!***

$$\sin(0°)=\sqrt{\frac{0}{4}}=0$$

$$\sin(30°)=\sqrt{\frac{1}{4}}=\frac{1}{2}$$

$$\sin(45°)=\sqrt{\frac{2}{4}}=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$

$$\sin(60°)=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$

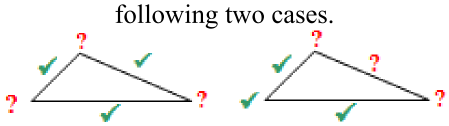
$$\sin(90°)=\sqrt{\frac{4}{4}}=1$$

Law of Cosines

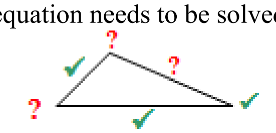
$$c^2=a^2+b^2-2ab\cos C$$

The law of cosines is a *generalization* of the Pythagorean Theorem.

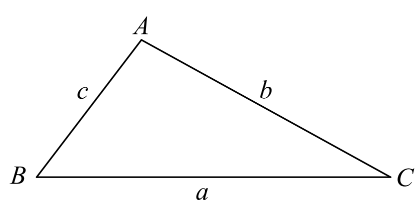
The law of cosines is most useful in the following two cases.



The law of cosines also can be used in the following case. However, a quadratic equation needs to be solved.



While the Pythagorean Theorem ***holds only for right triangles***, the Sine Law and the Cosine Law hold for ***all triangles***!

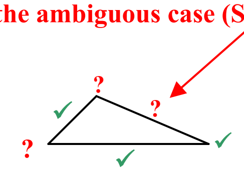


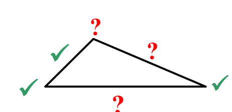
However, you ***should not*** use these laws when working with right triangles. It is much easier to use the basic trigonometric ratios and the Pythagorean Theorem.

Law of Sines

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

The law of sines is used in the following two cases. **However, you must beware of the ambiguous case (SSA).**





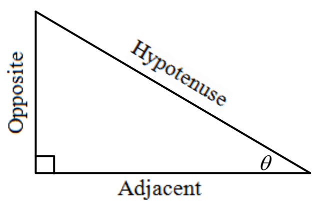
Important Exercise

Rearrange the equation for the law of cosines in such a way that you solve for $\cos C$. In what situation might you want to solve for $\cos C$?

The Reciprocal Trigonometric Ratios?

It is easy to show that sin and cos alone are sufficient for expressing all trigonometric relationships. However, using sin and cos alone can cause certain situations to be needlessly complicated. This is where the other trig ratios come into play. In addition to tan, the reciprocal trigonometric ratios ***cosecant*** (csc), ***secant*** (sec) and ***cotangent*** (cot) can help to simplify matters in certain cases.

The following table lists all the trigonometric ratios in modern usage.

	$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$	$\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}} = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \dfrac{1}{\cos\theta}$ $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{1}{\tan\theta}$	SOH CAH TOA CHO SHA COTAO

What is an Identity?

☺ An ***identity*** is an equation that expresses the ***equivalence of two expressions***.

e.g. $\cos^2\theta + \sin^2\theta = 1$, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

- The given equations are identities. ***For all values of θ that make sense***, the left-hand-side equal equals the right-hand-side. That is, the expression on the left side ***is equivalent to*** the expression on the right side.
- For the identity $\cos^2\theta + \sin^2\theta = 1$, there are no restrictions on the value of θ.
- For the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, we cannot allow θ to take on values that make $\cos\theta = 0$ because this would lead to the undefined operation of dividing by zero.

List of Identities that we Already Know

- To discourage the erroneous notion that θ is the only symbol that is allowed to express trigonometric identities or trigonometric functions, x will often be used in place of θ. It should be clear that any symbol whatsoever can be used as long as meaning is not compromised.
- Since the identities in the following list have already been proved to be true, they can be used to construct proofs of other identities. Examples are given below.

Pythagorean Identities	*Quotient Identity*	*Reciprocal Identities*		
For all $x \in \mathbb{R}$, $\cos^2 x + \sin^2 x = 1$ $\sin^2 x = 1 - \cos^2 x$ $\cos^2 x = 1 - \sin^2 x$	For all $x \in \mathbb{R}$ such that $\cos x \neq 0$, $\tan x = \dfrac{\sin x}{\cos x}$	For all $x \in \mathbb{R}$ such that $\sin x \neq 0$, $\csc x = \dfrac{1}{\sin x}$	For all $x \in \mathbb{R}$ such that $\cos x \neq 0$, $\sec x = \dfrac{1}{\cos x}$	For all $x \in \mathbb{R}$ such that $\tan x \neq 0$, $\cot x = \dfrac{1}{\tan x}$

Important Note about Notation

- $\sin^2 x$ is a shorthand notation for $(\sin x)^2$, which means that ***first*** $\sin x$ is evaluated, ***then*** the result is squared

e.g. $\sin^2 45° = (\sin 45°)^2 = \left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1}{2}$ $\cos^2 60° = (\cos 60°)^2 = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

- This notation is used to avoid the excessive use of parentheses
- $\sin x^2 \neq \sin^2 x$ The expression $\sin x^2$ means that ***first*** x is squared and ***then*** "sin" is applied

Characteristics of Sinusoidal Functions

1. Sinusoidal functions have the general form $f(x) = A\sin(k(x-p)) + d$, where A, d, p and k are the ***amplitude***, ***vertical displacement***, ***phase shift*** and ***angular frequency*** respectively.

