Unit 1 – Exponential and Logarithmic Functions

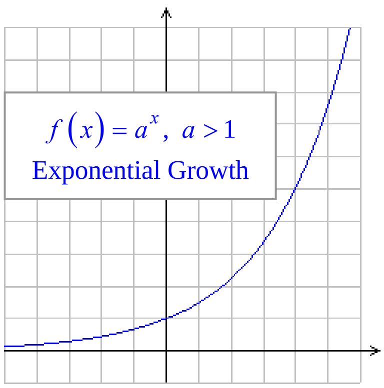
REVIEW OF EXPONENTIAL FUNCTIONS

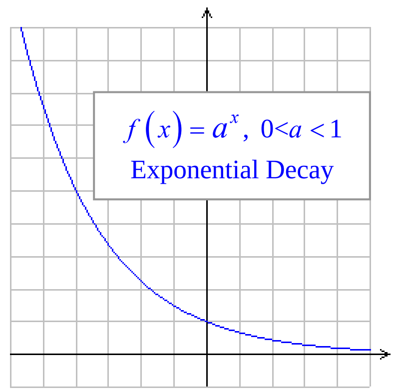
Exponential Growth and Exponential Decay

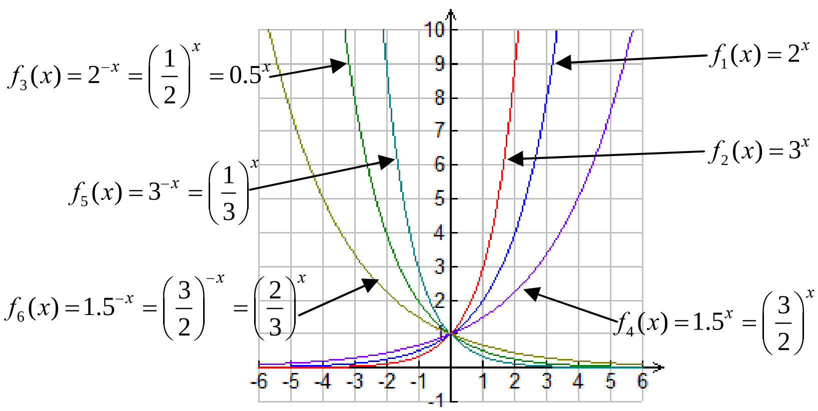
Exponential functions are used to model very fast growth or very fast decay. Specifically, exponential functions model

- growth that involves doubling, tripling, etc. at regular intervals
  e.g. Since 1950, the Earth's population has been doubling approximately every 40 years.
- decay that involves cutting in half, cutting in thirds, etc. at regular intervals
  e.g. Sodium-24 loses half of its mass every 14.9 hours. We say that the ***half-life*** of Sodium-24 is 14.9 hours.

Graphs of Exponential Functions







x	y1(x) 2^x	y2(x) 3^x	y3(x) 0.5^x	y4(x) 1.5^x	y5(x) 3^(-x)	y6(x) 1.5^(-x)
-5	0.03125	0.004115	32	0.131687	243	7.59375
-4	0.0625	0.012346	16	0.197531	81	5.0625
-3	0.125	0.037037	8	0.296296	27	3.375
-2	0.25	0.111111	4	0.444444	9	2.25
-1	0.5	0.333333	2	0.666667	3	1.5
0	1	1	1	1	1	1
1	2	3	0.5	1.5	0.333333	0.666667
2	4	9	0.25	2.25	0.111111	0.444444
3	8	27	0.125	3.375	0.037037	0.296296
4	16	81	0.0625	5.0625	0.012346	0.197531
5	32	243	0.03125	7.59375	0.004115	0.131687
6	64	729	0.015625	11.3906	0.001372	0.087791
7	128	2187	0.007813	17.0859	0.000457	0.058528
8	256	6.56E+03	0.003906	25.6289	1.52E-04	0.039018
9	512	1.97E+04	0.001953	38.4434	5.10E-05	0.026012
10	1024	5.90E+04	0.000977	57.665	1.70E-05	0.017342

Example

In 1950, the Earth's population was about 2.5 billion people. Since then, the world's population has been doubling roughly every 40 years. If the current trend continues, ***predict*** the world's population in 2100. Do you think that the current trend will continue?

Solution

Using a Table to help us Understand the Problem

Year	*Population*
1950	2.5×10^9
1990	$5\times10^9 = (2.5\times10^9)(2)$
2030	$1\times10^{10} = (2.5\times10^9)(2^2)$
2070	$2\times10^{10} = (2.5\times10^9)(2^3)$
2110	$4\times10^{10} = (2.5\times10^9)(2^4)$

Writing an Equation

To ***double*** means to ***multiply by 2***. Furthermore, we must multiply by 2 every 40 years. Thus, if we set $t = 0$ years to correspond to 1950, it is clear that the following exponential function models the given situation:

$$P(t) = 2.5\times10^9\left(2^{t/40}\right)$$

Solving the Problem

The year 2100 is 150 years after 1950. Therefore, $t = 150$.

$$P(150) = 2.5\times10^9\left(2^{150/40}\right)$$

$$\doteq 3.4\times10^{10}$$

If the current trend continues, in 2100 the Earth's population will be about 34 billion.

It is difficult to imagine that the current rate of population growth will continue indefinitely. First of all, it is very unlikely that the Earth's food supply can grow at a pace that matches or exceeds the growth of population. To compound matters, as the population size increases, the amount of arable land tends to decrease because extra space is required for residential, commercial and industrial purposes.

General Form of an Exponential Function

Algebraic Form	*Transformations expressed in Words*	*Transf. in Mapping Notation*
$f(x)=a^x$ $g(x)=Af(b(x-h))+k$ $\quad =Aa^{b(x-h)}+k$ ***Note*** Since a is being used to denote the base of the exponential function, A is used to denote the vertical stretch factor.	***Horizontal*** **1.** Stretch/compress by a factor of $1/b=b^{-1}$ depending on whether $0<b<1$ or $b>1$. If b is negative, there is also a reflection in the *y*-axis. **2.** Shift h units right if $h>0$ or h units left if $h<0$. ***Vertical*** **1.** Stretch/compress by a factor of A depending on whether $A>1$ or $0<A<1$. If A is negative, there is also a reflection in the *x*-axis. **2.** Shift k units up/down depending on whether k is positive or negative.	$(x,y)\rightarrow\left(\dfrac{1}{b}x+h,\ Ay+k\right)$

Example

$f(x)=2^x$

$g(x)=-5f(-1.5(x+1))+6$

$\quad =-5\left(2^{-1.5(x+1)}\right)+6$

To obtain the graph of g from the graph of f, do the following:

Horizontal

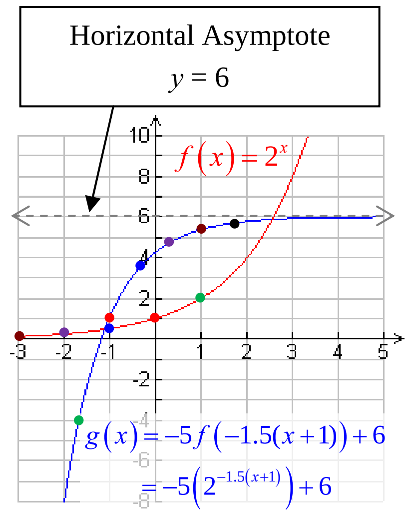
1. Compress horizontally by a factor of 1/1.5=2/3, reflect in the *y*-axis.
2. Translate 1 unit to the left.

Vertical

1. Stretch vertically by a factor of 5, reflect in the *x*-axis.
2. Translate 6 units up.

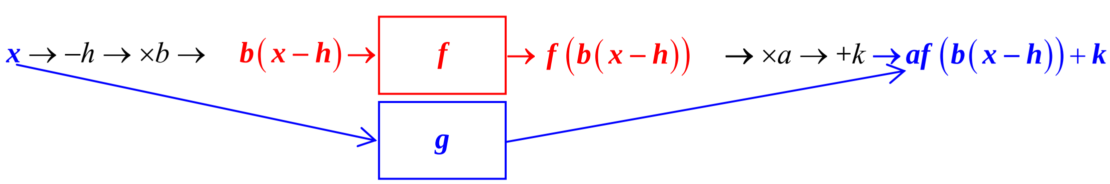
$$(x,y)\rightarrow\left(-\frac{2}{3}x-1,-5y+6\right)$$

Pre-image	*Image*
$(0,1)$	$(-1,1)$
$(1,2)$	$\left(-\frac{5}{3},-4\right)$
$\left(-1,\frac{1}{2}\right)$	$\left(-\frac{1}{3},\frac{7}{2}\right)$
$\left(-2,\frac{1}{4}\right)$	$\left(\frac{1}{3},\frac{19}{4}\right)$
$\left(-3,\frac{1}{8}\right)$	$\left(1,\frac{43}{8}\right)$
$\left(-4,\frac{1}{16}\right)$	$\left(\frac{5}{3},\frac{91}{16}\right)$



Why are the Horizontal Transformations the Opposite of the Operations Performed to the Independent Variable?

$$y=g(x)=af(b(x-h))+k$$



As can be seen very clearly from the above diagram,

- The operations that affect x are performed ***before*** the function f is applied.
- The operations that affect y are performed ***after*** the function f is applied.
- The input to f is $b(x-h)$. Recall that $f(b(x-h))$ means "the *y*-value obtained when $b(x-h)$ is the input given to *f*."
- The input to the function g, however, is x, not $b(x-h)$.
- To find out the output obtained when g is applied to x, it is first necessary to "see" what output is produced by applying f to $b(x-h)$. In other words, we first must "look ahead" to see what happens when f is applied to $b(x-h)$, then "***reverse our steps***" back to x, to determine what happens when g is applied to x.

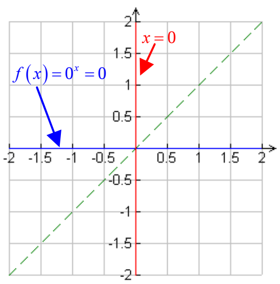
Very Important Restriction on Bases of Exponential Functions

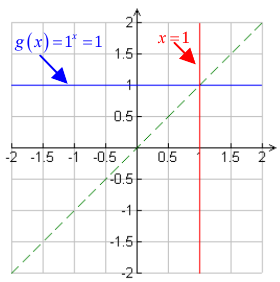
1. ***Negative bases are not allowed!*** When the bases are allowed to be negative, the resulting functions are "badly behaved" in the sense that they are not continuous and smooth. For instance, the table of values below (for the function $f(x)=(-2)^x$) illustrates some of the problems associated with allowing negative bases.

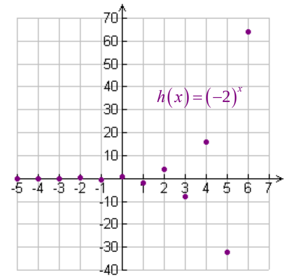
x	-2	-1	$-1/2$	0	$1/4$	$1/2$	1	$3/2$	2
$f(x)=(-2)^x$	$1/4$	$-1/2$	$1/\sqrt{-2}$ (undefined)	1	$\sqrt[4]{-2}$ (undefined)	$\sqrt{-2}$ (undefined)	-2	$\left(\sqrt{-2}\right)^3$ (undefined)	4

- $f(x)=(-2)^x$ is ***undefined*** at an infinite number of points
- $f(x)=(-2)^x$ "***jumps***" across the x-axis, from positive to negative values and vice versa, at an infinite number of points
- Functions like $f(x)=(-2)^x$ behave very erratically. They do not model natural processes such as radioactive decay or population growth.

2. ***The bases 0 and 1 are not allowed***. This is very easy to accept because $0^x=0$ and $1^x=1$ for all $x\in\mathbb{R}$. That is, the functions $f(x)=0^x$ and $g(x)=1^x$ are nothing more than ***constant functions***, which means that their graphs are horizontal straight lines. To make matters even worse, their inverses are not even functions (graphs are vertical lines).







x	(-2)^x
-4	0.0625
-3.5	undef
-3	-0.125
-2.5	undef
-2	0.25
-1.5	undef
-1	-0.5
-0.5	undef
0	1
0.5	undef
1	-2
1.5	undef
2	4
2.5	undef
3	-8
3.5	undef
4	16
4.5	undef
5	-32
5.5	undef
6	64

Summary

Functions that we call ***exponential must be*** of the form $f(x)=Aa^{b(x-h)}+k$, where $a>0$ and $a\neq 1$. If $a\le 0$ or $a=1$, the resulting functions do not exhibit the behaviour of natural processes such as population growth and are ***not called*** exponential. (In addition, A, b, k and h must all be real numbers but $A\neq 0$ and $b\neq 0$.)

Furthermore, if an exponential function A is used as a ***mathematical model*** for a process that depends on time, then the ***amount present at time t*** is given by $A(t)=A_0\left(a^{t/\tau}\right)$,where A_0 represents the ***initial amount*** (amount at time 0), the base of the exponential function a corresponds to the ***growth factor*** (**e.g.** if the amount doubles at regular intervals then a = 2), and τ represents how long it takes for the amount to increase by a factor of a.

Important Differences between Power Functions and Exponential Functions

Power Functions

$f(x)=x^n, n\in\mathbb{Z}$

e.g. $f(x)=x^2$

- The ***base*** is variable and the ***exponent*** is constant.
- Power functions grow more slowly than exponential functions.

x	$f(x)=x^2$
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49

Exponential Functions

$g(x)=a^x, a\in\mathbb{R}, a>0, a\neq 1$

e.g. $g(x)=2^x$

- The ***base*** is constant and the ***exponent*** is variable.
- Exponential functions grow more quickly than power functions.

x	$g(x)=2^x$
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128

Problems

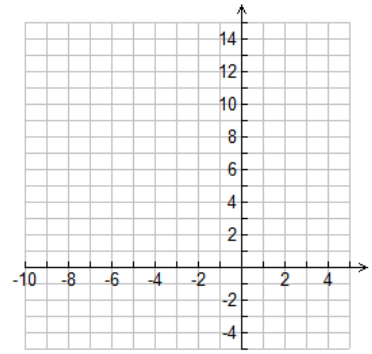
1. Given $f(x)=\left(\frac{1}{2}\right)^{x}$, sketch the graph of $g(x)=3f\left(-2(x+1)\right)-4$.

Equation of g

Transformations of f expressed in Words

Transformations in Mapping Notation

Pre-image	Image



2. Lemmings are small rodents usually found in or near the Arctic. Contrary to popular belief, lemmings ***do not commit mass suicide*** when they migrate. Driven by strong biological urges, they will migrate in large groups when population density becomes too great. During such migrations, lemmings may choose to swim across bodies of water in search of a new habitat. Many lemmings drown during such treks, which may in part explain the myth of mass suicide.

What is true about lemmings is that they reproduce at a very fast rate, causing populations to increase dramatically over a very short time. Possibly because of limited resources and the life cycles of their predators, lemming populations tend to plummet every four years. These periodic "boom-and-bust" cycles may also contribute to the mass suicide myth.

Using the data in the table at the right, model the lemming population for a four year cycle.

Time (Years)	Population Per Hectare
0	5
0.5	7.2
1	10.4
1.5	15
2	21.6
2.5	31.2
3	45
3.5	64.9
4	93.6

3. Since exponential growth is so fast, it usually cannot be sustained for very long. The rate of growth of any system is constrained by the availability of resources. Once the growth rate outstrips the rate of growth of resources, the system's growth is necessarily curbed. In such cases, a ***logistic function*** is likely a better mathematical model than an exponential function.

Construct both an ***exponential model*** and a ***logistic model*** for the following.

(a) According to each model, how long would it take to reach the maximum rate of infection?

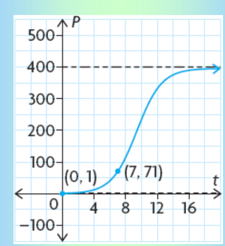
(b) Which model describes the given situation more realistically?

A town has a population of 5000 people. During a March Break trip, one of the town's residents contracted a virus. One week after her return to the town, 70 additional people had become infected with the same virus. Detailed scientific studies of the transmission of this virus determined that it infects approximately 8% of a given population.

The general equation of a ***logistic function*** is

$$f(x)=\frac{c}{1+ab^{x}}$$, where c

represents the ***carrying capacity*** (upper limit) of the function. The following is an example of a graph of a logistic function.

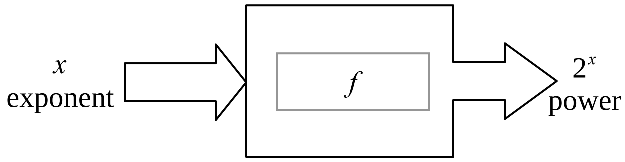


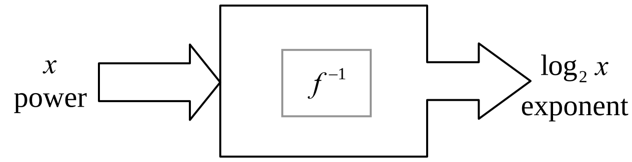
4. How long would it take for an investment of \$5000.00 to double if it is invested at a rate of 2.4% per annum (per year) compounded monthly?

