Unit 2 – Trigonometric Functions

Essential Concepts of Trigonometry

Introduction

To a great extent, this unit is just an extension of the trigonometry that you studied in grades 10 and 11. Given below is a list of the additional topics and concepts that are covered in this course.

- For the most part, angles will be measured in ***radians*** instead of degrees.
- Angles of rotation will be extended beyond the range $0° \le \theta \le 360°$
- The reciprocal trigonometric functions csc, sec and cot will be studied in much greater depth.
- Trigonometric identities will be studied in much greater depth.

What is Trigonometry?

Trigonometry (Greek *trigōnon* "triangle" + *metron* "measure") is a branch of mathematics that deals with the ***relationships among the interior angles and side lengths of triangles***, as well as with the study of trigonometric functions. Although the word "trigonometry" emerged in the mathematical literature only about 500 years ago, the origins of the subject can be traced back more than 4000 years to the ancient civilizations of Egypt, Mesopotamia and the Indus Valley. Trigonometry has evolved into its present form through important contributions made by, among others, the Greek, Chinese, Indian, Sinhalese, Persian and European civilizations.

Why Triangles?

Triangles are the basic building blocks from which any shape (with straight boundaries) can be constructed. A square, pentagon or any other polygon can be divided into triangles, for instance, using straight lines that radiate from one vertex to all the others.

Examples of Problems that can be solved using Trigonometry

☺ How tall is Mount Everest? How tall is the CN Tower?
☺ What is the distance from the Earth to the sun? How far is the Alpha Centauri star system from the Earth?
☺ What is the diameter of Mars? What is the diameter of the sun?
☺ At what times of the day will the tide come in?

General Applications of Trigonometric Functions

Trigonometry is one of the most widely applied branches of mathematics. A small sample of its myriad uses is given below. ***The power of trigonometry is that it relates angles to distances.*** Since it is much easier in general to measure angles than it is to measure distances, trigonometric relationships give us a method to calculate distances that are otherwise inaccessible.

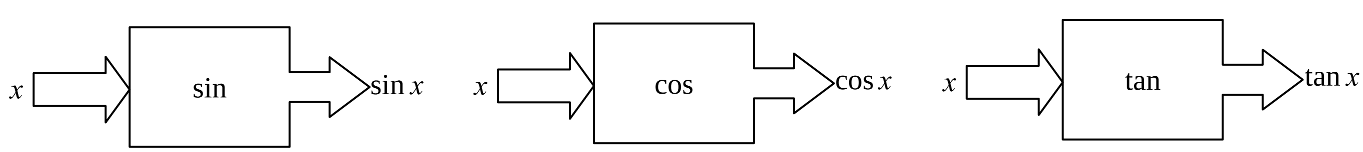
Application	*Examples*
Modelling of ***cyclic*** (periodic) processes	Orbits, Hours of Daylight, Tides
Measurement	Navigation, Engineering, Construction, Surveying
Electronics	Circuit Analysis (Modelling of Voltage Versus Time in AC Circuits, Fourier Analysis, etc)

Extremely Important Note on the Notation of Trigonometry

Input (Argument)

Name of Function $\rightarrow$ $$\sin x = \sin(x)$$ $\leftarrow$ Parentheses can be used but are usually omitted.

Output (Function Value)

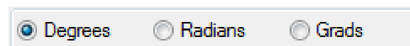


Radian Measure

Summary of Various Units for Measuring Angles

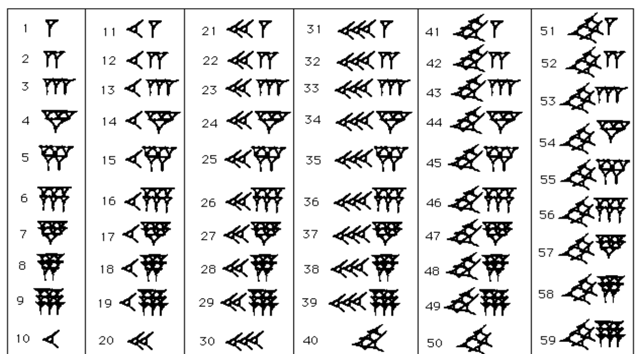
Degree Measure	*Radian Measure*	*Grad (also known as Gon, Grade, Gradian) Measure*
• 360 degrees in one full revolution • Very well suited and widely used for practical applications because one degree is a fairly small unit • For greater precision, one degree can be subdivided further into ***minutes*** (') and ***seconds*** (") • There are 60 *minutes* or *arcminutes* in a degree, 60 *seconds* or *arcseconds* in a minute **e.g.** Central Peel's location 43°, 41', 49" N (or 43.6969°) 79°, 44', 59" W (or –79.7496°)	• 2π radians in one full revolution • Not well suited to practical applications because one radian is a rather large unit • Very well suited to mathematical theory because the radian turns out to be dimensionless. As a result, trigonometric equations are greatly simplified, especially those for derivatives and integrals in calculus.	• 400 grads in one full revolution • Very well suited for practical applications because one grad is a fairly small unit • The creation of the grad was an attempt to bring angle measure in line with the metric system (i.e. based on ten) • This idea never gained much momentum but most scientific calculators support the grad

Calculator Use

Scientific Calculator	*Windows Calculator*
 This key is used to switch among degrees, radians and grads mode. Whenever you are working with angles, make sure that your calculator is in the correct mode.	

Why 360 Degrees in one Full Revolution?

The number 360 as the number of "degrees" in a circle, and hence the unit of a degree as a sub-arc of $^1/_{360}$ of the circle, was probably adopted because it approximates the number of days in a year. Its use is thought to originate from the methods of the ancient Babylonians, who used a sexagesimal number system (a number system with ***sixty*** as the base). Ancient astronomers noticed that the stars in the sky, which circle the celestial pole every day, seem to advance in that circle by approximately one-360th of a circle, that is, one degree, each day. Primitive calendars, such as the Persian Calendar used 360 days for a year. Its application to measuring angles in geometry can possibly be traced to Thales of Miletus, who popularized geometry among the Greeks and lived in Anatolia (modern western Turkey) among people who had dealings with Egypt and Babylon.



The 59 symbols used by the Babylonians. These symbols are built from the two basic symbols and , representing one and ten respectively.

Another motivation for choosing the number 360 is that it is readily divisible: 360 has 24 divisors (including 1 and 360), including every number from 1 to 10 except 7. For the number of degrees in a circle to be divisible by every number from 1 to 10, there would need to be 2520 in a circle, which is a much less convenient number.

Divisors of 360: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360

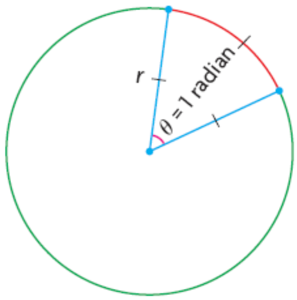
The division of the circle into 360 parts also occurred in ancient India, as evidenced in the Rig Veda:

> Twelve spokes, one wheel, navels three.
> Who can comprehend this?
> On it are placed together
> three hundred and sixty like pegs.
> They shake not in the least.
> (Dirghatama, Rig Veda 1.164.48)

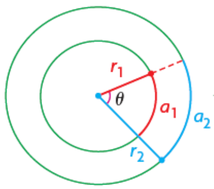
Definition of the Radian

radian

the size of an angle that is subtended at the centre of a circle by an arc with a length equal to the radius of the circle; both the arc length and the radius are measured in units of length (such as centimetres) and, as a result, the angle is a real number without any units



1 radian is defined as the angle subtended by an arc length, *l*, equal to the radius, *r*. It appears as though 1 radian should be a little less than 60°, since the sector formed resembles an equilateral triangle, with one side that is curved slightly.



It is important to note that the size of an angle in radians is not affected by the size of the circle. The diagram shows that a_1 and a_2 subtend the same angle θ, so $\theta = \frac{a_1}{r_1} = \frac{a_2}{r_2}$.

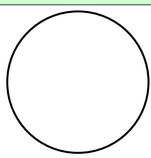
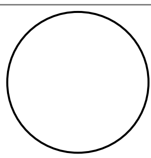
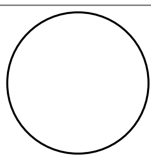
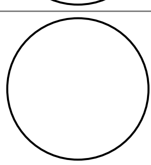
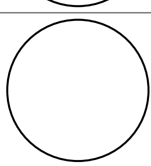
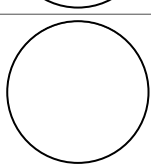
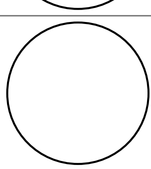
Verb: subtend sub'tend

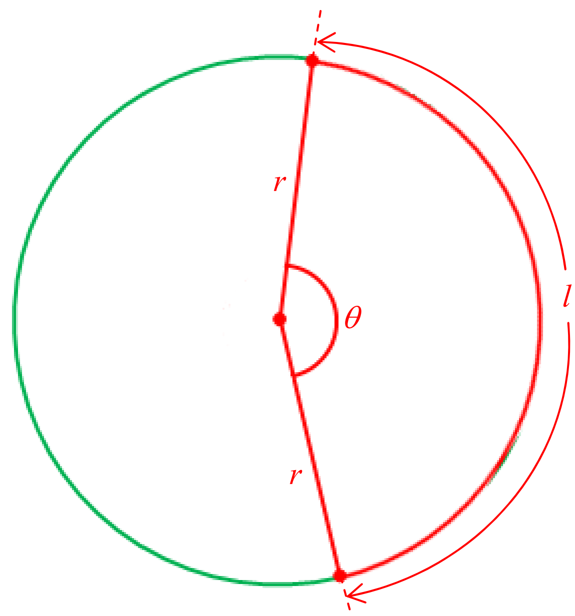
1. Be opposite to; of angles and sides, in geometry

> ***Important Note***
> Whenever the units of angle measure are not specified, the units are ***assumed to be radians***.

Investigation – The Relationship among θ, r and l

When θ is measured in radians, there is a very simple equation that relates r (the radius of the circle), θ (the angle at the centre of the circle) and l (the length of the arc that subtends the angle θ). The purpose of this investigation is to discover this relationship. Complete the table below and then answer the question at the bottom of the page.

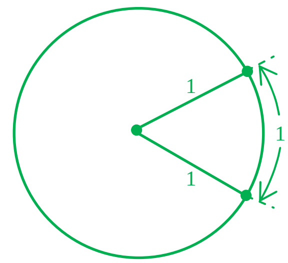
Diagram	*r*	*θ*	*l*
	1	1	
	1	2	
	1	3	
	2	1	
	2	2	
	2	3	
	3	1.5	

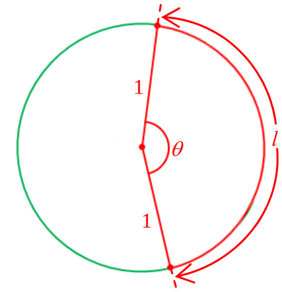


Now that you have completed the table, write an equation that relates r, l and θ.

A more Analytical Approach to finding the Relationship among θ, r and l

How many Radians are there in One Full Revolution?





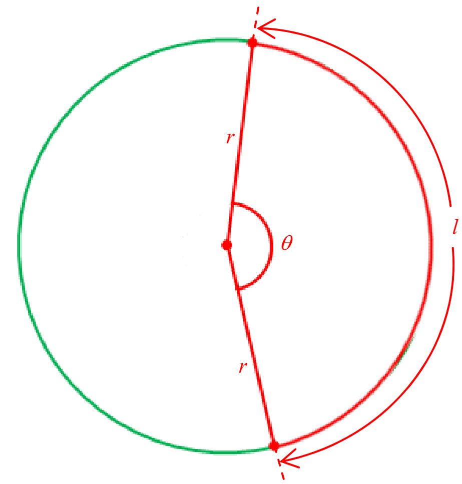
First, we need to establish the number of radians in one full revolution. We can accomplish this by considering a ***unit circle*** (a circle of radius 1). It is easy to see that for such a circle, $l=\theta$. For example, if $\theta=1$, then by the definition of a radian, $l=1$. Similarly, if $\theta=2$, then $l=2$. For an arc whose length is equal to the circumference of the circle, $l=C=2\pi r=2\pi(1)=2\pi$. Since $l=\theta$, for one complete revolution, $\theta=2\pi$.

Therefore, one full revolution = 2π radians.

How is θ related to the "Amount of Rotation?"

It should be fairly obvious that the angle θ determines the ***fraction*** of one full revolution. For instance, consider the examples in the table given below.

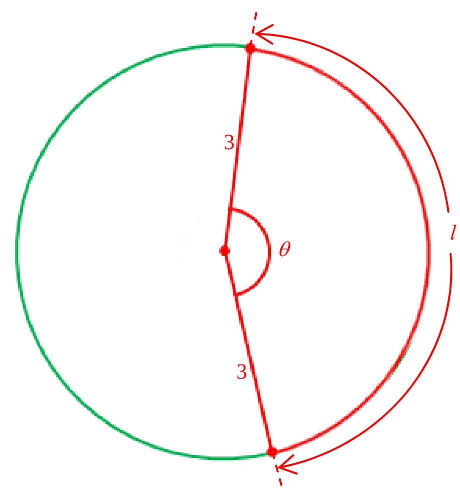
θ (rad)	*Fraction of One Revolution*	*How Fraction is Calculated*
$\frac{\pi}{4}$	$\frac{1}{8}$	$\frac{\theta}{2\pi}=\left(\frac{\pi}{4}\right)\Big/(2\pi)=\left(\frac{\pi}{4}\right)\left(\frac{1}{2\pi}\right)=\frac{1}{8}$
$\frac{\pi}{2}$	$\frac{1}{4}$	$\frac{\theta}{2\pi}=\left(\frac{\pi}{2}\right)\Big/(2\pi)=\left(\frac{\pi}{2}\right)\left(\frac{1}{2\pi}\right)=\frac{1}{4}$
π	$\frac{1}{2}$	$\frac{\theta}{2\pi}=(\pi)\Big/(2\pi)=\left(\frac{\pi}{2\pi}\right)=\frac{1}{2}$
$\frac{3\pi}{2}$	$\frac{3}{4}$	$\frac{\theta}{2\pi}=\left(\frac{3\pi}{2}\right)\Big/(2\pi)=\left(\frac{3\pi}{2}\right)\left(\frac{1}{2\pi}\right)=\frac{3}{4}$
θ	$\frac{\theta}{2\pi}$	$\frac{\theta}{2\pi}=\frac{\text{angle of rotation}}{\text{angle for one full rotation}}$



How is l related to the Circumference of a Circle?

It should also be obvious that l determines the ***fraction*** of the circumference of a circle. Consider the following table for a circle with r = 3 units and $C=2\pi r=2\pi(3)=6\pi$ units.

θ (rad)	*l*	*Fraction of the Circumference*
$\frac{\pi}{4}$	$\frac{6\pi}{8}=\frac{3\pi}{4}=3\theta$	$\frac{l}{C}=\left(\frac{3\pi}{4}\right)\Big/(6\pi)=\left(\frac{3\pi}{4}\right)\left(\frac{1}{6\pi}\right)=\frac{1}{8}$
$\frac{\pi}{2}$	$\frac{6\pi}{4}=\frac{3\pi}{2}=3\theta$	$\frac{l}{C}=\left(\frac{3\pi}{2}\right)\Big/(6\pi)=\left(\frac{3\pi}{2}\right)\left(\frac{1}{6\pi}\right)=\frac{1}{4}$
π	$\frac{6\pi}{2}=3\pi=3\theta$	$\frac{l}{C}=(3\pi)\Big/(6\pi)=\left(\frac{3\pi}{6\pi}\right)=\frac{1}{2}$
$\frac{3\pi}{2}$	$3\left(\frac{3\pi}{2}\right)=\frac{9\pi}{2}=3\theta$	$\frac{l}{C}=\left(\frac{9\pi}{2}\right)\Big/(6\pi)=\left(\frac{9\pi}{2}\right)\left(\frac{1}{6\pi}\right)=\frac{3}{4}$
θ	3θ	$\frac{l}{C}=\frac{3\theta}{6\pi}=\frac{\theta}{2\pi}$



An important observation to make at this point is that $\frac{l}{C}=\frac{\theta}{2\pi}$. That is, the ratio of the length of the arc to the circumference of the circle is equal to the ratio of the angle subtended by the arc to the number of radians in one full revolution. Now, by recalling that $C=2\pi r$, we can write the above proportion as

$$\frac{l}{2\pi r}=\frac{\theta}{2\pi}$$

By multiplying both sides by $2\pi r$, we obtain $l=r\theta$.

Summary: Calculating the Length of an Arc

Let r represent the radius of a circle and θ represent the measure of an angle at the centre of the circle. If θ is subtended by an arc whose length is l, then

$$l = r\theta$$

$C = 2\pi r$ is a Special Case of $l = r\theta$

Note that the equation $l = r\theta$ is a generalized form of $C = 2\pi r$. In the case of $C = 2\pi r$, $l = C$ and $\theta = 2\pi$.

Converting between Radians and Degrees

We know that one full revolution is equal to 2π radians ***and*** that one full revolution is equal to 360°.

$$\therefore 2\pi \text{ rad} = 360°$$

$$\therefore \pi \text{ rad} = 180°$$

By remembering that π rad = 180°, you will be able to convert easily between radians and degrees

Radians to Degrees	*Degrees to Radians*
$\pi \text{ rad} = 180°$ $\therefore 1 \text{ rad} = \dfrac{180°}{\pi}$ $\therefore x \text{ rad} = \dfrac{x(180°)}{\pi}$	$180° = \pi \text{ rad}$ $\therefore 1° = \dfrac{\pi}{180} \text{rad}$ $\therefore x° = \dfrac{x\pi}{180} \text{rad}$

Examples

1. Convert 6 radians to degrees.

Solution

$$\begin{aligned} \pi \text{ rad} &= 180° \\ \therefore 1 \text{ rad} &= \frac{180°}{\pi} \\ \therefore 6 \text{ rad} &= \frac{6(180°)}{\pi} \\ &= \frac{1080°}{\pi} \\ &\doteq 343.8° \end{aligned}$$

We can be confident that this answer is correct because 6 radians is just short of one full revolution as is 343.8°. //

2. Convert 972° to radians.

Solution

$$\begin{aligned} 180° &= \pi \text{ rad} \\ \therefore 1° &= \frac{\pi}{180} \text{rad} \\ \therefore 972° &= \frac{972\pi}{180} \text{rad} \\ &= \frac{27\pi}{5} \text{rad} \\ &\doteq 16.96 \text{ rad} \end{aligned}$$

We can be confident that this answer is correct because 972° falls short of 3 full revolutions by about 100°, as does 16.96 rad. (3 full revs $\doteq 18.85$ rad) //

Special Angles

As shown in the following table, it is very easy to convert between degrees and radians for certain special angles.

Angle in Degrees	30°	45°	60°	90°
Angle in Radians	$\dfrac{180°}{6} = \dfrac{\pi}{6}$	$\dfrac{180°}{4} = \dfrac{\pi}{4}$	$\dfrac{180°}{3} = \dfrac{\pi}{3}$	$\dfrac{180°}{2} = \dfrac{\pi}{2}$

In addition, it is also very easy to convert between radians and degrees for multiples of the special angles. Examples are shown below.

$$150° = 5(30°) = \frac{5\pi}{6} \qquad 240° = 4(60°) = \frac{4\pi}{3} \qquad 315° = 7(45°) = \frac{7\pi}{4} \qquad 270° = 3(90°) = \frac{3\pi}{2}$$

Radians are Dimensionless

Since $l=\theta r$, it follows that $\theta=\frac{l}{r}$. Because both l and r are measured in units of distance, the units “divide out.”

$$\theta=\frac{l}{r}$$

Both are measured in units of distance. Therefore, the units “divide out” and θ turns out to be dimensionless.

This means that when θ is measured in radians, it is a dimensionless number, that is, a pure real number. Because of this, the radian is very well suited to theoretical purposes since functions operate on real numbers and not on angles measured in degrees or any other unit.

Angular Frequency (Angular Speed)

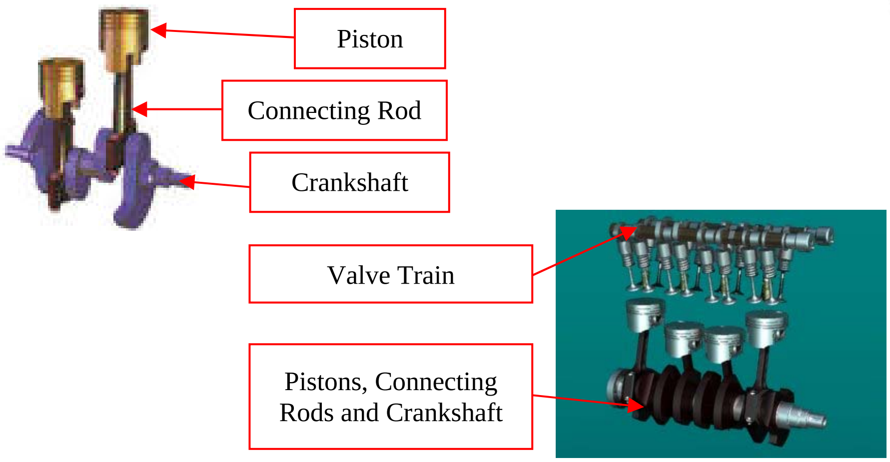
Angular frequency or ***angular speed*** is the rate at which an object rotates. The Greek letter ω (lowercase omega) is often used to denote angular frequency.

Example 1

The RPM gauge on a car measures the speed at which the crankshaft (see pictures below) rotates in ***revolutions per minute***. While Victor was driving through a school zone, his RPM gauge read 9500 RPM. Convert this value to radians/second.

Solution

$$\begin{aligned}\omega &= 9500\text{ RPM}\\ &=\frac{9500\text{ RPM}}{60\text{ s/min}}\\ &=\frac{475}{3}\text{ rot/s}\\ &=\left(\frac{475}{3}\text{ rot/s}\right)(2\pi\text{ rad/rot})\\ &=\frac{950\pi}{3}\text{ rad/s}\\ &\doteq 994.8\text{ rad/s}\end{aligned}$$



The angular frequency of Victor’s crankshaft is about 994.8 rad/s. //

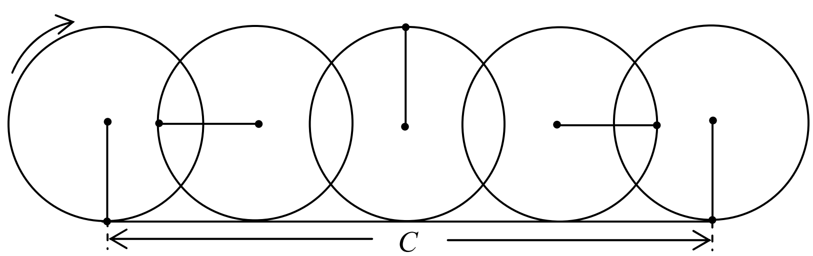
Example 2

While Victor was driving his turbo-charged Volvo on the 410, the wheels of his car were spinning with an angular frequency of 200 rad/s. If the radius of each wheel is 40 cm, how far will Victor’s Volvo travel in 15 minutes?

Solution

The ***crux*** of this problem is to make the ***connection*** between the ***circumference*** of the wheel and the ***distance travelled***.

As shown in the diagram at the right, the distance travelled after ***one rotation*** of the wheel ***is equal to*** the ***circumference*** of the wheel.



$C=2\pi r=2\pi(0.4\text{ m})=0.8\pi\text{ m}$

In one second, the wheel moves through 200 rad. Therefore, the distance travelled in one second can be calculated easily using the relation $l=r\theta$:

$l=r\theta=(0.4\text{ m})(200\text{ rad})=80\text{ m}$

Therefore, the car’s speed is 80 m/s.

Consequently, in 15 minutes, Victor’s car travels (80 m/s)(15 min)(60 s/min) = 72000 m = 72 km. //

Example 3

The London Eye Ferris wheel has a diameter of 135 m and completes one revolution in 30 min.

a) Determine the angular velocity, ω, in radians per second.

b) How far has a rider travelled at 10 min into the ride?

Solution

a) $30 \text{ min} = 30 \cancel{\text{min}} \times \dfrac{60 \text{ s}}{1 \cancel{\text{min}}}$

$= 1800 \text{ s}$

Since the question asks for angular velocity in radians per second, convert the time to seconds.

$$\text{Angular velocity, } \omega = \frac{2\pi}{1800} \text{ radians/s}$$

$$= \frac{\pi}{900} \text{ radians/s}$$

$$\doteq 0.003\ 49 \text{ radians/s}$$

Each revolution of the Ferris wheel represents an angular motion through an angle of 2π radians. Therefore, the Ferris wheel moves through 2π radians every 30 min.

b) $\text{Radius, } r = \dfrac{135}{2} \text{ m}$

$= 67.5 \text{ m}$

The rider moves in a circular motion on the edge of a circle that has a radius of 67.5 m.

$$\text{Number of revolutions, } n = \frac{10 \cancel{\text{min}}}{30 \cancel{\text{min}}}$$

$$= \frac{1}{3} \text{ revolution}$$

The wheel turns through one revolution every 30 min, so the rider has gone through $\frac{1}{3}$ of a revolution at 10 min.

$$\text{Distance travelled, } d = \frac{1}{3}(2\pi \times 67.5 \text{ m})$$

The rider travels $\frac{1}{3}$ of the circumference in 10 min.

$$= 45\pi \text{ m}$$

$$\doteq 141.4 \text{ m}$$

//

Homework

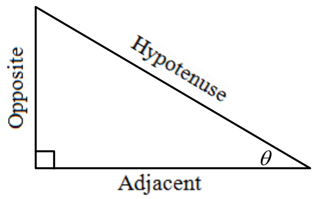
pp. 320 – 322

1 → 10, 12, 14, 15, 16

Radian Measure and Angles on the Cartesian Plane

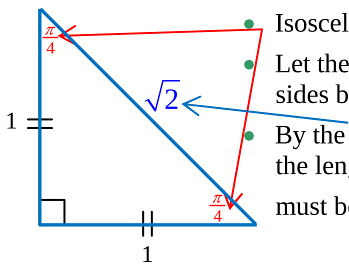
Trigonometry of Right Triangles – Trigonometric Ratios of Acute Angles

Right triangles can be used to define the ***trigonometric ratios*** of ***acute*** angles (angles that measure less than 90º).