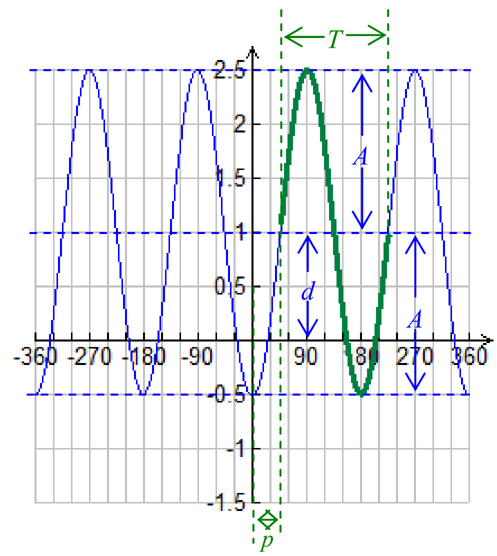
2. Sinusoidal functions are ***periodic***. This makes sinusoidal functions ideal for modelling ***periodic processes***. The letter T is used to denote the ***period*** (also called ***primitive period*** or ***wavelength***) of a sinusoidal function. As you have learned in previous courses, $T = \dfrac{360°}{k}$.

3. Sinusoidal functions ***oscillate*** (vary continuously, back and forth) between a maximum and a minimum value. This makes sinusoidal functions ideal for modelling ***oscillatory*** or ***vibratory*** motions. (**e.g.** a pendulum swinging back and forth, a playground swing, a vibrating string, a vibrating tuning fork, alternating current, quartz crystal vibrating in an electronic watch, light waves, radio waves, etc)

4. There is a horizontal line that exactly "cuts" a sinusoidal function "in half." The vertical distance from this horizontal line to the peak of the curve is called the ***amplitude***.

Example

The graph at the right shows a few ***cycles*** of the function $f(x) = 1.5\sin(2(x-45°)) + 1$. One of the cycles is shown as a thick green curve to make it stand out among the others. Notice the following:



- The ***maximum*** value of f is 2.5.
- The ***minimum*** value of f is –0.5.
- The function f oscillates between –0.5 and 2.5.
- The horizontal line with equation $y = 1$ exactly "cuts" the function "in half."
- The ***amplitude*** of this function is $A = 1.5$. This can be seen in a number of ways. Clearly, the vertical distance from the line $y = 1$ to the peak of the curve is 1.5. Also, the amplitude can be calculated by finding half the distance between the maximum and minimum values: $\dfrac{2.5-(-0.5)}{2} = \dfrac{3}{2} = 1.5$
- The ***period***, that is the length of one cycle, is $T = 180°$. This can be seen from the graph (225º – 45º = 180º) or it can be determined by applying your knowledge of transformations. The period of $y = \sin x$ is 360º. Since f has undergone a horizontal compression by a factor of $1/2$, its period should be half of 360º, which is 180º. In general, the period of a sinusoidal function is $T = \dfrac{360°}{k}$, where k is the ***angular frequency***.
- ***Angular frequency*** = # radians per second, or more generally, # radians per unit of the independent variable (to be investigated thoroughly in unit 2)
- It is customary to use the Greek letter ω (lowercase omega) to denote angular frequency. For any periodic function, T = (horizontal compression factor) (period of base function) = $\dfrac{1}{\omega}$(period of base function).
- The function $g(x) = 1.5\sin(2(x-45°))$ would be "cut in half" by the x-axis (the line $y = 0$). The function f has exactly the same shape as g except that it is ***shifted up by 1 unit***. This is the significance of the ***vertical displacement***. In this example, the vertical displacement $d = 1$.
- The function $g(x) = 1.5\sin(2(x-45°))$ has exactly the same shape as $h(x) = 1.5\sin 2x$ but is shifted 45º to the right. This horizontal shift is called the ***phase shift***.