Introduction to Logarithmic Functions

Introduction

As shown in the following examples, ***logarithmic*** and ***exponential*** functions to the same base are ***inverses*** of each other.





x (exponent)	$f(x)=2^x$ (power)
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

x (power)	$f^{-1}(x)=\log_2 x$ (exponent)
$1/8$	-3
$1/4$	-2
$1/2$	-1
1	0
2	1
4	2
8	3

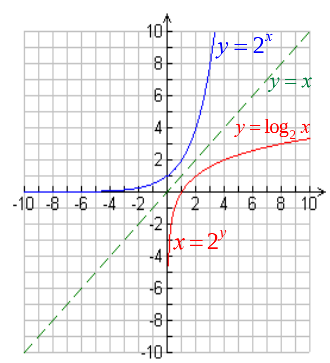
- 2^x is read "2 to the exponent *x*" or "the x^{th} power of 2"
- $2^x \to 2$ is the ***base***, *x* is the ***exponent***, 2^x is the ***power***
- Sometimes, the word ***power*** is used as if it were synonymous with ***exponent***. This is not strictly correct. However, this mistake is so common that we are forced to accept that statements such as "2 to the power *x*" mean the same thing as "2 to the exponent *x*."

- $\log_2 x$ is read "the logarithm of *x* to the base 2"
- If $y=\log_2 x$, then *y* is the ***exponent*** to which the ***base*** 2 must be raised to obtain the ***power*** *x*. Therefore, a logarithm is an ***exponent***.
- The functions $f(x)=2^x$ and $f^{-1}(x)=\log_2 x$ ***contain exactly the same information***. However, exponential functions grow very rapidly and can be hard to manage. Since logarithmic functions grow very slowly, they are often much easier to work with.

Definition of a Logarithmic Function

Consider the exponential function $y=a^x$, where $a>0$ and $a\neq 1$.

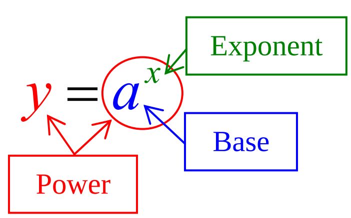
Since $y=a^x$ is one-to-one, its inverse (reflection in the line $y=x$) is also a function. We could write the equation of the inverse in the form $x=a^y$; however, it is preferable to write equations of functions in such a way that the ***value of the dependent variable*** is given by some expression that is specified ***only in terms of the independent variable***.

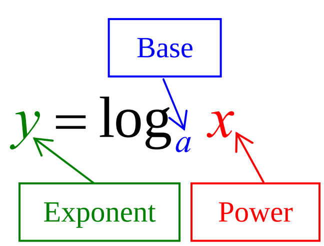


Definition

Thus, we ***define*** $g(x)=\log_a x$ to be the ***inverse*** of $f(x)=a^x$.

That is, $g(x)=\log_a x = f^{-1}(x)$. (Recall that the base *a* of an exponential function is a constant and that $a>0$ and $a\neq 1$.)





In words, if $y=\log_a x$, then *y* is the ***exponent*** to which *a* must be raised to obtain the power *x*.

Examples

1. Evaluate each logarithm.

(a) $\log_2 32 = 5$ because $2^5 = 32$ **(b)** $\log_2 \frac{1}{8} = -3$ because $2^{-3} = \frac{1}{8}$ **(c)** $\log_{\frac{1}{5}} 25 = -2$ because $\left(\frac{1}{5}\right)^{-2} = 25$

(d) $\log_{\sqrt{7}} 49 = 4$ because $\left(\sqrt{7}\right)^4 = 49$ **(e)** $\log_{10} 0.0001 = -4$ because $10^{-4} = 0.0001$

2. Express each exponential equation in logarithmic form.

(a) $2^y = 100$ (exponential form)
$\log_2 100 = y$ (logarithmic form)

(b) $a^y = x$ (exponential form)
$\log_a x = y$ (logarithmic form)

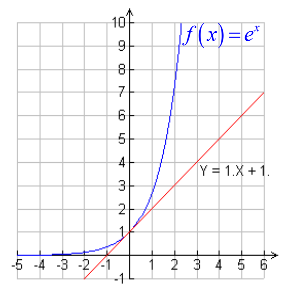
(c) $10^3 = 1000$ (exponential form)
$\log_{10} 1000 = 3$ (logarithmic form)

Important Note on Calculator Use

Scientific and graphing calculators usually have two buttons for computing logarithms, "log" and "ln." The "log" button computes the logarithm of a number to ***the base 10*** while the "ln" button evaluates the logarithm of a number ***to the base e***. (That is, $\log = \log_{10}$ and $\ln = \log_e$. See details below.) The function "ln" is called the ***natural logarithmic function*** (*logarithme naturel* in French, hence "ln" and not "nl") and is pronounced "lawn."

log → This button computes the logarithm of a number ***to the base 10***.

ln → This button computes the logarithm of a number ***to the base e***. Like π, e is an irrational number with a ***geometric significance***. The function $f(x) = e^x$ is the unique exponential function whose tangent line at the point $(0,1)$ has a slope of 1. The importance of e will become apparent when you study calculus. (Note that e is called "Euler's number" and that $e \doteq 2.718$.)

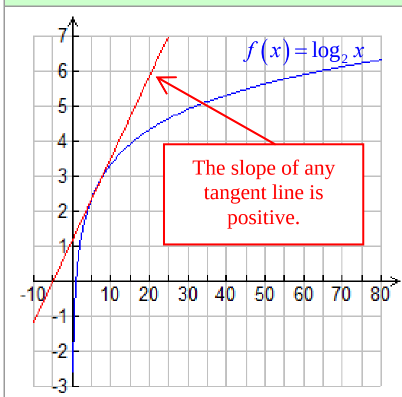


Important Note on Notation

In most cases, whenever the base is omitted, it is understood to be 10. That is, it is usually the case that $\log = \log_{10}$. In advanced mathematics, however, "log" is used to mean "$\log_e$" because the base 10 has no special significance in mathematical theory.

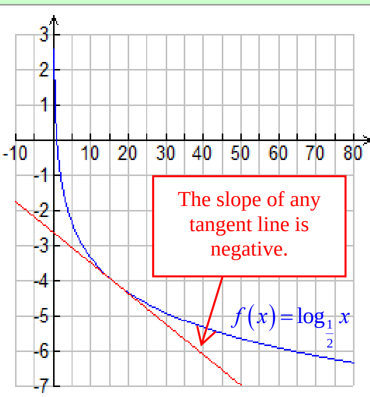
Characteristics of Logarithmic Functions

For $f(x) = \log_a x$



When a > 1

- The slope of any tangent line is ***positive***, meaning that the ***rate of change is positive***.
- The function ***increases*** as x increases but the rate of increase is extremely slow.
- The y-axis (i.e. the line with equation $x = 0$) is a ***vertical asymptote***.



When 0 < a < 1

- The slope of any tangent line is ***negative***, meaning that the ***rate of change is negative***.
- The function ***decreases*** as x increases but the rate of decrease is extremely slow.
- The y-axis (i.e. the line with equation $x = 0$) is a ***vertical asymptote***.

Domain of $f(x) = \log_a x$ *for any a > 0, a ≠ 1*

From the above, you should have noticed that $f(x) = \log_a x$ is only defined for positive values of x. That is, the domain of any such logarithmic function is $\{x \in \mathbb{R} : x > 0\}$. Another way of expressing this is that one can only "take" the logarithm of a positive number. To understand why this is the case, consider an example. Suppose that $y = \log_a(-6)$. Then $a^y = -6$, which is impossible because $a^y > 0$ for all $y \in \mathbb{R}$. (Remember that a must be positive.)

In Summary

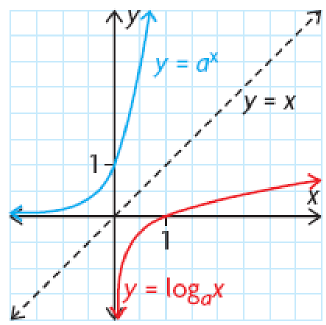
Key Ideas

- The inverse of the exponential function $y = a^x$ is also a function. It can be written as $x = a^y$. (This is the exponential form of the inverse.) An equivalent form of $x = a^y$ is $y = \log_a x$. (This is the logarithmic form of the inverse and is read as "the **logarithm** of x to the base a.") The function $y = \log_a x$ is called the logarithmic function.
- Since $x = a^y$ and $y = \log_a x$ are equivalent, a logarithm is an exponent. The expression $\log_a x$ means "the exponent that must be applied to base a to get the value of x." For example, $\log_2 8 = 3$ since $2^3 = 8$.

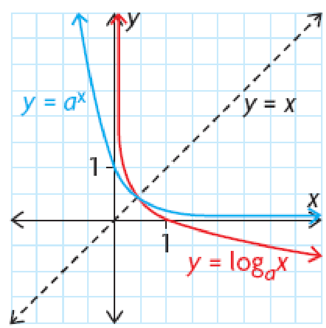
Need to Know

- The general shape of the graph of the logarithmic function depends on the value of the base.

  When $a > 1$, the exponential function is an increasing function, and the logarithmic function is also an increasing function.



  When $0 < a < 1$, the exponential function is a decreasing function and the logarithmic function is also a decreasing function.



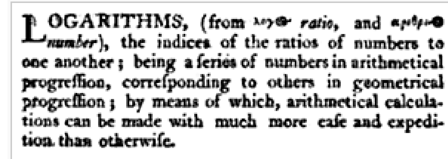
- The y-axis is the vertical asymptote for the logarithmic function. The x-axis is the horizontal asymptote for the exponential function.
- The x-intercept of the logarithmic function is 1, while the y-intercept of the exponential function is 1.
- The domain of the logarithmic function is $\{x \in \mathbf{R} \mid x > 0\}$, since the range of the exponential function is $\{y \in \mathbf{R} \mid y > 0\}$.
- The range of the logarithmic function is $\{y \in \mathbf{R}\}$, since the domain of the exponential function is $\{x \in \mathbf{R}\}$.

Homework

p. 451	1, 2, 4, 5, 6, 7, 9, 10, 11
p. 466	1, 2, 3, 5, 6, 7,

A Very Brief History of Logarithms

(Adapted from Wikipedia Article http://en.wikipedia.org/wiki/Logarithms *)*


LOGARITHMS, (from λογος *ratio*, and αριθμος *number*), the indices of the ratios of numbers to one another; being a ſeries of numbers in arithmetical progreſſion, correſponding to others in geometrical progreſſion; by means of which, arithmetical calculations can be made with much more eaſe and expedition than otherwiſe.

The 1797 Britannica explains logarithms as "a series of numbers in arithmetical progression, corresponding to others in geometrical progression; by means of which, arithmetical calculations can be made with much more ease and expedition than otherwise."

The method of logarithms was first publicly propounded in 1614, in a book entitled *Mirifici Logarithmorum Canonis Descriptio*, by John Napier, Baron of Merchiston, in Scotland. (Joost Bürgi independently discovered logarithms; however, he did not publish his discovery until four years after Napier.) Early resistance to the use of logarithms was muted by Kepler's enthusiastic support and his publication of a clear and impeccable explanation of how they worked.

Their use contributed to the advancement of science, especially of astronomy, by ***making some difficult calculations much easier to perform***. Prior to the advent of calculators and computers, they were used constantly in surveying, navigation, and other branches of practical mathematics. The methods of logarithms supplanted the more involved method of prosthaphaeresis, which relied on trigonometric identities as a quick method of computing products.

Questions

1. What is meant by the following reference from the 1797 Encyclopaedia Britannica?

"… being a series of numbers in arithmetical progression, corresponding to others in geometrical progression; by means of which, arithmetical calculations can be made with much more ease and expedition than otherwise."

The following example may help you to understand the above reference.

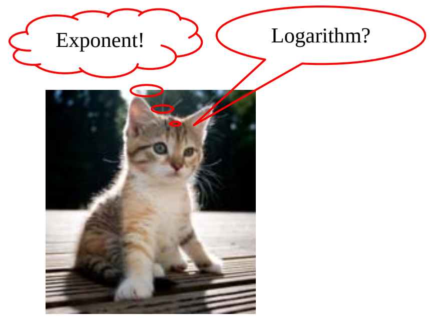
Geometric Progression (Geometric Sequence) Powers	$1=2^0$	$2=2^1$	$4=2^2$	$8=2^3$	$16=2^4$	$32=2^5$	$64=2^6$	$128=2^7$
Arithmetic Progression (Arithmetic Sequence) Exponents = Logarithms	0	1	2	3	4	5	6	7

2. What is the main advantage of using logarithms? Give some examples to support your answer.

The Laws of Logarithms

Introduction – The Meaning of "Logarithm"

It is very important to keep in mind that a logarithm is simply an ***alternative way of writing an exponent!*** Whenever you hear the word "logarithm," think "exponent." Whenever you hear the word "exponent," think "logarithm."



Review – Laws of Exponents

Law Expressed in Algebraic Form	*Law Expressed in Verbal Form*	*Example Showing why Law Works*
1. $a^x a^y = a^{x+y}$	To ***multiply*** two powers with the ***same base***, keep the base and ***add the exponents***.	$a^2a^4 = (a)(a)(a)(a)(a)(a) = a^6$
2. $\dfrac{a^x}{a^y} = a^{x-y}$	To ***divide*** two powers with the ***same base***, keep the base and ***subtract the exponents***.	$\dfrac{a^5}{a^2} = \dfrac{(a)(a)(a)(a)(a)}{(a)(a)} = a^3$
3. $(a^x)^y = a^{xy}$	To ***raise*** a power to an exponent, keep the base and ***multiply*** the exponents.	$(a^3)^4 = (a^3)(a^3)(a^3)(a^3) = a^{12}$

How the Laws of Exponents give Rise to Laws of Logarithms

Exponent Law Expressed in Verbal Form	*Equivalent Statement expressed in Language of Logarithms*	*Logarithmic Law Suggested by Law of Exponents*
To ***multiply*** two powers with the ***same base***, keep the base and ***add the exponents*** (i.e. add the logarithms).	The logarithm of the ***product*** of two powers is equal to the ***sum*** of the logarithm of one power and the logarithm of the other power.	$\log_a(xy) = \log_a x + \log_a y$
To ***divide*** two powers with the ***same base***, keep the base and ***subtract the exponents*** (i.e. subtract the logarithms).	The logarithm of the ***quotient*** of two powers is equal to the ***difference*** of the logarithm of one power and the logarithm of the other power.	$\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
To ***raise*** a power to an exponent, keep the base and ***multiply*** the exponents.	The logarithm of a ***power raised to an exponent*** is the ***product*** of the exponent and the logarithm of the power.	$\log_a(x^y) = y\log_a x$

Testing the Conjectures

The above table shows how logical reasoning can be used to ***suggest*** new mathematical laws. Although the reasoning appears to be sound, there is still some doubt as to whether the suggested laws are correct. To determine the plausibility of the laws, it is necessary to perform tests.

Suggested Law	*Tests*
$\log_a(xy) = \log_a x + \log_a y$	Let $x = 4$, $y = 8$, $a = 2$. L.S.$= \log_2(4\times 8) = \log_2 32 = 5$, R.S.$= \log_2 4 + \log_2 8 = 2+3 = 5$, $\therefore$ L.S.=R.S.
$\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$	Let $x = 4$, $y = 8$, $a = 2$. L.S.$= \log_2(4/8) = \log_2(1/2) = -1$, R.S.$= \log_2 4 - \log_2 8 = 2-3 = -1$, $\therefore$ L.S.=R.S.
$\log_a(x^y) = y\log_a x$	Let $x = 2$, $y = 3$, $a = 2$. L.S.$= \log_2(2^3) = \log_2 8 = 3$, R.S.$= 3\log_2 2 = 3(1) = 3$, $\therefore$ L.S.=R.S.

Proofs

The tests performed above all confirm the conjectures. Nevertheless, it is not possible to assert the validity of a mathematical statement solely on the basis of examples! The fact that a statement is valid for a particular example does not preclude the possibility that it could be invalid in other cases. Thus, a mathematical statement cannot be treated as "true" until a proof is found that ***demonstrates its validity in all possible cases!*** The second table below gives arguments that show irrefutably that the three conjectures given above are indeed true. Each of the proofs given below relies on another very important property of logarithms, namely $\log_a a^x = x$. This property must be proved ***before*** proofs of the laws of logarithms can be constructed. In addition, another property will be stated and proved because it is closely related to the required property.

Property	*Explanation*	*Proof*
$\log_a a^x = x$	The ***exponent*** to which the ***base*** a must be raised to obtain a^x is equal to x. Alternatively, since $y=\log_a x$ and $y=a^x$ are inverses of each other, $\log_a a^x$ must be equal to x. (In general, $f\left(f^{-1}(x)\right)=f^{-1}\left(f(x)\right)=x$.)	Let $x=\log_a y$. Then, $a^x=y$. Therefore, $\log_a a^x=\log_a y$. But $x=\log_a y$. Therefore, $\log_a a^x=x$.
$a^{\log_a x}=x$	The ***base*** a is raised to the ***exponent*** to which a must be raised to obtain x. Therefore, the result must be equal to x. Alternatively, since $y=\log_a x$ and $y=a^x$ are inverses of each other, $a^{\log_a x}$ must be equal to x.	Let $y=\log_a x$. Then, $a^y=x$. But $y=\log_a x$. Therefore, $a^{\log_a x}=x$.