	$\sin\theta=\frac{\text{opposite}}{\text{hypotenuse}}$ $\quad \csc\theta=\frac{\text{hypotenuse}}{\text{opposite}}=\frac{1}{\sin\theta}$ $\cos\theta=\frac{\text{adjacent}}{\text{hypotenuse}}$ $\quad \sec\theta=\frac{\text{hypotenuse}}{\text{adjacent}}=\frac{1}{\cos\theta}$ $\tan\theta=\frac{\text{opposite}}{\text{adjacent}}$ $\quad \cot\theta=\frac{\text{adjacent}}{\text{opposite}}=\frac{1}{\tan\theta}$	SOH CAH TOA **S**hout **O**ut **H**ey” “**C**anadians **A**re **H**ot” “**T**ight **O**iled **A**bs” Have fun by creating your own mnemonic! CHO SHA COTAO

The Special Triangles – Trigonometric Ratios of Special Angles

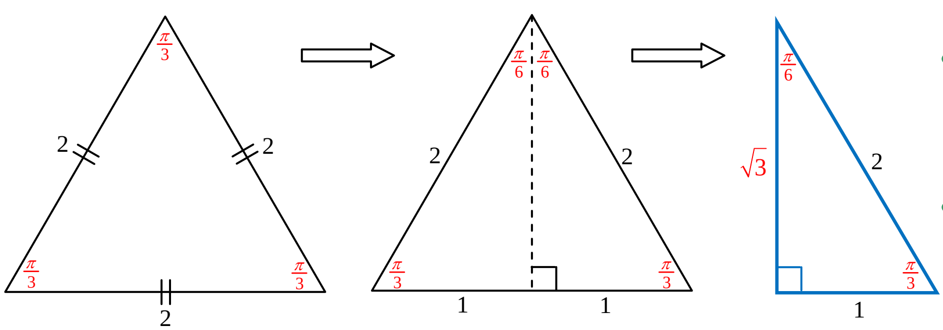
For certain ***special angles***, it is possible to calculate the ***exact value*** of the trigonometric ratios. As I have mentioned on many occasions, it is not advisable to memorize blindly! Instead, you can ***deduce*** the values that you need to calculate the trig ratios by ***understanding*** the following triangles!



- Isosceles Right Triangle
- Let the length of the equal sides be 1 unit.
- By the Pythagorean Theorem, the length of the hypotenuse must be $\sqrt{2}$.

$$\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{4}=\frac{1}{\sqrt{2}} \qquad \tan\frac{\pi}{4}=\frac{1}{1}=1$$

$$\csc\frac{\pi}{4}=\frac{\sqrt{2}}{1}=\sqrt{2} \qquad \sec\frac{\pi}{4}=\frac{\sqrt{2}}{1}=\sqrt{2} \qquad \cot\frac{\pi}{4}=\frac{1}{1}=1$$

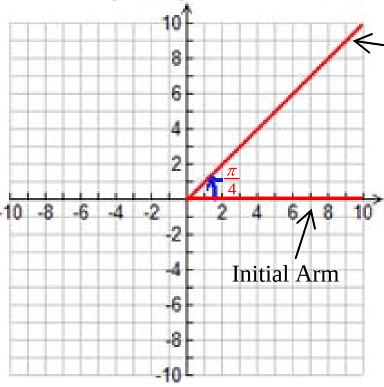


- Begin with an equilateral triangle having sides of length 2 units. Then cut it in half to form a $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ right triangle.
- Use the Pythagorean Theorem to calculate the height of the triangle. ($\sqrt{3}$)

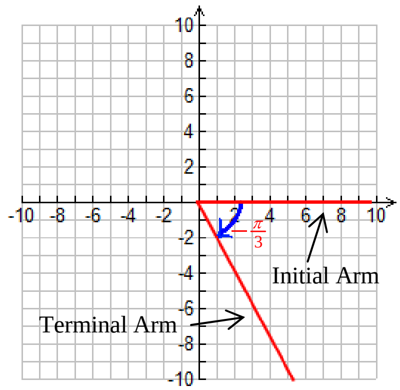
Trig Ratios of $\frac{\pi}{6}$			Trig Ratios of $\frac{\pi}{3}$		
$\sin\frac{\pi}{6}=\frac{1}{2}$	$\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$	$\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$	$\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$	$\cos\frac{\pi}{3}=\frac{1}{2}$	$\tan\frac{\pi}{3}=\sqrt{3}$
$\csc\frac{\pi}{6}=\frac{2}{1}=2$	$\sec\frac{\pi}{6}=\frac{2}{\sqrt{3}}$	$\cot\frac{\pi}{6}=\frac{\sqrt{3}}{1}=\sqrt{3}$	$\csc\frac{\pi}{3}=\frac{\sqrt{3}}{2}$	$\sec\frac{\pi}{3}=\frac{2}{1}=2$	$\cot\frac{\pi}{3}=\frac{1}{\sqrt{3}}$

Trigonometric Ratios of Angles of Rotation – Trigonometric Ratios of Angles of any Size

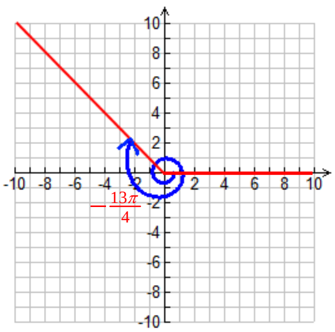
We can extend the idea of trigonometric ratios to angles of any size by introducing the concept of ***angles of rotation*** (also called ***angles of revolution***).

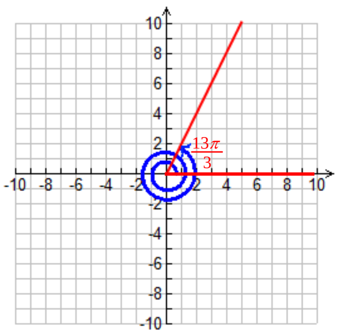


- A ***positive*** angle of revolution (rotation) in ***standard form***.
- "Standard form" means that the ***initial arm*** of the angle lies on the positive x-axis and the ***vertex*** of the angle is at the origin.
- A positive angle results from a ***counter-clockwise*** revolution. (The British say "anti-clockwise.")



- A ***negative*** angle of revolution (rotation) in ***standard form***.
- "Standard form" means that the ***initial arm*** of the angle lies on the positive x-axis and the ***vertex*** of the angle is at the origin.
- A negative angle results from a ***clockwise*** revolution.

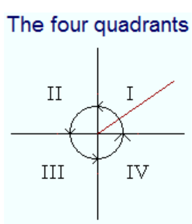


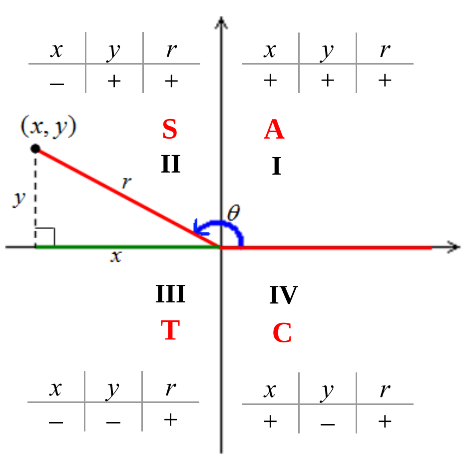


Why Angles of Rotation?

To describe motion that involves moving from one place to another, it makes sense to use units of distance. For instance, it is easy to find your destination if you are told that you need to move 2 km north and 1 km west of your current position.

Consider a spinning figure skater. It does not make sense to describe his/her motion using units of distance because he/she is fixed in one spot and rotating. However, it is very easy to describe the motion through angles of rotation.





$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{r}{y}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{r}{x}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$$

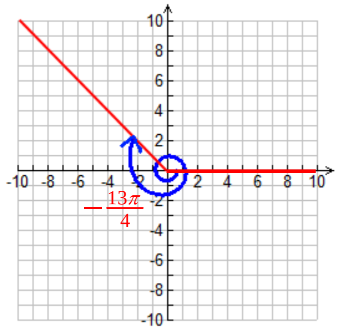
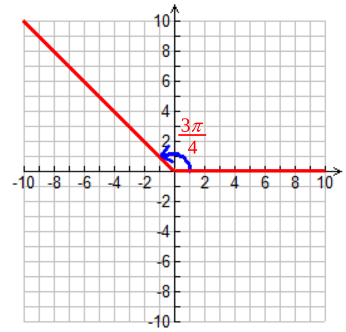
- Since r represents the length of the terminal arm, $r > 0$.
- In quadrant I, $x > 0$ and $y > 0$. Therefore **A**LL the trig ratios are positive.
- In quadrant II, $x < 0$ and $y > 0$. Therefore only **S**INE and cosecant are positive. The others are negative.
- In quadrant III, $x < 0$ and $y < 0$. Therefore only **T**ANGENT and cotangent are positive. The others are negative.
- In quadrant IV, $x > 0$ and $y < 0$. Therefore only **C**OSINE and secant are positive. The others are negative.
- Hence the ***mnemonic***, "**A**LL **S**TUDENTS **T**ALK on **C**ELLPHONES"

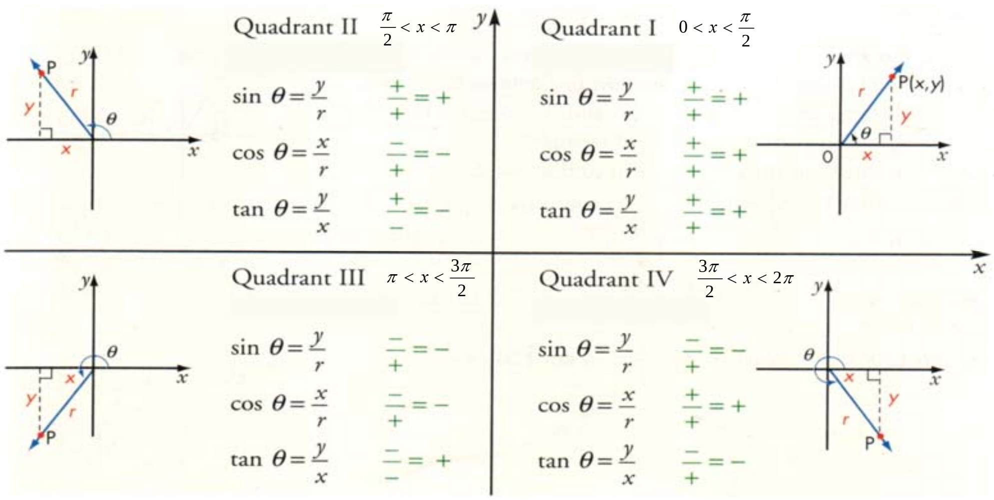
Coterminal Angles

Angles of revolution are called ***coterminal*** if, when in standard position, they share the same terminal arm. For example, $-\frac{\pi}{2}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$ are coterminal angles. An angle coterminal to a given angle can be found by adding or subtracting any multiple of 2π.

Principal Angle

Any angle θ satisfying $0 \le \theta < 2\pi$ is called a ***principal angle***. Every angle of rotation has a principal angle. To find the principal angle of an angle α, simply find the angle θ that is coterminal with α and that also satisfies $0 \le \theta < 2\pi$. An example is given below.

Angle of Rotation $\alpha = -\frac{13\pi}{4}$	*Principal Angle of* α, $\theta = \frac{3\pi}{4}$
	



Example – Evaluating Trig Ratios by using the Related First Quadrant Angle

Find the trigonometric ratios of $\frac{11\pi}{3}$.

> Every angle of rotation has a ***related (acute) first quadrant angle***. The related first quadrant angle is found by taking the ***acute angle between the terminal arm and the x-axis***.
>
> For $\frac{11\pi}{3}$, the ***related (acute) first quadrant angle*** is $\frac{\pi}{3}$

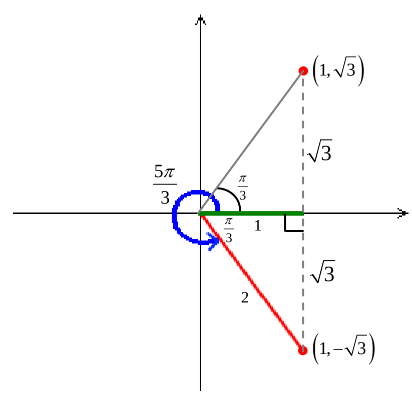
Solution

From the diagram at the right, we can see that the principal angle of $\frac{11\pi}{3}$ is $\frac{5\pi}{3}$. Furthermore, the terminal arm is in the fourth quadrant and we obtain a 30º-60º-90º right triangle in quadrant IV. By observing the ***acute angle*** between the ***terminal arm and the x-axis***, we find the ***related first quadrant angle***, $\frac{\pi}{3}$.

Therefore,

$$\sin\frac{11\pi}{3} = \sin\frac{5\pi}{3} = \frac{y}{r} = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$
$$\cos\frac{11\pi}{3} = \cos\frac{5\pi}{3} = \frac{x}{r} = \frac{1}{2}$$
$$\tan\frac{11\pi}{3} = \tan\frac{5\pi}{3} = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

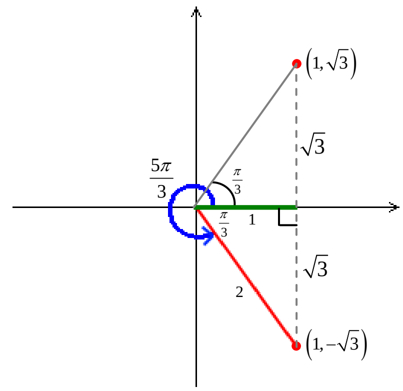
> Compare these answers to
>
> $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
>
> $\cos\frac{\pi}{3} = \frac{1}{2}$
>
> $\tan\frac{\pi}{3} = \sqrt{3}$



Question

How are the trigonometric ratios of the principal angle $\frac{5\pi}{3}$ ***related to*** the trigonometric ratios of $\frac{\pi}{3}$?

Answer



Notice that the right triangle formed for $\frac{5\pi}{3}$ is ***congruent*** to the right triangle for $\frac{\pi}{3}$.

Therefore, the ***magnitudes*** of the trig ratios of $\frac{5\pi}{3}$ are equal to the ***magnitudes*** of those of the related first quadrant angle $\frac{\pi}{3}$. However, due to the fact that the y-co-ordinate of any point in quadrant IV is negative, the ratios may differ in ***sign***. To determine the correct sign, use the ***ASTC*** rule. In case you forget how to apply the ASTC rule, just think about the ***signs*** of x and y in each quadrant. Don't forget that r is always positive because it represents the length of the terminal arm. Thus, the above ratios could have been calculated as follows:

Angle of Rotation: $\frac{5\pi}{3}$ (quadrant IV) **Related First Quadrant Angle:** $\frac{\pi}{3}$

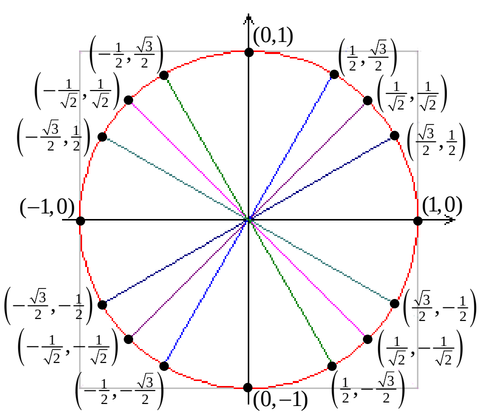
In quadrant IV, $\sin\theta = \frac{y}{r} < 0$ because $\frac{-}{+} = -$, $\cos\theta = \frac{x}{r} > 0$ because $\frac{+}{+} = +$ and $\tan\theta = \frac{y}{x} < 0$ because $\frac{-}{+} = -$.

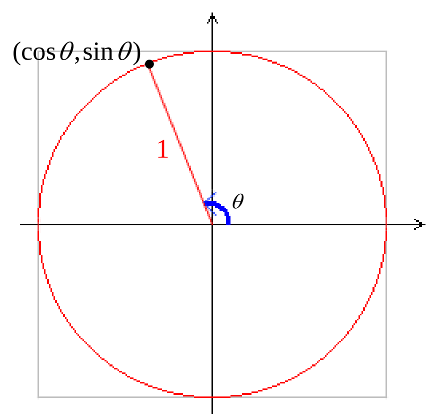
Hence, $\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$, $\cos\frac{5\pi}{3} = \cos\frac{\pi}{3} = \frac{1}{2}$ and $\tan\frac{5\pi}{3} = -\tan\frac{\pi}{3} = -\sqrt{3}$.

Additional Tools for Determining Trig Ratios of Special Angles

The Unit Circle

A ***unit circle*** is any circle having a radius of ***one unit***. If a unit circle is centred at the origin, it is described by the equation $x^2 + y^2 = 1$, meaning that for any point (x, y) lying on the circle, the value of $x^2 + y^2$ must equal 1. Furthermore, for any point (x, y) lying on the unit circle and for any angle θ, $r = 1$. Therefore, $\cos\theta = \frac{x}{r} = \frac{x}{1} = x$ and $\sin\theta = \frac{y}{r} = \frac{y}{1} = y$. In other words, for any point (x, y) lying on the unit circle, the ***x-co-ordinate is equal to the cosine of θ*** and the ***y-co-ordinate is equal to the sine of θ***.



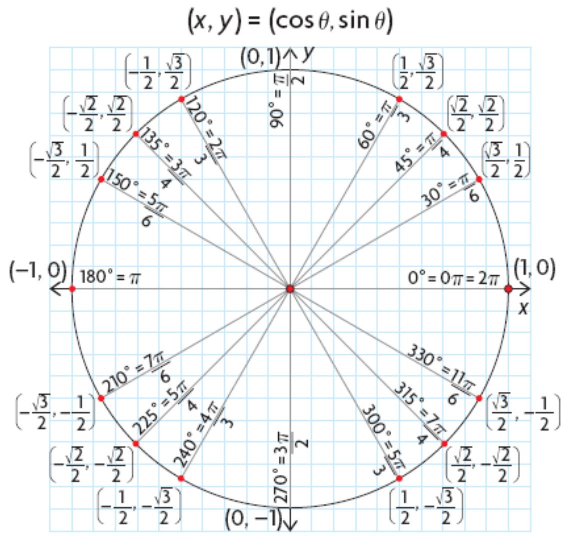


The Rule of Quarters (Beware of Blind Memorization!)

The rule of quarters makes it easy to remember the sine of special angles. ***Be aware, however, that this rule invites blind memorization!***

$$\sin(0^\circ) = \sqrt{\frac{0}{4}} = 0$$
$$\sin(30^\circ) = \sqrt{\frac{1}{4}} = \frac{1}{2}$$
$$\sin(45^\circ) = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$
$$\sin(60^\circ) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$
$$\sin(90^\circ) = \sqrt{\frac{4}{4}} = 1$$

Note that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. This follows by multiplying both the numerator and the denominator of $\frac{1}{\sqrt{2}}$ by $\sqrt{2}$.



By expressing $\frac{1}{\sqrt{2}}$ in the form $\frac{\sqrt{2}}{2}$, it is very easy to remember the unit circle.

Homework
pp. 330 – 332
5, 6, 10, 11, 14, 16, 21

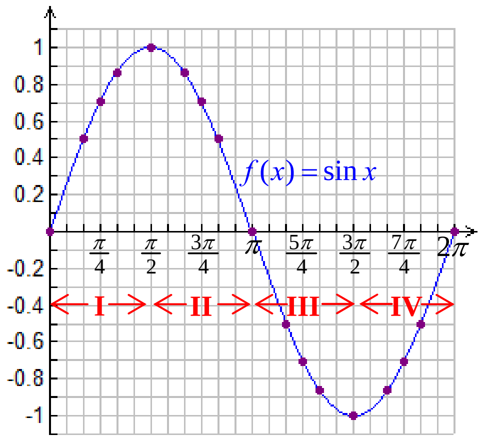
Introduction to Trigonometric Functions

Overview

Now that we have developed a thorough understanding of trigonometric ratios, we can proceed to our investigation of trigonometric functions. The old adage "a picture says a thousand words" is very fitting in the case of the graphs of trig functions. The curves summarize everything that we have learned about trig ratios. Answer the questions below to discover the details.

Graphs

x	$y=\sin x$
0	0
$\frac{\pi}{6}$	$1/2=0.5$
$\frac{\pi}{4}$	$1/\sqrt{2} \doteq 0.70711$
$\frac{\pi}{3}$	$\sqrt{3}/2 \doteq 0.86603$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\sqrt{3}/2 \doteq 0.86603$
$\frac{3\pi}{4}$	$1/\sqrt{2} \doteq 0.70711$
$\frac{5\pi}{6}$	$1/2=0.5$
π	0
$\frac{7\pi}{6}$	$-1/2=-0.5$
$\frac{5\pi}{4}$	$-1/\sqrt{2} \doteq -0.70711$
$\frac{4\pi}{3}$	$-\sqrt{3}/2 \doteq -0.86603$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\sqrt{3}/2 \doteq -0.86603$
$\frac{7\pi}{4}$	$-1/\sqrt{2} \doteq -0.70711$
$\frac{11\pi}{6}$	$-1/2=-0.5$
2π	0



Questions about sin x

1. State the domain and range of the sine function.

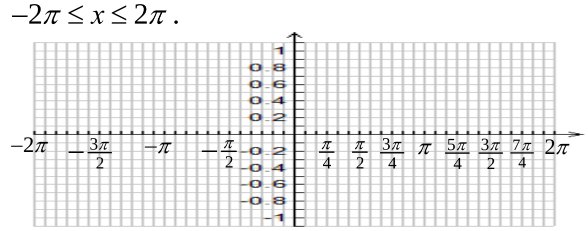
2. Is the sine function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x)=\sin x$ over which $f^{-1}(x)$ is defined.

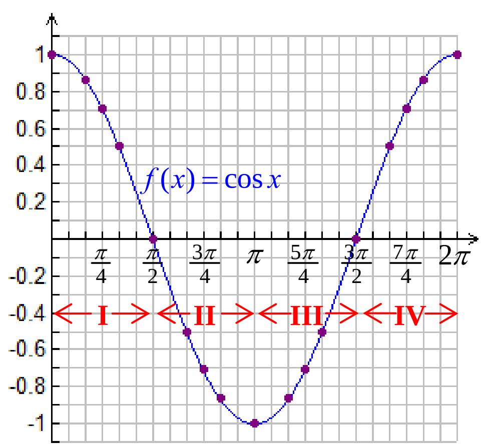
4. How can the graph of the sine function help you to remember the ***sign*** (i.e. + or –) of a sine ratio for any angle?

5. How can the graph of the sine function help you to remember the sine ratios of the special angles?

6. Sketch the graph of $f(x)=\sin x$ for $-2\pi \le x \le 2\pi$.



x	$y=\cos x$
0	1
$\frac{\pi}{6}$	$\sqrt{3}/2 \doteq 0.86603$
$\frac{\pi}{4}$	$1/\sqrt{2} \doteq 0.70711$
$\frac{\pi}{3}$	$1/2=0.5$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-1/2=-0.5$
$\frac{3\pi}{4}$	$-1/\sqrt{2} \doteq -0.70711$
$\frac{5\pi}{6}$	$-\sqrt{3}/2 \doteq -0.86603$
π	-1
$\frac{7\pi}{6}$	$-\sqrt{3}/2 \doteq -0.86603$
$\frac{5\pi}{4}$	$-1/\sqrt{2} \doteq -0.70711$
$\frac{4\pi}{3}$	$-1/2=-0.5$
$\frac{3\pi}{2}$	0
$\frac{5\pi}{3}$	$1/2=0.5$
$\frac{7\pi}{4}$	$1/\sqrt{2} \doteq 0.70711$
$\frac{11\pi}{6}$	$\sqrt{3}/2 \doteq 0.86603$
2π	1



Questions about cos x

1. State the domain and range of the cosine function.

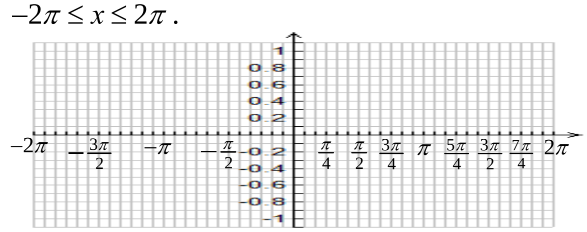
2. Is the cosine function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x)=\cos x$ over which $f^{-1}(x)$ is defined.

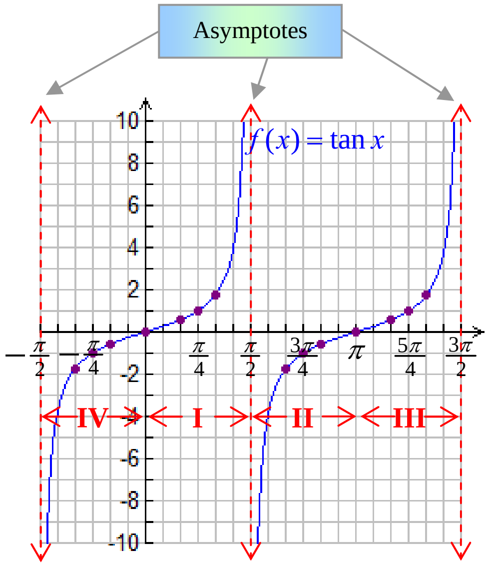
4. How can the graph of the cosine function help you to remember the ***sign*** (i.e. + or –) of a cosine ratio for any angle?

5. How can the graph of the cosine function help you to remember the cosine ratios of the special angles?

6. Sketch the graph of $f(x)=\cos x$ for $-2\pi \le x \le 2\pi$.



x	$y = \tan x$
$-\frac{\pi}{2}$	undefined
$-\frac{\pi}{3}$	$-\sqrt{3} \doteq -1.73205$
$-\frac{\pi}{4}$	-1
$-\frac{\pi}{6}$	$-1/\sqrt{3} \doteq -0.57735$
0	0
$\frac{\pi}{6}$	$1/\sqrt{3} \doteq 0.57735$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3} \doteq 1.73205$
$\frac{\pi}{2}$	undefined
$\frac{2\pi}{3}$	$-\sqrt{3} \doteq -1.73205$
$\frac{3\pi}{4}$	-1
$\frac{5\pi}{6}$	$-1/\sqrt{3} \doteq -0.57735$
π	0
$\frac{7\pi}{6}$	$1/\sqrt{3} \doteq 0.57735$
$\frac{5\pi}{4}$	1
$\frac{4\pi}{3}$	$\sqrt{3} \doteq 1.73205$
$\frac{3\pi}{2}$	undefined



Questions about tan x

1. State the domain and range of the tangent function.

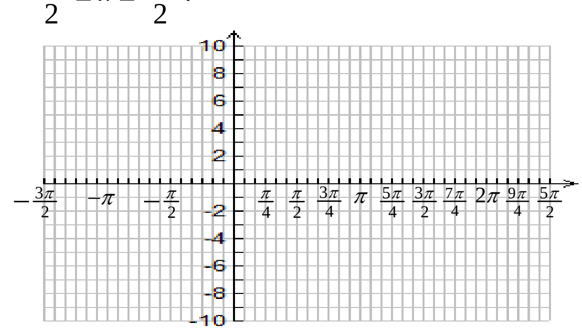
2. Is the tangent function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x) = \tan x$ over which $f^{-1}(x)$ is defined.