Example

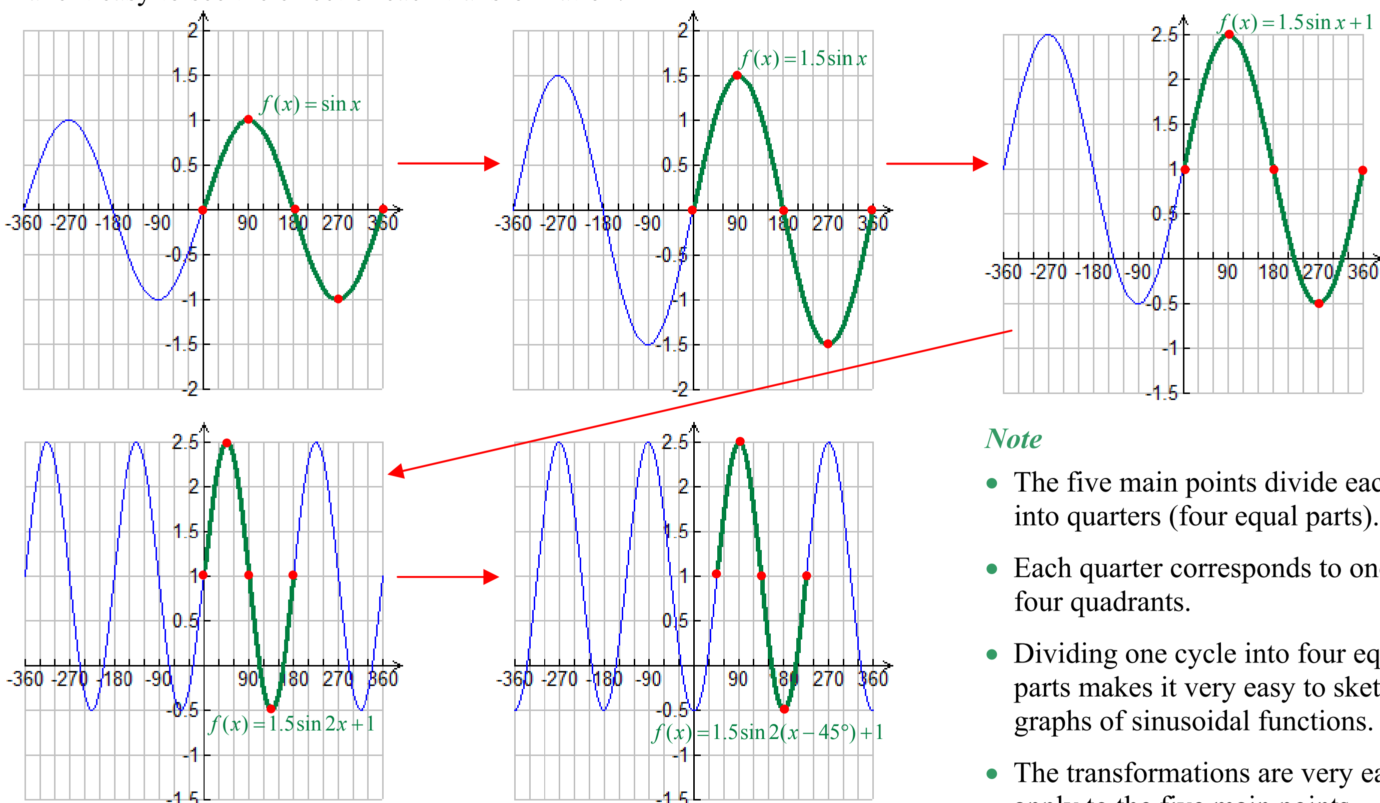
Sketch the graph of $f(x)=1.5\sin\left(2\left(x-45°\right)\right)+1$

Solution

Amplitude	*Vertical Displacement*	*Phase Shift*	*Period*	*Transformations*	
$A=1.5$	$d=1$	$p=45°$	$T=360°\left(\frac{1}{\omega}\right)$ $=360°\left(\frac{1}{2}\right)=180°$	***Vertical*** Stretch by a factor of 1.5. Translate 1 unit up.	***Horizontal*** Compress by a factor of $2^{-1}=\frac{1}{2}$ Translate 45° to the right.

Method 1 – The Long Way

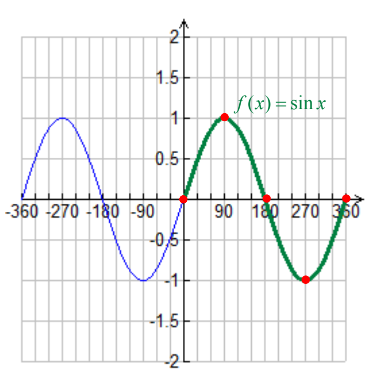
The following shows how the graph of $f(x)=1.5\sin\left(2\left(x-45°\right)\right)+1$ is obtained by beginning with the base function $f(x)=\sin x$ and applying the transformations one-by-one in the correct order. One cycle of $f(x)=\sin x$ is highlighted in green to make it easy to see the effect of each transformation. In addition, ***five main points*** are displayed in red to make it easy to see the effect of each transformation.