The Laws of Logarithms

Product Law	*Quotient Law*	*Power Law*
$\log_a(xy)=\log_a x+\log_a y$	$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$	$\log_a\left(x^y\right)=y\log_a x$
Proof Let $x=a^w$ and $y=a^z$. Then, $xy=a^wa^z=a^{w+z}$. Therefore, $\log_a xy=\log_a a^{w+z}=w+z$. But $w=\log_a x$ and $z=\log_a y$. Therefore, $\log_a xy=\log_a x+\log_a y$.	***Proof*** Let $x=a^w$ and $y=a^z$. Then, $\frac{x}{y}=\frac{a^w}{a^z}=a^{w-z}$. Therefore, $\log_a\frac{x}{y}=\log_a a^{w-z}=w-z$. But $w=\log_a x$ and $z=\log_a y$. Therefore, $\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$.	***Proof*** Let $x=a^w$. Then, $x^y=\left(a^w\right)^y=a^{wy}$. Therefore, $\log_a x^y=\log_a a^{wy}=wy$. But $w=\log_a x$. Therefore, $\log_a\left(x^y\right)=y\log_a x$.

Examples

1. Use the laws of logarithms to evaluate each of the following expressions.

(a) $\log_3 12+\log_3 6.75$
$=\log_3(12\times 6.75)$
$=\log_3 81$
$=4$

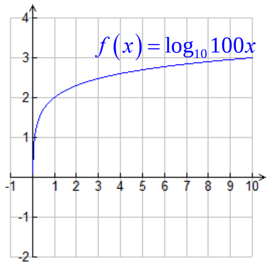
(b) $\log_2 96-\log_2 6$
$=\log_2\left(\frac{96}{6}\right)$
$=\log_2 16$
$=4$

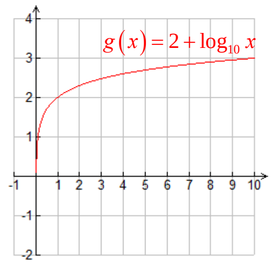
(c) $\log_7\sqrt[3]{343}$
$=\log_7 343^{\frac{1}{3}}$
$=\frac{1}{3}\log_7 343$
$=\frac{1}{3}(3)$
$=1$

(d) $\dfrac{\log_3 12}{\log_3 6.75}$

Unlike the others, this expression cannot be simplified because there is no law of logarithms that corresponds to the form of this expression. This expression can be evaluated with a calculator but first we need to have a method to convert to a base that a calculator can evaluate directly.

2. Graph the functions $f(x)=\log_{10}100x$ and $g(x)=2+\log_{10}x$. How do the graphs compare? Explain algebraically.





As we can see, the graphs of the two functions are identical. This is due to the fact that $\log_{10}100x$ and $2+\log_{10}x$ are ***equivalent expressions***:

$$\begin{aligned}&\log_{10}100x\\ &=\log_{10}100+\log_{10}x\\ &=2+\log_{10}x\end{aligned}$$

3. Use the laws of logarithms to write $\log_a\sqrt[3]{\dfrac{x^2y^5}{w^4}}$ in terms of $\log_a x$, $\log_a y$ and $\log_a w$.

Solution

$$\begin{aligned}\log_a\sqrt[3]{\frac{x^2y^5}{w^4}}&=\log_a\left(\frac{x^2y^5}{w^4}\right)^{\frac{1}{3}}\\ &=\log_a\left(\frac{x^{\frac{2}{3}}y^{\frac{5}{3}}}{w^{\frac{4}{3}}}\right)\\ &=\log_a\left(x^{\frac{2}{3}}y^{\frac{5}{3}}\right)-\log_a\left(w^{\frac{4}{3}}\right)\\ &=\log_a\left(x^{\frac{2}{3}}\right)+\log_a\left(y^{\frac{5}{3}}\right)-\frac{4}{3}\log_a w\\ &=\frac{2}{3}\log_a x+\frac{5}{3}\log_a y-\frac{4}{3}\log_a w\end{aligned}$$

OR

$$\begin{aligned}\log_a\sqrt[3]{\frac{x^2y^5}{w^4}}&=\log_a\left(\frac{x^2y^5}{w^4}\right)^{\frac{1}{3}}\\ &=\frac{1}{3}\log_a\left(\frac{x^2y^5}{w^4}\right)\\ &=\frac{1}{3}\left(\log_a\left(x^2y^5\right)-\log_a\left(w^4\right)\right)\\ &=\frac{1}{3}\left(\log_a\left(x^2\right)+\log_a\left(y^5\right)-4\log_a w\right)\\ &=\frac{1}{3}\left(2\log_a x+5\log_a y-4\log_a w\right)\\ &=\frac{2}{3}\log_a x+\frac{5}{3}\log_a y-\frac{4}{3}\log_a w\end{aligned}$$

The Change of Base Formula

The Formula	*Explanation*	*Proof*
$\log_b x=\dfrac{\log_a x}{\log_a b}$	The logarithm of a value to the base b is equal to the quotient of the logarithm of the value to the base a and the logarithm of b to the base a. (This is called the ***change of base*** formula. It is used to convert a logarithm expressed in a given base to a more convenient base such as 10.)	Let $y=\log_b x$. Then, $b^y=x$. Therefore, $\log_a b^y=\log_a x$. Thus, $y\log_a b=\log_a x$, which means that $y=\dfrac{\log_a x}{\log_a b}$. But $y=\log_b x$. Hence, $\log_b x=\dfrac{\log_a x}{\log_a b}$

Example

Evaluate $\dfrac{\log_3 12}{\log_3 6.75}$ correct to one decimal place.

Solution

$$\frac{\log_3 12}{\log_3 6.75}=\frac{\left(\dfrac{\log 12}{\log 3}\right)}{\left(\dfrac{\log 6.75}{\log 3}\right)}=\left(\frac{\log 12}{\log 3}\right)\left(\frac{\log 3}{\log 6.75}\right)=\frac{\log 12}{\log 6.75}\doteq 1.3$$

Homework

pp. 467-468 9, 11, 12, 15, 16, 18, 19, 20, 23
pp. 475-476 4, 5, 6, 7, 9, 10, 11, 12, 13, 16, 17, 18

Graphs and Transformations of Logarithmic Functions

General Form of a Logarithmic Function

Algebraic Form	*Transformations expressed in Words*	*Transf. in Mapping Notation*
$f(x)=\log_a x$ $g(x)=Af\left(b(x-h)\right)+k$ $\quad=A\log_a\left(b(x-h)\right)+k$ ***Note*** Since a is being used to denote the base of the logarithmic function, A is used to denote the vertical stretch factor.	***Horizontal*** **1.** Stretch/compress by a factor of $1/b=b^{-1}$ depending on whether $0<b<1$ or $b>1$. If b is negative, there is also a reflection in the y-axis. **2.** Shift h units right if $h>0$ or h units left if $h<0$. ***Vertical*** **1.** Stretch/compress by a factor of A depending on whether $A>1$ or $0<A<1$. If A is negative, there is also a reflection in the x-axis. **2.** Shift k units up/down depending on whether k is positive or negative.	$(x,y)\to\left(\frac{1}{b}x+h, Ay+k\right)$

Example

$f(x)=\log_2 x$

$g(x)=-\frac{1}{2}f\left(\tfrac{1}{4}(x-1)\right)+3$

$\quad=-\frac{1}{2}\log_2\left(\tfrac{1}{4}(x-1)\right)+3$

To obtain the graph of g from the graph of f, do the following:

Horizontal

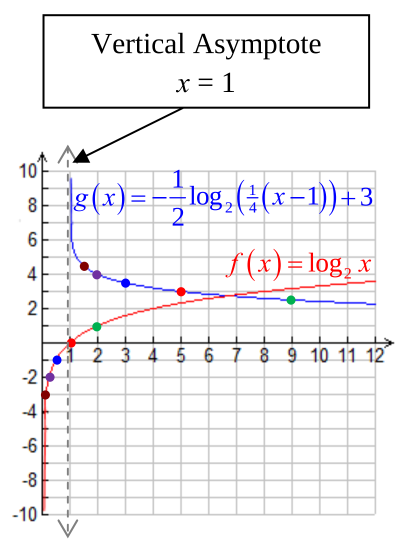
1. ***Stretch*** horizontally by a factor of $1/\left(\tfrac{1}{4}\right)=4$.
2. Translate 1 unit to the right.

Vertical

1. ***Compress*** vertically by a factor of 1/2, reflect in the x-axis.
2. Translate 3 units up.

$$(x,y)\to\left(4x+1,-\frac{1}{2}y+3\right)$$

Pre-image	*Image*
$(1,0)$	$(5,3)$
$(2,1)$	$\left(9,\frac{5}{2}\right)$
$\left(\frac{1}{2},-1\right)$	$\left(3,\frac{7}{2}\right)$
$\left(\frac{1}{4},-2\right)$	$(2,4)$
$\left(\frac{1}{8},-3\right)$	$\left(\frac{3}{2},\frac{9}{2}\right)$



Exercise

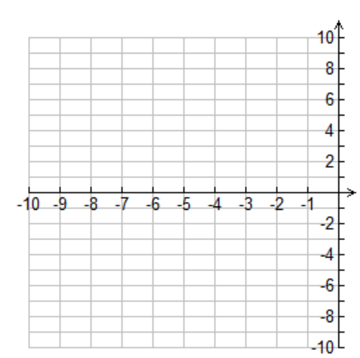
Given $f(x)=\log_2 x$, sketch the graph of $g(x)=3f\left(-2(x+1)\right)-4$.

Equation of g

Transformations of f expressed in Words

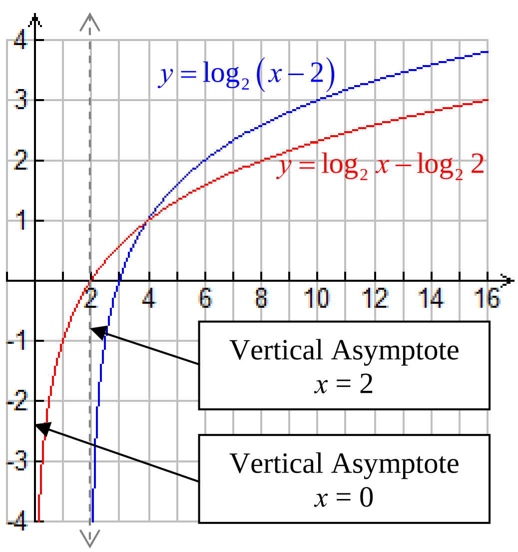
Transformations in Mapping Notation

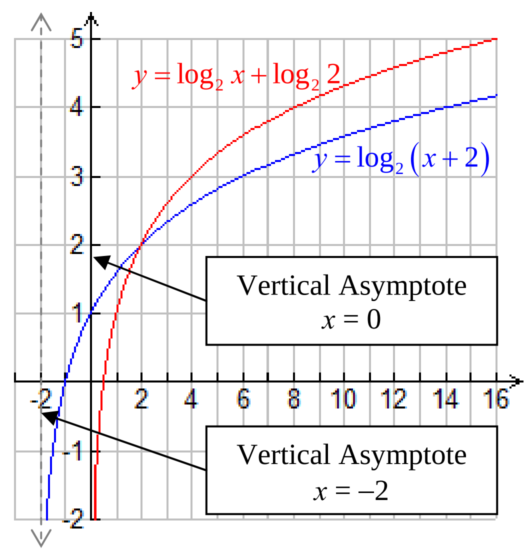
Pre-image	*Image*



Using Transformations to gain a better Understanding of the Laws of Logarithms

Law of Logarithms Each of these equations is an identity	*Common Errors* Each of these equations is *not* an identity.	*Explanation using Counterexample*	*Graphical Explanation*
$\log_a(xy)=\log_a x+\log_a y$	$\cancel{\log_a(x+y)=\log_a x+\log_a y}$	Let $x=y=1$, $a=2$. $\text{L.S.}=\log_2(1+1)$ $=\log_2 2$ $=1$ $\text{R.S.}=\log_2 1+\log_2 1$ $=0+0$ $=0$ $\therefore \text{L.S.}\neq\text{R.S.}$	**e.g.** $y=\log_2(x+2)$ The graph of $y=\log_2 x$ is shifted two units to the left. **e.g.** $y=\log_2 x+\log_2 2$ $=\log_2 x+1$ The graph of $y=\log_2 x$ is shifted one up. $\therefore \log_2(x+2)\neq\log_2 x+\log_2 2$
$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$	$\cancel{\log_a(x-y)=\log_a x-\log_a y}$	Let $x=16$, $y=8$, $a=2$. $\text{L.S.}=\log_2(16-8)$ $=\log_2 8$ $=3$ $\text{R.S.}=\log_2 16-\log_2 8$ $=4-3$ $=1$ $\therefore \text{L.S.}\neq\text{R.S.}$	$y=\log_2(x-2)$ The graph of $y=\log_2 x$ is shifted two units to the right. $y=\log_2 x-\log_2 2$ $=\log_2 x-1$ The graph of $y=\log_2 x$ is shifted one down. $\therefore \log_2(x-2)\neq\log_2 x-\log_2 2$





As can be seen from the graphs, $\log_2(x-2)$ and $\log_2 x-\log_2 2$ ***are not equivalent*** expressions. In addition, $\log_2(x+2)$ and $\log_2 x+\log_2 2$ ***are not equivalent*** expressions.

Homework
pp. 457-458 3, 4, 5ef, 6, 7, 9, 10, 11

Solving Exponential and Logarithmic Equations

Introduction – Advantages of using Logarithms

1. Logarithms simplify calculations. The laws of logarithms show us how logarithms turn products into sums, quotients into differences and powers into products.
2. Since logarithmic and exponential functions are inverses of each other, exponential equations can be solved by applying logarithmic functions and logarithmic equations can be solved by applying exponential functions.

General Principles of Solving Equations

1. ***Algebraically speaking***, all equations can be solved (in principle at least) by ***applying inverse operations***. That is, an equation of the form

$$f(x)=c\,,$$

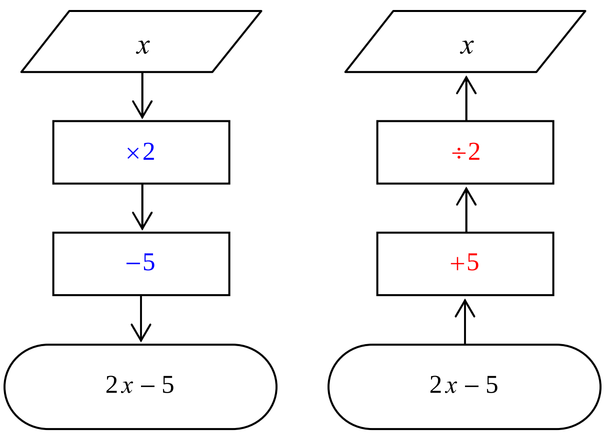
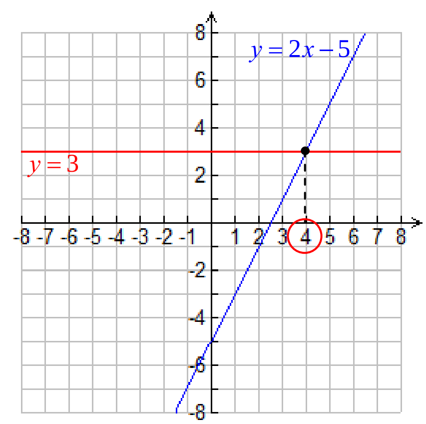
where *f* is some function and *c* is some real number, can be solved by applying f^{-1} to both sides:

$$f(x)=c$$
$$\therefore f^{-1}(f(x))=f^{-1}(c)$$
$$\therefore x=f^{-1}(c)$$

2. ***Geometrically (graphically) speaking***, all equations are solved by ***finding point(s) of intersection***. That is, an equation of the form

$$f(x)=g(x)$$

where *f* and g are any two functions, can be solved by finding the point(s) of intersection of the graphs of $y=f(x)$ and $y=g(x)$.

Example of Algebraic Solution	*Example of Graphical Solution*
$2x-5=3$ $\therefore 2x-5+5=3+5$ $\therefore 2x=8$ $\therefore \frac{2x}{2}=\frac{8}{2}$ $\therefore x=4$  	

Applying these Principles to Exponential and Logarithmic Equations

To solve an ***exponential equation***, ***take the logarithm of both sides***. This works because logarithms and exponentials are ***inverses*** of each other.

Example

$$2^x=234$$
$$\therefore \log(2^x)=\log 234$$
$$\therefore x\log 2=\log 234$$
$$\therefore x=\frac{\log 234}{\log 2}$$
$$\therefore x \doteq 7.87$$

Note that taking the logarithm of both sides to the base 2 gives a more direct solution. However, this is impractical because most calculators do not have a "$\log_2$" function.

$$2^x=234$$
$$\therefore \log_2(2^x)=\log_2 234$$
$$\therefore x=\log_2 234$$

At this point, the change of base formula is required.

To solve a ***logarithmic equation***, ***express the equation in exponential form***. This works because logarithms and exponentials are ***inverses*** of each other.

Example

$$\log_3\frac{x}{4}=7$$
$$\therefore \frac{x}{4}=3^7$$
$$\therefore x=4(3^7)$$
$$\therefore x=8748$$

An alternative approach that highlights the importance of inverses is to raise 3 to each side of the equation:

$$\log_3\frac{x}{4}=7$$
$$\therefore 3^{\log_3\frac{x}{4}}=3^7$$
$$\therefore \frac{x}{4}=3^7$$
$$\therefore x=8748$$

Since 3^x and $\log_3 x$ are inverses of each other, $3^{\log_3 y}=y$ for all $y>0$.