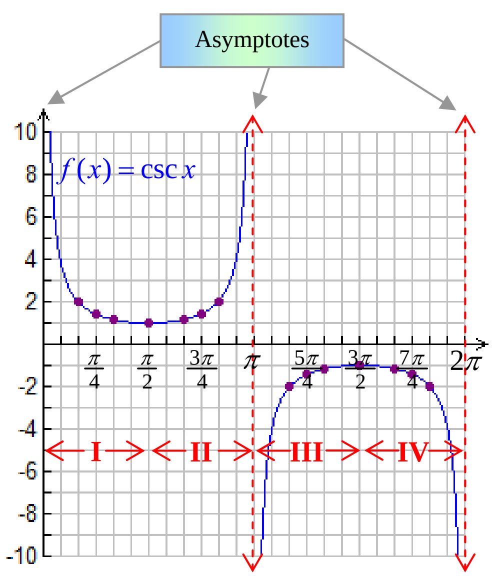
4. How can the graph of the tangent function help you to remember the sign (i.e. + or –) of a tangent ratio for any angle?

5. How can the graph of the tangent function help you to remember the tangent ratios of the special angles?

6. Sketch the graph of $f(x) = \tan x$ for $-\frac{3\pi}{2} \le x \le \frac{5\pi}{2}$.



x	$y = \csc x$
0	undefined
$\frac{\pi}{6}$	2
$\frac{\pi}{4}$	$\sqrt{2} \doteq 1.41421$
$\frac{\pi}{3}$	$2/\sqrt{3} \doteq 1.15470$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$2/\sqrt{3} \doteq 1.15470$
$\frac{3\pi}{4}$	$\sqrt{2} \doteq 1.41421$
$\frac{5\pi}{6}$	2
π	undefined
$\frac{7\pi}{6}$	-2
$\frac{5\pi}{4}$	$-\sqrt{2} \doteq -1.41421$
$\frac{4\pi}{3}$	$-2/\sqrt{3} \doteq -1.15470$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-2/\sqrt{3} \doteq -1.15470$
$\frac{7\pi}{4}$	$-\sqrt{2} \doteq -1.41421$
$\frac{11\pi}{6}$	-2
2π	undefined



Questions about csc x

1. State the domain and range of the cosecant function.

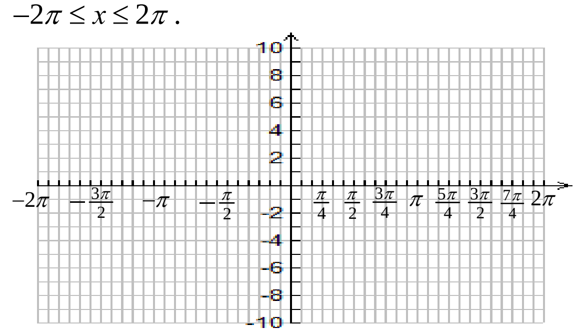
2. Is the cosecant function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x) = \csc x$ over which $f^{-1}(x)$ is defined.

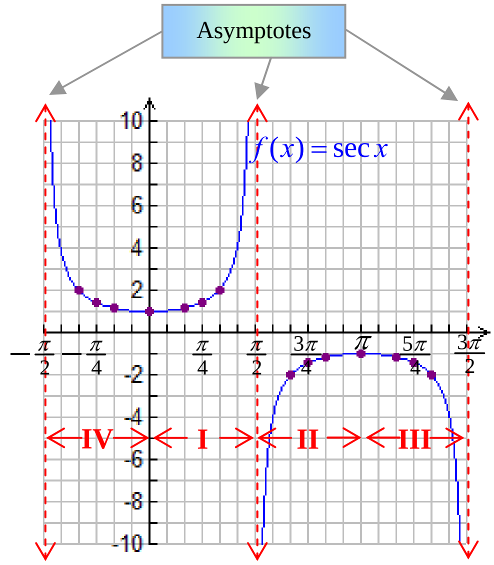
4. How can the graph of the cosecant function help you to remember the sign (i.e. + or –) of a cosecant ratio for any angle?

5. How can the graph of the cosecant function help you to remember the cosecant ratios of the special angles?

6. Sketch the graph of $f(x) = \csc x$ for $-2\pi \le x \le 2\pi$.



x	$y=\sec x$
$-\frac{\pi}{2}$	undefined
$-\frac{\pi}{3}$	2
$-\frac{\pi}{4}$	$\sqrt{2} \doteq 1.41421$
$-\frac{\pi}{6}$	$2/\sqrt{3} \doteq 1.15470$
0	1
$\frac{\pi}{6}$	$2/\sqrt{3} \doteq 1.15470$
$\frac{\pi}{4}$	$\sqrt{2} \doteq 1.41421$
$\frac{\pi}{3}$	2
$\frac{\pi}{2}$	undefined
$\frac{2\pi}{3}$	-2
$\frac{3\pi}{4}$	$-\sqrt{2} \doteq -1.41421$
$\frac{5\pi}{6}$	$-2/\sqrt{3} \doteq -1.15470$
π	-1
$\frac{7\pi}{6}$	$-2/\sqrt{3} \doteq -1.15470$
$\frac{5\pi}{4}$	$-\sqrt{2} \doteq -1.41421$
$\frac{4\pi}{3}$	-2
$\frac{3\pi}{2}$	undefined



Questions about sec x

1. State the domain and range of the secant function.

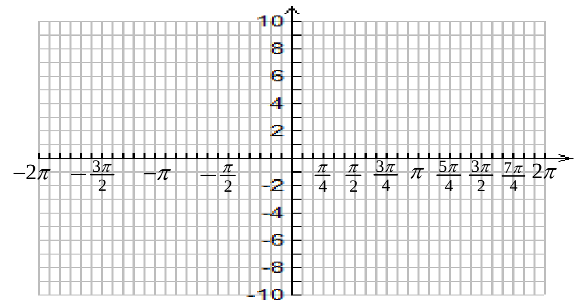
2. Is the secant function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x)=\sec x$ over which $f^{-1}(x)$ is defined.

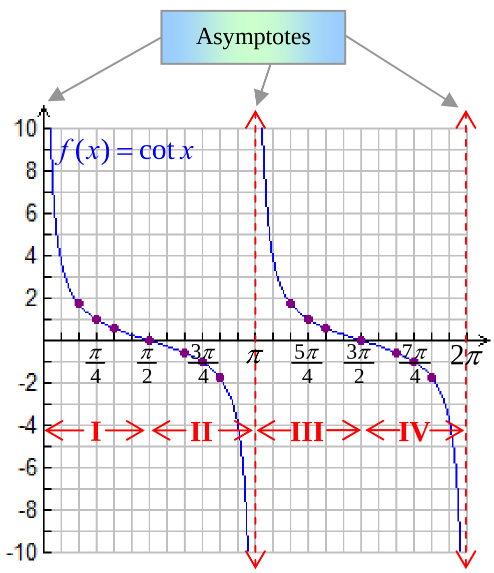
4. How can the graph of the secant function help you to remember the sign (i.e. + or –) of a secant ratio for any angle?

5. How can the graph of the secant function help you to remember the secant ratios of the special angles?

6. Sketch the graph of $f(x)=\sec x$ for $-2\pi \le x \le 2\pi$.



x	$y=\cot x$
0	undefined
$\frac{\pi}{6}$	$\sqrt{3} \doteq 1.73205$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$1/\sqrt{3} \doteq 0.57735$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-1/\sqrt{3} \doteq -0.57735$
$\frac{3\pi}{4}$	-1
$\frac{5\pi}{6}$	$-\sqrt{3} \doteq -1.73205$
π	undefined
$\frac{7\pi}{6}$	$\sqrt{3} \doteq 1.73205$
$\frac{5\pi}{4}$	1
$\frac{4\pi}{3}$	$1/\sqrt{3} \doteq 0.57735$
$\frac{3\pi}{2}$	0
$\frac{5\pi}{3}$	$-1/\sqrt{3} \doteq -0.57735$
$\frac{7\pi}{4}$	-1
$\frac{11\pi}{6}$	$-\sqrt{3} \doteq -1.73205$
2π	undefined



Questions about cot x

1. State the domain and range of the cotangent function.

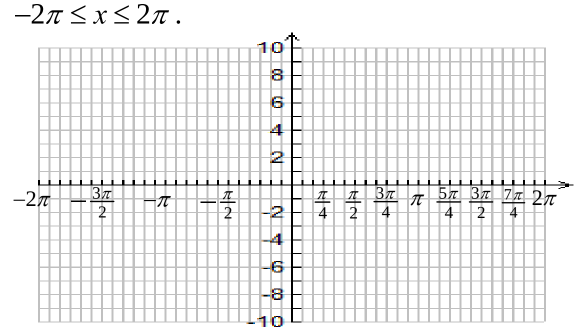
2. Is the cotangent function one-to-one or many-to-one?

3. State a suitable subset of the domain of $f(x)=\cot x$ over which $f^{-1}(x)$ is defined.

4. How can the graph of the cotangent function help you to remember the sign (i.e. + or –) of a cotangent ratio for any angle?

5. How can the graph of the cotangent function help you to remember the cotangent ratios of the special angles?

6. Sketch the graph of $f(x)=\cot x$ for $-2\pi \le x \le 2\pi$.



TRANSFORMATIONS OF TRIGONOMETRIC FUNCTIONS

What on Earth is a Sinusoidal Function?

A ***sinusoidal function*** is simply any function that can be obtained by ***stretching*** (compressing) and/or ***translating*** the function $f(x) = \sin x$. That is, a sinusoidal function is any function of the form $g(x) = A\sin\left(\omega\left(x-p\right)\right) + d$. Since we have already investigated transformations of logarithmic and exponential and logarithmic functions, we can immediately state the following:

Transformation of $f(x) = \sin x$ expressed in Function Notation	*Transformation of $f(x) = \sin x$ expressed in Mapping Notation*
$g(x) = A\sin\left(\omega\left(x-p\right)\right) + d$	$(x,y) \to \left(\omega^{-1}x + p, Ay + d\right)$
Vertical Transformations (Apply Operations following Order of Operations)	***Horizontal Transformations (Apply Inverse Operations opposite the Order of Operations)***
1. Stretch or compress vertically by a factor of A. If A <0, then this includes a reflection in the x-axis. **2.** Translate vertically by d units. $(x,y) \to (x, Ay + d)$	**1.** Stretch or compress horizontally by a factor of $\omega^{-1} = 1/\omega$. If ω <0, then this includes a reflection in the y-axis. **2.** Translate horizontally by p units. $(x,y) \to \left(\omega^{-1}x + p, y\right)$

Since sinusoidal functions look just like ***waves*** and are perfectly suited to modelling wave or wave-like phenomena, special names are given to the quantities A, d, p and k.

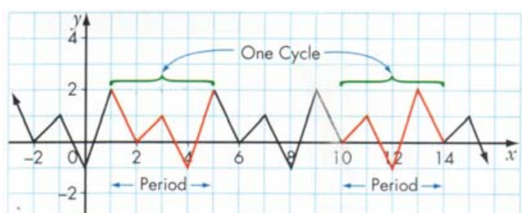
- $|A|$ is called the ***amplitude*** (absolute value is needed because amplitude is a distance, which must be positive)
- d is called the ***vertical displacement***
- p is called the ***phase shift***
- ω (also written as k) is called the ***angular frequency***

These quantities are described in detail on the next page.

Periodic Functions

There are many naturally occurring and artificially produced phenomena that undergo repetitive cycles. We call such phenomena ***periodic***. Examples of such processes include the following:

- orbits of planets, moons, asteroids, comets, etc
- rotation of planets, moons, asteroids, comets, etc
- phases of the moon
- the tides
- changing of the seasons
- hours of daylight on a given day
- light waves, radio waves, etc
- alternating current (**e.g.** household alternating current has a frequency of 60 Hz, which means that it changes direction 60 times per second)



An example of a ***periodic*** function.

Intuitively, a function is said to be ***periodic*** if the graph consists of a "basic pattern" that is repeated over and over at ***regular intervals***. One complete pattern is called a ***cycle***.

Formally, if there is a number T such that $f(x+T) = f(x)$ for all values of x, then we say that f is ***periodic***. The smallest possible positive value of T is called the ***period*** of the function. The ***period*** of a periodic function is equal to the ***length of one cycle***.

Exercise

Suppose that the periodic function shown above is called f. Evaluate each of the following.

(a) $f(2)$ **(b)** $f(4)$ **(c)** $f(1)$ **(d)** $f(0)$ **(e)** $f(16)$ **(f)** $f(18)$ **(g)** $f(33)$ **(h)** $f(-16)$

(i) $f(-31)$ **(j)** $f(-28)$ **(k)** $f(-27)$ **(l)** $f(-11)$ **(m)** $f(-6)$ **(n)** $f(-9)$ **(o)** $f(-5)$ **(p)** $f(-101)$

Characteristics of Sinusoidal Functions

1. Sinusoidal functions have the general form $f(x) = A\sin(\omega(x-p)) + d$, where A, d, p and ω are as described above.

   (Those of you who prefer to use k may write this is $f(x) = A\sin(k(x-p)) + d$.)

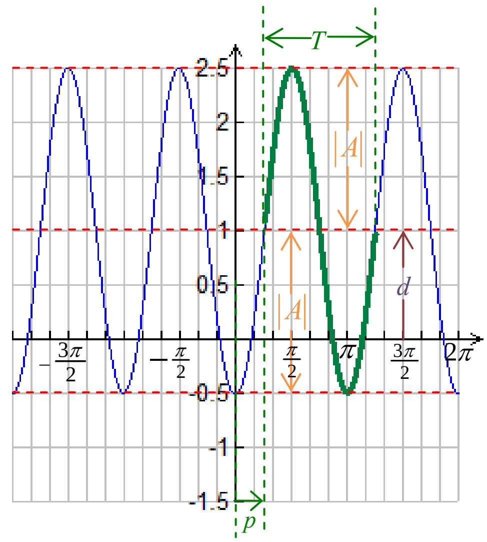
2. Sinusoidal functions are ***periodic***. This makes sinusoidal functions ideal for modelling ***periodic processes*** such as those described on page 19. The letter T is used to denote the ***period*** (also called ***primitive period*** or ***wavelength***) of a sinusoidal function.

3. Sinusoidal functions ***oscillate*** (vary continuously, back and forth) between a maximum and a minimum value. This makes sinusoidal functions ideal for modelling ***oscillatory*** or ***vibratory*** motions. (**e.g.** a pendulum swinging back and forth, a playground swing, a vibrating string, a tuning fork, alternating current, quartz crystal vibrating in a watch, light waves, radio waves, etc)

4. There is a horizontal line called the ***horizontal axis*** that exactly "cuts" a sinusoidal function "in half." The vertical distance (maximum displacement) from this horizontal line to the peak of the curve is called the ***amplitude***.

Example

The graph at the right shows a few ***cycles*** of the function $f(x) = 1.5\sin\left(2\left(x-\frac{\pi}{4}\right)\right)+1$. One of the cycles is shown as a thick green curve to make it stand out among the others. Notice the following:

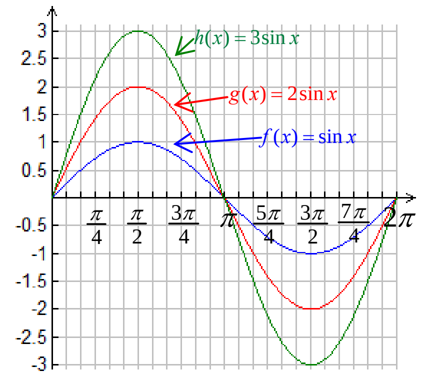
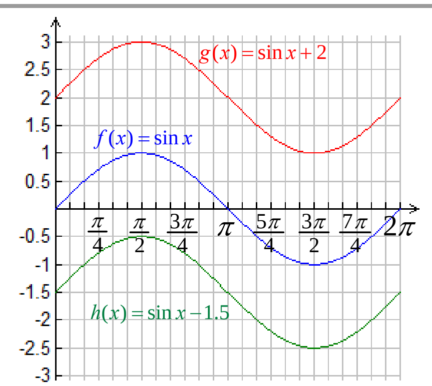
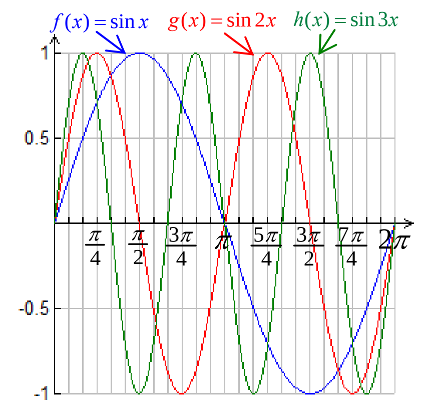
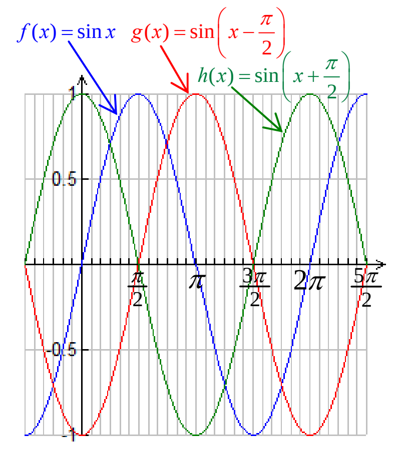


- The ***maximum*** value of f is 2.5.
- The ***minimum*** value of f is –0.5.
- The function f ***oscillates*** between –0.5 and 2.5.
- The horizontal line with equation $y=1$ exactly "cuts" the function "in half." This line is called the ***horizontal axis***.
- The ***amplitude*** of this function is $|A| = 1.5$. This can be seen in a number of ways. Clearly, the vertical distance from the line $y=1$ to the peak of the curve is 1.5. Also, the amplitude can be calculated by finding half the distance between the maximum and minimum values: $\dfrac{2.5-(-0.5)}{2} = \dfrac{3}{2} = 1.5$
- The ***period***, that is the length of one cycle, is $T=\pi$. This can be seen from the graph ($\frac{5\pi}{4} - \frac{\pi}{4} = \pi$) or it can be determined by applying your knowledge of transformations. The period of $y=\sin x$ is 2π. Since f has undergone a horizontal compression by a factor of $1/2$, its period should be half of 2π, which is π. In general,

  T = (period of base function)(absolute value of the horizontal compression factor) or $T = 2\pi\left|\frac{1}{\omega}\right|$. The reason that absolute value is needed here is that the period is a distance and hence, must be positive. (Those of you who prefer to use k to represent the angular frequency may write this as $T = 2\pi\left|\frac{1}{k}\right|$.)
- The ***absolute value of the angular frequency*** determines the number of cycles in 2π radians. In this example, $|\omega| = 2$, which means that there are 2 cycles in 2π radians = 1 cycle in π radians = $\frac{1}{\pi}$ cycles in 1 radian = $\frac{1}{\pi}$ cycles/radian.
- The function $g(x) = 1.5\sin\left(2\left(x-\frac{\pi}{4}\right)\right)$ would be "cut in half" by the x-axis (i.e. the horizontal axis $y=0$). The function f has exactly the same shape as g except that it is ***shifted up by 1 unit***. This is the significance of the ***vertical displacement***. In this example, the vertical displacement $d=1$.
- The function $g(x) = 1.5\sin\left(2\left(x-\frac{\pi}{4}\right)\right)$ has exactly the same shape as $h(x) = 1.5\sin 2x$ but is shifted $\frac{\pi}{4}$ to the right. This horizontal shift is called the ***phase shift***.

Important Exercises

Complete the following table. The first one is done for you.

Function	A	d	p	ω=k	T	Description of Transformation	
f	1	0	0	1	2π	None	
g	2	0	0	1	2π	The graph of $f(x)=\sin x$ is stretched vertically by a factor of 2. The amplitude of g is 2.	
h	3	0	0	1	2π	The graph of $f(x)=\sin x$ is stretched vertically by a factor of 3. The amplitude of g is 3.	
f							
g							
h							
f							
g							
h							
f							
g							
h							

Example

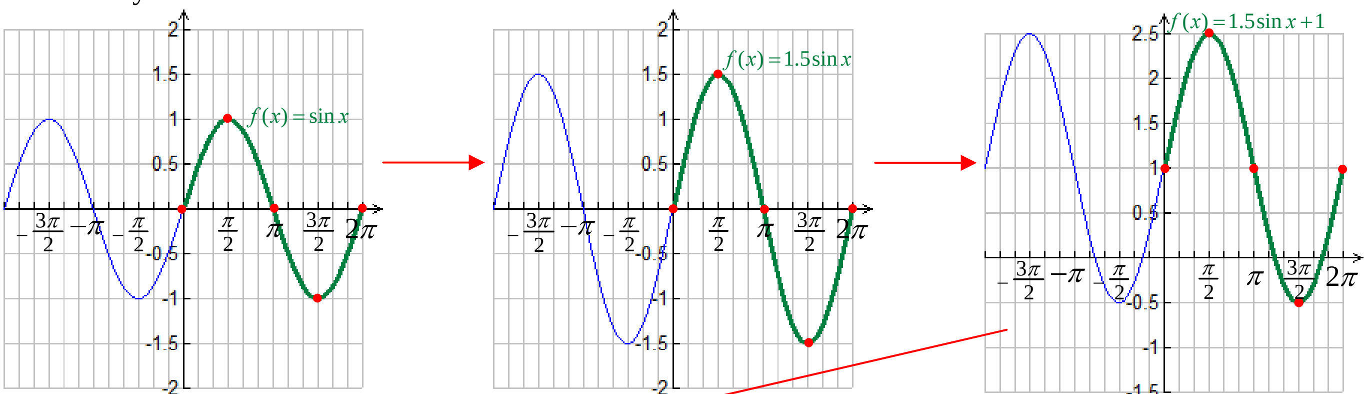
Sketch the graph of $f(x)=1.5\sin\left(2\left(x-\frac{\pi}{4}\right)\right)+1$

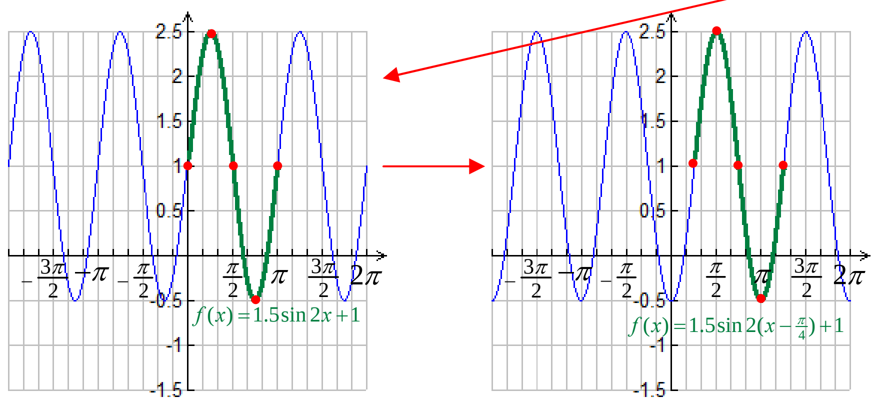
Solution

Amplitude	Vertical Displacement	Phase Shift	Period	Angular Frequency	Transformations	
$\lvert A\rvert=1.5$	$d=1$	$p=\dfrac{\pi}{4}$	$T=2\pi\left\lvert\dfrac{1}{\omega}\right\rvert$ $=2\pi(1/2)$ $=\pi$	$\omega=2$ $\lvert\omega\rvert=2$, which means 2 cycles per 2π radians	***Vertical*** Stretch by a factor of 1.5. Translate 1 unit up.	***Horizontal*** Compress by a factor of $2^{-1}=\frac{1}{2}$ Translate $\dfrac{\pi}{4}$ to the right.

Method 1 – The Long Way

The following shows how the graph of $f(x)=1.5\sin\left(2\left(x-\frac{\pi}{4}\right)\right)+1$ is obtained by beginning with the base function $f(x)=\sin x$ and applying the transformations one-by-one in the correct order. One cycle of $f(x)=\sin x$ is highlighted in green to make it easy to see the effect of each transformation. In addition, ***five main points*** are displayed in red to make it easy to see the effect of each transformation.

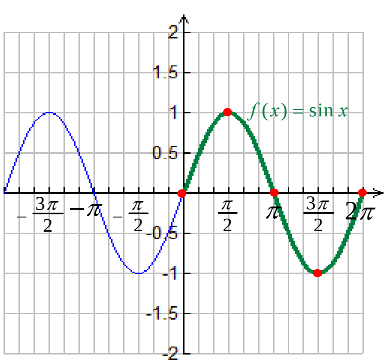




Note

- The five main points divide each cycle into quarters (four equal parts).
- Each quarter corresponds to one of the four quadrants.
- Dividing one cycle into four equal parts makes it very easy to sketch the graphs of sinusoidal functions.
- The transformations are very easy to apply to the five main points.