Note

- The five main points divide each cycle into quarters (four equal parts).
- Each quarter corresponds to one of the four quadrants.
- Dividing one cycle into four equal parts makes it very easy to sketch the graphs of sinusoidal functions.
- The transformations are very easy to apply to the five main points.

Method 2 – A Much Faster Approach



$f(x)=1.5\sin\left(2\left(x-45°\right)\right)+1$

$A=1.5,\ d=1,\ k=\omega=2,\ p=45°$

$(x,y)\to\left(k^{-1}x+p, Ay+d\right)$

$\therefore(x,y)\to\left(\frac{1}{2}x+45°,1.5y+1\right)$

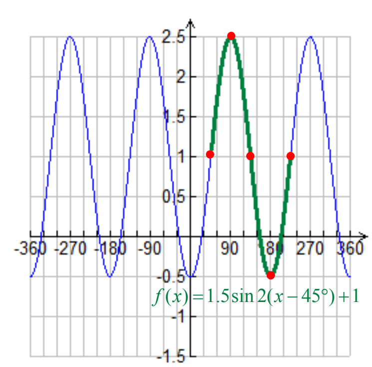
$\therefore(0°,0)\to\left(\frac{1}{2}(0)+45°,1.5(0)+1\right)=(45°,1)$

$\therefore(90°,1)\to\left(\frac{1}{2}(90°)+45°,1.5(1)+1\right)=(90°,2.5)$

$\therefore(180°,0)\to\left(\frac{1}{2}(180°)+45°,1.5(0)+1\right)=(135°,1)$

$\therefore(270°,-1)\to\left(\frac{1}{2}(270°)+45°,1.5(-1)+1\right)=(180°,-0.5)$

$\therefore(360°,0)\to\left(\frac{1}{2}(360°)+45°,1.5(0)+1\right)=(225°,1)$



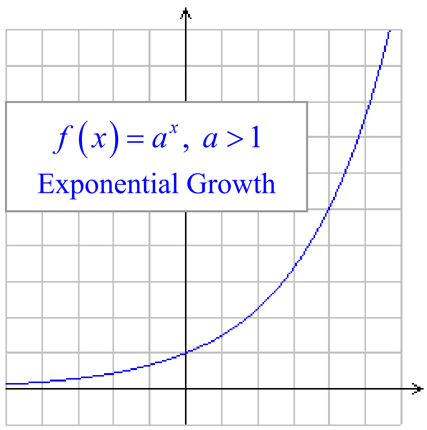
Exponential Functions

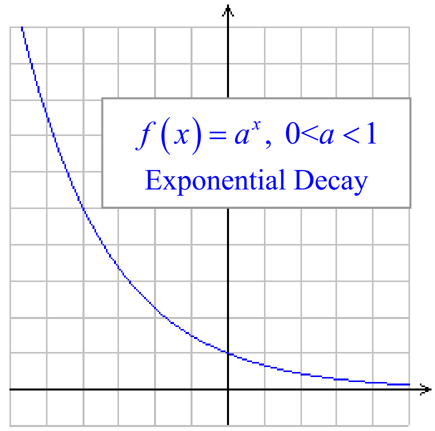
Exponential Growth and Exponential Decay

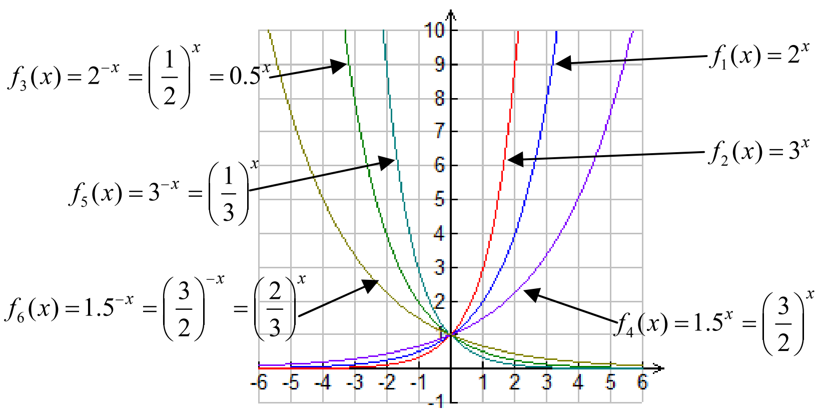
Exponential functions are used to model very fast growth or very fast decay. Specifically, exponential functions model

- growth that involves doubling, tripling, etc. at regular intervals
  e.g. Since 1950, the Earth's population has been doubling approximately every 40 years.
- decay that involves cutting in half, cutting in thirds, etc. at regular intervals
  e.g. Sodium-24 loses half of its mass every 14.9 hours. We say that the ***half-life*** of Sodium-24 is 14.9 hours.