A Cornucopia of Examples

1. Solve each of the following equations.

(a) $3^{x+2} - 3^{x-2} = 720$

$\therefore 3^{x-2}\left(3^4 - 1\right) = 720$

$\therefore 3^{x-2}(80) = 720$

$\therefore 3^{x-2} = 9$

$\therefore 3^{x-2} = 3^2$

$\therefore x - 2 = 2$

$\therefore x = 4$

(b) $4^{3x-1} = 6^{3x-2}$

$\therefore \log\left(4^{3x-1}\right) = \log\left(6^{3x-2}\right)$

$\therefore (3x-1)\log 4 = (3x-2)\log 6$

$\therefore (3\log 4)x - \log 4 = (3\log 6)x - 2\log 6$

$\therefore (3\log 4)x - (3\log 6)x = \log 4 - 2\log 6$

$\therefore x(3\log 4 - 3\log 6) = \log 4 - 2\log 6$

$\therefore x = \dfrac{\log 4 - 2\log 6}{3\log 4 - 3\log 6}$

$\therefore x \doteq 1.8$

(c) $10^{4x} - 15\left(10^{2x}\right) = -56$

$\therefore \left(10^{2x}\right)^2 - 15\left(10^{2x}\right) + 56 = 0$

Let $y = 10^{2x}$.

$\therefore y^2 - 15y + 56 = 0$

$\therefore (y-7)(y-8) = 0$

$\therefore y = 7$ or $y = 8$

$\therefore 10^{2x} = 7$ or $10^{2x} = 8$

$\therefore \log\left(10^{2x}\right) = \log 7$ or $\log\left(10^{2x}\right) = \log 8$

$\therefore 2x = \log 7$ or $2x = \log 8$

$\therefore x = \dfrac{\log 7}{2}$ or $x = \dfrac{\log 8}{2}$

$\therefore x \doteq 0.4226$ or $x \doteq 0.4515$

(d) $\log_x 0.125 = -3$

$\therefore x^{-3} = 0.125$

$\therefore \dfrac{1}{x^3} = 0.125$

$\therefore x^3 = \dfrac{1}{0.125}$

$\therefore x^3 = 8$

$\therefore \left(x^3\right)^{\frac{1}{3}} = 8^{\frac{1}{3}}$

$\therefore x = 2$

(e) $\log_5 30x - \log_5 10 = 4$

$\therefore \log_5 \dfrac{30x}{10} = 4$

$\therefore \log_5 3x = 4$

$\therefore 3x = 5^4$

$\therefore x = \dfrac{5^4}{3}$

$\therefore x = \dfrac{625}{3}$

(f) $\log_7(x+1) + \log_7(x-5) = 1$

$\therefore \log_7\left[(x+1)(x-5)\right] = 1$

$\therefore (x+1)(x-5) = 7^1$

$\therefore x^2 - 4x - 5 = 7$

$\therefore x^2 - 4x - 12 = 0$

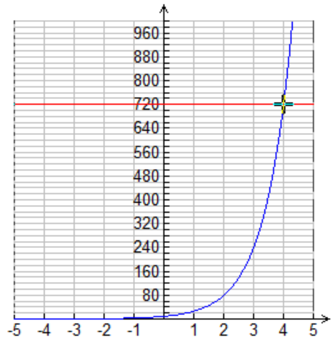
$\therefore (x-6)(x+2) = 0$

$\therefore x = 6$ or $\cancel{x = -2}$

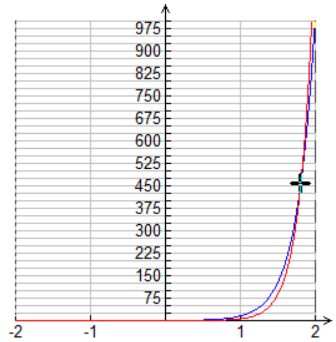
The root $x = -2$ is called ***inadmissable*** because it does not satisfy the original equation. This ***extraneous*** root occurs because the functions $g(x) = \log_7\left[(x+1)(x-5)\right]$ and $f(x) = \log_7(x+1) + \log_7(x-5)$ have different domains.

2. Give geometric (graphical) interpretations of each of the equations in question 1. Use the graphs to verify that the algebraic solutions are correct.

(a)



(b)



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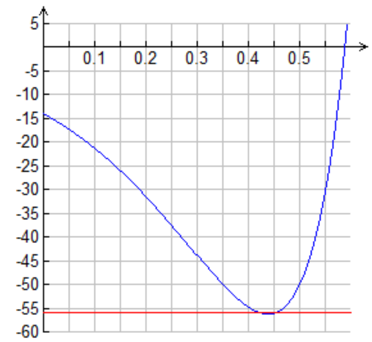

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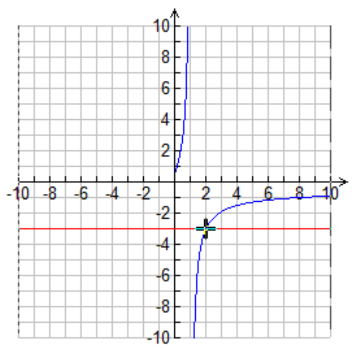

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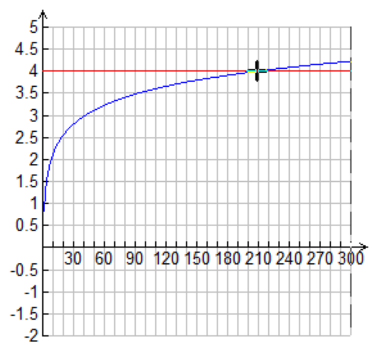

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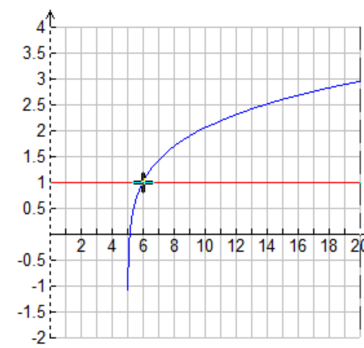
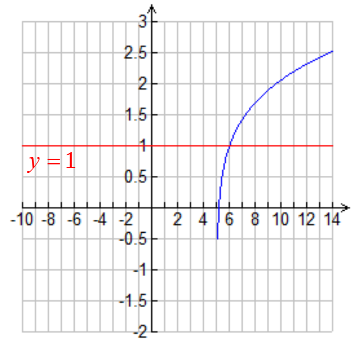


3. Explain why the ***extraneous root*** $x=-2$ was obtained in the solution of the equation in 1(f). (A root is called ***extraneous*** if it satisfies ***at least one*** of the equations produced by the method used to solve a given equation but it does not satisfy the original equation.)

Explanation

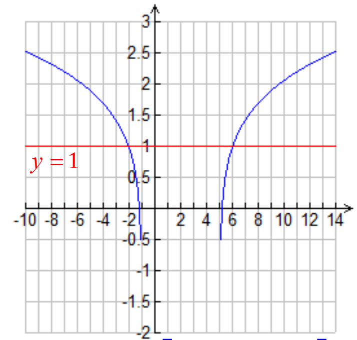
Recall that logarithmic functions cannot be applied to negative numbers. Therefore, the function $f(x)=\log_7(x+1)+\log_7(x-5)$ is defined only for values for which $x+1>0$ ***and*** $x-5>0$. Therefore, $x>-1$ ***and*** $x>5$, which implies that $x>5$.

On the other hand, the function $g(x)=\log_7\left[(x+1)(x-5)\right]$ is defined for all values for which $(x+1)(x-5)>0$. Now $(x+1)(x-5)>0$ as long as both $x+1$ and $x-5$ have the same sign (i.e. both positive or both negative). This implies that either $x>5$ or $x<-1$.



$f(x)=\log_7(x+1)+\log_7(x-5)$

$D=\{x\in\mathbb{R}:x>5\}$



$g(x)=\log_7\left[(x+1)(x-5)\right]$

$D=\{x\in\mathbb{R}:x>5 \text{ or } x<-1\}$

4. How long would it take for an investment of $5000.00 to double if it is invested at a rate of 2.4% per annum (per year) compounded monthly?

Solution

The monthly interest rate is 2.4%/12 = 0.024/12 = 0.002. Each month, the investment grows by a factor of 1.002.

Using a Table to help us Understand the Problem

Time (Months)	*Value of Investment ($)*
0	5000 $=5000(1.002)^0$
1	$5000(1.002)$ $=5000(1.002)^1$
2	$5000(1.002)(1.002)$ $=5000(1.002)^2$
3	$5000(1.002)^2(1.002)$ $=5000(1.002)^3$
t	$5000(1.002)^t$

Writing an Equation

As we can see from the table, in any given month, the value of the investment is 1.002 times greater than the value of the investment in the previous month. After t months, the original investment is multiplied by 1.002 t times. A shorter way to write this is $5000(1.002)^t$. Therefore, the value $V(t)$ of the investment after t months is given by

$$V(t)=5000(1.002)^t$$

Solving the Problem

Double the original investment is $10000. Therefore, $V(t)=10000$.

$\therefore 5000(1.002)^t=10000$

$\therefore (1.002)^t=2$

$\therefore \log(1.002)^t=\log 2$

$\therefore t\log 1.002=\log 2$

$\therefore t=\dfrac{\log 2}{\log 1.002}$

$\therefore t\doteq 346.9$

It would take about 347 months (28 years, 11 months) for the investment to double.

5. Radioactive elements have unstable nuclei, which causes them to give off radiation as the nuclei tend toward a state of greater stability. For example, carbon-14 (also called ***radiocarbon***) is a radioactive isotope of carbon that occurs in trace amounts on Earth. It decays into nitrogen-14, a stable and extremely abundant isotope of nitrogen. Carbon-14 (^{14}C) is very useful in determining the age of carbonaceous materials (materials rich in carbon) up to about 60000 years old. (See http://en.wikipedia.org/wiki/Carbon_14 for a more detailed description.)

A Brief Description of How Radiocarbon Dating Works

Organisms acquire carbon during their lifetime. Plants acquire it through photosynthesis and animals acquire it from consumption of plants and other animals. When an organism dies, it ceases to take in new carbon. Since carbon-12 (^{12}C) is not radioactive, the amount of ^{12}C in the remains of the organism will stay constant over time. However, since ^{14}C ***is*** radioactive, the amount of ^{14}C in the remains of the organism will decrease over time. The proportion of ^{14}C left when the remains of the organism are examined provides an indication of the time elapsed since its death. (See http://en.wikipedia.org/wiki/Radiometric_dating for a more detailed description of radiometric dating.)

Limitations of Carbon Dating

- Carbon-14 has a half-life of about 5730 years. This means that after 5730 years, half of the ^{14}C in the original sample would have decayed into ^{14}N. Eventually, there would be so little ^{14}C left in the sample that it would be impossible to measure the ^{14}C to ^{12}C ratio. Thus, radiocarbon dating is limited to organic material with a maximum age of about 58,000 to 62,000 years.
- ^{14}C dating only works for organisms that acquire most of their carbon, either directly or indirectly, from the atmosphere. It does not work on aquatic organisms because they acquire much of their carbon from minerals dissolved in water.
- ^{14}C does dating not work on organic material of very recent origin. The widespread emission of CO_2 into the atmosphere due to the burning of fossil fuels has caused the ratio of ^{14}C to ^{12}C to decrease since the beginnings of the Industrial Age. To complicate matters even further, the above-ground nuclear tests that occurred in several countries between 1955 and 1963 dramatically increased the amount of carbon-14 in the atmosphere and subsequently in the biosphere; after the tests ended, the atmospheric concentration of the isotope once again began to decrease.

Disclaimer

In the so-called real world, the process of radiometric dating is somewhat more complicated than might be suggested by high school math problems. Please be aware that the math problems are presented in an intentionally simplified manner to highlight important mathematical principles. To obtain highly reliable and accurate results, scientists must also take into account factors that could have a significant effect on their calculations. For instance, it is known that the atmospheric concentration of carbon-14 varies over time. Failing to take this into consideration could severely affect the accuracy of calculated ages.

Problem

An ancient wooden carving was discovered by a group of archaeologists in an excavation of an early Mayan settlement dating back to about 1800 BC. After some time, a debate arose among archaeologists concerning the age of the carving. Some archaeologists claimed that the artifact was made, out of freshly cut wood, by the people who inhabited the settlement while others asserted that it was much older. Since wood is rich in carbon, radiocarbon dating was done to settle the argument.

Through detailed studies, scientists have determined that in ***living*** carbonaceous material, the ratio of ^{14}C atoms to ^{12}C atoms is $1:10^{12}$. That is, there is only 1 atom of ^{14}C for every trillion atoms of ^{12}C. (For the purposes of this problem, we shall ignore the fact that this ratio varies slightly over time.) After performing mass spectrometry on several small samples of the Mayan carving, the average ratio of ^{14}C to ^{12}C in the samples was found to be $1:1.79\times10^{12}$. Estimate the age of the wood used to make the carving. (Recall that the half-life of ^{14}C is 5730 years.)

Solution

The ratio $1:10^{12}$ can be written as $\frac{1}{10^{12}}$, which equals 10^{-12}.

Time (Years)	0	5730	11460	17190	t
Ratio of ^{14}C to ^{12}C	10^{-12} $=10^{-12}\left(\frac{1}{2}\right)^0$	$10^{-12}\left(\frac{1}{2}\right)$ $=10^{-12}\left(\frac{1}{2}\right)^1$	$10^{-12}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ $=10^{-12}\left(\frac{1}{2}\right)^2$	$10^{-12}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)$ $=10^{-12}\left(\frac{1}{2}\right)^3$	$10^{-12}\left(\frac{1}{2}\right)^{\frac{t}{5730}}$

As we can see from the table, the ratio of ^{14}C to ^{12}C is cut in half every 5730 years. An equivalent way of stating this is that that the ratio is multiplied by $1/2$ every 5730 years. Thus, if we let $R(t)$ represent the ratio of ^{14}C to ^{12}C t years after the death of the organism, then

$$R(t)=10^{-12}\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

To determine the approximate age of the wood used to make the Mayan carving, all that we need to do is solve the equation $R(t)=\dfrac{1}{1.79\times10^{12}}$ (i.e. the ratio of ^{14}C to ^{12}C at time t is equal to $\dfrac{1}{1.79\times10^{12}}$).

- Note that 1e–12 ***means*** 1×10^{-12}, 5.58659e–13 ***means*** 5.58659×10^{-13}.
- Programming languages use this format for scientific notation.
- Many calculators use this format for scientific notation.
- This format is also used in TI-Interactive and other math programs.

$$\therefore 10^{-12}\left(\frac{1}{2}\right)^{\frac{t}{5730}}=\frac{1}{1.79\times10^{12}}$$

$$\therefore \left(\frac{1}{2}\right)^{\frac{t}{5730}}=\frac{10^{12}}{1.79\times10^{12}}$$

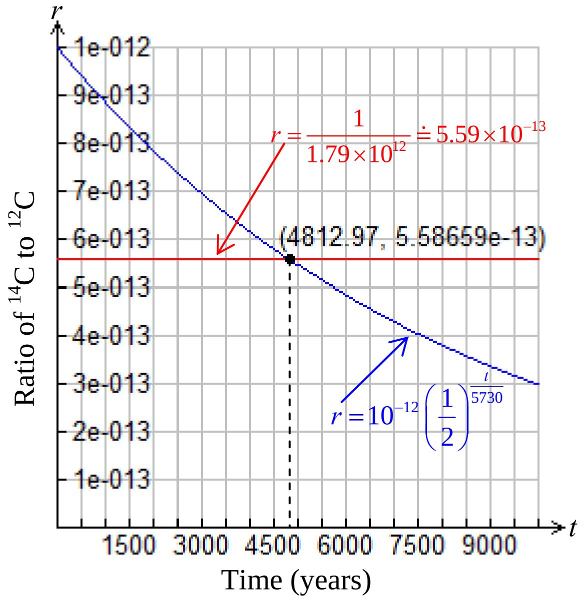
$$\therefore \left(\frac{1}{2}\right)^{\frac{t}{5730}}=\frac{1}{1.79}$$

$$\therefore \log\left(\left(\frac{1}{2}\right)^{\frac{t}{5730}}\right)=\log\left(\frac{1}{1.79}\right)$$

$$\therefore \frac{t}{5730}\log\left(\frac{1}{2}\right)=\log\left(\frac{1}{1.79}\right)$$

$$\therefore t=\frac{5730\log\left(\frac{1}{1.79}\right)}{\log\left(\frac{1}{2}\right)}$$

$$\therefore t\doteq 4813$$



According to radiocarbon dating, the wood used to make the carving is approximately 4800 years old. Since the Mayan settlement was dated to 1800 BC, which is only about 3800 years ago, it is not possible that the carving was made by the inhabitants of the settlement out of freshly cut wood. If the carving was indeed made by the inhabitants of the settlement, they would have to have used 1000-year-old wood. Since it is unlikely that such old wood would have been readily available, it's more likely that the carving was made at an earlier time.

Homework

pp. 485-486 4, 6, 7, 8, 11, 15, 16
pp. 491-492 4,5,6,7, 9, 10, 13, 17, 20

Applications of Exponential and Logarithmic Functions

Introduction – Advantage of Using Logarithmic Scales

- Recall that $y=a^x$ and $y=\log_a x$ are inverses of each other.
- Since an inverse is formed simply by interchanging the x and y co-ordinates of each ordered pair of a function f, f and f^{-1} contain ***exactly the same information***. Therefore, $y=a^x$ and $y=\log_a x$ are interchangeable in this sense.
- In the case of $y=a^x$ and $y=\log_a x$, it is often inconvenient to use $y=a^x$ because exponential functions increase/decrease so rapidly. In such cases, it is often much easier to use $y=\log_a x$.
- Presentation of data on a logarithmic scale can be helpful when the data cover a large range of values – the logarithm reduces this to a more manageable range.