Method 2 – A Much Faster Approach



$f(x)=1.5\sin\left(2\left(x-\frac{\pi}{4}\right)\right)+1$, $A=1.5$, $d=1$, $\omega=2$, $p=\frac{\pi}{4}$

$(x,y)\to\left(\omega^{-1}x+p, Ay+d\right)$

$\therefore (x,y)\to\left(\frac{1}{2}x+\frac{\pi}{4}, 1.5y+1\right)$

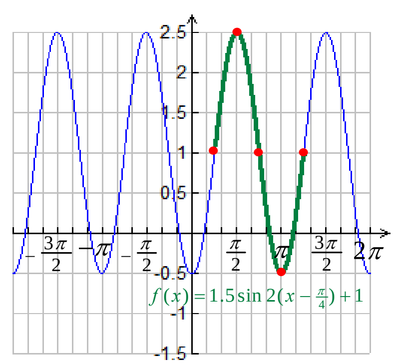
$\therefore (0,0)\to\left(\frac{1}{2}(0)+\frac{\pi}{4}, 1.5(0)+1\right)=\left(\frac{\pi}{4},1\right)$

$\therefore \left(\frac{\pi}{2},1\right)\to\left(\frac{1}{2}\left(\frac{\pi}{2}\right)+\frac{\pi}{4}, 1.5(1)+1\right)=\left(\frac{\pi}{2},2.5\right)$

$\therefore (\pi,0)\to\left(\frac{1}{2}\pi+\frac{\pi}{4}, 1.5(0)+1\right)=\left(\frac{3\pi}{4},1\right)$

$\therefore \left(\frac{3\pi}{2},-1\right)\to\left(\frac{1}{2}\left(\frac{3\pi}{2}\right)+\frac{\pi}{4}, 1.5(-1)+1\right)=(\pi,-0.5)$

$\therefore (2\pi,0)\to\left(\frac{1}{2}(2\pi)+\frac{\pi}{4}, 1.5(0)+1\right)=\left(\frac{5\pi}{4},1\right)$



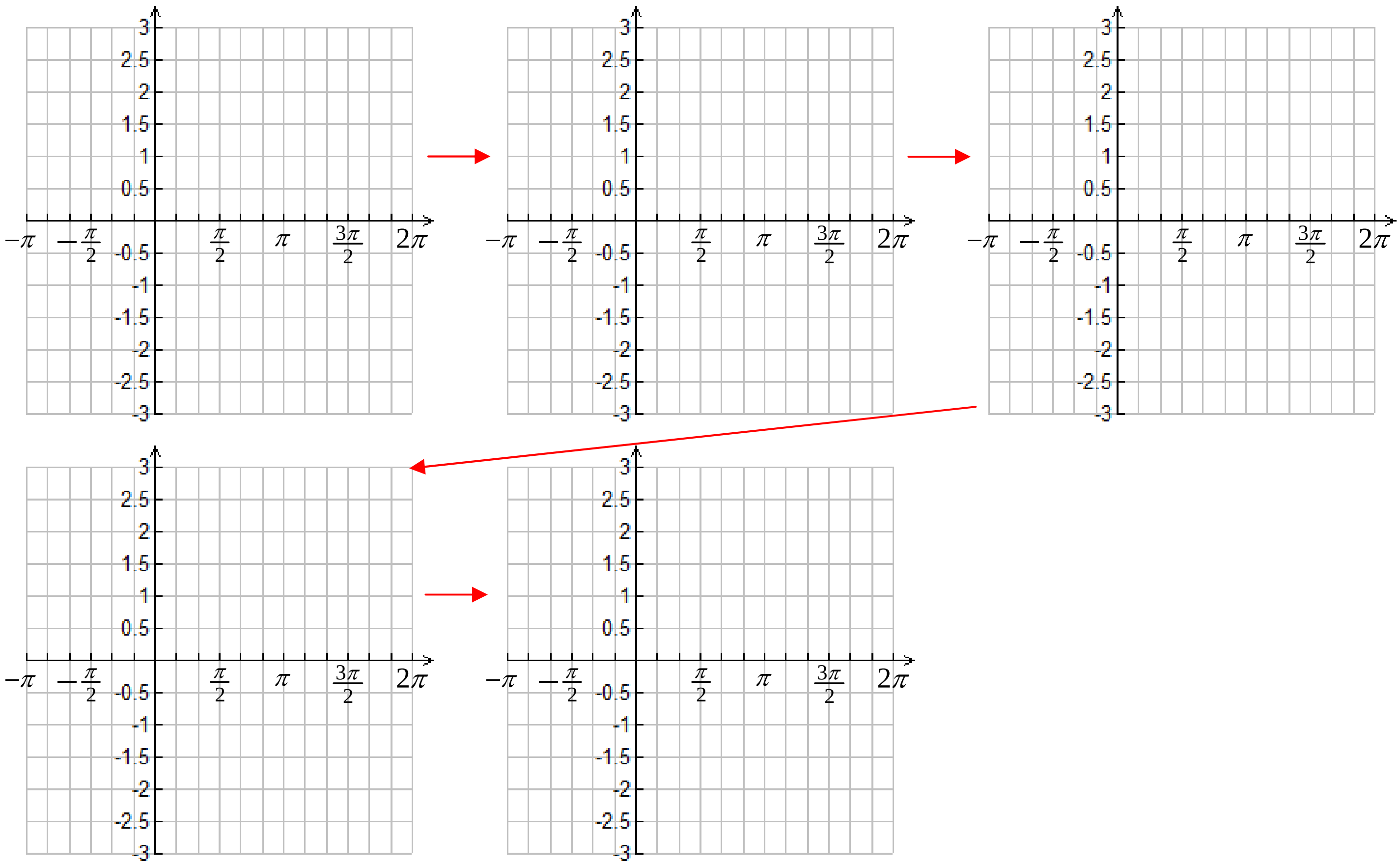
Exercise 1

Using ***both*** of the approaches shown in the previous example, sketch a few cycles of the graph of $f(x)=-2\cos\left(-3\left(x+\frac{\pi}{3}\right)\right)-1$.

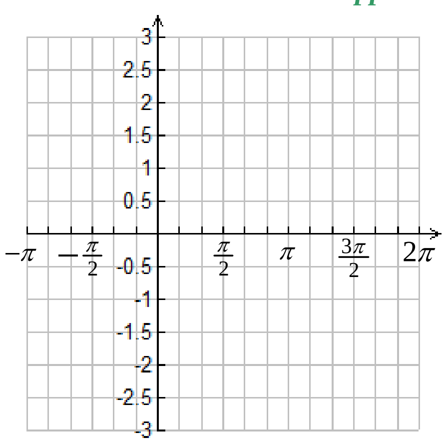
Solution

Amplitude	*Vertical Displacement*	*Phase Shift*	*Period*	*Angular Frequency*	*Transformations*	
					Vertical	*Horizontal*

Method 1 – The Long Way



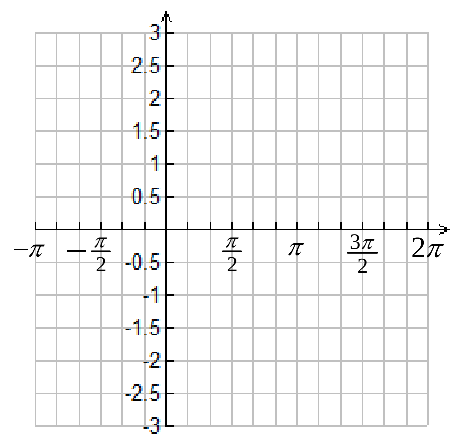
Method 2 – A Much Faster Approach



$f(x)=-2\cos\left(-3\left(x+\frac{\pi}{3}\right)\right)-1$

$A=$ ____, $d=$ ____, $\omega=k=$ ____, $p=$ ____

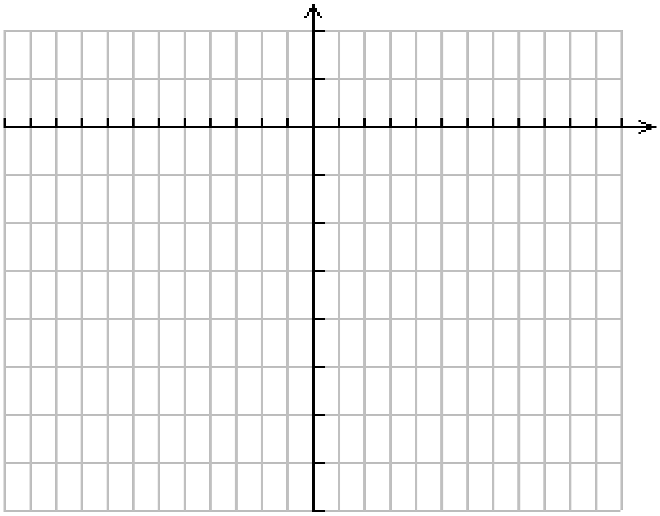
$(x,y)\rightarrow$



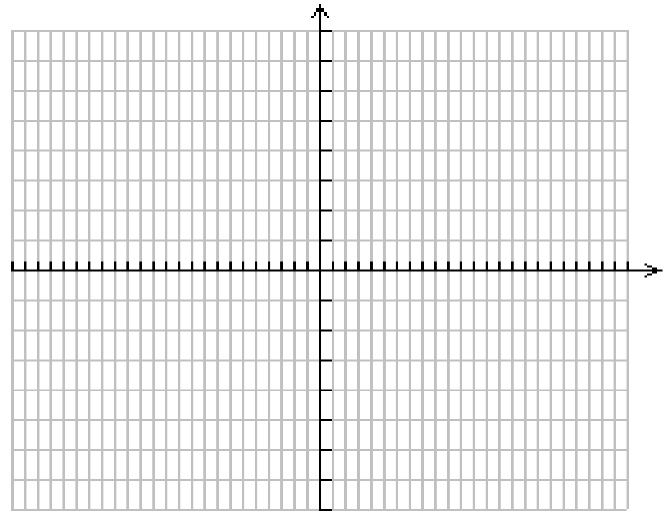
Homework Exercises

Sketch at least three cycles of each of the following functions. In addition, state the domain and range of each, as well as the amplitude, the vertical displacement, the phase shift and the period.

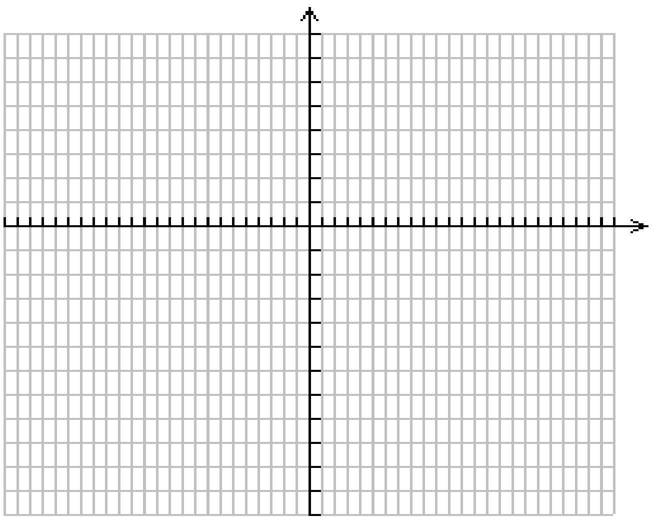
(a) $f(x) = -\cos 3x - 2$



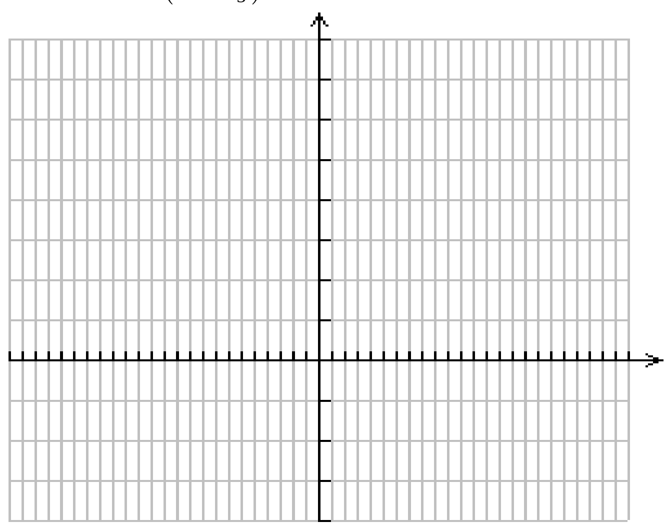
(b) $g(x) = 3\cos\left(x - \frac{\pi}{6}\right)$



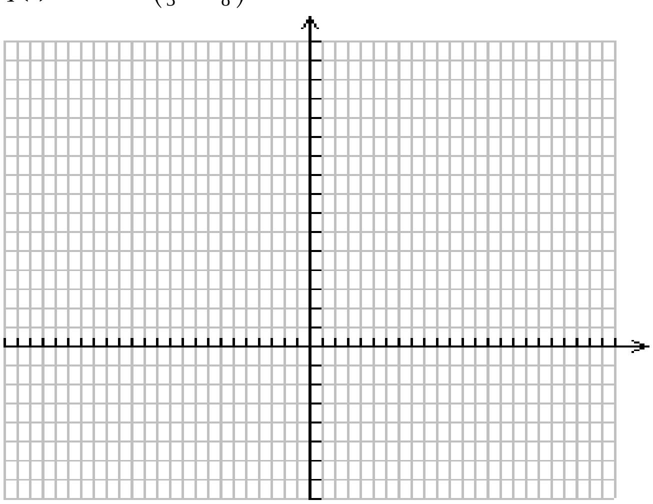
(c) $h(x) = 4\sin\left(2\left(x + \frac{\pi}{4}\right)\right) - 1$



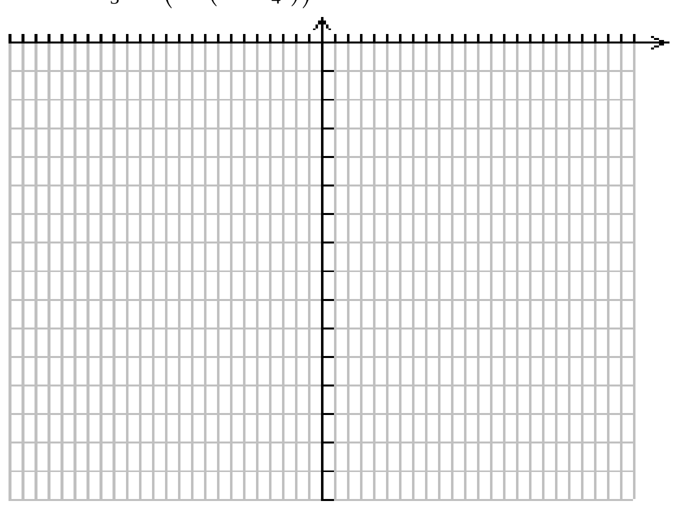
(d) $p(x) = -2\sin\left(2x + \frac{\pi}{3}\right) + 1$



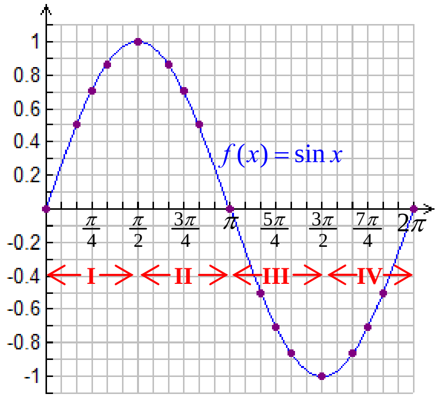
(e) $q(t) = -5\cos\left(\frac{1}{3}t + \frac{\pi}{8}\right) + 2$



(f) $r(\theta) = -\frac{2}{3}\sin\left(-2\left(\theta - \frac{3\pi}{4}\right)\right) - 3$

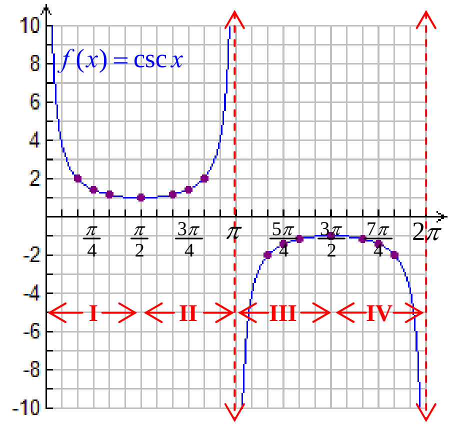


One Cycle of each of the Trigonometric Base/Parent/Mother Functions



$D = \mathbb{R}$, $R = \{y \in \mathbb{R} : -1 \le y \le 1\}$

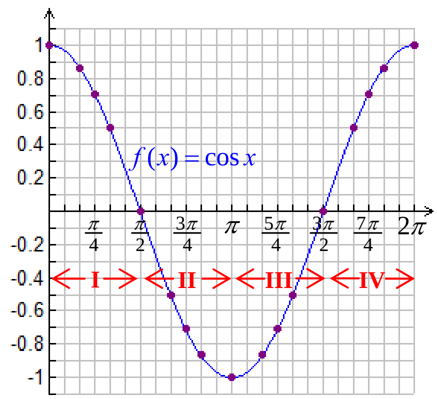
$A = 1,\ d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$



$D = \{x \in \mathbb{R} : x \ne n\pi, n \in \mathbb{Z}\}$, $R = \{y \in \mathbb{R} : y \le -1 \text{ or } y \ge 1\}$

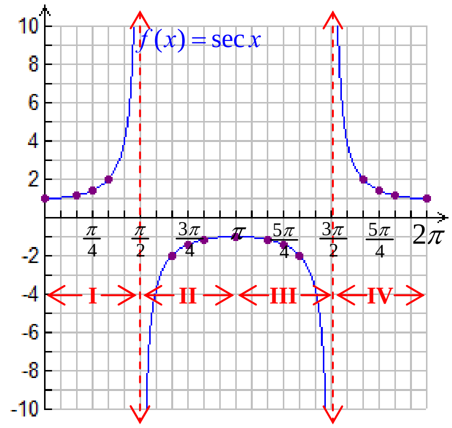
A = 1 but the amplitude is undefined, $d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$

(The vertical stretch factor is 1 but there is no feature corresponding to amplitude.)



$D = \mathbb{R}$, $R = \{y \in \mathbb{R} : -1 \le y \le 1\}$

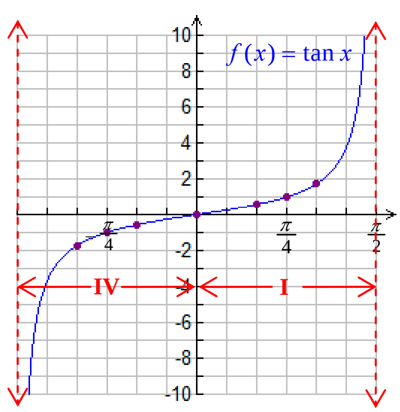
$A = 1,\ d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$



$D = \left\{x \in \mathbb{R} : x \ne \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}\right\}$, $R = \{y \in \mathbb{R} : y \le -1 \text{ or } y \ge 1\}$

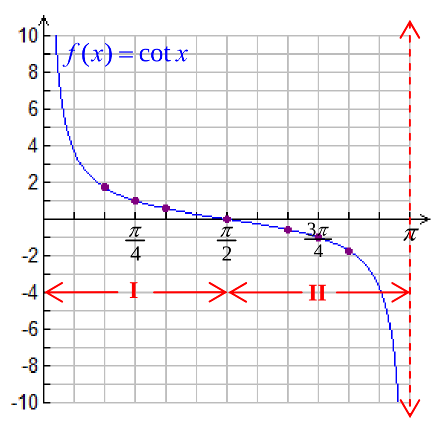
A = 1 but the amplitude is undefined, $d = 0,\ p = 0,\ \omega = 1,\ T = 2\pi$

(The vertical stretch factor is 1 but there is no feature corresponding to amplitude.)



$D = \left\{x \in \mathbb{R} : x \ne \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}\right\}$, $R = \mathbb{R}$

A = 1 but the amplitude is undefined,
$d = 0,\ p = 0,\ \omega = 1,\ T = \pi$

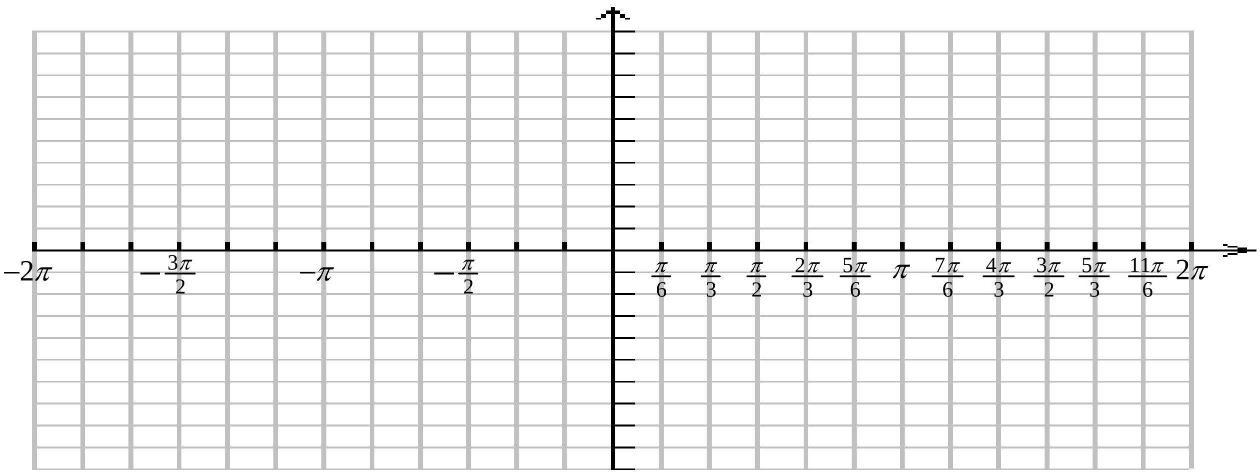


$D = \{x \in \mathbb{R} : x \ne n\pi, n \in \mathbb{Z}\}$, $R = \mathbb{R}$

A = 1 but the amplitude is undefined,
$d = 0,\ p = 0,\ \omega = 1,\ T = \pi$

Suggestions for Graphing Trigonometric Functions

1. Identify the transformations and express using ***mapping notation***.
2. Think carefully about the effect of the transformations on the features of the base graph
   - (a) ***Horizontal stretches/compressions*** affect the ***period*** and the ***locations of the vertical asymptotes***.
   - (b) ***Horizontal translations*** affect the ***locations of the vertical asymptotes*** and the ***phase shift***.
   - (c) ***Vertical stretches/compressions*** affect the ***amplitude*** (if applicable) and the ***y-co-ordinates of maximum/minimum points***.
   - (d) ***Vertical translations*** simply cause all the points on the graph to move up or down by some constant amount.
3. Apply the transformations to a few special points on the base function.
4. Sometimes it is easier to apply the stretches/compressions first to obtain the final "shape" of the curve. Then it is a simple matter to translate the curve into its final position.
5. To find a suitable scale for the x-axis, divide the period by a number that is divisible by 4. The number 12 works particularly well because it divides evenly into 360°, giving increments of 30° or $\frac{\pi}{6}$ radians (see diagram).



Graphing Exercises

Now sketch graphs of each of the following functions by applying appropriate transformations to one of the base functions given above. Once you are done, use TI-Interactive or a graphing calculator to check whether your graphs are correct. Detailed solutions are also available at http://www.misternolfi.com/courses.htm under "Unit 2 - Trigonometric Functions."

a) $y = 18\cos\left(\frac{\pi x}{4}\right) - 14$

b) $y = -\frac{4}{5}\sin\left(\frac{2}{7}\left(x + \frac{3\pi}{4}\right)\right) + 10$

c) $y = 101\cos\left(x - \frac{7\pi}{4}\right) - \frac{9}{10}$

d) $y = 6\sin(\pi x + 13) + 22$

e) $y = -\cos\left(\frac{5\pi}{3}(x - 1)\right) + 1$

f) $y = 15\sin\left(\frac{x}{15}\right) - 5$

g) $y = -2\tan\left(3\left(x + \frac{\pi}{4}\right)\right) - 3$

h) $y = -2\sec\left(\frac{2}{\pi}\left(x + \frac{\pi}{6}\right)\right) + 5$

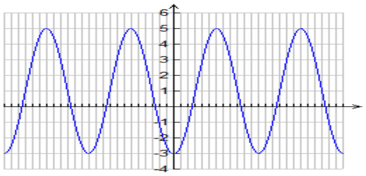
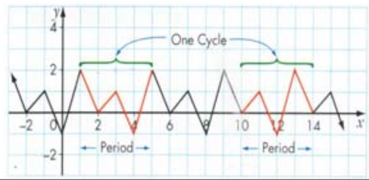
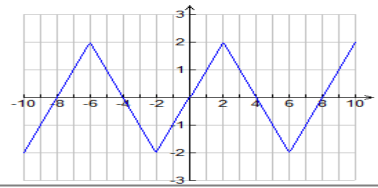
i) $y = 3\cot\left(\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right) + 2$

j) $y = \frac{5}{3}\csc\left(1.5x + \frac{\pi}{4}\right) + \frac{3}{2}$

Using Sinusoidal Functions to Model Periodic Phenomena

Summary

A sinusoidal function is any function of the form $f(x) = \mathbf{A}\sin\left(\boldsymbol{\omega}\left(x-\mathbf{p}\right)\right)+\mathbf{d}$. Such functions can be used to model ***periodic phenomena*** that involve some quantity alternately increasing and decreasing, “smoothly” and at regular intervals, between a maximum and minimum value. In addition to exhibiting this smooth and regular “up and down” behaviour, a sinusoidal function “spends” exactly the same amount of “time” increasing as it does decreasing. Note also that the horizontal axis of a sinusoidal function is located exactly at the average of the maximum and minimum values.