Graphs of Exponential Functions







x	y1(x) 2^x	y2(x) 3^x	y3(x) 0.5^x	y4(x) 1.5^x	y5(x) 3^(-x)	y6(x) 1.5^(-x)
-5	0.03125	0.004115	32	0.131687	243	7.59375
-4	0.0625	0.012346	16	0.197531	81	5.0625
-3	0.125	0.037037	8	0.296296	27	3.375
-2	0.25	0.111111	4	0.444444	9	2.25
-1	0.5	0.333333	2	0.666667	3	1.5
0	1	1	1	1	1	1
1	2	3	0.5	1.5	0.333333	0.666667
2	4	9	0.25	2.25	0.111111	0.444444
3	8	27	0.125	3.375	0.037037	0.296296
4	16	81	0.0625	5.0625	0.012346	0.197531
5	32	243	0.03125	7.59375	0.004115	0.131687
6	64	729	0.015625	11.3906	0.001372	0.087791
7	128	2187	0.007813	17.0859	0.000457	0.058528
8	256	6.56E+03	0.003906	25.6289	1.52E-04	0.039018
9	512	1.97E+04	0.001953	38.4434	5.10E-05	0.026012
10	1024	5.90E+04	0.000977	57.665	1.70E-05	0.017342

Example

In 1950, the Earth's population was about 2.5 billion people. Since then, the world's population has been doubling roughly every 40 years. If the current trend continues, predict the world's population in 2100. Do you think that the current trend will continue?

Solution

Using a Table to help us Understand the Problem

Year	*Population*
1950	2.5×10^9
1990	$5\times10^9 = (2.5\times10^9)(2)$
2030	$1\times10^{10} = (2.5\times10^9)(2^2)$
2070	$2\times10^{10} = (2.5\times10^9)(2^3)$
2110	$4\times10^{10} = (2.5\times10^9)(2^4)$

Writing an Equation

To ***double*** means to ***multiply by 2***. Furthermore, we must multiply by 2 every 40 years. Thus, if we set $t = 0$ years to correspond to 1950, it is clear that the following exponential function perfectly models the given situation:

$$P(t) = 2.5\times10^9\left(2^{t/40}\right)$$

Solving the Problem

The year 2100 is 150 years after 1950. Therefore, $t = 150$.

$$P(150) = 2.5\times10^9\left(2^{150/40}\right)$$

$$\doteq 3.4\times10^{10}$$

If the current trend continues, in 2100 the Earth's population will be about 34 billion.

It is difficult to imagine that the current rate of population growth will continue indefinitely. First of all, it is very unlikely that the Earth's food supply can grow at a pace that matches or exceeds the growth of population. To compound matters, as the population size increases, the amount of arable land tends to decrease as a result of the extra space required for housing.

Critically Important Logical Principles and Mathematical Terminology

Logic

The study of the ***principles of reasoning***, especially of the ***structure*** of propositions as distinguished from their ***content*** and of ***method***, and ***validity*** in deductive reasoning.

Premise, Conclusion, Logical Implication

- A ***premise*** is a statement that is known or assumed to be true and from which a ***conclusion*** can be drawn.
- A ***conclusion*** is a position, opinion or judgment reached after consideration.
- A ***logical implication*** or ***conditional statement*** is a statement that takes the form "If ***premise*** then ***conclusion***." In such a statement, the truth of the premise ***guarantees*** the truth of the conclusion.

Example

Premise → "If he has been injured"
Conclusion → "then he will not be able to play in tonight's hockey game"

Logical Implication → "If he has been injured, then he will not be able to play in tonight's hockey game."

Extremely Important Logical Implications in Mathematics

If $x = y$ then $x + a = y + a$, $x - a = y - a$, $ax = ay$, $\frac{x}{a} = \frac{y}{a}$ (if $a \neq 0$), $x^2 = y^2$, $\sqrt{x} = \sqrt{y}$, $\sin x = \sin y$, etc If the same operation is performed to both sides of an equation, then equality is preserved. ***Example*** $2x - 7 = 9$ $\therefore 2x - 7 + 7 = 9 + 7$ $\therefore 2x = 16$ $\therefore \frac{2x}{2} = \frac{16}{2}$ $\therefore x = 8$	If $a = b$ ***and*** $a = c$, then $b = c$. If two quantities are each equal to the same quantity, then they are equal to each other. ***Example*** $h(t) = -4.9t^2 + 49t$ and $h(t) = 0$ $\therefore -4.9t^2 + 49t = 0$	If $xy = 0$, then $x = 0$ or $y = 0$. If the product of two numbers is equal to zero, then at least one of the numbers must be zero. ***Example*** $(x-3)(x+7) = 0$ $\therefore x - 3 = 0$ or $x + 7 = 0$

Logical Fallacy: Ad Hominem Argument – An Example of Studying the Structure of an Argument

Basic Structure of an Ad Hominem Argument	*What is wrong with an ad hominem argument?*
A (fallacious) ad hominem argument has the basic form: Person A makes claim X There is something objectionable about Person A Therefore claim X is false	

Deductive Reasoning

Deductive arguments take the form "If ***Cause*** Then ***Effect***" or "If ***Premise*** Then ***Conclusion***"

In a deductive argument, we know that a ***cause*** (premise) produces a certain ***effect*** (conclusion). If we observe the cause, we can ***deduce*** (conclude by reasoning) that the effect ***must*** occur. These arguments always produce ***definitive*** conclusions; ***general principles*** are applied to reach specific conclusions. We must keep in mind, however, that a false premise can lead to a false result and an inconclusive premise can yield an inconclusive conclusion.