Example 1 – How Acidic or Basic is a Solution?

The acidity of a solution is determined by the concentration of positive hydrogen ions. As can be seen from the table below, the actual hydrogen ion concentrations are cumbersome numbers that vary dramatically. Working with such numbers would be nightmarishly awkward. Therefore, a logarithmic scale known as the "pH" scale was devised to simplify matters.

Actual Positive Hydrogen Ion Concentration [H^+] (mol/L)	Concentration of hydrogen ions compared to distilled water		Examples of solutions at this pH
$1=10^0$	10 000 000	pH = 0	battery acid, strong hydrofluoric acid
$0.1=10^{-1}$	1 000 000	pH = 1	hydrochloric acid secreted by stomach lining
$0.01=10^{-2}$	100 000	pH = 2	lemon juice, gastric acid, vinegar
$0.001=10^{-3}$	10 000	pH = 3	grapefruit, orange juice, soda
$0.0001=10^{-4}$	1000	pH = 4	tomato juice, acid rain
$0.00001=10^{-5}$	100	pH = 5	soft drinking water, black coffee
$0.000001=10^{-6}$	10	pH = 6	urine, saliva
$0.0000001=10^{-7}$	1	pH = 7	"pure" water
$0.00000001=10^{-8}$	$\frac{1}{10}$	pH = 8	seawater
$0.000000001=10^{-9}$	$\frac{1}{100}$	pH = 9	baking soda
$0.0000000001=10^{-10}$	$\frac{1}{1000}$	pH = 10	Great Salt Lake, milk of magnesia
$0.00000000001=10^{-11}$	$\frac{1}{10\,000}$	pH = 11	ammonia solution
$0.000000000001=10^{-12}$	$\frac{1}{100\,000}$	pH = 12	soapy water
$0.0000000000001=10^{-13}$	$\frac{1}{1\,000\,000}$	pH = 13	bleaches, oven cleaner
$0.00000000000001=10^{-14}$	$\frac{1}{10\,000\,000}$	pH = 14	liquid drain cleaner

Points of Chemical Interest

- "pH" stands for "potential of Hydrogen"
- The mole is the ***amount of substance of a system*** which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12.
- In this definition, it is understood that the carbon 12 atoms are unbound, at rest and in their ground state.
- When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles or specified groups of such particles.
- ***Examples***

  one mole of iron contains the same number of ***atoms*** as one mole of gold

  one mole of benzene contains the same number of ***molecules*** as one mole of water

  the number of ***atoms*** in one mole of iron is equal to the number of ***molecules*** in one mole of water

In chemistry, the symbol [H^+] is used to denote the concentration of positive hydrogen ions in a solution. The reader can use the table to verify that pH is related to [H^+] according to the following logarithmic equation:

$$\mathrm{pH} = -\log[H^+]$$

Note

- pH < 7 Solution is called ***acidic***.
- pH = 7 Solution is called ***neutral***.
- pH > 7 Solution is called ***alkaline*** or ***basic***.

Example

(a) Calculate the pH of a solution with a hydrogen ion concentration of 2.74×10^{-8} mol/L. Is the solution acidic or basic?

Solution

$$\begin{aligned}\text{pH} &= -\log\left[\text{H}^+\right]\\ &= -\log\left(2.74\times10^{-8}\right)\\ &\doteq -(-7.56)\\ &= 7.56\end{aligned}$$

The pH of such a solution is approximately 7.56. Therefore, the solution is basic.

(b) Calculate the hydrogen ion concentration of a solution with a pH of 3.7.

Solution

$$\begin{aligned}&\text{pH} = -\log\left[\text{H}^+\right]\\ &\therefore 3.7 = -\log\left[\text{H}^+\right]\\ &\therefore \log\left[\text{H}^+\right] = -3.7\\ &\therefore \left[\text{H}^+\right] = 10^{-3.7}\\ &\therefore \left[\text{H}^+\right] \doteq 2.0\times10^{-4}\end{aligned}$$

The hydrogen ion concentration is about 2.0×10^{-4} mol/L

(c) How much "stronger" is an acid with a pH of 4.2 than an acid with a pH of 6.1?

Solution

$$\begin{aligned}&\text{pH} = -\log\left[\text{H}^+\right]\\ &\therefore 4.2 = -\log\left[\text{H}^+\right]\\ &\therefore \log\left[\text{H}^+\right] = -4.2\\ &\therefore \left[\text{H}^+\right] = 10^{-4.2}\end{aligned}$$

$$\begin{aligned}&\text{pH} = -\log\left[\text{H}^+\right]\\ &\therefore 6.1 = -\log\left[\text{H}^+\right]\\ &\therefore \log\left[\text{H}^+\right] = -6.1\\ &\therefore \left[\text{H}^+\right] = 10^{-6.1}\end{aligned}$$

$$\frac{10^{-4.2}}{10^{-6.1}} = 10^{-4.2-(-6.1)} = 10^{1.9} \doteq 79$$

An acid with a pH of 4.2 is about 79 times "stronger" than an acid with a pH of 6.1.

Example 2 – The Richter Scale

The ***Richter magnitude scale***, or more correctly ***local magnitude M_L scale***, assigns a single number to quantify the amount of seismic energy released by an earthquake. It is a base-10 logarithmic scale obtained by calculating the logarithm of the combined horizontal amplitude of the largest displacement from zero on a Wood–Anderson torsion seismometer output. So, for example, an earthquake that measures 5.0 on the Richter scale has a shaking amplitude 10 times larger than one that measures 4.0. The effective limit of measurement for local magnitude is about M_L = 6.8.

True Intensity	Richter Scale Magnitude
10^1	$\log_{10}10^1 = 1$
10^4	$\log_{10}10^4 = 4$
$10^{5.8}$	$\log_{10}10^{5.8} = 5.8$

Though still widely reported, the ***Richter scale*** has been superseded by moment magnitude scale which gives generally similar values.

Example

How much more intense is an earthquake that measures 8.5 on the Richter scale than one that measures 3.7?

Solution

$$\frac{10^{8.5}}{10^{3.7}} = 10^{8.5-3.7} = 10^{4.8} \doteq 63000$$

An earthquake that measures 8.5 on the Richter scale is about 63000 times more intense than an earthquake that measures 3.7.

More Examples

Read examples 3 and 4 on pages 496 – 498 of the textbook.

Homework

pp. 499-501 4, 5, 6d, 8, 10, 12, 13, 14, 15, 17, 18

A Detailed Investigation of Rates of Change

The Concept of Rate of Change

- The ***rate of change*** of a quantity measures ***how fast the quantity changes***.
- We are already familiar with many quantities that change with respect to time (i.e. over time, as time changes).

  e.g. How fast does position change with respect to time? (This rate of change is what we normally call velocity.)
  How fast does the mass of a radioactive substance change with respect to time?
  How fast does population change with respect to time?
  How fast does the cost of petroleum change with respect to time?

In general, a ***rate of change*** measures how fast the ***dependent variable*** changes with respect to the ***independent variable***.

Examples of Rates of Change not Involving Time

- How fast does the volume of a cube change with respect to the length of one of its sides?
- How fast does the volume of a box change with respect to its surface area?
- How does the temperature of a gas change with respect to its volume?

A more General and Abstract Look at Rates of Change – Average and Instantaneous Rates of Change

(a) ***Average Rate of Change***

The ***average rate of change*** of a dependent variable with respect to an independent variable measures the rate of change ***over an interval*** of the independent variable.

e.g. How fast does the distance change from $t = 0$ s to $t = 30$ s? That is, what is the average rate of change of distance with respect to time from $t = 0$ s to $t = 30$ s?

Average rate of change of $y = f(x)$ with respect to x when x changes from x_1 to x_2

$= \dfrac{\text{change in } y}{\text{change in } x}$

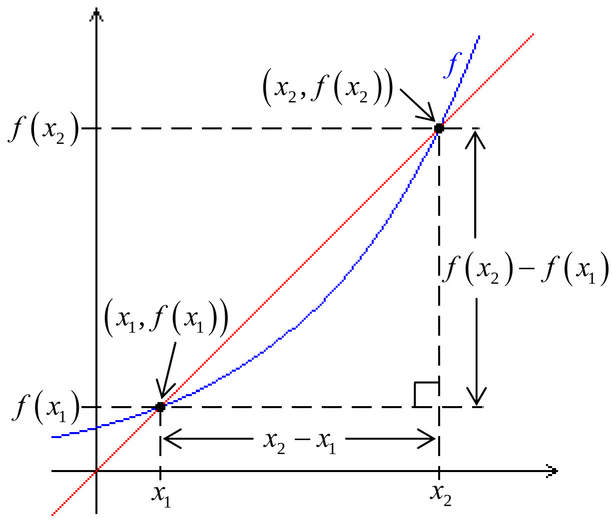
$= \dfrac{\Delta y}{\Delta x}$

$= \dfrac{\text{rise}}{\text{run}}$

$=$ slope of ***secant line*** through $(x_1, f(x_1))$ and $(x_2, f(x_2))$

$=$ average steepness of the curve from x_1 to x_2

$= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$



(b) ***Instantaneous Rate of Change***

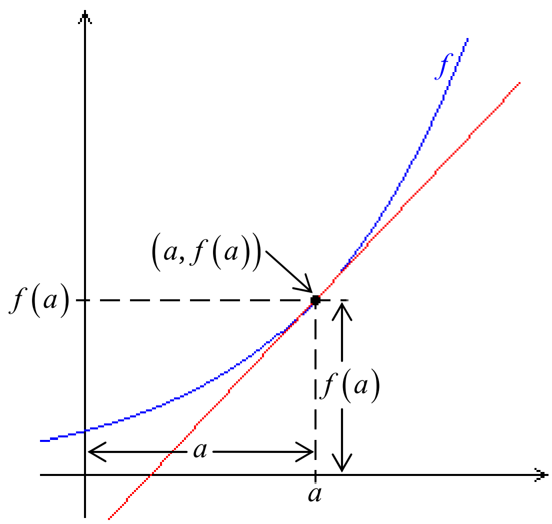
The ***instantaneous*** rate of change of a dependent variable with respect to an independent variable measures the rate of change ***at a single point***.

e.g. How fast does the distance change at $t = 15$ s? That is, what is the instantaneous rate of change of distance with respect to time at $t = 15$ s?

Instantaneous rate of change of $y = f(x)$ with respect to x when x is equal to a

$=$ slope of ***tangent line*** at the point $(a, f(a))$

$=$ steepness of the curve at the point $(a, f(a))$



Summary

Slope = Steepness of Curve = Rate of Change

Slope of Secant Line = Average Steepness of Curve between Two Points = Average Rate of Change between Two Points

Slope of Tangent Line = Steepness of Curve at a Point = Instantaneous Rate of Change at a Point

Instantaneous → occurring, done, or completed in an instant
→ existing at or pertaining to a particular instant
→ present or occurring at a specific instant

> Instant
> an infinitesimal or very short space of time; a moment: *They arrived not an instant too soon.*

Important Note on Calculating Instantaneous Rates of Change

- An average rate of change can be calculated very easily because ***two points*** are known. The slope of a secant line is calculated very easily by evaluating $\dfrac{\Delta y}{\Delta x}$ using the two known points.
- An instantaneous rate of change is much more difficult to calculate because ***only one point is known***. It is not possible to use $\dfrac{\Delta y}{\Delta x}$ to calculate the slope of a line when only one point is known.
- The branch of mathematics known as ***calculus*** was developed precisely for the purpose of calculating instantaneous rates of change. Calculus provides us with tools that can be used to calculate the ***exact*** slope of a tangent line. The slope of the tangent to the function $y = f(x)$ at $x = a$ is written $f'(a)$, $\left.\dfrac{dy}{dx}\right|_{x=a}$ or $\left.\dfrac{df(x)}{dx}\right|_{x=a}$
- Without calculus, we can ***estimate*** the slope of a tangent line (i.e. the instantaneous rate of change) by
  - Using software such as TI-Interactive.
  - Using a graphing calculator.
  - Calculating the slope of a secant line over a ***very small interval*** of the independent variable.

Example

A sample of ^{14}C has an initial mass of 100 mg. (Recall that the half-life of ^{14}C is about 5730 years.)

(a) How fast does the mass of the ^{14}C sample change over the first 10000 years?

(b) How fast does the mass of the ^{14}C sample change at the instant that 10000 years have passed?

Solution

Let $M(t)$ represent the mass of ^{14}C remaining, in mg, after t years. Then clearly, $M(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$. (If necessary, use a table to verify that this equation is correct.)

(a) rate of change of mass over first 10000 years

= average rate of change of mass with respect to time from $t = 0$ years to $t = 10000$ years

$$= \frac{\Delta M}{\Delta t}$$

$$= \frac{M(10000) - M(0)}{10000 - 0}$$

$$= \frac{100\left(\frac{1}{2}\right)^{\frac{10000}{5730}} - 100\left(\frac{1}{2}\right)^{\frac{0}{5730}}}{10000}$$

$$\doteq -0.007 \text{ mg/year}$$

(b) rate of change of mass exactly at 10000 years

= instantaneous rate of change of mass with respect to time exactly at $t = 10000$ years

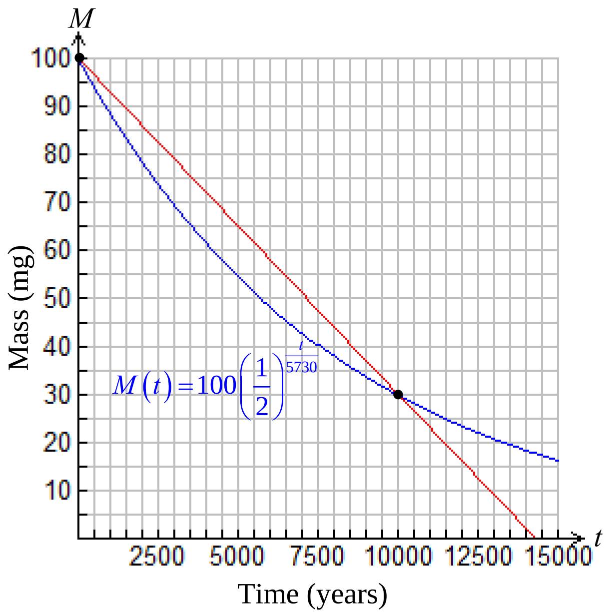
$$= M'(10000)$$

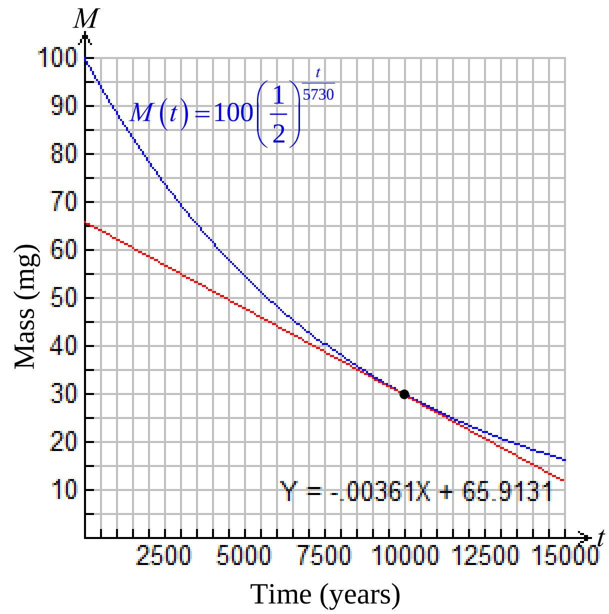
$$\doteq \frac{M(10000.1) - M(9999.9)}{10000.1 - 9999.9}$$

$$= \frac{100\left(\frac{1}{2}\right)^{\frac{10000.1}{5730}} - 100\left(\frac{1}{2}\right)^{\frac{9999.9}{5730}}}{0.2}$$

$$\doteq -0.0036 \text{ mg/year}$$

> Approximate the slope of the tangent line by using a secant line over a tiny interval near $t = 10000$.





The average rate of change of mass with respect to time over the first 10000 years is equal to the slope of the secant line passing through the points $(0,100)$ and $(10000,30)$.

The instantaneous rate of change of mass with respect to time exactly at $t = 10000$ years is equal to the slope of the tangent line at the point $(10000,30)$.

Rates of Change Activity

The following table lists the population of the United States, to the nearest million, from 1900 to 2000 in ten year intervals. (*Source: U.S. Census Bureau*)

Year	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000
Population (millions)	76	92	106	123	132	151	179	203	227	249	281

Use TI-Interactive to do the following:

1. Enter the given data. (Think carefully about the independent and dependent variables. To simplify matters, set $t = 0$ at the year 1900. Then t represents the number of years *since* 1900.)
2. Create a scatter plot. DO NOT CONNECT THE DOTS!
3. Use regression to find a curve of best fit. Superimpose the graph of your function on the scatter plot to see how closely it fits the data given in the table.
4. Find the ***average rate of change*** of the population between 1910 and 1960.