A Sinusoidal Periodic Function	***Periodic Functions that are NOT Sinusoidal***
	 

Vertical Transformations

$A =$ the factor by which the base function $y=\sin x$ is ***stretched vertically*** (may include reflection)

$|A| =$ the ***amplitude*** of the sinusoidal function (must be positive since length/distance cannot be negative)

$= \dfrac{\text{max-min}}{2}$

$d =$ the amount by which the base function $y=\sin x$ is ***translated vertically***

= the ***vertical displacement***

$= \dfrac{\text{max+min}}{2}$ = the average of the max and the min

=the average value of the sinusoidal function

$y = d \rightarrow$ the equation of the ***horizontal axis*** of the sinusoidal function

= the image of the x-axis under the transformation

$|d| =$ absolute value of the vertical displacement

= distance from the x-axis to the horizontal axis

Horizontal Transformations

$\dfrac{1}{\omega} =$ the factor by which the base function $y=\sin x$ is ***stretched horizontally*** (may include reflection)

$\omega =$ the ***angular frequency***

= how fast an object rotates (positive value→counterclockwise, negative value→clockwise)

$|\omega| =$ the absolute value of the ***angular frequency***

= number of cycles in 2π radians

$T =$ ***period***

= length of one cycle (must be positive since length/distance cannot be negative)

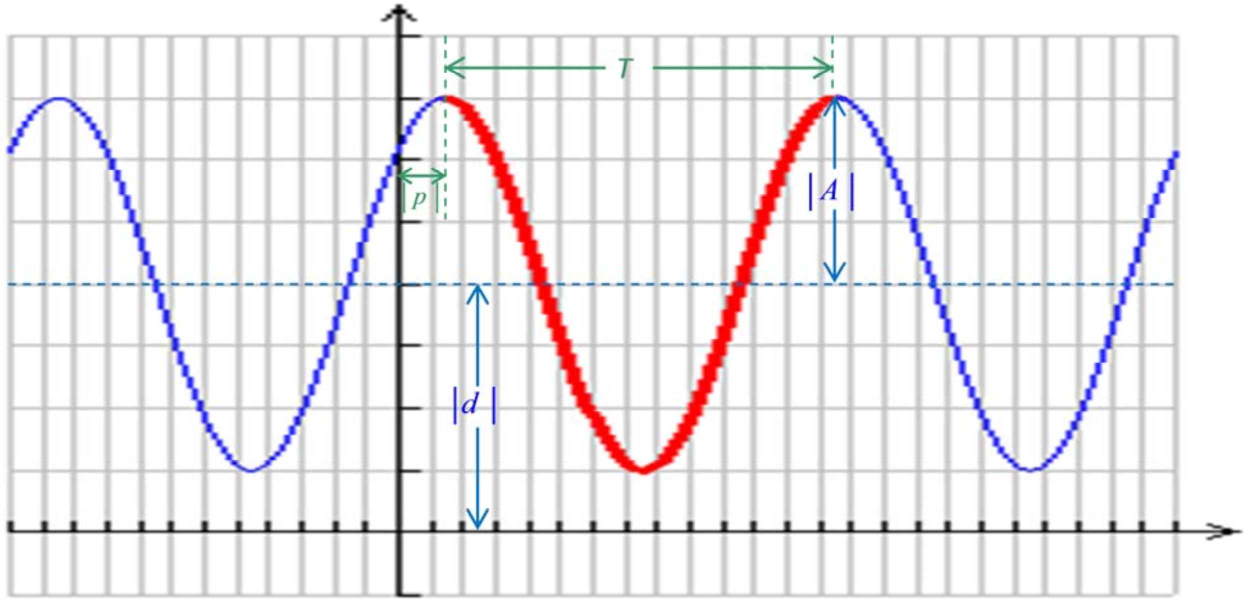
= (period of base function $y=\sin x$)(absolute value of the horizontal stretch/compression factor)

$= 2\pi\left|\dfrac{1}{\omega}\right|$

$p =$ the amount by which the base function $y=\sin x$ is ***translated horizontally***

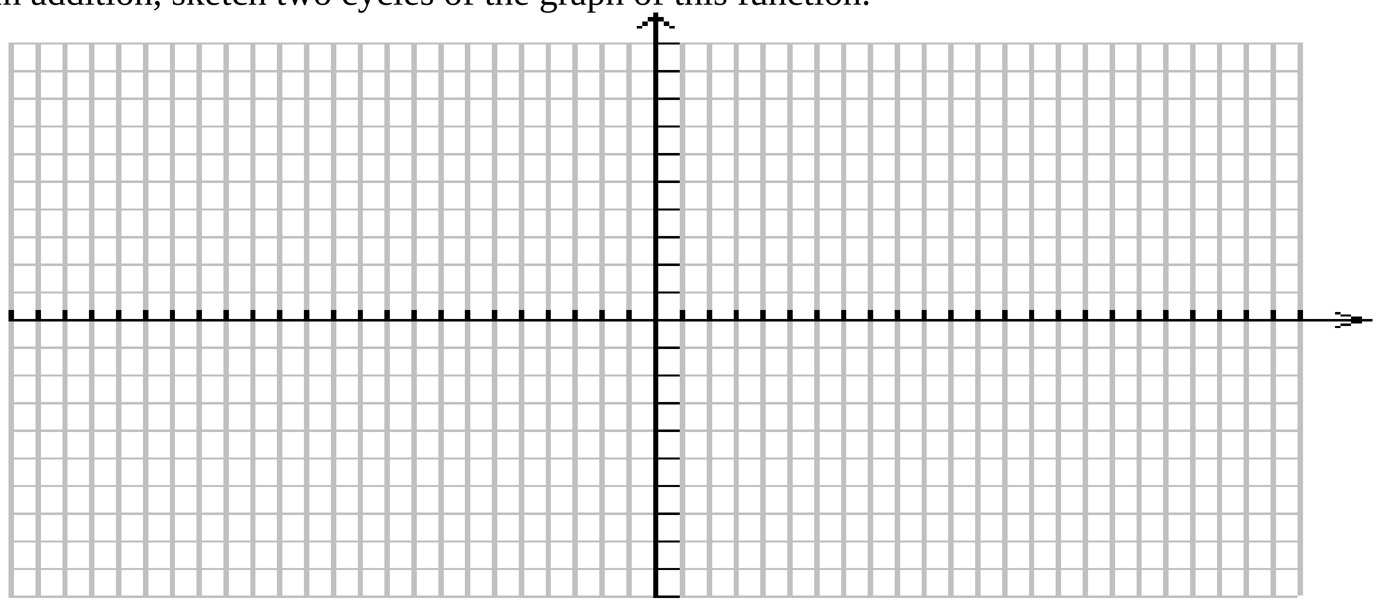
= the ***phase shift***

If it is difficult to determine the phase shift graphically, it can be ***calculated*** once A, d and ω are known. Simply choose a point known to be on the curve, substitute into the equation and solve for p. Alternatively, p can be determined by using the image of the point $(0,0)$, if $y=\sin x$ is used as the base function, or the image of the point $(0,1)$, if $y=\cos x$ is used as the base function. (See the example on page 33 for more details.)



Activity 1

1. A cosine curve has an amplitude of 3 units and a period of 3π radians. The equation of the axis is $y = 2$, and a horizontal shift of $\frac{\pi}{4}$ radians to the left has been applied. Write the equation of this function. In addition, sketch two cycles of the graph of this function.



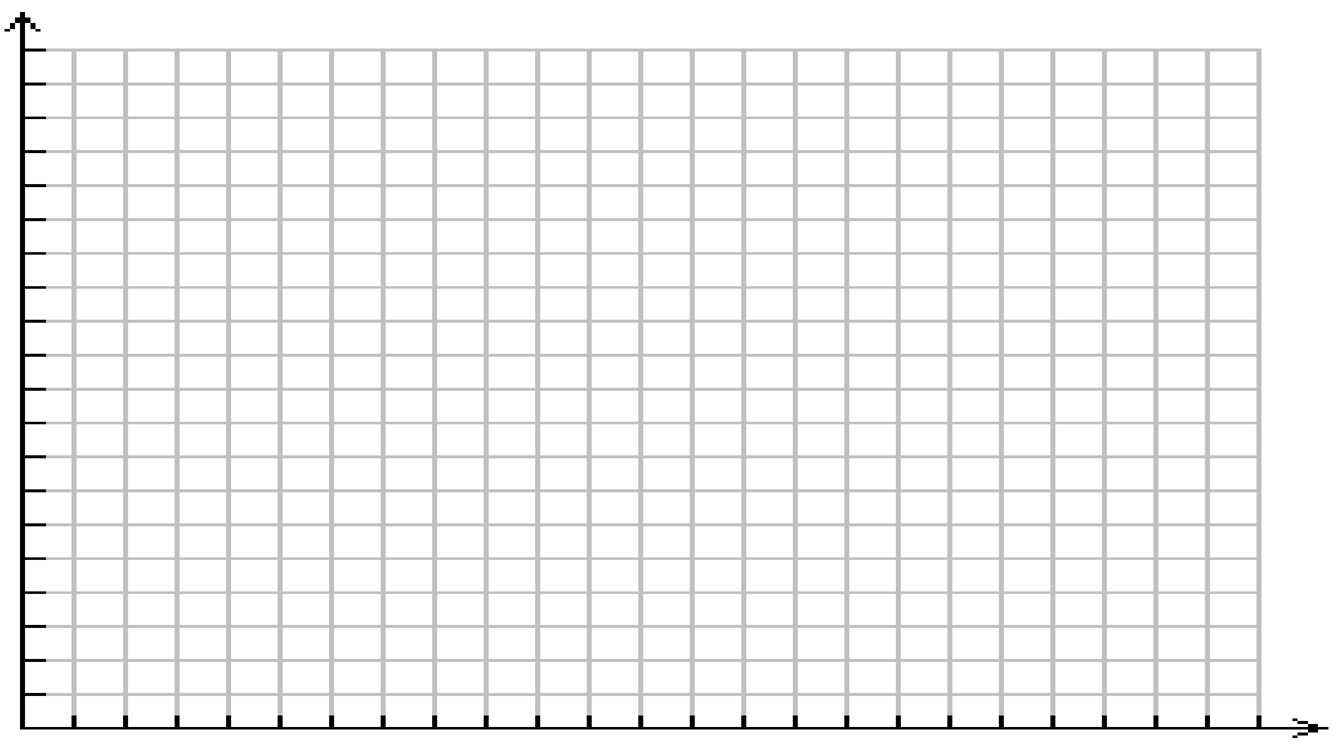
2. Determine the value of the function in question 1 if $x = \frac{\pi}{2}, \frac{3\pi}{4}$, and $\frac{11\pi}{6}$.

3. Use your graph to estimate the x-value(s) in the domain $0 < x < 2$, where $y = 2.5$, to one decimal place.

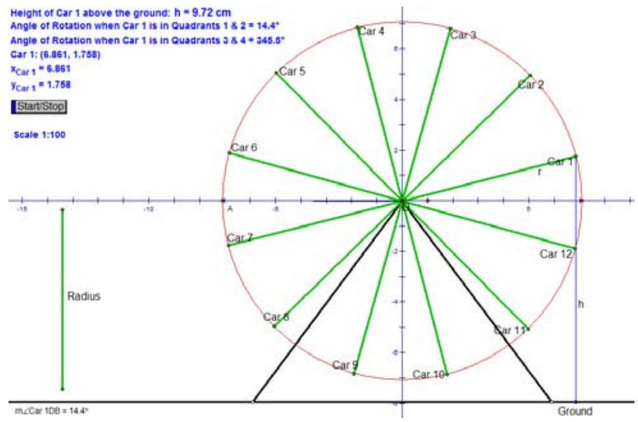
4. The number of hours of daylight in Vancouver can be modelled by a sinusoidal function of time, in days. The longest day of the year is June 21, with 15.7 h of daylight. The shortest day of the year is December 21, with 8.3 h of daylight.

a) Find an equation for $h(t)$, the number of hours of daylight on the tth day of the year. In addition, sketch one cycle of the graph of this function.

b) Use your equation to predict the number of hours of daylight in Vancouver on January 30th.



Activity 2 – Ferris Wheel Simulation



Either from www.misternolfi.com or from the "I:" drive, load the Geometer's Sketchpad Ferris wheel simulation. Once you understand how to start and stop the animation and how to interpret the given information, answer the questions found below.

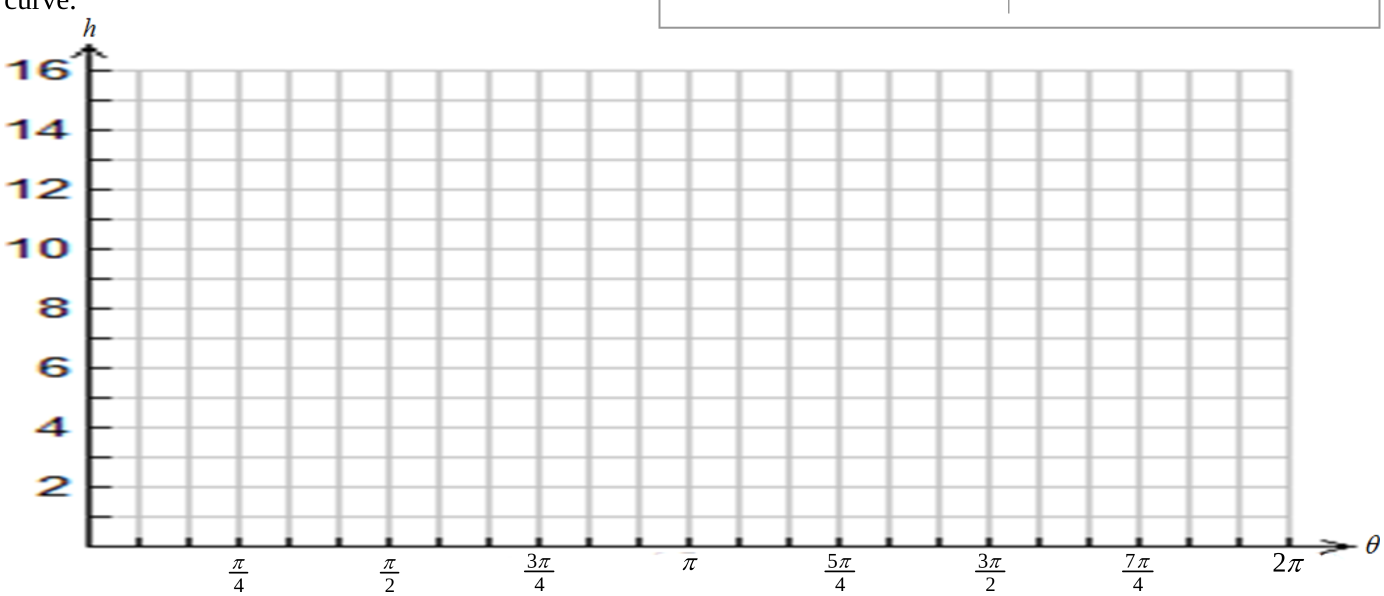
Questions

This simulation involves finding out how the height of "Car 1" above the ground ***is related to*** the angle of rotation of the line segment joining "Car 1" to the axis of rotation of the Ferris wheel.

1. Complete the table at the right. Stop the animation each time that a car reaches the x-axis (the car ***does not*** need to be exactly on the x-axis). Each time that you stop the animation, record the angle of rotation of "Car 1" and its height above the ground. (***Note***: Because of limitations of Geometer's Sketchpad, it was not possible to display the angle of rotation using a single formula. A different formula was used for quadrants III and IV so be careful when recording the data!)

Angle of Rotation of Car 1	*Height of Car 1*

2. Now use the given grid to plot the data that you recorded in question 1. Once you have plotted all the points, join them by sketching a smooth curve that passes through all the points. Does your curve look familiar? Try to write an equation that describes the curve.



3. For this question, you may use either a graphing calculator or TI-Interactive. First, take the data from the above table and create two lists (e.g. L1 and L2). Then perform a ***sinusoidal regression***. (Performing a regression means that the data are "fit" to a mathematical function. A ***sinusoidal regression*** finds the sinusoidal function that ***best fits*** the data.) How does the equation produced by the regression compare to the equation that you wrote in question 2?

4. Now use a graphing calculator or TI-Interactive to graph the function produced by the regression. How does it compare to the graph that you sketched in question 2?

5. Use the equation obtained in question 4 to predict the height of "Car 1" when its angle of rotation is 2 radians.

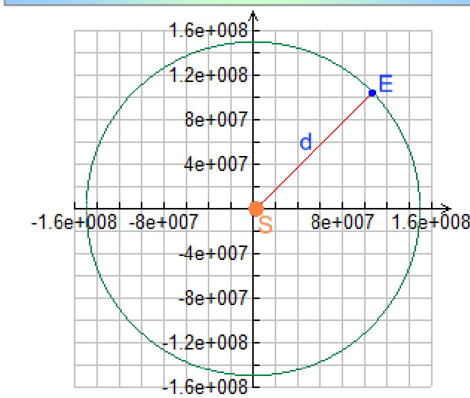
Activity 3 – Earth’s Orbit

The table below gives the approximate distance from the Earth to the Sun on certain days of a particular year.

Date	*Day of the Year*	*Earth’s Distance (d) from Sun (km)*
January 3	2	1.47098×10^{8}
February 2	32	1.47433×10^{8}
March 5	63	1.48349×10^{8}
April 4	93	1.49599×10^{8}
May 5	124	1.50848×10^{8}
June 4	154	1.51763×10^{8}
July 5	185	1.52098×10^{8}
August 4	215	1.51763×10^{8}
September 4	246	1.50848×10^{8}
October 4	276	1.49599×10^{8}
November 4	307	1.48349×10^{8}
December 4	337	1.47433×10^{8}

Perihelion is the point in the Earth’s orbit at which it is closest to the sun. Perihelion occurs in early January.

Aphelion is the point in the Earth’s orbit at which it is farthest from the sun. Aphelion occurs in early July.

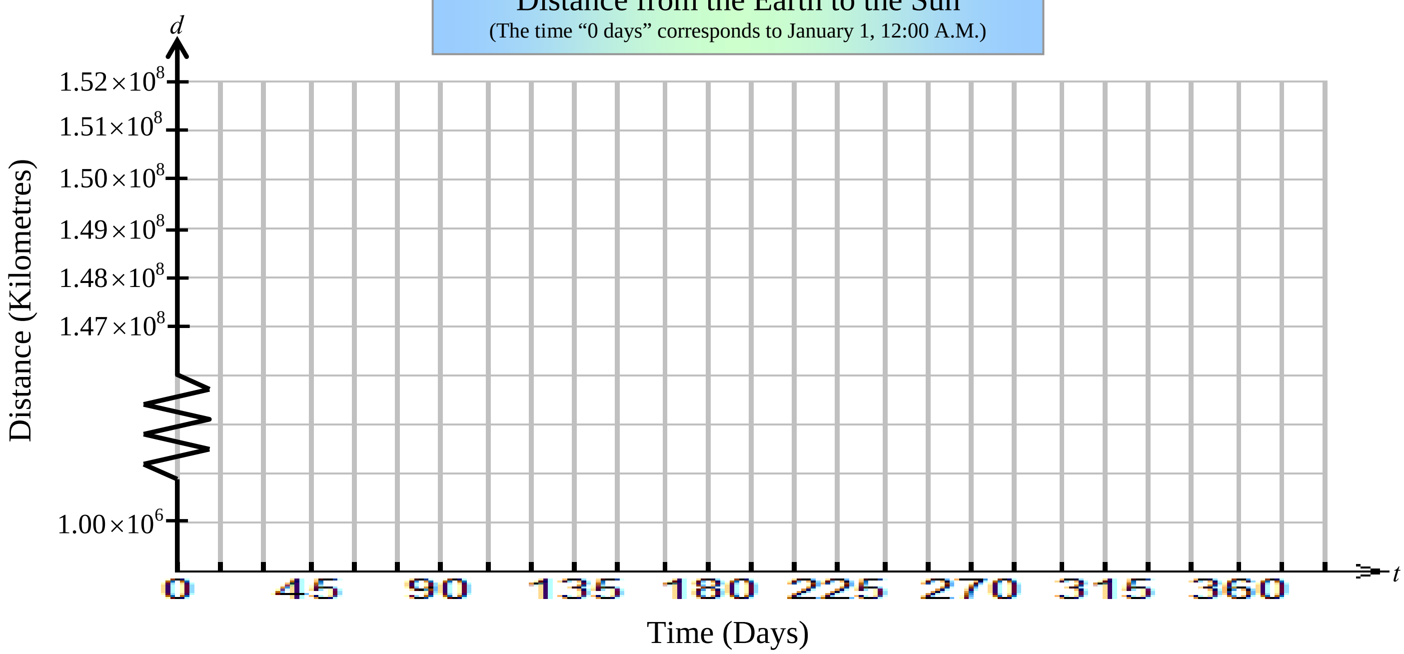


The Earth’s orbit around the Sun is an ellipse that is very close to a perfect circle. The Sun is located at one of the two ***foci*** (singular ***focus***) of the ellipse.

Questions

1. Use the grid below to plot the data in the above table. Once you have done so, join the points with a smooth curve. Use your knowledge of trigonometric functions to write an equation of the curve.

Distance from the Earth to the Sun
(The time “0 days” corresponds to January 1, 12:00 A.M.)



2. Now use TI-Interactive or a graphing calculator to perform a sinusoidal regression on the data in the above table. Compare the equation obtained by regression to the one that you wrote in question 1.
3. Are you surprised that perihelion occurs in early January and that aphelion occurs in early July? Explain.
4. Use the equation obtained in question 2 to predict the distance from the Earth to the Sun on Valentine’s Day.
5. Suppose that the Earth’s orbit were highly elliptical instead of being nearly a perfect circle. Do you think that life as we know it would still exist? Explain.

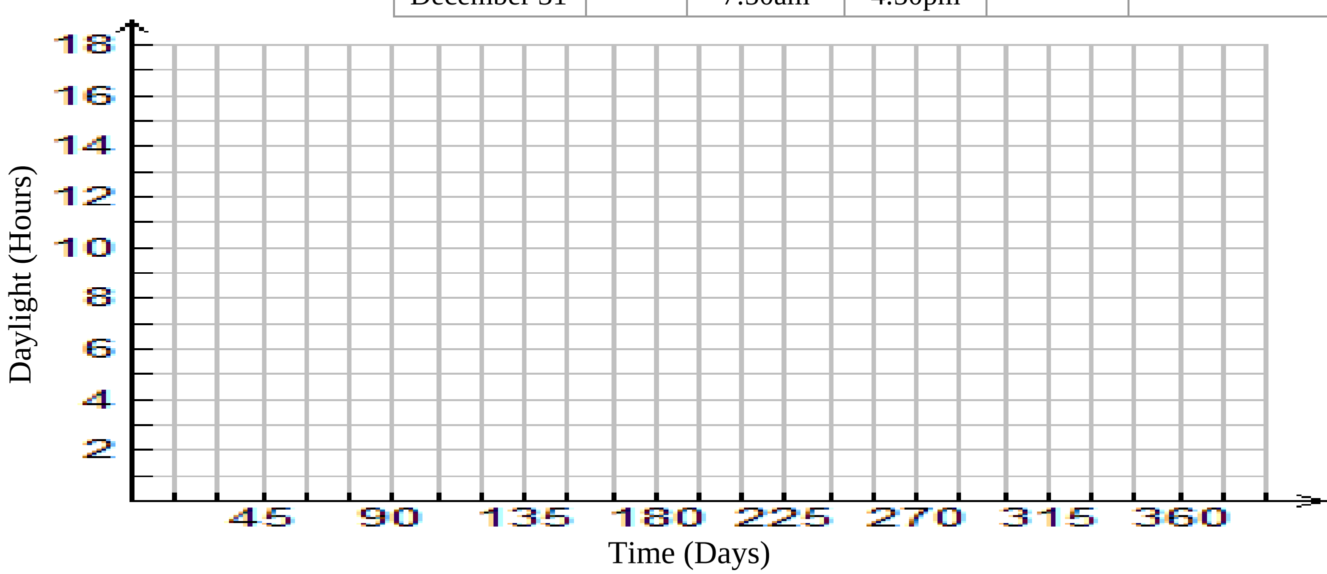
Activity 4 – Sunrise/Sunset

The table at the right contains sunrise and sunset data for Toronto, Ontario for the year 2007. (Data obtained from www.sunrisesunset.com .)

Questions

1. Complete the table.
2. Use the provided grid to plot a graph of ***number of daylight hours versus the day of the year***. First plot the points and then draw a smooth curve through the points.
3. Write an equation that describes the curve that you obtained in question 2.
4. Use TI-Interactive or a graphing calculator to perform a sinusoidal regression. Enter the values for "day of the year" in L1 and "number of daylight hours" in L2. Compare the equation obtained by the regression to the one that you wrote in question 3.
5. Use your equation to predict the number of daylight hours on December 25.
6. Suppose that you lived in a town situated exactly on the equator. How would the graph of number of hours of daylight versus day of the year differ from the one for Toronto?

Date	*Day of the Year*	*Sunrise (hh:mm)*	*Sunset (hh:mm)*	*Daylight (hh:mm)*	*Number of daylight hours to the nearest 100th of an hour*
January 1	0	7:51am	4:51pm	9:00	9
January 15	14	7:48am	4:58pm	9:10	9.17
January 29		7:38am	5:23pm		
February 12		7:21am	5:42pm		
February 26		7:00am	6:01pm		
March 12		7:36am	7:19pm		
March 26		7:11am	7:36pm		
April 9		6:46am	7:52pm		
April 23		6:23am	8:09pm		
May 7		6:03am	8:25pm		
May 21		5:48am	8:40pm		
June 4		5:38am	8:53pm		
June 18		5:36am	9:01pm		
July 2		5:40am	9:02pm		
July 16		5:51am	8:56pm		
July 30		6:04am	8:44pm		
August 13		6:19am	8:26pm		
August 27		6:35am	8:04pm		
September 10		6:50am	7:39pm		
September 24		7:06am	7:14pm		
October 8		7:22am	6:48pm		
October 22		7:39am	6:25pm		
November 5		6:57am	5:05pm		
November 19		7:15am	4:50pm		
December 3		7:32am	4:42pm		
December 17		7:44am	4:42pm		
December 31		7:50am	4:50pm		



Example

On a particular Ferris wheel, the maximum height of a passenger above the ground is 35 m. The wheel takes 2 minutes to complete one revolution and the passengers board the Ferris wheel 2 m above the ground at the bottom of its rotation.

(a) Sketch two cycles of the graph of height of passenger (in metres) versus time (in seconds).

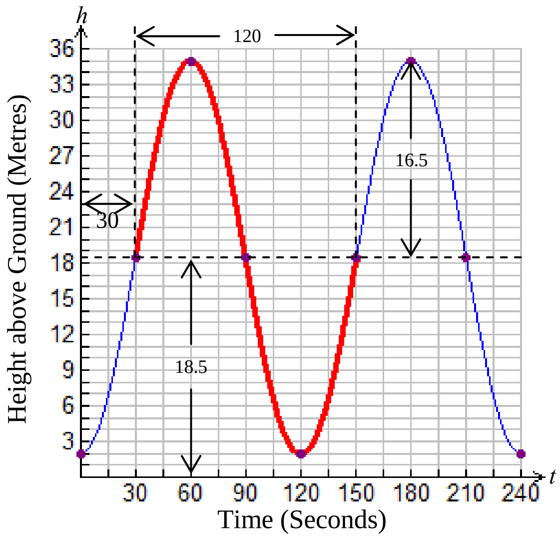
(b) Write ***an*** equation of the graph that you obtained in part (a).

(c) How high is the passenger after 25 s?

(d) If the ride lasts six minutes, at what times will the passenger be at the maximum height?

Solution

(a) For this question, we shall assume that the passenger is 2 m above the ground at time $t = 0$.



(b) Maximum Height = 35 m, Minimum Height = 2 m

$A = (35-2) \div 2 = 16.5$

$d = (35+2) \div 2 = 18.5$ (the average of the max and min)

Since it takes 120 seconds to complete one rotation, $T = 120$.