Examples of Deductive Reasoning from Everyday Life

1. If I spill my drink on the floor, the floor will get wet.
2. When students "forget" to do homework, Mr. Nolfi gets angry!
3. If a student is caught cheating, Mr. Nolfi will assign a mark of zero to him/her, ridicule the student publicly, turn red in the face and yell like a raving madman whose underwear are on fire!
4. Drinking too much alcohol causes drunkenness.

Exercise

Rephrase examples 2 and 4 in "If … then" form.

The Meaning of π – An Example of Deductive Reasoning

The following is an example of a typical conversation between Mr. Nolfi and a student who blindly memorizes formulas:

Student: Sir, I can't remember whether the area of a circle is πr^2 or $2\pi r$. Which one is it?

Mr. Nolfi: If you remember the meaning of π, you should be able to figure it out.

Student: How can 3.14 help me make this decision? It's only a number!

Mr. Nolfi: How dare you say something so disrespectful about one of the most revered numbers in the mathematical lexicon! (Just kidding. I wouldn't really say that.) It's true that the number 3.14 is an approximate value of π. But I asked you for its ***meaning***, not its value.

Student: I didn't know that π has a meaning. I thought that it was just a "magic" number.

Mr. Nolfi: Leave magic to the magicians. In mathematics, every term (except for primitive terms) has a very precise definition. Read the following carefully and you'll never need to ask your original question ever again!

In ***any*** circle, the ***ratio*** of the ***circumference*** to the ***diameter*** is equal to a ***constant*** value that we call π. That is,

$$C : d = \pi .$$

Alternatively, this may be written as

$$\frac{C}{d} = \pi$$

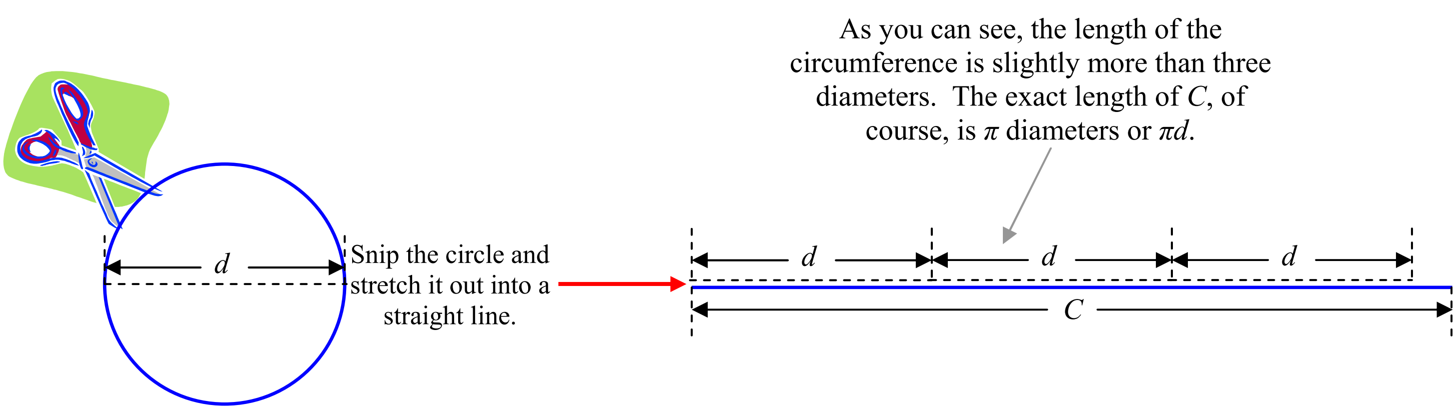
or in the more familiar form

$$C = \pi d .$$

If we recall that $d = 2r$, then we finally arrive at the most common form of this ***relationship***,

$$C = 2\pi r .$$

> This is an example of a ***deductive argument***. Each statement ***follows logically*** from the previous statement.
>
> That is, the argument takes the form "If P is true then Q must also be true" or more concisely, "P implies Q."