   Given a function $y = f(x)$, we measure the ***average rate of change*** from x_1 to x_2 by finding the quotient of the change in y and the change in x. That is, the ***average rate of change*** from x_1 to x_2 is given by $\dfrac{f(x_1)-f(x_2)}{x_1-x_2}=\dfrac{\Delta y}{\Delta x}$.
5. Estimate the ***instantaneous rate of change*** at the start of 1950 ($t = 50$) by using a ***very small centred interval***. That is, select a time slightly less than $t = 50$ and another time slightly more than $t = 50$. Then calculate the average rate of change of population between the two times that you selected.
6. Estimate the ***instantaneous rate of change*** at the start of 1950 ($t = 50$) by using a ***very small interval*** that begins at $t = 50$ and ends just slightly above $t = 50$.
7. Estimate the ***instantaneous rate of change*** at the start of 1950 ($t = 50$) by using TI-Interactive to create a tangent line exactly at $t = 50$.
8. Compare your estimates from questions 5, 6 and 7. Which is the best estimate? Explain.

Homework

pp. 507-508 4, 5, 6, 8, 9, 10, 11

End Behaviours of Exponential and Logarithmic Functions

What do we mean by End Behaviour?

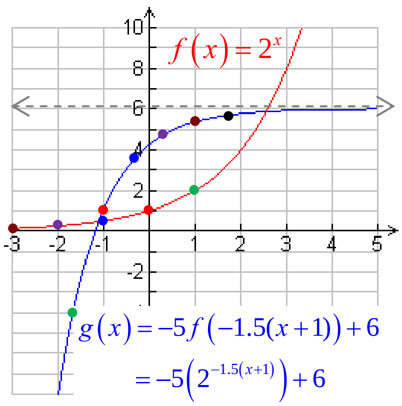
The end behaviour of a function refers to the manner in which it behaves at the extreme ends of its domain.

e.g. $f(x)=2^x$, $D=\mathbb{R}$: End behaviour refers to how f behaves as x gets larger and larger in the positive direction and how it behaves as x gets smaller and smaller (or larger and larger, depending on your perspective) in the negative direction.

e.g. $f(x)=\log_2 x$, $D=\{x\in\mathbb{R}: x>0\}$: End behaviour refers to how f behaves as x gets larger and larger in the positive direction and how it behaves as x gets smaller and smaller in the negative direction (i.e. as x gets closer and closer to zero).

Example – Using Graphs to Understand End Behaviours

Using TI-Interactive or a Similar Graphing Program



Graph of $f(x)=2^x$

1. As x increases (gets larger and larger without bound), we observe that 2^x also increases without bound. We say that as x approaches infinity, 2^x also approaches infinity. Symbolically we write this as follows.

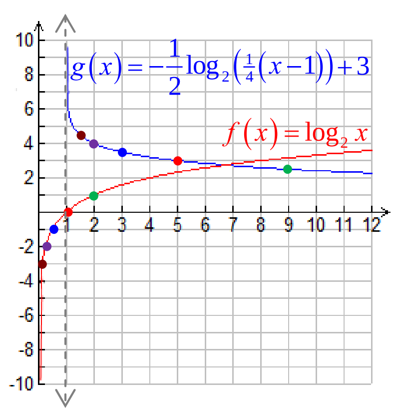
   As $x\to\infty$, $2^x\to\infty$

   Alternatively, we can also write this as shown below.

   As $x\to\infty$, $f(x)\to\infty$ OR As $x\to\infty$, $y\to\infty$

2. If x decreases in the negative direction without bound, then 2^x gets closer and closer to zero. We write this as follows.

   As $x\to-\infty$, $2^x\to 0$



Other Graphs

$f(x)=-5\left(2^{-1.5(x+1)}\right)+6$

$D=\mathbb{R}$

As $x\to\infty$, $f(x)\to 6$

As $x\to-\infty$, $f(x)\to-\infty$

$f(x)=\log_2 x$

$D=\{x\in\mathbb{R}: x>0\}$

As $x\to\infty$, $f(x)\to\infty$

As $x\to 0$, $f(x)\to-\infty$

$g(x)=-\frac{1}{2}\log_2\left(\frac{1}{4}(x-1)\right)+3$

$D=\{x\in\mathbb{R}: x>1\}$

As $x\to\infty$, $g(x)\to-\infty$

As $x\to 1$, $g(x)\to\infty$

Using Geometer's Sketchpad

Geometer's Sketchpad has some interesting features that can help you to determine the end behaviours of functions. Since the features are somewhat complicated, this will be demonstrated in class.

Example – Using Tables of Values to Understand End Behaviours

$y = 2^x$

Notice that as x gets larger, $y = 2^x$ very rapidly increases. There is no limit to how large $y = 2^x$ can grow as x increases. Therefore, as $x \to \infty$, $y \to \infty$.

x	y1(x) 2^x
100	1.27E+30
101	2.54E+30
102	5.07E+30
103	1.01E+31
104	2.03E+31
105	4.06E+31
106	8.11E+31
107	1.62E+32
108	3.25E+32
109	6.49E+32
110	1.30E+33
111	2.60E+33
112	5.19E+33
113	1.04E+34

Notice that as x becomes more and more negative (read the table from bottom to top), $y = 2^x$ gets closer and closer to zero. Therefore, as $x \to -\infty$, $y \to 0$.

x	y1(x) 2^x
-15	3E-005
-14	6E-005
-13	0.00012
-12	0.00024
-11	0.00049
-10	0.00098
-9	0.00195
-8	0.00391
-7	0.00781
-6	0.01563
-5	0.03125
-4	0.0625
-3	0.125
-2	0.25

$y = \log_2 x = \dfrac{\log x}{\log 2}$

Notice that as x gets larger, $y = \log_2 x$ very slowly increases. There is no limit to how large $y = \log_2 x$ can grow as x increases. Therefore, as $x \to \infty$, $y \to \infty$.

x	y1(x) log(x)/log(2)
1.00E+300	996.578
2.00E+300	997.578
3.00E+300	998.163
4.00E+300	998.578
5.00E+300	998.9
6.00E+300	999.163
7.00E+300	999.386
8.00E+300	999.578
9.00E+300	999.748
1.00E+301	999.9
1.10E+301	1000.04
1.20E+301	1000.16
1.30E+301	1000.28
1.40E+301	1000.39

Notice that as x becomes closer and closer to zero (read the table from bottom to top), $y = \log_2 x$ gets more and more negative. Since there is no limit to how "negative" y can become, as $x \to 0$, $y \to -\infty$.

x	y1(x) log(x)/log(2)
0	undef
1.00E-200	-664.386
2.00E-200	-663.386
3.00E-200	-662.801
4.00E-200	-662.386
5.00E-200	-662.064
6.00E-200	-661.801
7.00E-200	-661.578
8.00E-200	-661.386
9.00E-200	-661.216
1.00E-199	-661.064
1.10E-199	-660.926
1.20E-199	-660.801
1.30E-199	-660.685
1.40E-199	-660.578
1.50E-199	-660.479

Exercises

1. Equations of several exponential and logarithmic functions are given below. Use TI-Interactive to

(a) create a graph of each function

(b) create tables of values that help to reveal the end behaviours of each function

Then use your graphs and tables to determine the end behaviours of each function. Do not forget to save your work as it will prove to be a valuable study aid.

(i) $R(t) = 10^{-12}\left(\dfrac{1}{2}\right)^{\frac{t}{5730}}$

(ii) $g(x) = 3\log_2\left(-2(x+1)\right) - 4$

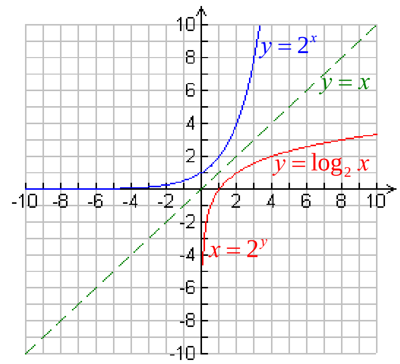
(iii) $h(t) = 5000(1.002)^{3t} - 5$

(iv) $p(u) = -2\log_{\frac{1}{3}}\left(-\tfrac{2}{3}(u-5)\right) + 1$

2. Explain how transformations can affect the end behaviours of exponential and logarithmic functions.

Review of Logarithmic and Exponential Functions

Review of the Properties of Logarithms

Symbolic Representation	*Verbal Representation*	*Proofs, Explanations, Visuals*
1. If $f(x)=\log_a x$ and $g(x)=a^x$ then $f(x)=g^{-1}(x)$ and $g(x)=f^{-1}(x)$. Recall that $a>0$ and that $a\neq 1$.	The functions $f(x)=\log_a x$ and $g(x)=a^x$ are ***inverses of each other***. This means that f does the ***opposite*** of g and g does the ***opposite*** of f. (Formally, if the ordered pair (x,y) belongs to f then (y,x) belongs to g.)	
2. $\log_a x$	The ***exponent*** to which the ***base*** a must be raised to obtain the ***power*** x (i.e. the result x).	
3. $y=\log_a x$	The value y is equal to the ***exponent*** to which the ***base*** a must be raised to obtain the ***power*** x.	
4. $a^{\log_a x}=x$	The ***base*** a is raised to the ***exponent*** to which a must be raised to obtain x. Therefore, the result must be equal to x.	Let $y=\log_a x$. Then, $a^y=x$. But $y=\log_a x$. Therefore, $a^{\log_a x}=x$.
5. $\log_a a^x=x$	The ***exponent***, to which the ***base*** a must be raised to obtain a^x, must be equal to x.	Let $x=\log_a y$. Then, $a^x=y$. Therefore, $\log_a a^x=x$.
6. $\log_a xy=\log_a x+\log_a y$	The logarithm of a ***product*** is equal to the ***sum*** of the logarithms. This law is a direct consequence of the exponent law $a^x a^y=a^{x+y}$. In other words, the ***exponents are added*** when ***two powers with the same base are multiplied***.	Let $x=a^w$ and $y=a^z$. Then, $xy=a^w a^z=a^{w+z}$. Therefore, $\log_a xy=\log_a a^{w+z}=w+z$. But $w=\log_a x$ and $z=\log_a y$. Therefore, $\log_a xy=\log_a x+\log_a y$.
7. $\log_a \dfrac{x}{y}=\log_a x-\log_a y$	The logarithm of a ***quotient*** is equal to the ***difference*** of the logarithms. This law is a direct consequence of the exponent law $\dfrac{a^x}{a^y}=a^{x-y}$. In other words, the ***exponents are subtracted*** when ***two powers with the same base are divided***.	Let $x=a^w$ and $y=a^z$. Then, $\dfrac{x}{y}=\dfrac{a^w}{a^z}=a^{w-z}$. Therefore, $\log_a \dfrac{x}{y}=\log_a a^{w-z}=w-z$. But $w=\log_a x$ and $z=\log_a y$. Therefore, $\log_a \dfrac{x}{y}=\log_a x-\log_a y$.
8. $\log_a x^y=y\log_a x$	The logarithm of a ***power*** is equal to the ***product*** of the exponent and the logarithm of the base. This law is a direct consequence of the exponent law $\left(a^x\right)^y=a^{xy}$. In other words, the ***exponents are multiplied*** when a ***power is raised to an exponent***.	Let $x=a^w$. Then, $x^y=\left(a^w\right)^y=a^{wy}$. Therefore, $\log_a x^y=\log_a a^{wy}=wy$. But $w=\log_a x$. Therefore, $\log_a x^y=y\log_a x$.
9. $\log_b x=\dfrac{\log_a x}{\log_a b}$	The logarithm of a value to the base b is equal to the quotient of the logarithm of the value to the base a and the logarithm of b to the base a. (This is called the ***change of base*** formula. It is used to convert a logarithm expressed in a given base to a more convenient base such as 10.)	Let $y=\log_b x$. Then, $b^y=x$. Therefore, $\log_a b^y=\log_a x$. Thus, $y\log_a b=\log_a x$ and $y=\dfrac{\log_a x}{\log_a b}$. But $y=\log_b x$. Hence, $\log_b x=\dfrac{\log_a x}{\log_a b}$

Review Questions

1. Label the following.

$$y = a^x \qquad\qquad y = \log_a x$$

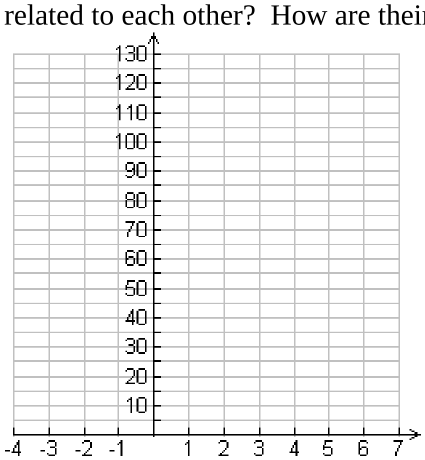
2. Complete the following table.

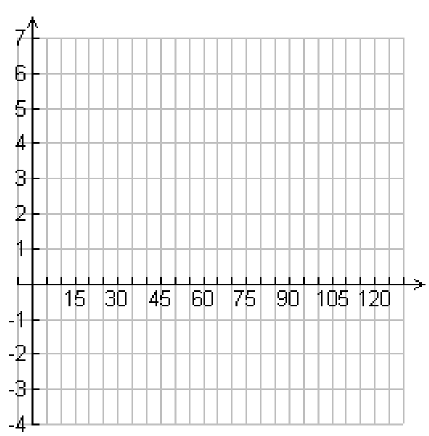
Exponential Form	*Logarithmic Form*
$10^6 = 1000000$	
	$\log_3 \frac{1}{81} = -4$
$y = 6^x$	
	$y = \log_4 x$
$a = b^c$	
	$m = \log_n p$

3. In the equation $y = \log_a x$, what is the ***meaning*** of $\log_a$? Does it represent a number? If not, what does it represent?

4. Explain the ***meaning*** of the expression $\log_n p$.

5. Use the provided grids to sketch the graphs of the ***functions*** $f(x) = 2^x$ and $g(x) = \log_2 x$. How are the two functions related to each other? How are their graphs related





The relationship between f and g is ____________________________

____

____

The relationship between the graphs of f and g is ___________

____

____

6. After consuming 16 energy drinks and 22 hamburgers, Andrew decided that he had enough energy to tackle his math homework. The excessive food and drink made Andrew so hyper that he hurried through his work without giving it much thought. The following are samples of his work. Has Andrew applied valid mathematical reasoning? Explain.

$$\frac{\log_{10} x}{\log_7 x} = \frac{\log_{10} \cancel{x}}{\log_7 \cancel{x}} = \frac{\log_{10}}{\log_7} \qquad \frac{\log_{10} x}{\log_5 x} = \frac{\cancel{\log}_{10} \cancel{x}}{\cancel{\log}_5 \cancel{x}} = \frac{10}{5} = 2$$

7. Suppose that $f(x)=2^x$ and $g(x)=\log_2 x$. Use the provided grids to sketch the graphs of

$$p(x)=-1.5g(-2(x-3))+1 \text{ and } q(x)=\frac{1}{2}f\left(\frac{1}{3}x+\frac{4}{3}\right)+5$$

(a) Write equations of p and q without using the symbols f and g.

$p(x)=$ $\qquad\qquad$ $q(x)=$

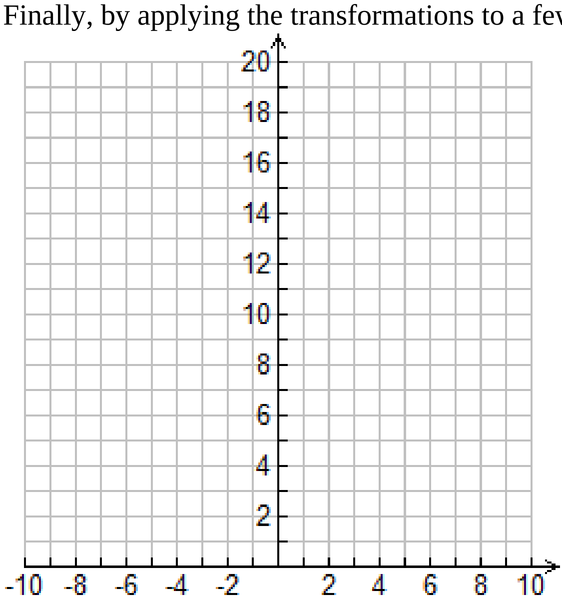
(b) State the transformations required to obtain q from f and p from g.

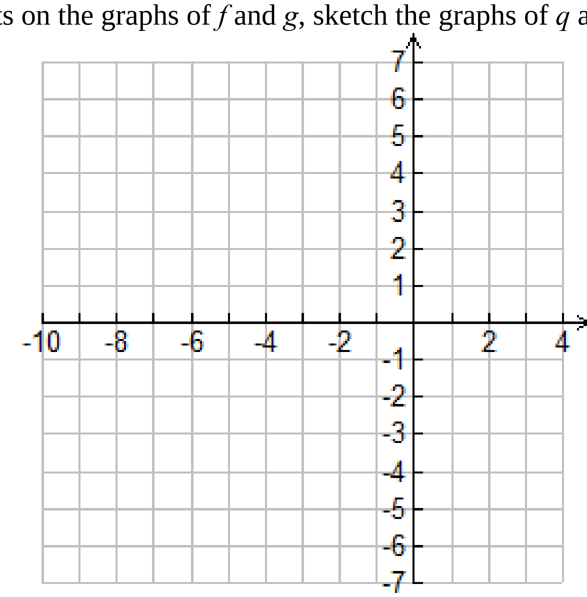
$g \to p$		$f \to q$	
Horizontal	***Vertical***	***Horizontal***	***Vertical***

(c) Now express both transformations in ***mapping notation***.

$g \to p$ $\qquad (x,y)\to$ $\qquad\qquad$ $f \to q$ $\qquad (x,y)\to$

(d) Finally, by applying the transformations to a few key points on the graphs of f and g, sketch the graphs of q and p.