But $T = 2\pi\left(\dfrac{1}{\omega}\right)$, which implies that $2\pi\left(\dfrac{1}{\omega}\right) = 120$.

$$\therefore \omega = \frac{\pi}{60}$$

Finally, it's obvious from the graph that if we use $y = \sin x$ as the base function, $p = 30$.

$$\therefore h(t) = 16.5\sin\left(\tfrac{\pi}{60}(t-30)\right) + 18.5$$

Alternatively, the value of p can be found by using either of the following two methods:

Method 1

$\because h(t) = 16.5\sin\left(\tfrac{\pi}{60}(t-p)\right) + 18.5$ and $h(0) = 2$

$\therefore 16.5\sin\left(\tfrac{\pi}{60}(0-p)\right) + 18.5 = 2$

$\therefore \sin\left(\dfrac{-p\pi}{60}\right) = \dfrac{2-18.5}{16.5}$

$\therefore \sin\left(\dfrac{-p\pi}{60}\right) = -1$

$\therefore \dfrac{-p\pi}{60} = \dfrac{-\pi}{2}$

$\therefore p = 30$

Method 2

In mapping notation, the transformation is expressed as $(x, y) \to (\omega^{-1}x + p, Ay + d)$. As we can see from the graph, the image of $(0,0)$ is $(30, 18.5)$.

$\therefore \omega^{-1}(0) + p = 30 \Rightarrow p = 30$

(c) $h(25) = 16.5\sin\left(\tfrac{\pi}{60}(25-30)\right) + 18.5 \doteq 14.2$

At 25 seconds, the passenger was about 14.2 m above the ground. (Make sure that your calculator is in radians mode!)

(d) The passenger is at the maximum height whenever $h(t) = 35$. From the graph we can see that this occurs at $t = 60$ s and $t = 180$ s. Since h is periodic, $h(180+120) = h(180) = 35$ (120 radians is the period). Therefore, the passenger is at the maximum height at 60 s, 180 s and 300 s.

Note on Angular Frequency

In the above problem we determined that $\omega = \frac{\pi}{60}$. Since the angular frequency ω is equal to the number of cycles in 2π radians, the Ferris wheel completes $\frac{\pi}{60}$ cycles in a span of 2π radians. Since the independent variable is time and is measured in units of seconds, the Ferris wheel completes $\frac{\pi}{60}$ revolutions in 2π seconds or $\frac{1}{120}$ of a revolution in 1 second. Now $\frac{1}{120}$ revolutions/s = $\frac{1}{120}(2\pi)$ radians/s $= \frac{\pi}{60}$ radians/s. Therefore, the Ferris wheel turns at a rate of $\frac{\pi}{60}$ radians/s. Hence, the angular frequency $\omega = \frac{\pi}{60}$ determines the rate of rotation of the Ferris wheel.

Homework

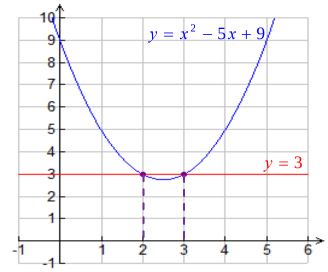
pp. 360 – 362: 1, 4, 6, 7, 9, 10, 11, 13

Trigonometric Identities

Important Prerequisite Information – Different Types of Equations

☺ *Equations that are Solved for the Unknown*

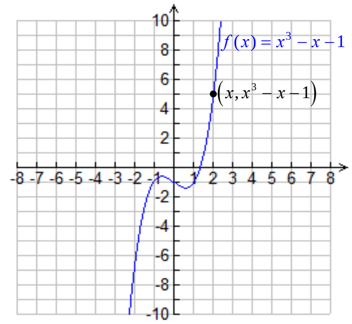
e.g. Solve $x^2 - 5x + 9 = 3$



- This means that we need to ***find*** the value(s) of x that make the left-hand-side equal to the right-hand-side.
- Geometrically, this equation describes the ***x-co-ordinates*** of the ***points of intersection*** of the graphs of $y = x^2 - 5x + 9$ and $y = 3$.
- As can be seen in the graph at the right, there are only two points of intersection and hence, only two solutions $x = 2$ and $x = 3$.

☺ *Equations that Express Mathematical Relationships (i.e. Functions, Relations)*

e.g. $f(x) = x^3 - x - 1$, $\quad x^2 + y^2 = 16$, $\quad c^2 = a^2 + b^2$ (Pythagorean Theorem)



- Such equations express a ***relationship*** between an ***independent variable*** (or a group of independent variables) and a ***dependent variable***. For instance, in the graph of $f(x) = x^3 - x - 1$ shown at the right, any point lying on the curve must have co-ordinates $(x, x^3 - x - 1)$. Once the value of the independent variable x is chosen, the dependent variable y ***must*** have a value of $x^3 - x - 1$.
- Equations that express relationships cannot be solved in the same sense as equations such as $x^2 - 5x + 9 = 3$ are solved because there are two or more unknowns.
- However, it does make sense to use algebraic manipulations to rewrite them in a different form.
- If x is allowed to vary continuously, such equations usually describe (piecewise) continuous curves.
- If x is restricted to integral or rational values (i.e. whole numbers or fractions), the graphs of such functions will be a discrete collection of points in the Cartesian plane.

☺ *Identities*

An ***identity*** is an ***equation that expresses the equivalence of two expressions***.

e.g. $(a+b)^2 = a^2 + 2ab + b^2$ $\quad \cos^2\theta + \sin^2\theta = 1$, $\quad \tan\theta = \dfrac{\sin\theta}{\cos\theta}$

- The given equations are identities. ***For all values of the unknown(s) that make sense***, the left-hand-side ***equals*** the right-hand-side. That is, the expression on the left side ***is equivalent to*** the expression on the right side.
- For the identity $(a+b)^2 = a^2 + 2ab + b^2$, there are no restrictions on the values of a and b.
- For the identity $\cos^2\theta + \sin^2\theta = 1$, there are no restrictions on the value of θ.
- For the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, we cannot allow θ to take on values that make $\cos\theta = 0$ because this would lead to the undefined operation of dividing by zero.

Note

- Identities ***need not*** involve trigonometry!
- To discourage the erroneous notion that θ is the only symbol that is allowed to be used to represent the independent variable of a trigonometric function, x will often be used in place of θ. It should be clear that any symbol whatsoever can be used as long as meaning is not compromised.
- Once an equation is proved to be an identity, it can be used to construct proofs of other identities. Examples are given below.

List of Identities that we already Know

Pythagorean Identities	*Quotient Identities*	*Reciprocal Identities*
For all $x \in \mathbb{R}$, $\cos^2 x + \sin^2 x = 1$ The following can be derived very easily from the above identity. In mathematical terms, we say that they are ***corollaries*** of $\cos^2 x + \sin^2 x = 1$. $\sin^2 x = 1 - \cos^2 x$ $\cos^2 x = 1 - \sin^2 x$ $1 + \tan^2 x = \sec^2 x$ $1 + \cot^2 x = \csc^2 x$	For all $x \in \mathbb{R}$ such that $\cos x \neq 0$, $\tan x = \dfrac{\sin x}{\cos x}$ The following identity can be derived very easily from the above identity. For all $x \in \mathbb{R}$ such that $\sin x \neq 0$, $\cot x = \dfrac{\cos x}{\sin x}$	The following identities can be derived easily from the definitions of csc, sec and cot. For all $x \in \mathbb{R}$ such that $\sin x \neq 0$, $\csc x = \dfrac{1}{\sin x}$ For all $x \in \mathbb{R}$ such that $\cos x \neq 0$, $\sec x = \dfrac{1}{\cos x}$ For all $x \in \mathbb{R}$ such that $\tan x \neq 0$, $\cot x = \dfrac{1}{\tan x}$

Important Note about Notation

- $\sin^2 x$ is a shorthand notation for $(\sin x)^2$, which means that ***first*** $\sin x$ is evaluated, ***then*** the result is squared

  e.g. $\sin^2 \dfrac{\pi}{4} = \left(\sin \dfrac{\pi}{4}\right)^2 = \left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1}{2}$ $\qquad$ $\cos^2 \dfrac{\pi}{3} = \left(\cos \dfrac{\pi}{3}\right)^2 = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

- This notation is used to avoid the excessive use of parentheses
- $\sin x^2 \neq \sin^2 x$ $\qquad$ The expression $\sin x^2$ means that ***first*** x is squared and ***then*** "sin" is applied

Proofs of the Pythagorean and Quotient Identities

Prove the following identities. (Here L.S. means "left side" and R.S. means "right side.")

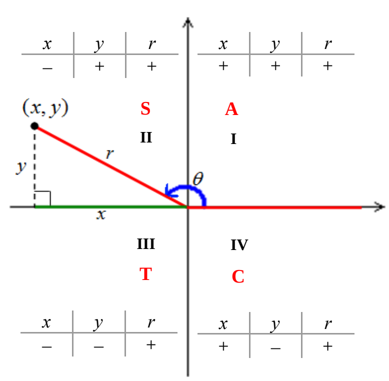
1. $\sin^2 \theta + \cos^2 \theta = 1$ (Assertion)

Proof (Justification)

$$\begin{aligned}
\text{L.S.} &= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 \quad \text{(definitions of sin, cos)} \\
&= \frac{y^2}{r^2} + \frac{x^2}{r^2} \\
&= \frac{y^2 + x^2}{r^2} \\
&= \frac{r^2}{r^2} \quad \text{(Pyth. Theorem)} \\
&= 1 \\
&= \text{R.S.}
\end{aligned}$$

$\therefore$ L.S. = R.S.

$\therefore \sin^2 \theta + \cos^2 \theta = 1$



CRUX of the problem

crux: a vital, basic, decisive or pivotal point

Once you have identified the crux of a problem and found a way to proceed, the rest of the solution is usually straightforward.

2. $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$ (Assertion)

Proof (Justification)

$$\begin{aligned}
\text{L.S.} &= \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)} \quad \text{(definitions of sin, cos)} \\
&= \left(\frac{y}{r}\right)\left(\frac{r}{x}\right) \\
&= \frac{y}{x} \\
&= \tan \theta \quad \text{(definition of tan)} \\
&= \text{R.S.}
\end{aligned}$$

$\therefore$ L.S. = R.S.

$\therefore \tan \theta = \dfrac{\sin \theta}{\cos \theta}$

Examples

Prove that each of the following equations is an identity.

Assertions

1. $\cot x = \dfrac{\cos x}{\sin x}$

Proof (Justification)

$$\text{L.S.} = \cot x$$

Assertions

$$= \frac{1}{\tan x} \quad \text{(reciprocal identity)}$$

$$= \frac{1}{\left(\dfrac{\sin x}{\cos x}\right)} \quad \text{(quotient identity)}$$

Justifications

$$= \frac{1}{1} \times \left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{\cos x}{\sin x}$$

$$\text{R.S.} = \frac{\cos x}{\sin x}$$

$\therefore$ L.S. = R.S.

$\therefore \cot x = \dfrac{\cos x}{\sin x}$ //

2. $1 + \tan^2 x = \sec^2 x$

Proof (Justification)

$\because \cos^2 x + \sin^2 x = 1$ (proved above)

$$\therefore \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\therefore \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$\therefore 1 + \tan^2 x = \sec^2 x$ //

Note

- The final step is justified by one of the quotient identities and one of the reciprocal identities.
- The nature of this proof is a little different from the others because it does not begin with the left side or right side of the given equation. Instead, it begins with an identity that has already been proved and through algebraic manipulations, the desired equation is derived!

3. $\dfrac{\sin^2 x}{1-\cos x} = 1 + \cos x$

Proof (Justification)

$$\text{L.S.} = \frac{\sin^2 x}{1-\cos x}$$

$$= \frac{1-\cos^2 x}{1-\cos x} \quad \text{(Pyth identity)}$$

$$= \frac{(1-\cos x)(1+\cos x)}{1-\cos x}$$

$$= 1 + \cos x$$

$$\text{R.S.} = 1 + \cos x$$

$\therefore$ L.S. = R.S

$\therefore \dfrac{\sin^2 x}{1-\cos x} = 1 + \cos x$ //

Exercises

1. Prove the rest of the Pythagorean identities (i.e. the ones that have not been proved on pages 35-36).

2. Prove the reciprocal identities by using the definitions of sin, cos, tan, csc, sec and cot (i.e. $\sin\theta = \dfrac{y}{r}$, $\cos\theta = \dfrac{x}{r}$, etc).

Proofs that make use solely of definitions are known as ***proofs from first principles*** because they do not rely upon any "facts" that are derived.

Logical and Notational Pitfalls – Please Avoid Absurdities!

1. The purpose of a proof is to ***establish*** the "truth" of a mathematical statement. ***Therefore, you must never assume what you are trying to prove!*** A common error is shown at the right. ***The series of steps shown is wrong and would be assigned a mark of zero!*** To write a correct proof, the left and right sides of the equation must be treated separately. Only once you have demonstrated that the left side is equal to the right side are you allowed to declare their equality.

"Proof"

$$\cot x = \frac{\cos x}{\sin x}$$

$$\therefore \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

$$\therefore \frac{1}{\left(\dfrac{\sin x}{\cos x}\right)} = \frac{\cos x}{\sin x}$$

$$\therefore \frac{1}{1} \times \left(\frac{\cos x}{\sin x}\right) = \frac{\cos x}{\sin x}$$

$$\therefore \frac{\cos x}{\sin x} = \frac{\cos x}{\sin x}$$

2. Keep in mind that words like "sin," "cos" and "tan" ***are function names, not numerical values!*** Therefore, you must not treat them as numbers. For example, it makes sense to write $\dfrac{\sin 2x}{\sin x}$ but it makes no sense whatsoever to "cancel" the sines. Many students will write statements such as $\cancel{\dfrac{\sin 2x}{\sin x} = \dfrac{2x}{x}} = 2$, which are completely nonsensical. First of all, dividing the numerator and the denominator by "sin" is invalid because "sin" is not a number. Furthermore, a simple test reveals that $\dfrac{\sin 2x}{\sin x} \neq 2$: $\quad \dfrac{\sin 2(\frac{\pi}{4})}{\sin \frac{\pi}{4}} = \dfrac{\sin \frac{\pi}{2}}{\sin \frac{\pi}{4}} = \dfrac{1}{1/\sqrt{2}} = \sqrt{2}$. Clearly, $\sqrt{2} \neq 2$. Therefore, the assertion that was made is entirely false!

Suggestions for Proving Trig Identities

1. Write the given expressions in terms of sin and cos.
2. Begin with the more complicated side and try to simplify it.
3. Keep a list of important identities in plain view while working.
4. Expect to make mistakes! If one approach seems to lead to a dead end, try another. Don't give up!

Homework

Do a representative selection of questions 1 to 20.

1. State an equivalent expression for each.

a) $\cos x \tan x$ **b)** $\sin^2 x$
c) $\cos^2 x$ **d)** $\tan^2 x$
e) $\tan x \sin x$ **f)** $1 - \sin^2 x$
g) $\sin x \tan x \cos x$ **h)** $1 - \cos^2 x$
i) $\sin^2 x + \cos^2 x$

2. Prove each identity.

a) $\dfrac{\sin x}{\tan x} = \cos x$

b) $\sin x \cos x \tan x = 1 - \cos^2 x$

c) $\dfrac{1-\cos^2 x}{\sin x} = \sin x$

d) $\sin^2 x + \dfrac{\sin x \cos x}{\tan x} = 1$

e) $1 + \dfrac{1}{\tan^2 x} = \dfrac{1}{\sin^2 x}$

f) $2\sin^2 x - 1 = \sin^2 x - \cos^2 x$

g) $\dfrac{1}{\cos x} - \cos x = \sin x \tan x$

h) $\sin x + \tan x = \tan x\,(1 + \cos x)$

i) $\dfrac{1}{1-\sin^2 x} = 1 + \tan^2 x$

j) $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$

k) $\sin^2 x + \cos^2 x + \tan^2 x = \dfrac{1}{\cos^2 x}$

l) $\dfrac{\sin x}{\sin x + \cos x} = \dfrac{\tan x}{1 + \tan x}$

m) $\dfrac{1+\tan^2 x}{1-\tan^2 x} = \dfrac{1}{\cos^2 x - \sin^2 x}$

3. Use a graphing calculator to show that each equation appears to be an identity. Then, prove that the equation is an identity.

a) $\cos x \tan x = \sin x$

b) $\sin x + \tan x = \tan x\,(1 + \cos x)$

c) $1 + \tan^2 x = \dfrac{1}{\cos^2 x}$

d) $\cos^2 x = \sin^2 x + 2\cos^2 x - 1$

e) $\dfrac{1}{1+\sin x} + \dfrac{1}{1-\sin x} = \dfrac{2}{\cos^2 x}$

f) $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$

4. Prove each identity.

a) $\dfrac{1}{\sin^2 x} + \dfrac{1}{\cos^2 x} = \dfrac{1}{\sin^2 x \cos^2 x}$

b) $\tan x + \dfrac{1}{\tan x} = \dfrac{1}{\sin x \cos x}$

c) $\dfrac{1}{1-\cos x} + \dfrac{1}{1+\cos x} = \dfrac{2}{\sin^2 x}$

d) $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

e) $(1 - \cos^2 x)\left(1 + \dfrac{1}{\tan^2 x}\right) = 1$

f) $\dfrac{1 + 2\sin x \cos x}{\sin x + \cos x} = \sin x + \cos x$

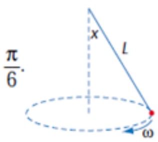
g) $\dfrac{\sin x}{1-\cos x} - \dfrac{1+\cos x}{\sin x} = 0$

h) $\sin^2 x - \sin^4 x = \cos^2 x - \cos^4 x$

i) $(1 + \tan^2 x)(1 - \cos^2 x) = \tan^2 x$

j) $\dfrac{\sin x - 1}{\sin x + 1} = \dfrac{-\cos^2 x}{(\sin x + 1)^2}$

5. Conical pendulum A conical pendulum is so named because of the cone-shaped path traced by the bob and the wire. The length of the pendulum wire, L, is related to the angle, x, that the wire makes with the vertical by the formula $L = \dfrac{g}{\omega^2 \cos x}$, where g is the acceleration due to gravity and ω is the angular velocity of the bob about the vertical, in radians per second. Another way of expressing the relationship is the formula $L = \dfrac{g \tan x}{\omega^2 \sin x}$.



a) Verify that the two formulas are equivalent when $x = \dfrac{\pi}{6}$.

b) Prove that $\dfrac{g}{\omega^2 \cos x} = \dfrac{g \tan x}{\omega^2 \sin x}$ is an identity.

6. Kicking a ball When a ball is kicked from the ground, the time of flight of the ball can be determined by the formula $t = \dfrac{2v_0 \sin x}{g}$.

In this formula, t seconds is the time of flight, v_0 metres per second is the initial velocity of the ball, x is the angle that the path of the ball makes with the ground when the ball is kicked, and g is the acceleration due to gravity.

a) Write another formula that determines the time of flight of the ball.

b) Equate the trigonometric expressions from the given formula and the formula you found in part a) to write an equation.

c) Use a graphing calculator to check if the equation appears to be an identity.

d) If the equation appears to be an identity, prove that it is an identity.

e) The formula for the horizontal distance, d metres, travelled by a ball kicked from the ground is $d = \dfrac{2v_0^2 \sin x \cos x}{g}$.

Write another formula for the horizontal distance.

f) Equate the trigonometric expressions from the two formulas in part e) to write an equation. Use a graphing calculator to check if the equation appears to be an identity.

g) If the equation appears to be an identity, prove that it is an identity.

7. Prove each identity.

a) $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

b) $\sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x = 1$

c) $\dfrac{4}{\cos^2 x} - 5 = 4\tan^2 x - 1$

d) $\dfrac{\cos x - \sin x - \cos^3 x}{\cos x} = \sin^2 x - \tan x$

e) $\dfrac{\sin^2 x - 6\sin x + 9}{\sin^2 x - 9} = \dfrac{\sin x - 3}{\sin x + 3}$

8. Find a counterexample to show that each equation is not an identity.

a) $\sin x = \sqrt{\sin^2 x}$ **b)** $\cos x = \sqrt{\cos^2 x}$

9. Technology Use radian measure for the following.

a) In the same viewing window, graph $y = \sin x$ and $y = x$ for $-0.2 \le x \le 0.2$ and $-0.2 \le y \le 0.2$. Do the graphs suggest that $\sin x = x$ is an identity?

b) Repeat part a) for $-2 \le x \le 2$ and $-2 \le y \le 2$.

c) Write a conclusion about verifying identities graphically.

10. Application Use the x, y, and r definitions of $\sin x$ and $\cos x$ to prove the following identity.

$$\frac{\cos\theta}{1-\sin\theta} = \frac{1+\sin\theta}{\cos\theta}$$

11. Inquiry/Problem Solving Determine if each of the following equations is an identity or not.

a) $\dfrac{1}{\tan x} + \cos x = \dfrac{\cos x(1 + \sin x)}{\sin x}$ **b)** $\dfrac{1}{\tan x} + \cos x = \tan x + \sin x$

c) $\dfrac{1}{\tan x} + \cos x = \dfrac{2\cos x}{\sin x}$

12. Communication Explain why you think that the equation $(a + b)^2 = a^2 + 2ab + b^2$ can be called an algebraic identity.

13. Algebra If $x = a\cos\theta - b\sin\theta$ and $y = a\sin\theta + b\cos\theta$, show that $x^2 + y^2 = a^2 + b^2$.

14. Since $1 - \cos^2 x = \sin^2 x$ is an identity, is $\sqrt{1-\cos^2 x} = \sin x$ also an identity? Explain how you know.

15. a) Show graphically that $\sin^2 x + \cos^2 x = (\sin x + \cos x)^2$ is not an identity. Explain your reasoning.

b) Explain how the graph shows if there any values of x for which the equation is true.

16. Write a list of helpful strategies for proving trigonometric identities, and describe situations in which you would try each strategy. Compare your list with your classmates'.

17. Formulating problems **a)** Create a trigonometric identity that has not appeared in this section.
b) Have a classmate check graphically that your equation may be an identity. If so, have your classmate prove your identity.

18. Technology **a)** Use a graph to show that the equation $\frac{\cos^2 x - 1}{\cos x + 1} = \cos x - 1$ appears to be an identity.
b) Compare the functions defined by each side of the equation by displaying a table of values. Find a value of x for which the values of the two functions are not the same. Have you shown that the equation is not an identity? Explain.

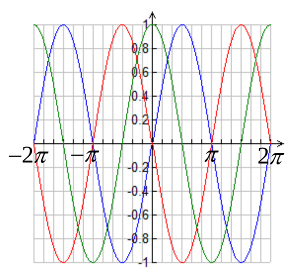
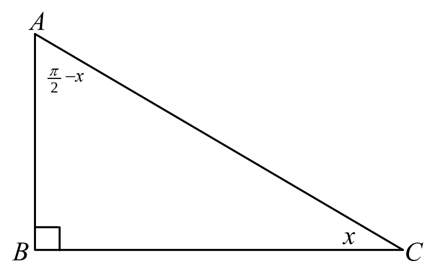
20. Prove that $\frac{\tan x \sin x}{\tan x + \sin x} = \frac{\tan x - \sin x}{\tan x \sin x}$.

Selected Answers

1. Answers may vary. **a)** $\sin\theta$ **b)** $1 - \cos^2\theta$ **c)** $1 - \sin^2\theta$ **d)** $\frac{\sin^2\theta}{\cos^2\theta}$ **e)** $\frac{\sin^2\theta}{\cos\theta}$ **f)** $\cos^2\theta$ **g)** $\sin^2\theta$ **h)** $\sin^2\theta$ **i)** 1 **5. a)** Each formula gives $\frac{2g}{\sqrt{3g^2}}$. **8. a)** $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$, $\sqrt{\sin^2\left(-\frac{\pi}{6}\right)} = \frac{1}{2}$; LHS ≠ RHS **b)** $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\sqrt{\cos^2\frac{2\pi}{3}} = \frac{1}{2}$; LHS ≠ RHS **11. a)** an identity **b)** not an identity **c)** an identity **14.** No; the left-hand side is never negative.

Exercises on Equivalence of Trigonometric Expressions

Complete the following table. The first row is done for you.

Identity	*Graphical Justification*	*Justification using Right Triangle or Angle of Rotation*
$\sin(\frac{\pi}{2} - x) = \cos x$	Since $\sin\left(\frac{\pi}{2} - x\right) = \sin\left(-1\left(x - \frac{\pi}{2}\right)\right)$, the graph of $y = \sin\left(\frac{\pi}{2} - x\right)$ can be obtained by reflecting $y = \sin x$ in the y-axis, followed by a shift to the right by $\frac{\pi}{2}$. Once these transformations are applied, lo and behold, the graph of $y = \cos x$ is obtained! 	 $\cos x = \frac{BC}{AC}$ $\sin\left(\frac{\pi}{2} - x\right) = \frac{BC}{AC}$ $\therefore \cos x = \sin\left(\frac{\pi}{2} - x\right)$
$\cos(\frac{\pi}{2} - x) = \sin x$		
$\cos(\frac{\pi}{2} + \theta) = -\sin\theta$		

Identity	*Graphical Justification*	*Justification using Angles of Rotation*
$\sin(\pi-\theta)=\sin\theta$		
$\cos(\pi-\theta)=-\cos\theta$		
$\sin(-\theta)=-\sin\theta$		
$\cos(-\theta)=\cos\theta$		

List of Important Identities that can be Discovered/Justified using Transformations

1. First complete "Exploring Equivalent Trigonometric Functions" on page 388-390 in the textbook.

2. Read the summary on page 40 (i.e. the next page).

3. Do the following questions for homework:

Homework
pp. 392 – 393: 1, 2, 3, 4, 5, 6, 7

In Summary

Key Ideas

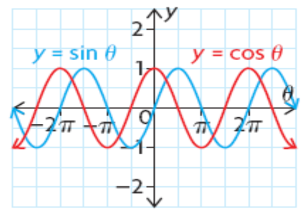
- Because of their periodic nature, there are many equivalent trigonometric expressions.
- Two expressions may be equivalent if the graphs created by a graphing calculator of their corresponding functions coincide, producing only one visible graph over the entire domain of both functions. To demonstrate equivalency requires additional reasoning about the properties of both graphs.

Need to Know

- Horizontal translations that involve multiples of the period of a trigonometric function can be used to obtain two equivalent functions with the same graph. For example, the sine function has a period of 2π, so the graphs of $f(\theta) = \sin \theta$ and $f(\theta) = \sin(\theta + 2\pi)$ are the same. Therefore, $\sin \theta = \sin(\theta + 2\pi)$.

- Horizontal translations of $\frac{\pi}{2}$ that involve both a sine function and a cosine function can be used to obtain two equivalent functions with the same graph. Translating the cosine function $\frac{\pi}{2}$ to the right $\left(f(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)\right)$ results in the graph of the sine function, $f(\theta) = \sin \theta$.