Mr. Nolfi: So you see, by understanding the meaning of π, you can ***deduce*** that $C = 2\pi r$. Therefore, the formula for the area must be $A = \pi r^2$. Furthermore, it is not possible for the expression $2\pi r$ to yield units of area. The number 2π is dimensionless and r is measured in units of distance such as metres. Therefore, the expression $2\pi r$ must result in a value measured in units of distance. On the other hand, the expression πr^2 must give a value measured in units of area because $r^2 = r(r)$, which involves multiplying a value measured in units of distance by itself. Therefore, by considering units alone, we are drawn to the inescapable conclusion that the area of a circle must be πr^2 and ***not*** $2\pi r$!

Examples

$2\pi r \doteq 2(3.14)(3.6\text{ cm}) = 22.608\text{ cm} \rightarrow$ This answer cannot possibly measure area because cm is a unit of distance.

Therefore, πr^2 must be the correct expression for calculating the area of a circle.

$\pi r^2 \doteq 3.14(3.6\text{ cm})^2 = 3.14(3.6\text{ cm})(3.6\text{ cm}) = 40.6944\text{ cm}^2 \rightarrow$ Notice that the unit cm^2 is appropriate for area.

Using the Language of Mathematics to Describe Quantitative Relationships

Introduction

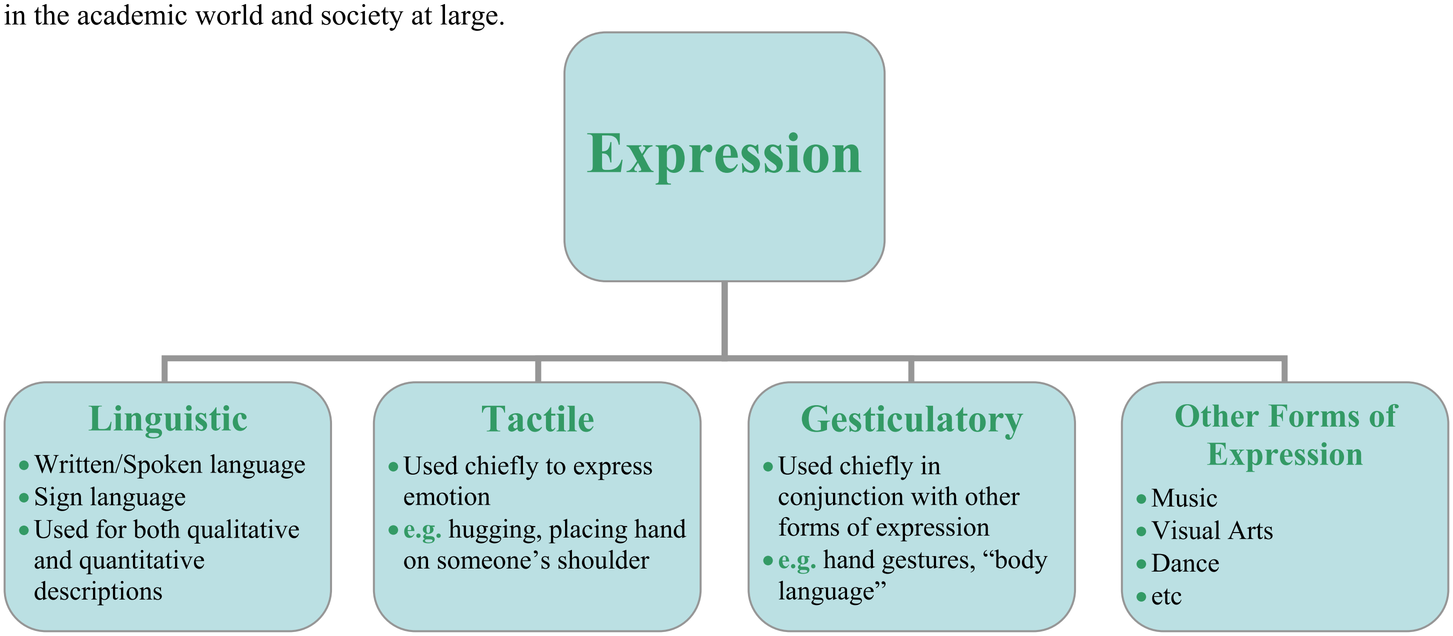
Over the course of my career, I have learned that there is a sure-fire way to put a damper on a lively party. All I have to do is to reveal that I am a mathematics teacher. Consequently, at social gatherings I do my very best to steer conversations away from my livelihood. Inevitably, however, *it* happens every time. While engaging in a riveting conversation with a few newfound acquaintances, somebody always pops the dreaded question: "You're a teacher, aren't you? What subject do you teach?" Realizing that my efforts to avoid the topic have been in vain, I always respond, in a manifestly ironic tone, "I teach *everyone's favourite* subject!" Invariably, the response to my cute retort is something to the effect of, "Ooh! You teach *math*? I never understood math. It was always my worst subject and I have always hated it!" Aside from confessing to being an axe murderer, I cannot imagine another revelation that could so quickly transform a look of delightful revelry into one of painful revulsion.