8. Complete the following table. Remember that a counterexample is sufficient to demonstrate that a statement is false. However, a general proof is required to demonstrate that a statement is true. Also recall that when the base of a logarithm is omitted, it is usually assumed to mean "log to the base 10."

Statement	*True or False?*	*Proof, Counterexample or Explanation*
$\log 5b^2 = 2\log 5b$		
$\log 3x^2 = \log 3x + \log x$		
The graphs of $y = \log 3x^2$ and $y = \log 3x + \log x$ are identical.		
To obtain the graph of $y = \log_a \sqrt{x}$, compress the graph of $y = \log_a x$ vertically by a factor of $\frac{1}{2}$.		
$\log_a a^{n+1} = n + 1$		
$a^{3\log_a(5b)} = 125b^3$		
$\log \frac{x}{10} = \frac{\log x}{\log 10} = \frac{\log x}{1} = \log x$		
To obtain the graph of $y = \log \frac{x}{10}$, translate the graph of $y = \log x$ down 1 unit.		

9. Why is it not possible to evaluate the logarithm of zero or a negative number? Give examples to illustrate your answer.

10. Does it make sense to write expressions such as $\log_{-2}(-32)$? Explain.

11. You are given a solution of hydrochloric acid with a pH of 1.7 and are asked to increase its pH by 1.4.

(a) Determine the factor by which you would need to dilute the solution.

(b) If the solution originally had a pH of 2.2 and you were asked to increase its pH by 1.4, would you dilute by the same factor that you calculated in (a)? Explain.

More Review

pp. 510-511 Answer a representative selection of the questions found on these pages.

Practice Test

Multiple Choice
Identify the choice that best completes the statement or answers the question.

____ 1. Which of the following statements is true?
- a. The domain of a transformed logarithmic function is always $\{x \in \mathbf{R}\}$.
- b. Vertical and horizontal translations must be performed before horizontal and vertical stretches/compressions.
- c. A transformed logarithmic function always has a horizontal asymptote.
- d. The vertical asymptote changes when a horizontal translation is applied.

____ 2. Express $27^{\frac{1}{3}} = 3$ in logarithmic form.
- a. $\log_3 27 = 3$
- b. $\log_{\frac{1}{3}} 3 = 27$
- c. $\log_{27} 3 = \frac{1}{3}$
- d. $\log_3 3 = 27$

____ 3. Solve $\log_x 81 = 4$ for x.
- a. 3
- b. 9
- c. 20.25
- d. 324

____ 4. Evaluate $\log_m m^{2n}$.
- a. n
- b. n^2
- c. mn
- d. $2n$

____ 5. The function $S(d) = 300 \log d + 65$ relates $S(d)$, the speed of the wind near the centre of a tornado in miles per hour, to d, the distance that the tornado travels, in miles. If winds near the centre of tornado reach speeds of 400 mph, estimate the distance it can travel.
- a. 130 miles
- b. 13 miles
- c. 13000 miles
- d. 1.1666 miles

____ 6. Evaluate $\log_2 4^5$.
- a. 4
- b. 5
- c. 7
- d. 10

____ 7. Which of the following statements will NOT be true regarding the graphs of $f(x) = \log_3(3x)$, $f(x) = \log_3(9x)$, and $f(x) = \log_3\left(\frac{x}{3}\right)$?
- a. They will all have the same vertical asymptote
- b. The will all have the same x-intercept
- c. They will all curve in the same direction
- d. They will all have the same domain

____ 8. Evaluate $\log_2 \sqrt[3]{64}$.
- a. 2
- b. 3
- c. 8
- d. 16

____ 9. Which does not help to explain why you cannot use the laws of logarithms to expand or simplify $\log_4(3y-4)$?

a. The expression $3y - 4$ cannot be factored.
b. The expression $3y - 4$ is not raised to a power.
c. $3y$ and 4 are neither multiplied together, nor are they divided into each other
d. Each term in the expression does not have the same variable.

____ 10. Solve $5^{2-x} = \frac{1}{125}$ for x.

a. $\frac{5}{3}$
b. -1
c. 5
d. $\frac{7}{3}$

____ 11. Solve $\log(3x+1) = 5$.

a. $\frac{4}{3}$
b. 8
c. 300
d. 33 333

____ 12. Which of the following is NOT a strategy that is often used to solve logarithmic equations?

a. Express the equation in exponential form and solve the resulting exponential equation.
b. Simplify the expressions in the equation by using the laws of logarithms.
c. Represent the sums or differences of logs as single logarithms.
d. Square all logarithmic expressions and solve the resulting quadratic equation.

____ 13. Solve $\log_x 8 = -\frac{1}{2}$.

a. -64
b. -16
c. $\frac{1}{64}$
d. 4

____ 14. Describe the strategy you would use to solve $\log_6 x = \log_6 4 + \log_6 8$.

a. Use the product rule to turn the right side of the equation into a single logarithm. Recognize that the resulting value is equal to x.
b. Express the equation in exponential form, set the exponents equal to each other and solve.
c. Use the fact that the logs have the same base to add the expressions on the right side of the equation together. Express the results in exponential form, set the exponents equal to each other and solve.
d. Use the fact that since both sides of the equations have logarithms with the same base to set the expressions equal to each other and solve.

____ 15. Given the formula for magnitude of an earthquake, $R = \log\left(\frac{a}{T}\right) + B$, determine the how many times larger the amplitude a is in an earthquake with $R = 6.9$, $B = 3.2$, and $T = 1.9s$ compared to one with $R = 5.7$, $B = 2.9$, and $T = 1.6s$

a. 1.2 times as large
b. 1.6 times as large
c. 9.4 times as large
d. 15.8 times as large

____ 16. Solve $\log(x+3) + \log(x) = 1$.

a. $-5, 2$
b. 10
c. 2
d. 7

____ 17. Which of the following does not describe the use of logarithmic scales?
a. When the range of values vary greatly, using a logarithmic scale with powers of 10 makes comparisons between values more manageable.
b. Scales that measure a wide range of values, such as the pH scale, the Richter scale and decibel scales are logarithmic scales.
c. Logarithmic scales more effectively describe and compare vast or large quantities than they do small or microscopic quantities.
d. To compare concentrations modelled with logarithmic scales, determine the quotient of the values being compared.

____ 18. A radioactive substance has a half-life of 7 h. If a sample of the substance has an initial mass of 2000 g, estimate the instantaneous rate of change in mass 1.5 days later.
a. −5.6 g/ h
b. −56 g/h
c. −707 g/h
d. −0.845 g/h

____ 19. Which of the following statements regarding rates of change of exponential and logarithmic functions is NOT true?
a. The average rate of change is not constant for exponential and logarithmic functions.
b. The methods for finding the instantaneous rate of change at a particular point for logarithmic functions are different than those used for finding the instantaneous rate of change at a point for a rational function.
c. The graph of an exponential or logarithmic function can be used to determine when the average rate of change is the least or greatest.
d. The graph of an exponential or logarithmic function can be used to predict the greatest and least instantaneous rates of change and when they occur.

____ 20. Suppose the population of a given town is increasing for a given period of time. What can you tell about its instantaneous rate of change of the population during that period?
a. The instantaneous rate of change continues to get larger during the entire interval.
b. The instantaneous rate of change will be positive at each point in the interval.
c. The instantaneous rate of change may be zero, but cannot be negative.
d. The instantaneous rate of change at any point in the interval will be larger than the average rate of change for the interval.

Short Answer

21. State the domain and range of the transformed function $f(x)=6\log_{10} -2(x-5)$.

22. The parent function $f(x)=\log_{10}x$ is vertically stretched by a factor of 3, reflected in the y-axis, horizontally transformed 4 units to the left and vertically transformed 2.5 units up. What is the equation of the vertical asymptote of the transformed function?

23. State which of the values in the transformed function $f(x)=2\log_{10}\left[-\frac{1}{4}(x-1.5)\right]+5$ must be changed, and what they must be changed to, so that the resulting function has an asymptote at $x = 6$ with the curve of the graph to left of the vertical asymptote.

24. Estimate the value of $\log_3 91$ to two decimals places.

25. Simplify $4^{\log_4 64} + 10^{\log 100}$.

26. Evaluate $\log_5 625 + \log_2 32$.

27. Put the following in order from smallest to largest:
$\log_2 16, \log 100, \log_3 30, \log_5 40, \log_{20} 200$

28. State the product law of logarithms and the exponent law it is related to.

29. Write $4\log 2 + \log 6 - \log 3$ as a single logarithm.

30. Rewrite $x = \log_2\left(\frac{1}{\sqrt{8}}\right)$ in exponential form.

31. If you invested money into an account that pays 9%/a compounded weekly, how many years would it take fo your deposit to double?

32. Solve $10^{x+2} - 10^x = 9900$ for x.

33. Solve $3^{2x} = 7^{3x-1}$ for x. Round your answer to two decimal places.

34. Solve $2^{4x} = \frac{1}{32}$ for x.

35. What are the restrictions on the variable in the equation $\log(3x-5) - \log(x-2) = \log(x^2-5)$?

36. Solve $2\log x - \log 4 = 3\log 4$.

37. Solve $\log_2 x + \log(x-7) = 3$.

38. The population of a town is increasing at a rate of 6.2% per year. The city council believes they will have to add another elementary school when the population reaches 100 000. If there are currently 76 000 people living in the town, how long do they have before the new school will be needed?

39. If $f(x) = a(b+1)^x$ models an exponential growth situation, write an equation that models an exponential decay situation.

40. If the annual cost of a given good rises 2.3% per year for the next 20 years, write an equation to model the approximate cost of the good during any year in the next 20.

Problem

41. Describe two characteristics of the graph of the function $f(x) = \log_{10} x$ that are changed and two that remain the same under the following transformation: a horizontal compression by a factor of 2, a reflection in the y-axis and a vertical translation 3 units up.

42. Without graphing, compare the vertical asymptotes and domains of the functions $f(x) = 3\log_{10}(x-5) + 2$ and $f(x) = 3\log_{10}[-(x+5)] + 2$.

43. The half-life of radium is 1620 years. If a laboratory has 12 grams of radium, how long will it take before it has 8 grams of radium left?

44. Describe the transformations that take the graph of $f(x)=\log_4 x$ to the graph of $g(x)=\log_4 x^3-\log_4 8$. Justify your response algebraically

45. Write $\frac{1}{3}\log_a x+\frac{1}{2}\log_a 2y-\frac{1}{6}\log_a 4z$ as a single logarithm. Assume that all variables represent positive numbers.

46. Explain the difference in the process of solving exponential equations where both sides are written as powers of the same base and solving exponential equations where both sides are not written as powers of the same base.

47. If $\log\left(\frac{x-y}{3}\right)=\frac{1}{2}(\log x+\log y)$, show that $x^2+y^2=11xy$.

48. How many years will it take for a \$400 investment to grow to \$1000 with a interest rate of 12%/a compounded monthly?

49. The function $S(d)=86\log d+112$ relates the speed of the wind, S, in miles per hour, near the centre of a tornado to the distance the tornado travels, d, in miles. Estimate the rate at which the speed of the wind at the centre of the tornado is changing the moment it has travelled its 50th mile.

50. Discuss why exponential equations of the form $f(x)=ab^x$ always have positive instantaneous rates of change when a is positive and b is greater than one, and why they always have negative instantaneous rates of change when a is positive and b is between 0 and 1.

Practice Test Answers

Multiple Choice

1. d	**2.** c	**3.** a	**4.** d	**5.** b
6. d	**7.** b	**8.** a	**9.** d	**10.** c
11. d	**12.** d	**13.** c	**14.** a	**15.** c
16. c	**17.** c	**18.** a	**19.** b	**20.** b

Short Answer

21. $D=\{x\in\mathbb{R}:x<5\}$, $R=\mathbb{R}$

22. $x=-4$

23. Change 1.5 to 6. The curve is already to the left of the vertical asymptote.

24. 4.11

25. 164

26. 9

27. $\log_{20}200$, $\log 100$, $\log_5 40$, $\log_3 30$, $\log_2 16$

28. $\log_a(xy)=\log_a x+\log_a y$, $a^x a^y=a^{x+y}$

29. log 32

30. $2^x=\frac{1}{\sqrt{8}}$

31. 7.7 years

32. 2

33. 0.53

34. $-\frac{5}{4}$

35. $x\ge\sqrt{5}$

36. 16

37. 8

38. 4.6 years

39. $f(x)=a(b-1)^x, 1<b<2$

40. $C(t)=C_0(1.023)^t$

Problems

41. *Solution*

The transformed function $f(x)=\log(-2x)+3$ has the same range as the parent function, since the range of all transformed logarithmic functions have a range of all real numbers. The y-intercept is the vertical asymptote of both the parent and transformed functions.

The transformed function curves to the left, the original function curve to the right. The two functions will have different x-intercepts, the intercepts being reflected over the y-axis.

42. *Solution*

The vertical asymptote helps define the domain of a function. The vertical asymptote changes when a horizontal translation is applied.
The vertical asymptote of $f(x)=3\log_{10}(x-5)+2$ is $x=5$.
The vertical asymptote of $f(x)=3\log_{10}[-(x+5)]+2$ is $x=-5$.
The graph of the first function curves to the right of the asymptote. The domain of $f(x)=3\log_{10}(x-5)+2$ is $\{x\in\mathbf{R}\mid x>5\}$.
Since the expression $(x+5)$ is multiplied by -1, the graph is reflected in the y-axis and curves to the left of the asymptote. The domain of $f(x)=3\log_{10}[-(x+5)]+2$ is $\{x\in\mathbf{R}\mid x<-5\}$.

43. *Solution*

The equation for relating the amount of radium, r, in grams and the amount of time, t, in years is

$$r=12\times\left(\frac{1}{2}\right)^{(t\div1620)}.$$

Substituting 8 in for r gives $8=12\times(\frac{1}{2})^{(t\div1620)}$

$$\frac{2}{3}=\left(\frac{1}{2}\right)^{(t\div1620)}$$

Using guess and check gives $\frac{t}{1620}=0.59$

44. *Solution*

Using the laws of logarithms, $\log_4x^3-\log_48$ can be rewritten as the single logarithm $3\log_4\left(\frac{1}{2}x\right)$ by first applying the quotient law and then the product law of logarithms. Comparing the new form of $g(x)$ to $f(x)$ produces a vertical stretch by a factor of 3 and a horizontal stretch by a factor of 2.

45. *Solution*

$$\frac{1}{3}\log_ax+\frac{1}{2}\log_a2y-\frac{1}{6}\log_a4z$$
$$=\log_a\sqrt[3]{x}+\log_a\sqrt{2y}-\log_a\sqrt[6]{4z}$$
$$=\log_a\frac{\sqrt[3]{x}\sqrt{2y}}{\sqrt[6]{4z}}$$

46. *Solution*

We use the fact that when two exponential expressions with the same base are equal their exponents are equal to set the exponents equal to one another and solve. If $a^m=a^n$, then $m=n$.
When we have two exponential expressions with different bases set equal to each other, we use the fact that taking the log of equal expressions maintains their equality to start the solution process. If $M=N$, then $\log M=\log N$. Given that M and N are powers, we use the power rule to continue the solution process.

47. *Solution*

$$\log\left(\frac{x-y}{3}\right)=\frac{1}{2}(\log x+\log y)$$
$$2\log\left(\frac{x-y}{3}\right)=\log x+\log y$$
$$2\log\left(\frac{x-y}{3}\right)=\log xy$$
$$\log\left(\frac{x-y}{3}\right)^2=\log xy$$
$$\left(\frac{x-y}{3}\right)^2=xy$$
$$\frac{x^2-2xy+y^2}{9}=xy$$
$$x^2-2xy+y^2=9xy$$
$$x^2+y^2=11xy$$

48. *Solution*

$$1000=400(1.01)^{12t}$$
$$2.5=(1.01)^{12t}$$
$$\log 2.5=\log(1.01)^{12t}$$
$$\log 2.5=12t\log 1.01$$
$$0.3979\doteq 12t(0.00432)$$
$$t\doteq 7.7\text{ years}$$

49. *Solution*

$S(d)=86\log d+112$

$S(d)=86\log 49.9+112$
$=258.0366$

$S(d)=86\log 50.1+112$
$=258.1860$

Instantaneous Rate of Change

$$\doteq\frac{258.1860-259.0366}{50.1-49.9}=0.747\text{ mph/mi}$$

50. *Solution*

The graph of $f(x)=ab^x$ is constantly increasing when a is positive and b is greater than 1. The graph rises slowly and then more rapidly, but at no point does its direction change. Similarly, the graph of $f(x)=ab^x$ is constantly decreasing when a is positive and b is between 0 and 1. The graph first decreases rapidly and then much more slowly but, again, at no point does its direction change.

Appendix 1 – Review of Inverses of Functions

Introduction – The Notion of an Inverse

On an intuitive level, the ***inverse of a function*** is simply its ***opposite***. The following table lists some common operations and their opposites.