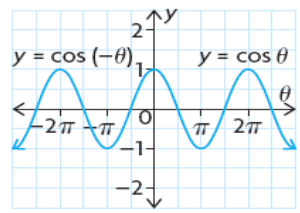
  Similarly, translating the sine function $\frac{\pi}{2}$ to the left $\left(f(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)\right)$ results in the graph of the cosine function, $f(\theta) = \cos \theta$.



$\sin \theta = \cos\left(\theta - \frac{\pi}{2}\right)$

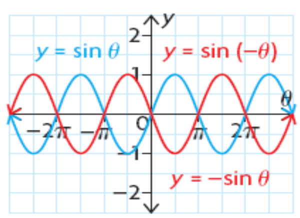
$\sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$

- Since $f(\theta) = \cos \theta$ is an even function, reflecting its graph across the y-axis results in two equivalent functions with the same graph.

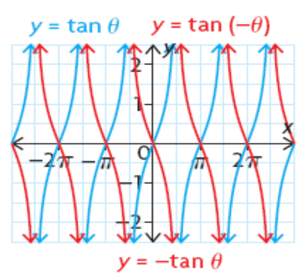


$\cos \theta = \cos(-\theta)$

- $f(\theta) = \sin \theta$ and $f(\theta) = \tan \theta$ are odd and have the property of rotational symmetry about the origin. Reflecting these functions across both the x-axis and the y-axis produces the same effect as rotating the function through 180° about the origin. Thus, the same graph is produced.

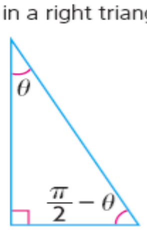


$\sin(-\theta) = -\sin \theta$



$\tan(-\theta) = -\tan \theta$

- The cofunction identities describe trigonometric relationships between the complementary angles θ and $\left(\frac{\pi}{2} - \theta\right)$ in a right triangle.



$$\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\tan \theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

- You can identify equivalent trigonometric expressions by comparing principal angles drawn in standard position in quadrants II, III, and IV with their related acute angle, θ, in quadrant I.

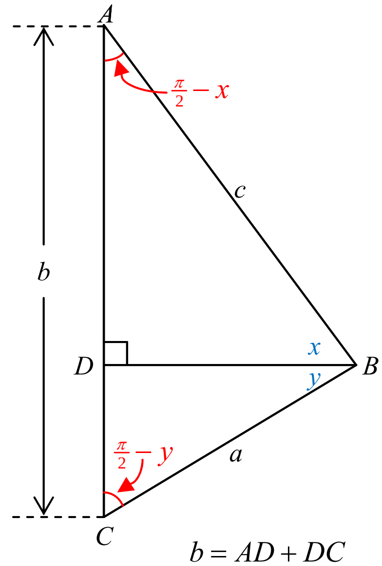
Principal Angle in Quadrant II	Principal Angle in Quadrant III	Principal Angle in Quadrant IV
$\sin(\pi - \theta) = \sin \theta$	$\sin(\pi + \theta) = -\sin \theta$	$\sin(2\pi - \theta) = -\sin \theta$
$\cos(\pi - \theta) = -\cos \theta$	$\cos(\pi + \theta) = -\cos \theta$	$\cos(2\pi - \theta) = \cos \theta$
$\tan(\pi - \theta) = -\tan \theta$	$\tan(\pi + \theta) = \tan \theta$	$\tan(2\pi - \theta) = -\tan \theta$

Compound Angle Identities

Question

If we know how to evaluate the trig ratios of the angles x and y, can we use these values to evaluate quantities such as $\sin(x+y)$ and $\sin(x-y)$? With a little hard work, we shall see that he answer to this question is "yes!"

Expressing sin(x+y) in terms of sin x, sin y, cos x and cos y



By the Law of Sines,

$$\frac{\sin(x+y)}{b}=\frac{\sin(\pi/2-x)}{a} \quad \text{and} \quad \frac{\sin(x+y)}{b}=\frac{\sin(\pi/2-y)}{c}$$

Using the cofunction identity $\sin(\pi/2-\theta)=\cos\theta$, the above equations can be written

$$\frac{\sin(x+y)}{b}=\frac{\cos x}{a} \quad (1) \qquad \text{and} \qquad \frac{\sin(x+y)}{b}=\frac{\cos y}{c} \quad (2)$$

By multiplying both sides of equations (1) and (2) by b, we obtain

$$\sin(x+y)=\frac{b\cos x}{a} \qquad (3)$$

$$\sin(x+y)=\frac{b\cos y}{c} \qquad (4)$$

Adding equations (3) and (4), we obtain $2\sin(x+y)=\dfrac{b\cos x}{a}+\dfrac{b\cos y}{c}$

$$\therefore \sin(x+y)=\frac{b\cos x}{2a}+\frac{b\cos y}{2c}$$

$$\therefore \sin(x+y)=\frac{bc\cos x}{2ac}+\frac{ab\cos y}{2ac} \qquad \text{(expressing with a common denominator)}$$

$$\therefore \sin(x+y)=\frac{b(c\cos x)}{2ac}+\frac{b(a\cos y)}{2ac}$$

$$\therefore \sin(x+y)=\frac{1}{2}\left(\frac{(AD+DC)DB}{ac}+\frac{(AD+DC)DB}{ac}\right) \qquad (\text{since } b=AD+DC \text{ and } c\cos x=a\cos y=DB)$$

$$\therefore \sin(x+y)=\frac{1}{2}\left(\frac{(AD)(DB)}{ac}+\frac{(DC)(DB)}{ac}+\frac{(AD)(DB)}{ac}+\frac{(DC)(DB)}{ac}\right)$$

$$\therefore \sin(x+y)=\frac{1}{2}\left[2\left(\frac{(AD)(DB)}{ac}\right)+2\left(\frac{(DC)(DB)}{ac}\right)\right]$$

$$\therefore \sin(x+y)=\frac{(AD)(DB)}{ac}+\frac{(DC)(DB)}{ac}$$

$$\therefore \sin(x+y)=\left(\frac{AD}{c}\right)\left(\frac{DB}{a}\right)+\left(\frac{DB}{c}\right)\left(\frac{DC}{a}\right)$$

$$\therefore \sin(x+y)=\sin x\cos y+\cos x\sin y$$

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$

Using sin(x+y)=sin x cos y + cos x sin y to Derive many other Compound Angle Identities

$$\begin{aligned}&\sin(x-y)\\&=\sin(x+(-y))\\&=\sin x\cos(-y)+\cos x\sin(-y)\\&=\sin x\cos y+\cos x(-\sin y)\\&=\sin x\cos y-\cos x\sin y\end{aligned}$$

$$\begin{aligned}&\cos(x+y)\\&=\sin(\pi/2-(x+y))\\&=\sin((\pi/2-x)-y)\\&=\sin(\pi/2-x)\cos(-y)+\cos(\pi/2-x)\sin(-y)\\&=\cos x\cos y+\sin x(-\sin y)\\&=\cos x\cos y-\sin x\sin y\end{aligned}$$

$$\begin{aligned}&\cos(x-y)\\&=\cos(x+(-y))\\&=\cos x\cos(-y)-\sin x\sin(-y)\\&=\sin x\cos y-\sin x(-\sin y)\\&=\cos x\cos y+\sin x\sin y\end{aligned}$$

$$\begin{aligned}&\tan(x+y)\\&=\frac{\sin(x+y)}{\cos(x+y)}\\&=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}\\&=\frac{\left(\dfrac{\sin x\cos y+\cos x\sin y}{\cos x\cos y}\right)}{\left(\dfrac{\cos x\cos y-\sin x\sin y}{\cos x\cos y}\right)}\\&=\frac{\left(\dfrac{\sin x\cos y}{\cos x\cos y}+\dfrac{\cos x\sin y}{\cos x\cos y}\right)}{\left(\dfrac{\cos x\cos y}{\cos x\cos y}-\dfrac{\sin x\sin y}{\cos x\cos y}\right)}\\&=\frac{\left(\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}\right)}{\left(1-\left(\dfrac{\sin x}{\cos x}\right)\left(\dfrac{\sin y}{\cos y}\right)\right)}\\&=\frac{\tan x+\tan y}{1-\tan x\tan y}\end{aligned}$$

$$\begin{aligned}&\tan(x-y)\\&=\tan(x+(-y))\\&=\frac{\tan x+\tan(-y)}{1-\tan x\tan(-y)}\\&=\frac{\tan x+(-\tan y)}{1-\tan x(-\tan y)}\\&=\frac{\tan x-\tan y}{1+\tan x\tan y}\end{aligned}$$

$$\begin{aligned}&\cot(x+y)\\&=\frac{1}{\tan(x+y)}\\&=\frac{1}{\left(\dfrac{\tan x+\tan y}{1-\tan x\tan y}\right)}\\&=\frac{1-\tan x\tan y}{\tan x+\tan y}\\&=\frac{\left(1-\dfrac{1}{\cot x\cot y}\right)}{\left(\dfrac{1}{\cot x}+\dfrac{1}{\cot y}\right)}\\&=\frac{\left(\dfrac{\cot x\cot y-1}{\cot x\cot y}\right)}{\left(\dfrac{\cot x+\cot y}{\cot x\cot y}\right)}\\&=\frac{\cot x\cot y-1}{\cot x+\cot y}\end{aligned}$$

$$\begin{aligned}&\cot(x-y)\\&=\cot(x+(-y))\\&=\frac{\cot x\cot(-y)-1}{\cot x+\cot(-y)}\\&=\frac{\cot x(-\cot y)-1}{\cot x+(-\cot y)}\\&=\frac{-\cot x\cot y-1}{\cot x-\cot y}\\&=\frac{-1(\cot x\cot y+1)}{\cot x-\cot y}\\&=\frac{\cot x\cot y+1}{-1(\cot x-\cot y)}\\&=\frac{\cot x\cot y+1}{\cot y-\cot x}\end{aligned}$$

Summary

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$

$$\sin(x-y)=\sin x\cos y-\cos x\sin y$$

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$

$$\cos(x-y)=\cos x\cos y+\sin x\sin y$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

$$\cot(x+y)=\frac{\cot x\cot y-1}{\cot x+\cot y}$$

$$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x}$$

Using Compound Angle Identities to Derive Double Angle Identities

$$\begin{aligned}&\sin 2x\\&=\sin(x+x)\\&=\sin x\cos x+\cos x\sin x\\&=2\sin x\cos x\end{aligned}$$

$$\begin{aligned}&\cos 2x\\&=\cos(x+x)\\&=\cos x\cos x-\sin x\sin x\\&=\cos^2 x-\sin^2 x\end{aligned}$$

$$\begin{aligned}&\cos 2x\\&=\cos^2 x-\sin^2 x\\&=(1-\sin^2 x)-\sin^2 x\\&=1-2\sin^2 x\end{aligned}$$

$$\begin{aligned}&\cos 2x\\&=\cos^2 x-\sin^2 x\\&=\cos^2 x-(1-\cos^2 x)\\&=2\cos^2 x-1\end{aligned}$$

$$\begin{aligned}&\tan 2x\\&=\tan(x+x)\\&=\frac{\tan x+\tan x}{1-\tan x\tan x}\\&=\frac{2\tan x}{1-\tan^2 x}\end{aligned}$$

$$\begin{aligned}&\cot 2x\\&=\cot(x+x)\\&=\frac{\cot x\cot x-1}{\cot x+\cot x}\\&=\frac{\cot^2 x-1}{2\cot x}\end{aligned}$$

Summary

$\sin 2x=2\sin x\cos x$

$\cos 2x=\cos^2 x-\sin^2 x$

$\cos 2x=1-2\sin^2 x$

$\cos 2x=2\cos^2 x-1$

$\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$

$\cot 2x=\dfrac{\cot^2 x-1}{2\cot x}$

Examples

1. Use compound angle identities to evaluate each of the following. Exact values are required. Do not use calculators!

(a) $\sin 75°$

$$\begin{aligned}&\sin 75°\\&=\sin(45°+30°)\\&=\sin(\pi/4+\pi/6)\\&=\sin(\pi/4)\cos(\pi/6)+\cos(\pi/4)\sin(\pi/6)\\&=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)\\&=\frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

(b) $\cos 255°$

$$\begin{aligned}&\cos 255°\\&=\cos(315°-60°)\\&=\cos(7\pi/4-\pi/3)\\&=\cos(7\pi/4)\cos(\pi/3)+\sin(\pi/3)\sin(7\pi/4)\\&=\cos(\pi/4)\cos(\pi/3)+\sin(\pi/3)(-\sin(\pi/4))\\&=\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)+\frac{\sqrt{3}}{2}\left(-\frac{1}{\sqrt{2}}\right)\\&=\frac{1-\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

(c) $\tan 105°$

$$\begin{aligned}&\tan 105°\\&=\tan(45°+60°)\\&=\tan(\pi/4+\pi/3)\\&=\frac{\tan(\pi/4)+\tan(\pi/3)}{1-\tan(\pi/4)\tan(\pi/3)}\\&=\frac{1+\sqrt{3}}{1-1(\sqrt{3})}\\&=\frac{1+\sqrt{3}}{1-\sqrt{3}}\end{aligned}$$

Quick Check

$\dfrac{\sqrt{3}+1}{2\sqrt{2}}>0$ as we would expect for $\sin 75°$ | $\dfrac{\sqrt{3}+1}{2\sqrt{2}}\doteq 0.9659$ | $\sin 75°\doteq 0.9659$

$\dfrac{1-\sqrt{3}}{2\sqrt{2}}<0$ as we would expect for $\cos 255°$ | $\dfrac{1-\sqrt{3}}{2\sqrt{2}}\doteq -0.2588$ | $\cos 255°\doteq -0.2588$

$\dfrac{1+\sqrt{3}}{1-\sqrt{3}}<0$ as we would expect for $\tan 105°$ | $\dfrac{1+\sqrt{3}}{1-\sqrt{3}}\doteq -3.732$ | $\tan 105°\doteq -3.732$

2. Use double angle identities to evaluate each of the following. Exact values are required. Do not use calculators!

(a) $\sin 15°$

Setting $\theta = \frac{x}{2}$ in $\cos 2\theta = 1 - 2\sin^2\theta$, we obtain

$$\cos x = 1 - 2\sin^2\frac{x}{2}$$

Solving for $\sin\frac{x}{2}$, we obtain

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

Therefore,

$$\sin 15° = \pm\sqrt{\frac{1-\cos 30°}{2}}$$
$$= \pm\sqrt{\frac{1-\sqrt{3}/2}{2}}$$
$$= \pm\sqrt{\frac{1}{2}\left(\frac{2-\sqrt{3}}{2}\right)}$$
$$= \pm\sqrt{\frac{2-\sqrt{3}}{4}}$$
$$= \pm\frac{\sqrt{2-\sqrt{3}}}{2}$$

Since 15° is in quadrant I,

$$\sin 15° = \frac{\sqrt{2-\sqrt{3}}}{2} = \frac{1}{2}\sqrt{2-\sqrt{3}}$$

(b) $\cos 22.5°$

Setting $\theta = \frac{x}{2}$ in $\cos 2\theta = 2\cos^2\theta - 1$, we obtain

$$\cos x = 2\cos^2\frac{x}{2} - 1$$

Solving for $\cos\frac{x}{2}$, we obtain

$$\cos\frac{x}{2} = \pm\sqrt{\frac{\cos x + 1}{2}}$$

Therefore,

$$\cos 22.5° = \pm\sqrt{\frac{\cos 45° + 1}{2}}$$
$$= \pm\sqrt{\frac{1/\sqrt{2}+1}{2}}$$
$$= \pm\sqrt{\frac{1}{2}\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)}$$
$$= \pm\sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$$

Since 22.5° is in quadrant I, $\cos 22.5° = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$

3. Prove that the following equations are identities.

(a) $\frac{\sin 2x}{1+\cos 2x} = \tan x$

Proof

$$\text{L.S.} = \frac{\sin 2x}{1+\cos 2x}$$
$$= \frac{2\sin x\cos x}{1+2\cos^2 x - 1}$$
$$= \frac{2\sin x\cos x}{2\cos^2 x}$$
$$= \frac{\sin x}{\cos x}$$
$$= \tan x$$

There are 3 different identities for $\cos 2x$. $\cos 2x = 2\cos^2 x - 1$ was chosen because it leads to the most convenient simplification of the denominator.

Divide top and bottom by $2\cos x$.

R.S. $= \tan x$

$\therefore$ L.S. = R.S.

$\therefore \frac{\sin 2x}{1+\cos 2x} = \tan x$ ***is*** an identity. //

(b) $\cos^4\theta - \sin^4\theta = \cos 2\theta$

Proof

L.S. $= \cos^4\theta - \sin^4\theta$

$= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$ (Factor diff. of squares)

$= 1(\cos^2\theta - \sin^2\theta)$ (Pythagorean identity)

$= \cos^2\theta - \sin^2\theta$

$= \cos 2\theta$

R.S. $= \cos 2\theta$

$\therefore$ L.S. = R.S.

$\therefore \cos^4\theta - \sin^4\theta = \cos 2\theta$ ***is*** an identity. //

4. Use counterexamples to prove that the following equations are ***not*** identities.

(a) $\sin(x+y)=\sin x+\sin y$

Proof

Let $x=y=\dfrac{\pi}{4}$

$$\begin{aligned}\text{L.S.}&=\sin(x+y)\\&=\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right)\\&=\sin\left(\frac{\pi}{2}\right)\\&=1\end{aligned}$$

$$\begin{aligned}\text{R.S.}&=\sin x+\sin y\\&=\sin\frac{\pi}{4}+\sin\frac{\pi}{4}\\&=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\\&=\frac{2}{\sqrt{2}}\\&=\sqrt{2}\end{aligned}$$

$\because 1\neq\sqrt{2}$

$\therefore \text{L.S.}\neq\text{R.S.}$

$\therefore \sin(x+y)=\sin x+\sin y$ is ***not*** an identity. //

(b) $\cos 4\theta-\cos\theta=\cos 3\theta$

Proof

Let $\theta=\dfrac{\pi}{2}$

$$\begin{aligned}\text{L.S.}&=\cos 4\theta-\cos\theta\\&=\cos\left(4\left(\frac{\pi}{2}\right)\right)-\cos\frac{\pi}{2}\\&=\cos 2\pi-\cos\frac{\pi}{2}\\&=\cos 2\pi-\cos\frac{\pi}{2}\\&=1-0\\&=1\end{aligned}$$

$$\begin{aligned}\text{R.S.}&=\cos 3\theta\\&=\cos\left(3\left(\frac{\pi}{2}\right)\right)\\&=\cos\frac{3\pi}{2}\\&=0\end{aligned}$$

$\because 1\neq 0$

$\therefore \text{L.S.}\neq\text{R.S.}$

$\therefore \cos 4\theta-\cos\theta=\cos 3\theta$ is ***not*** an identity. //

5. Use identities that we have learned to derive an identity for $\sin 3\theta$ that is expressed entirely in terms of $\sin\theta$.

$$\begin{aligned}&\sin 3\theta\\&=\sin(2\theta+\theta)\\&=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta\\&=(2\sin\theta\cos\theta)\cos\theta+(1-2\sin^2\theta)\sin\theta\\&=2\sin\theta\cos^2\theta+\sin\theta-2\sin^3\theta\\&=2\sin\theta(1-\sin^2\theta)+\sin\theta-2\sin^3\theta\\&=2\sin\theta-2\sin^3\theta+\sin\theta-2\sin^3\theta\\&=3\sin\theta-4\sin^3\theta\end{aligned}$$

$\therefore \sin 3\theta=3\sin\theta-4\sin^3\theta$ is an identity.

In Summary

Key Ideas

- A trigonometric identity states the equivalence of two trigonometric expressions. It is written as an equation that involves trigonometric ratios, and the solution set is all real numbers for which the expressions on both sides of the equation are defined. As a result, the equation has an infinite number of solutions.
- Some trigonometric identities are the result of a definition, while others are derived from relationships that exist among trigonometric ratios.

Need to Know

- The following trigonometric identities are important for you to remember:

Identities Based on Definitions

Reciprocal Identities

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Identities Derived from Relationships

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Addition and Subtraction Formulas

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin (x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos (x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- You can verify the truth of a given trigonometric identity by graphing each side separately and showing that the two graphs are the same.
- To prove that a given equation is an identity, the two sides of the equation must be shown to be equivalent. This can be accomplished using a variety of strategies, such as
  - simplifying the more complicated side until it is identical to the other side, ormanipulating both sides to get the same expression
  - rewriting expressions using any of the identities stated above
  - using a common denominator or factoring, where possible

Homework

pp. 400 – 401: 4, 5, 6, 7, 8bdf, 9bdf, 11, 12, 16, 18

pp. 407 – 408: 1, 2, 3, 5, 7, 8, 11, 12, 13, 16

Trig Identities – Summary and Extra Practice

1. Complete the following statements:

(a) An ***equation*** is an ***identity*** if ____

____.

(b) There are many different ways to confirm whether an equation is an identity. List ***at least three*** such ways.

____

____

____

(c) There is a very simple way to confirm that an equation is ***not*** an identity. In fact, this method can be used to show the falsity of any invalid mathematical statement. Describe the method and use it to demonstrate that the equation $\sqrt{x+y}=\sqrt{x}+\sqrt{y}$ is ***not*** an identity.

2. Mr. Nolfi asked Andrew to prove that the equation $\dfrac{\sin 2x}{1+\cos 2x}=\tan x$ is an identity. What mark would Andrew receive for the following response? Explain.

$$\frac{\sin 2x}{1+\cos 2x}=\tan x$$
$$\therefore \frac{2\sin x\cos x}{1+2\cos^2 x-1}=\tan x$$
$$\therefore \frac{2\sin x\cos x}{2\cos^2 x}=\tan x$$
$$\therefore \left(\frac{2}{2}\right)\left(\frac{\sin x}{\cos x}\right)\left(\frac{\cos x}{\cos x}\right)=\tan x$$
$$\therefore 1(\tan x)(1)=\tan x$$
$$\therefore \tan x=\tan x$$

3. List several strategies that can help you to prove that an equation is an identity.

____

____

____

____

____

____

____

4. Justify each of the following identities by using transformations and by using angles of rotation.

(a) $\sin(-x) = -\sin x$ (b) $\sin(\pi/2 - x) = \cos x$ (c) $\sin(x + \pi) = -\sin x$

(d) $\cos(-x) = \cos x$ (e) $\cos(\pi/2 - x) = \sin x$ (f) $\cos(x + \pi) = -\cos x$

(g) $\tan(-x) = -\tan x$ (h) $\tan(\pi/2 - x) = \cot x$ (i) $\tan(x + \pi) = \tan x$

5. Prove that each of the following equations is an identity:

a) $\dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin\theta\cos\theta} = 1 - \tan\theta$

b) $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$

c) $\tan^2 x - \cos^2 x = \dfrac{1}{\cos^2 x} - 1 - \cos^2 x$

d) $\dfrac{1}{1 + \cos\theta} + \dfrac{1}{1 - \cos\theta} = \dfrac{2}{\sin^2\theta}$

6. Prove that each of the following equations is an identity:

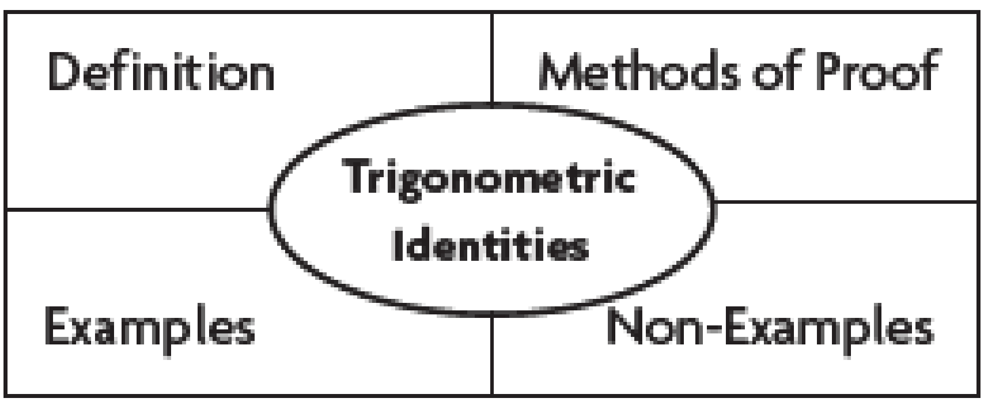
a) $\cos x \tan^3 x = \sin x \tan^2 x$

b) $\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta$

c) $(\sin x + \cos x)\left(\dfrac{\tan^2 x + 1}{\tan x}\right) = \dfrac{1}{\cos x} + \dfrac{1}{\sin x}$

d) $\tan^2\beta + \cos^2\beta + \sin^2\beta = \dfrac{1}{\cos^2\beta}$

7. Copy and complete the following Frayer diagram:



8. Express $8\cos^4 x$ in the form $a\cos 4x + b\cos 2x + c$. State the values of the constants a, b and c.

9. Give a counterexample to demonstrate that each of the following equations is not an identity.

a) $\cos x = \dfrac{1}{\cos x}$ c) $\sin(x + y) = \cos x \cos y + \sin x \sin y$

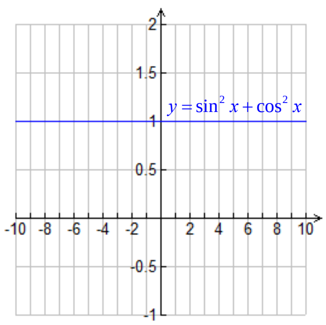
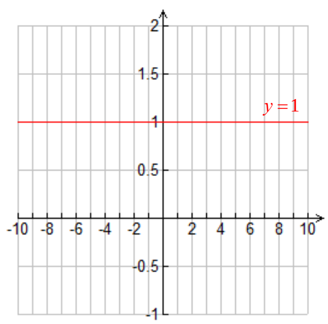
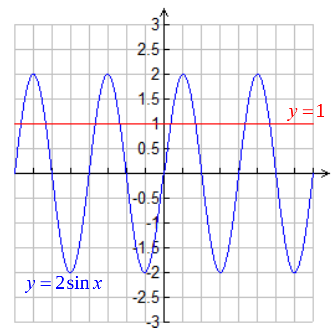
b) $1 - \tan^2 x = \sec^2 x$ d) $\cos 2x = 1 + 2\sin^2 x$

10. Demonstrate graphically that each of the equations in 9 is not an identity.

Solving Trigonometric Equations

Introduction – A Graphical Look at Equations that are not Identities

Let "L.S." represent the expression on the left side of an equation and let "R.S." represent the expression on the right side.