So why does the mere mention of math so often conjure up such harrowing images? Clearly, there are myriad sources of mathematics anxiety but I suspect that one of the main causes is the disconnect between mathematical symbols and their ***meaning***. This results directly from the mechanical, rote treatment of mathematics, particularly in elementary and secondary schools. In far too many math classes, students are taught to follow recipes (known as ***algorithms*** in mathematical jargon) that, when applied correctly, produce answers that match those listed at the end of the textbook but have as much significance to them as cuneiform. While such an approach may help students learn how to follow instructions, it accomplishes little else. As long as "algorithm execution" remains the focus of mathematics education, the vast majority of math students will continue to shudder whenever they cast their eyes upon a blackboard filled with cryptic and seemingly meaningless math symbols.

To remedy this situation, it is necessary to supplant this ridiculously single-minded, recipe-oriented pursuit of "the right answer" with an approach that reflects the true spirit of mathematics. Admittedly, this is a lofty goal but if it is to be achieved, there is a vital first step that must be taken. Students must be taught to understand the ***language*** of mathematics. Once this is accomplished, students will be able to appreciate mathematics as a form of ***expression*** whose ultimate purpose is to describe quantitative relationships.

Forms of Expression

The following diagram is a summary of various forms of expression. It is by no means exhaustive, but it serves as a convenient framework for the ensuing discussion. Where mathematics fits into this model will help to elucidate its place in the academic world and society at large.



Academic institutions tend to focus on the linguistic aspects of expression, particularly expression involving written language. What is unknown to many students and even some instructors is that written language is not confined to natural languages such as English. The language of mathematics, among others, also fits into this category! The examples in the next section will make this abundantly clear.

Quantitative versus Qualitative Description

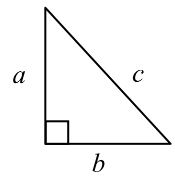
Imagine if William Shakespeare had been forced to write his sonnets and soliloquies using mathematical symbols. Obviously, this is a ludicrous proposition but it very clearly reveals the limitations of the language of mathematics. It cannot possibly be used to convey the broad spectrum of human experience. English, or any other natural language for that matter, is far better suited to descriptions of a ***qualitative*** nature.

Clearly, the language of math fails miserably (at least for the time being) in describing the beauty of a work of art, the feelings evoked by a haunting passage of music or the joy of being reunited with a loved one. When considering descriptions of a ***quantitative*** nature, however, math is decidedly the victor. The examples given below point out the limitations of natural languages in quantitative investigations.

Example 1 – The Pythagorean Theorem

The Pythagorean Theorem describes a relationship that exists among the sides of a right triangle. The table below shows how this theorem can be described using both English and the language of algebra.

English
In ***any*** right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.



Algebra
In ***any*** right triangle, $c^2 = a^2 + b^2$

As outlined below, a number of difficulties arise when English is used to describe mathematical relationships.

1. The wording tends to be long and cumbersome, which can easily cause confusion. Even a relationship as simple as the Pythagorean Theorem is quite difficult to describe in English.
2. In the absence of a translation into other languages, the description is accessible only to those who have a reading knowledge of English.
3. It is extremely difficult to manipulate relationships expressed in English or any other natural language.

By using an algebraic approach, however, these difficulties can be overcome quite easily.

1. The use of algebraic symbols results in very concise descriptions of relationships.
2. Algebraic descriptions are accessible to all people who have a rudimentary understanding of mathematics. Ethnic background is irrelevant.
3. Relationships can be manipulated very easily as shown in the following example:

$$c^2 = a^2 + b^2$$
$$\therefore b^2 = c^2 - a^2$$
$$\therefore a^2 = c^2 - b^2$$

Example 2 – Using Equations to Solve Problems

There are one hundred and forty coins in a collection of dimes and nickels. If the total value of the coins is \$10.90, how many dimes and how many nickels are there?

Solution

The main difficulty encountered in solving this type of problem is the ***translation*** from English into the language of algebra. Once this is done, the relationship between the number of coins and the total value of the coins is expressed in a form that is easy to manipulate. The following approach is usually not found in textbooks but it clearly shows how the equation that is obtained is nothing more than a ***mathematical restatement of the English description of the problem***.

The total value of the coins **is** \$10.90.

The total value of the coins = \$10.90

The value of all the nickels + the value of all the dimes = \$10.90

0.05×(the number of nickels) + 0.10×(the number of dimes) = \$10.90

Now we are ready to apply the language of algebra. If n represents the number of nickels, then the number of dimes must be equal to $140 - n$ (since there are 140 coins altogether). Finally, it is possible to write the following mathematical representation of the original English description:

$$0.05n + 0.10(140 - n) = \$10.90$$

By solving this equation, the desired result is obtained.