Some Operations and their Inverses		*Example*		*Observations*		*Conclusion*
$+$	$-$	$3+5=8$	$8-5=3$	$+: 3 \to 8$	$-: 8 \to 3$	The ***inverse of an operation*** "gets you back to where you started." It ***undoes the operation***.
$\times$	$\div$	$3 \times 5 = 15$	$15 \div 5 = 3$	$\times: 3 \to 15$	$\div: 15 \to 3$	
square a number	square root	$5^2 = 25$	$\sqrt{25} = 5$	${}^2: 5 \to 25$	$\sqrt{\ }: 25 \to 5$	

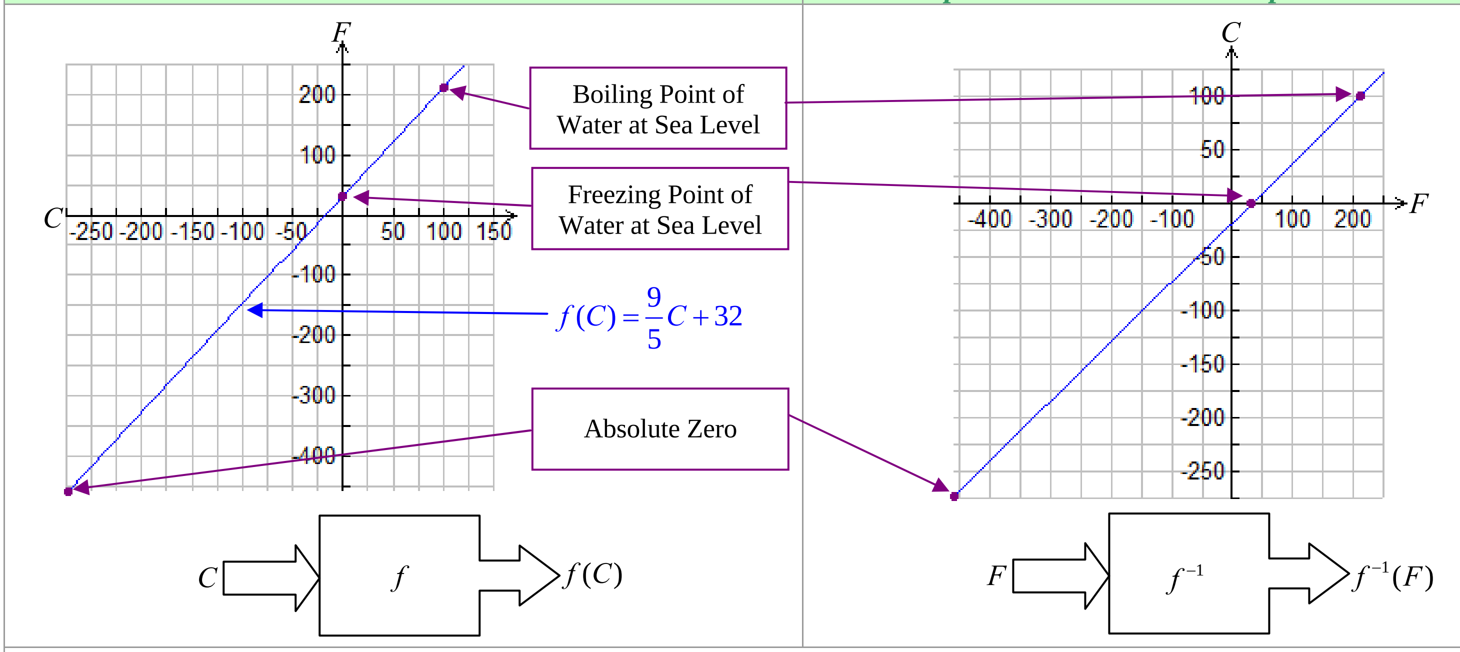
A Classic Example of a Function and its Inverse

With the exception of the United States and perhaps a very small number of other countries, the Celsius scale is used to measure temperature for weather forecasts and many other purposes. In the United States, however, the Fahrenheit scale is still used for most non-scientific purposes. The following shows you how the two scales are related.

C = degrees Celsius, F = degrees Fahrenheit

f = function that "outputs" the Fahrenheit temperature when given the Celsius temperature C as input

A Function f that Converts Celsius Temperatures into Fahrenheit Temperatures	*The Inverse of f, written f^{-1}, converts Fahrenheit Temperatures into Celsius Temperatures*



To obtain the equation of f^{-1}, apply the inverse operations in the reverse order:

Forward: $C \to \times(9/5) \to +32 \to F$ ***Reverse:*** $F \to -32 \to \div(9/5) \to C$ $\therefore C = f^{-1}(F) = \frac{5}{9}(F-32)$

(Note that dividing by 9/5 is the same as multiplying by 5/9.)

Understanding the Inverse of a Function from a Variety of Perspectives

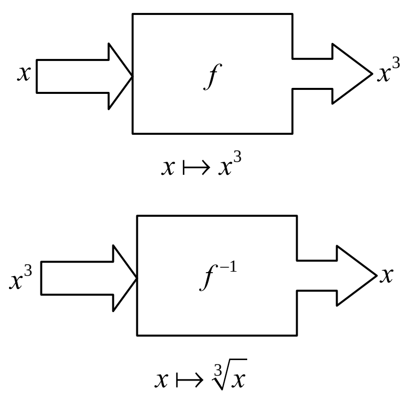
- It is critical that you understand that the ***inverse*** of a function is its ***opposite***. That is, the inverse of a function must ***undo*** whatever the function does.
- The inverse of a function is denoted f^{-1}. It is important to comprehend that the "–1" in this notation is ***not an exponent***. The symbol f^{-1} means "the inverse of the function f," ***not*** $\frac{1}{f}$.
- The notation $x \mapsto f(x)$, called ***mapping notation***, can be used to convey the same idea as a function machine.

Example 1

Does $f(x)=x^3$ have an inverse? If so, what is the inverse function of $f(x)=x^3$?

Solution

By examining the ***various perspectives of functions*** that we have considered, we can easily convince ourselves that $f(x)=x^3$ does have an inverse, namely $f^{-1}(x)=\sqrt[3]{x}$.



$f=\{(-3,-27),(-2,-8),(2,8),(3,27),\ldots\}$

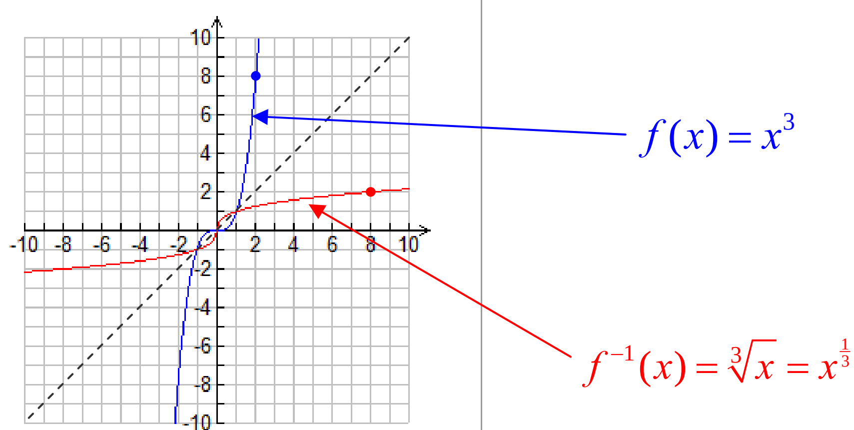
$f^{-1}=\{(-27,-3),(-8,-2),(8,2),(27,3),\ldots\}$

Note that the ordered pairs of f^{-1} are "***reversed***." That is, the *x* and *y*-co-ordinates of the ordered pairs of *f* are interchanged to obtain the ordered pairs of f^{-1}.

x	$f(x)$
−3	−27
−2	−8
−1	−1
0	0
1	1
2	8
3	27
⋮	⋮

x	$f^{-1}(x)$
−27	−3
−8	−2
−1	−1
0	0
1	1
8	2
27	3
⋮	⋮

−3	→	−27	−27	→	−3
−2	→	−8	−8	→	−2
−1	→	−1	−1	→	−1
0	→	0	0	→	0
1	→	1	1	→	1
2	→	8	8	→	2
3	→	27	27	→	3
⋮		⋮	⋮		⋮



Example 2

Does $f(x)=x^2$ have an inverse? If so, what is the inverse function of $f(x)=x^2$?

Solution

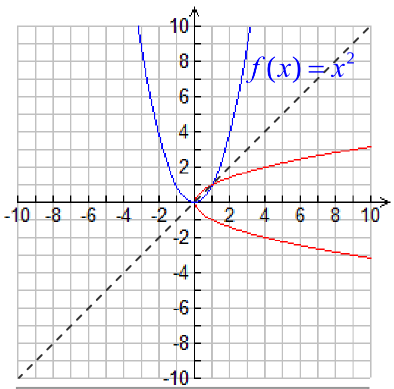
In the last example we learned that the inverse of a function *f* is obtained by interchanging the *x* and *y*-co-ordinates of the ordered pairs of *f*. Let's try this on a few of the ordered pairs of the function $f(x)=x^2$.

$$f=\{(x,y): x\in\mathbb{R}, y\in\mathbb{R}, y=x^2\}=\{\ldots,(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9),\ldots\}$$

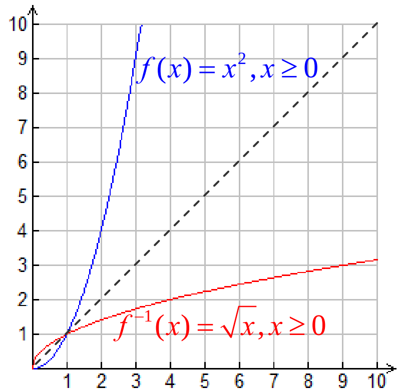
The inverse of *f* ***should be*** the following relation:

$$\{(x,y):(y,x)\in f\}=\{\ldots,(9,-3),(4-2),(1,-1),(0,0),(1,1),(4,2),(9,3),\ldots\}$$

It is readily apparent that there is something wrong, however. This relation is ***not*** a function. Therefore, $f(x)=x^2$ ***does not*** have an inverse function ***unless we restrict its domain***.

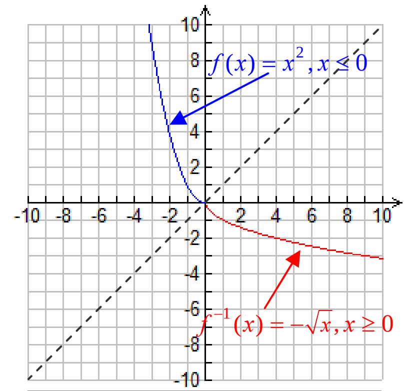


The relation obtained by interchanging the x and y co-ordinates fails the vertical line test. It is ***not*** a function.



The inverse of f is a function if the domain of f is restricted to the set of all real numbers $x\geq 0$.

Clearly, $f^{-1}(x)=\sqrt{x}$.



Alternatively, the domain of f can be restricted to the set of all real numbers $x\leq 0$. In this case, the inverse is $f^{-1}(x)=-\sqrt{x}$.

Observations

1. $f(x)=x^3$ is ***one-to-one*** and has inverse function $f^{-1}(x)=\sqrt[3]{x}$
2. $f(x)=x^2$ is ***many-to-one***; the inverse of f is not a function unless its domain is restricted to a "piece" of f that is one-to-one (e.g. $x\ge 0$ or $x\le 0$)
3. The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

Summary

We can extend the results of the above examples to all functions.

1. The inverse function f^{-1} of a function f exists ***if and only*** f is ***one-to-one***. (Technically, f must be a bijection. For our purposes, however, it will suffice to require that f be one-to-one.)
2. The inverse relation of a ***many-to-one*** function ***is not a function***. However, if the domain of a many-to-one function is restricted in such a way that it is one-to-one for a certain set of "x-values," then the inverse relation defined for this "piece" ***is a*** function.
3. Geometrically, the inverse function f^{-1} of a function f is the ***reflection of f in the line*** $y=x$.

APPENDIX 2 – ONTARIO MINISTRY OF EDUCATION GUIDELINES

A. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of the relationship between exponential expressions and logarithmic expressions, evaluate logarithms, and apply the laws of logarithms to simplify numeric expressions;
2. identify and describe some key features of the graphs of logarithmic functions, make connections among the numeric, graphical, and algebraic representations of logarithmic functions, and solve related problems graphically;
3. solve exponential and simple logarithmic equations in one variable algebraically, including those in problems arising from real-world applications.

SPECIFIC EXPECTATIONS

1. Evaluating Logarithmic Expressions

By the end of this course, students will:

1.1 recognize the logarithm of a number to a given base as the exponent to which the base must be raised to get the number, recognize the operation of finding the logarithm to be the inverse operation (i.e., the undoing or reversing) of exponentiation, and evaluate simple logarithmic expressions

Sample problem: Why is it not possible to determine $\log_{10}(-3)$ or $\log_2 0$? Explain your reasoning.

1.2 determine, with technology, the approximate logarithm of a number to any base, including base 10 (e.g., by reasoning that $\log_3 29$ is between 3 and 4 and using systematic trial to determine that $\log_3 29$ is approximately 3.07)

1.3 make connections between related logarithmic and exponential equations (e.g., $\log_5 125 = 3$ can also be expressed as $5^3 = 125$), and solve simple exponential equations by rewriting them in logarithmic form (e.g., solving $3^x = 10$ by rewriting the equation as $\log_3 10 = x$)

1.4 make connections between the laws of exponents and the laws of logarithms [e.g., use the statement $10^{a+b} = 10^a 10^b$ to deduce that $\log_{10} x + \log_{10} y = \log_{10}(xy)$], verify the laws of logarithms with or without technology (e.g., use patterning to verify the quotient law for logarithms by evaluating expressions such as $\log_{10} 1000 - \log_{10} 100$ and then rewriting the answer as a logarithmic term to the same base), and use the laws of logarithms to simplify and evaluate numerical expressions

2. Connecting Graphs and Equations of Logarithmic Functions

By the end of this course, students will:

2.1 determine, through investigation with technology (e.g., graphing calculator, spreadsheet) and without technology, key features (i.e., vertical and horizontal asymptotes, domain and range, intercepts, increasing/decreasing behaviour) of the graphs of logarithmic functions of the form $f(x) = \log_b x$, and make connections between the algebraic and graphical representations of these logarithmic functions

Sample problem: Compare the key features of the graphs of $f(x) = \log_2 x$, $g(x) = \log_4 x$, and $h(x) = \log_8 x$ using graphing technology.

2.2 recognize the relationship between an exponential function and the corresponding logarithmic function to be that of a function and its inverse, deduce that the graph of a logarithmic function is the reflection of the graph of the corresponding exponential function in the line $y = x$, and verify the deduction using technology

Sample problem: Give examples to show that the inverse of a function is not necessarily a function. Use the key features of the graphs of logarithmic and exponential functions to give reasons why the inverse of an exponential function is a function.

2.3 determine, through investigation using technology, the roles of the parameters d and c in functions of the form $y = \log_{10}(x - d) + c$ and the roles of the parameters a and k in functions of the form $y = a\log_{10}(kx)$, and describe these roles in terms of transformations on the graph of $f(x) = \log_{10}x$ (i.e., vertical and horizontal translations; reflections in the axes; vertical and horizontal stretches and compressions to and from the x- and y-axes)

Sample problem: Investigate the graphs of $f(x) = \log_{10}(x) + c$, $f(x) = \log_{10}(x - d)$, $f(x) = a\log_{10}x$, and $f(x) = \log_{10}(kx)$ for various values of c, d, a, and k, using technology, describe the effects of changing these parameters in terms of transformations, and make connections to the transformations of other functions such as polynomial functions, exponential functions, and trigonometric functions.

2.4 pose problems based on real-world applications of exponential and logarithmic functions (e.g., exponential growth and decay, the Richter scale, the pH scale, the decibel scale), and solve these and other such problems by using a given graph or a graph generated with technology from a table of values or from its equation

Sample problem: The pH or acidity of a solution is given by the equation $\text{pH} = -\log C$, where C is the concentration of $[\text{H}^+]$ ions in multiples of $M = 1$ mol/L. Use graphing software to graph this function. What is the change in pH if the solution is diluted from a concentration of $0.1M$ to a concentration of $0.01M$? From $0.001M$ to $0.0001M$? Describe the change in pH when the concentration of any acidic solution is reduced to $\frac{1}{10}$ of its original concentration. Rearrange the given equation to determine concentration as a function of pH.

3. Solving Exponential and Logarithmic Equations

By the end of this course, students will:

3.1 recognize equivalent algebraic expressions involving logarithms and exponents, and simplify expressions of these types

Sample problem: Sketch the graphs of $f(x) = \log_{10}(100x)$ and $g(x) = 2 + \log_{10}x$, compare the graphs, and explain your findings algebraically.

3.2 solve exponential equations in one variable by determining a common base (e.g., solve $4^x = 8^{x+3}$ by expressing each side as a power of 2) and by using logarithms (e.g., solve $4^x = 8^{x+3}$ by taking the logarithm base 2 of both sides), recognizing that logarithms base 10 are commonly used (e.g., solving $3^x = 7$ by taking the logarithm base 10 of both sides)

Sample problem: Solve $300(1.05)^n = 600$ and $2^{x+2} - 2^x = 12$ either by finding a common base or by taking logarithms, and explain your choice of method in each case.

3.3 solve simple logarithmic equations in one variable algebraically [e.g., $\log_3(5x + 6) = 2$, $\log_{10}(x + 1) = 1$]

3.4 solve problems involving exponential and logarithmic equations algebraically, including problems arising from real-world applications

Sample problem: The pH or acidity of a solution is given by the equation $\text{pH} = -\log C$, where C is the concentration of $[\text{H}^+]$ ions in multiples of $M = 1$ mol/L. You are given a solution of hydrochloric acid with a pH of 1.7 and asked to increase the pH of the solution by 1.4. Determine how much you must dilute the solution. Does your answer differ if you start with a pH of 2.2?