An Equation that is an Identity: $\sin^2 x + \cos^2 x = 1$	*An Equation that is not an Identity:* $2\sin x = 1$
• If an equation is an identity, the expression on the L.S. is ***equivalent*** to the expression on the R.S. of the equation. That is, the equation is satisfied for all real numbers for which the expressions are defined. • If an equation is an identity, then the graph of "y = L.S." is ***identical*** to the graph of "y = R.S." The graphs intersect at all real values for which the expressions are defined. In other words, every such value is a solution to the equation.	• If an equation is ***not*** an identity, the expression on the L.S. is ***not equivalent*** to the expression on the R.S. The expressions may agree for some real values but they ***do not agree*** for ***all*** values. • If an equation is ***not*** an identity, then the graph of "y = L.S." is ***not identical*** to the graph of "y = R.S." The expressions agree only at the point(s) of intersection of the two graphs. The number of points of intersection is equal to the number of solutions of the equation.
 	

Examples

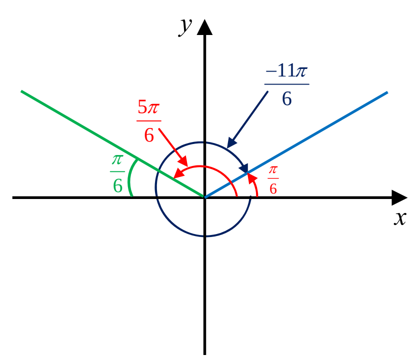
1. Use an algebraic method to solve the trigonometric equation $2\sin x - 1 = 0$. State all solutions in the interval $-4\pi \le x \le 4\pi$. Verify the solutions graphically. (**Note:** An alternative notation for writing the interval $-4\pi \le x \le 4\pi$ is $[-4\pi, 4\pi]$. The square brackets indicate that the endpoints are included in the interval.)

$$2\sin x - 1 = 0$$
$$\therefore 2\sin x = 1$$
$$\therefore \sin x = \frac{1}{2}$$
$$\therefore x = \sin^{-1}\left(\frac{1}{2}\right)$$
$$\therefore x = \frac{\pi}{6}$$

If your calculator is in "degrees mode," this will produce an answer of 30°. In "radians mode," an answer of about 0.5326 is obtained. Naturally, since $1/2$ is a trig ratio of a special angle, you should be able to state the exact answer $x = \pi/6$. To state the other solutions in the interval $[-4\pi, 4\pi]$, use the concept of related angles, the ASTC rule and a graph.

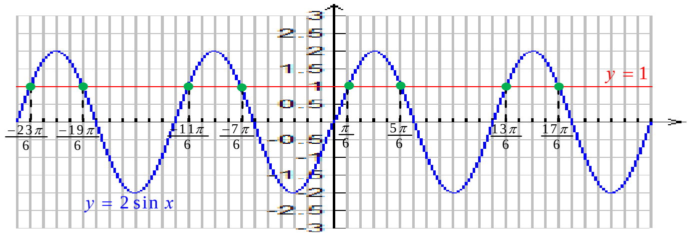
As shown in the diagram at the right,

- $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ are the ***principal angle*** solutions to the equation (since the sine function is positive in quadrants I and II)
- All the other solutions in the interval $[-4\pi, 4\pi]$ are found by taking all angles in this range that are ***coterminal*** with $\frac{\pi}{6}$ and $\frac{5\pi}{6}$



Therefore, the solutions in the interval $[-4\pi, 4\pi]$ are $\frac{-23\pi}{6}, \frac{-19\pi}{6}, \frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$ and $\frac{17\pi}{6}$.

The following is a graphical verification of the solutions given above.



2. Solve for x given that $x\in[0,2\pi]$.

(a) $2\sec^2 x-3+\tan x=0$

$\therefore 2(\tan^2 x+1)-3+\tan x=0$ (Pyth. identity)

$\therefore 2\tan^2 x+\tan x-1=0$

$\therefore (2\tan x-1)(\tan x+1)=0$

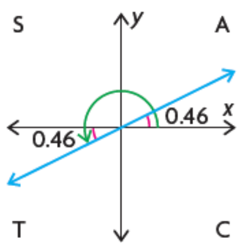
$\therefore 2\tan x-1=0$ or $\tan x+1=0$

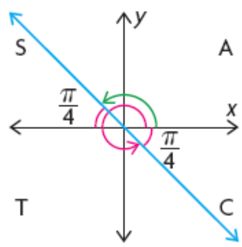
$\therefore \tan x=\dfrac{1}{2}$ or $\tan x=-1$

$\therefore x=\tan^{-1}\left(\dfrac{1}{2}\right)$ or $x=\tan^{-1}(-1)$

$\therefore x\doteq 0.46$ or $x=\dfrac{3\pi}{4}$ (calculator gives $-\dfrac{\pi}{4}$)

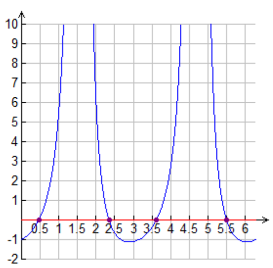
These solutions are in quadrants I and II. Because of the ASTC rule, there are also solutions in quadrants III and IV:





$\therefore x\doteq 0.46+\pi\doteq 3.6$ or $x=2\pi-\dfrac{\pi}{4}=\dfrac{7\pi}{4}$

$\therefore x\doteq 0.46$, $x\doteq 3.6$, $x=\dfrac{3\pi}{4}$ or $x=\dfrac{7\pi}{4}$



TI-Interactive was used to sketch the graphs of $y=2\sec^2 x-3+\tan x$ and $y=0$. The points of intersection agree with the solutions obtained above.

(b) $3\sin x+3\cos 2x=2$

$\therefore 3\sin x+3(1-2\sin^2 x)=2$ (double angle identity)

$\therefore 6\sin^2 x-3\sin x-1=0$

A quick check of the discriminant of this quadratic equation in $\sin x$ demonstrates that it ***does not*** factor:

$b^2-4ac=(-3)^2-4(6)(-1)=33$, which is not a perfect square. Therefore, the quadratic formula must be used.

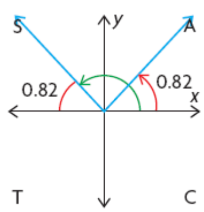
$$\sin x=\frac{-(-3)\pm\sqrt{(-3)^2-4(6)(-1)}}{2(6)}=\frac{3\pm\sqrt{33}}{12}$$

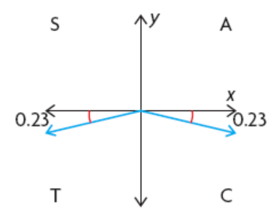
$\therefore \sin x=\dfrac{3+\sqrt{33}}{12}$ or $\sin x=\dfrac{3-\sqrt{33}}{12}$

$\therefore x=\sin^{-1}\left(\dfrac{3+\sqrt{33}}{12}\right)$ or $\sin x=\dfrac{3-\sqrt{33}}{12}$

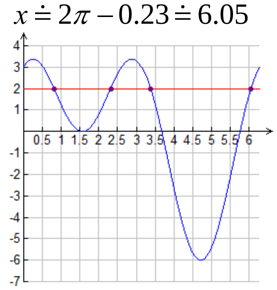
$\therefore x=\sin^{-1}\left(\dfrac{3+\sqrt{33}}{12}\right)$ or $x=\sin^{-1}\left(\dfrac{3-\sqrt{33}}{12}\right)$

$\therefore x\doteq 0.82$ or $x\doteq -0.23$ (solutions given by calculator)





Therefore, the solutions in the interval $[0,2\pi]$ are $x\doteq 0.82$, $x\doteq\pi-0.82\doteq 2.32$, $x\doteq\pi+0.23\doteq 3.37$ and $x\doteq 2\pi-0.23\doteq 6.05$



TI-Interactive was used to sketch the graphs of $y=3\sin x+3\cos 2x$ and $y=2$. The points of intersection agree with the solutions obtained above.

Homework

pp. 427 – 428: 6, 8bd, 9cf, 13, 14

pp. 436 – 437: 6, 7, 8def, 9bd, 11, 12, 14, 15

Rates of Change in Trigonometric Functions

Introductory Investigation

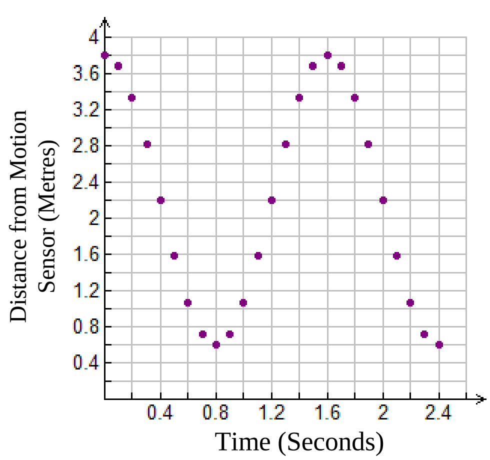
Gareth was walking through a playground minding his own business when all of a sudden, he felt little Niroj tugging at his pants. “Gareth, Gareth!” little Niroj exclaimed. “Please push me on a swing!” Being in a hurry, Gareth was a little reluctant to comply with little Niroj’s request at first. Upon reflection, however, Gareth remembered that he had to collect some data for his math homework. He reached into his knapsack and pulled out his very handy portable motion sensor. “Get on the swing Niroj!” Gareth bellowed. “I’ll set up the motion sensor in front of you and it will take some measurements as I push.” Gleefully, little Niroj hopped into the seat of the swing and waited for Gareth to start pushing.

The data collected by Gareth’s motion sensor are shown in the following tables. Time is measured in seconds and the distance, in metres, is measured from the motion sensor to little Niroj on the swing.

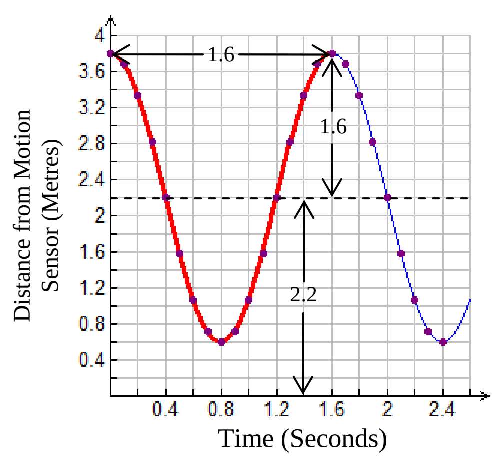
Time (s)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1
Distance (m)	3.8	3.68	3.33	2.81	2.2	1.59	1.07	0.72	0.6	0.72	1.07	1.59

Time (s)	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4
Distance (m)	2.2	2.81	3.33	3.68	3.8	3.68	3.33	2.81	2.2	1.59	1.07	0.72	0.6

Scatter Plot



Curve of Best Fit



Mathematical Model

$|A| =$ Amplitude $= 1.6$

$T =$ Period $= 1.6 = \left|\dfrac{1}{\omega}\right| 2\pi$

$\omega =$ Angular Frequency

$= \left(\dfrac{1}{T}\right)2\pi = \left(\dfrac{1}{1.6}\right)2\pi = \dfrac{5\pi}{4}$

$d =$ vertical displacement $= 2.2$

$p =$ phase shift $= 0$

$\therefore d(t) = 1.6\cos\left(\dfrac{5\pi}{4}t\right) + 2.2$

(Alternatively, we could write $d(t) = 1.6\sin\left(\dfrac{5\pi}{4}(t-1.2)\right) + 2.2$)

Questions

1. What quantity is measured by

(a) the slope of the secant line through the points $(t_1, d(t_1))$ and $(t_2, d(t_2))$?

(b) the slope of the tangent line at $(t, d(t))$?

2. Complete the following table.

Intervals of Time over which Niroj approaches the Motion Sensor	*Intervals of Time over which Niroj recedes from the Motion Sensor*	*Intervals of Time over which Niroj’s Speed Increases*	*Intervals of Time over which Niroj’s Speed Decreases*

3. Explain the difference between speed and velocity.

4. Describe the "shape" of the curve over the intervals of time during which

(a) Niroj's velocity is increasing

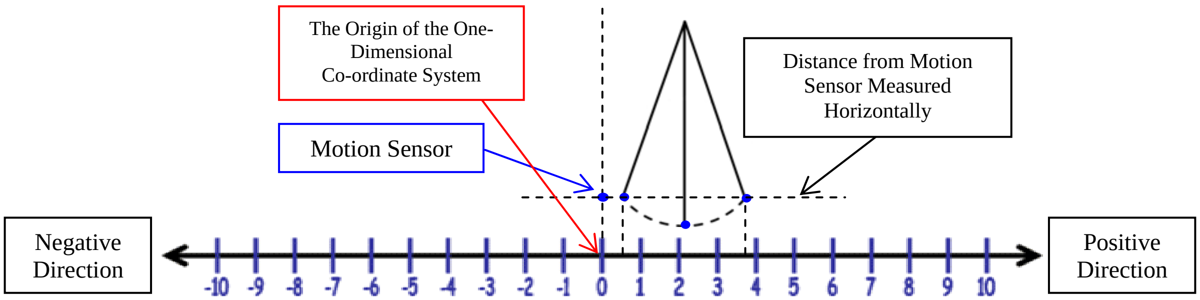
(b) Niroj's velocity is decreasing

5. Use the function given above to ***calculate*** the ***average*** rate of change of distance from the motion sensor with respect to time between 0.2 s and 1.0 s. Is your answer negative or positive? Interpret your result geometrically (i.e. as a slope) and physically (i.e. as a velocity).

6. Use the function given above to ***estimate*** the ***instantaneous*** rate of change of distance from the motion sensor with respect to time at 0.6 s. Is your answer negative or positive? Interpret your result geometrically (i.e. as a slope) and physically (i.e. as a velocity).

Rectilinear (Linear) Motion

- ***Rectilinear*** or ***linear*** motion is motion that occurs along a ***straight line***.
- Rectilinear motion can be described fully using a ***one-dimensional co-ordinate system***.
- Strictly speaking, Niroj's swinging motion is not rectilinear because he moves along a curve (see diagram).
- However, since only the horizontal distance to the motion sensor is measured, we can imagine that Niroj is moving along the horizontal line that passes through the motion sensor (see diagram). A more precise interpretation is that the equation given above models the position of Niroj's x-co-ordinate with respect to time.



The table below lists the meanings of various quantities that are used to describe one-dimensional motion.

Quantity	*Meaning and Description*	*Properties*
Position	The ***position*** of an object measures ***where*** the object is located at any given time. In linear motion, the position of an object is simply a number that indicates ***where it is*** with respect to a number line like the one shown above. Usually, the position function of an object is written as $s(t)$.	At any time t, if the object is located **(a)** ***at*** the origin, then $s(t)=0$ **(b)** to the ***right*** of the origin, then $s(t)>0$ **(c)** to the ***left*** of the origin, then $s(t)<0$ Also, $\lvert s(t)\rvert$ is the distance from the object to the origin.
Displacement	The ***displacement*** of an object between the times t_1 and t_2 is equal to its ***change in position*** between t_1 and t_2. That is, **displacement** $=\Delta s = s(t_2)-s(t_1)$.	If $\Delta s>0$, the object is to the ***right*** of its initial position. If $\Delta s<0$, the object is to the ***left*** of its initial position. If $\Delta s=0$, the object is ***at*** its initial position.
Distance	***Distance*** measures ***how far*** an object has travelled. Since an object undergoing linear motion can change direction, the distance travelled is found by summing (adding up) the absolute values of all the displacements for which there is a ***change in direction***.	The position of an object undergoing linear motion is tracked between times t_0 and t_n. In addition, the object changes direction at times $t_1, t_2, \ldots, t_{n-1}$ (and at no other times), where $t_0 < t_1 < \ldots < t_{n-1} < t_n$. If Δs_i represents the displacement from time t_{i-1} to time t_i, then the total distance travelled is equal to $d=\lvert\Delta s_1\rvert+\lvert\Delta s_2\rvert+\cdots+\lvert\Delta s_{n-1}\rvert+\lvert\Delta s_n\rvert$
Velocity	***Velocity*** is the ***instantaneous rate of change of position with respect to time***. Velocity measures ***how fast*** an object moves as well as its ***direction*** of travel. In one-dimensional rectilinear motion, velocity can be negative or positive, depending on the direction of travel.	At any time t, if the object is **(a)** moving in the ***positive*** direction, then $v(t)>0$ **(b)** moving in the ***negative*** direction, then $v(t)<0$ **(c)** at ***rest***, then $v(t)=0$ Also, $\lvert v(t)\rvert$ is the ***speed*** of the object.
Speed	***Speed*** is simply a measure of how fast an object moves ***without regard to its direction of travel***.	**speed** $=\lvert v(t)\rvert$

Table continued from previous page…

Quantity	*Meaning and Description*	*Properties*
Acceleration	***Acceleration*** is the ***instantaneous rate of change of velocity with respect to time***. In one-dimensional rectilinear motion, acceleration can be negative or positive, depending on the direction of the force causing the acceleration.	At any time t, if the object is **(a)** moving in the ***positive*** direction and ***speeding up*** or moving in the ***negative*** direction and ***slowing down***, then $a(t)>0$ **(b)** moving in the ***positive*** direction and ***slowing down*** or moving in the ***negative*** direction and ***speeding up***, then $a(t)<0$ **(c)** moving with a ***constant*** velocity, then $a(t)=0$

Example

Determine the quantities listed in the following table. (All times are specified in seconds.)

(a) Niroj's ***position*** at $t=2$

(b) Niroj's ***displacement*** over the interval $[0,2.6]$

(c) Niroj's ***average velocity*** over the interval $[0,2.6]$

(d) Niroj's ***average speed*** over the interval $[0,2.6]$

(e) The total ***distance*** travelled by Niroj over the interval $[0,2.6]$

(f) An estimate of Niroj's ***instantaneous velocity*** at $t=2$

Solution

(a) $s(2)=1.6\cos\left(\frac{5\pi}{4}(2)\right)+2.2=1.6\cos\left(\frac{5\pi}{2}\right)+2.2=1.6(0)+2.2=2.2$

Niroj's position at $t=2$ is 2.2 m to the right of the origin.

(b) $\Delta s=s(2.6)-s(0)$

$$=1.6\cos\left(\frac{5\pi}{4}(2.6)\right)+2.2-\left[1.6\cos\left(\frac{5\pi}{4}(0)\right)+2.2\right]$$

$$=1.6\cos\frac{13\pi}{4}-1.6\cos 0$$

$$=1.6\left(-\frac{1}{\sqrt{2}}\right)-1.6(1)$$

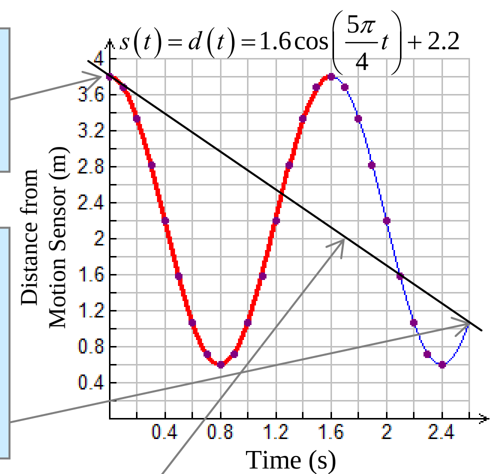
$$=-1.6\left(\frac{1}{\sqrt{2}}+1\right)$$

$$=-1.6\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)$$

$$\doteq -2.73$$

Position at time 0
$=s(0)$
$=3.8$

Position at time 2.6
$=s(2.6)$
$=1.6\left(-\frac{1}{\sqrt{2}}\right)+2.2$
$\doteq 1.07$



Over the interval $[0,2.6]$, Niroj's ***displacement*** is –2.73 m. This means that at 2.6 s, his position was 2.73 m to the ***left*** of his initial position (i.e. his position at 0 s).

(c) $v_{\text{avg}}=\frac{\Delta s}{\Delta t}$

$$=\frac{s(2.6)-s(0)}{2.6-0}$$

$$=\frac{-1.6\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)}{2.6}$$

$$\doteq -1.05$$

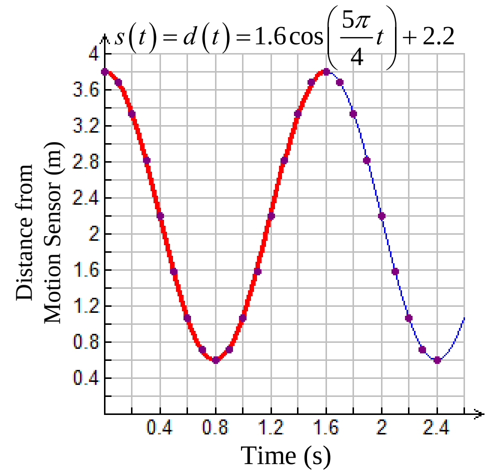
Average velocity over the interval $[0,2.6]$
= slope of secant line between $(0,s(0))$ and $(2.6,s(2.6))$
$\doteq -1.05$ m/s

Over the interval $[0,2.6]$, Niroj's ***average velocity*** is –1.05 m/s. This means that Niroj's position ***decreases*** at an average rate of 1.05 m/s over the interval $[0,2.6]$. That is, Niroj ***moves to the left*** with an average speed of 1.05 m/s.

(d) average speed $= |v_{avg}| = \left|\dfrac{\Delta s}{\Delta t}\right| = |-1.05| = 1.05$ m/s.

Over the interval $[0, 2.6]$, Niroj's average speed was 1.05 m/s.

(e) Examine the graph at the right, showing Niroj's position over time. Notice that over the interval $[0, 2.6]$, Niroj changes direction at 0.8 s, 1.6 s and 2.4 s. Therefore,



$$
\begin{aligned}
d &= |\Delta s_1| + |\Delta s_2| + |\Delta s_3| + |\Delta s_4| \\
&= |s(0.8) - s(0)| + |s(1.6) - s(0.8)| + |s(2.4) - s(1.6)| + |s(2.6) - s(2.4)| \\
&\doteq |0.6 - 3.8| + |3.8 - 0.6| + |0.6 - 3.8| + |1.07 - 0.6| \\
&= 3.2 + 3.2 + 3.2 + 0.47 \\
&= 10.07
\end{aligned}
$$

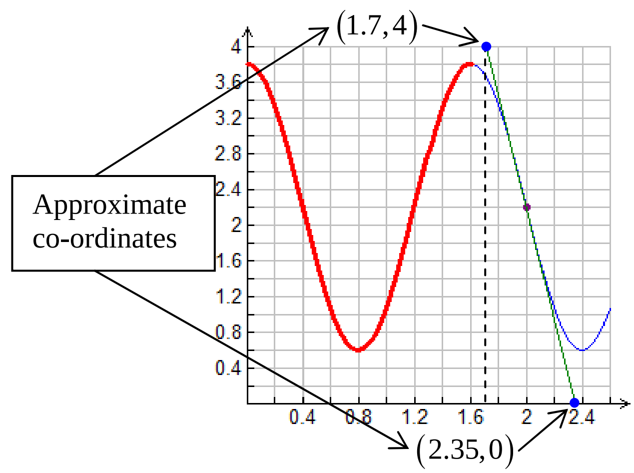
Over the interval $[0, 2.6]$, Niroj travelled a distance of about 10.07 m.

(f) $v(2)$ = Instantaneous velocity at 2 seconds

= slope of tangent of position-time graph at 2 seconds (see graph at right)

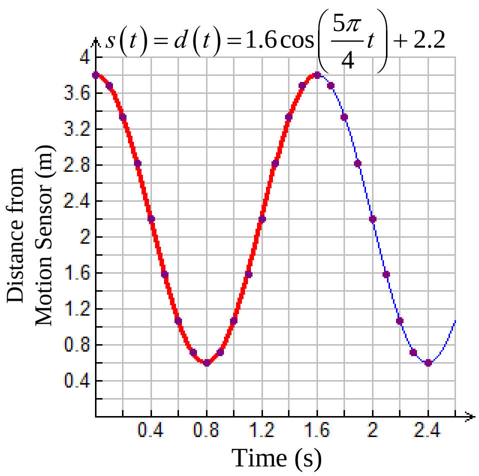
Since we do not yet have the tools of calculus at our disposal, the best we can do is to approximate the slope of the required tangent line by using a secant line that passes through two points that are very close to $t = 2$. If we use the centered interval $[1.99, 2.01]$, then

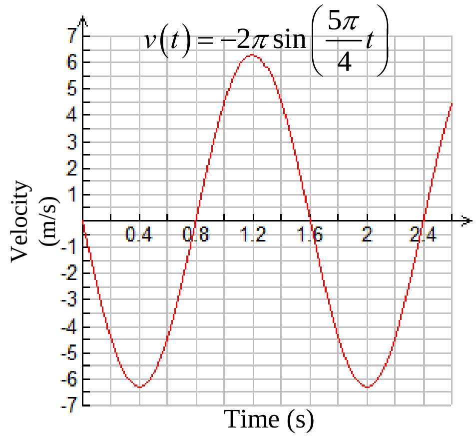
$$
\begin{aligned}
v(2) &\doteq \frac{\Delta s}{\Delta t} \quad \text{(provided that } \Delta t \text{ is very small)} \\
&= \frac{s(2.01) - s(1.99)}{2.01 - 1.99} \\
&= \frac{1.6\cos\left(\frac{5\pi}{4}(2.01)\right) + 2.2 - \left(1.6\cos\left(\frac{5\pi}{4}(1.99)\right) + 2.2\right)}{0.02} \\
&= \frac{1.6\cos\left(\frac{10.05\pi}{4}\right) - 1.6\cos\left(\frac{9.95\pi}{4}\right)}{0.02} \\
&\doteq -6.28
\end{aligned}
$$

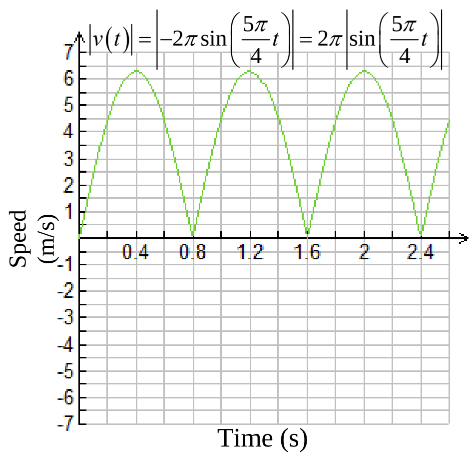


At $t = 2$ seconds, Niroj's instantaneous velocity is approximately –6.28 m/s, which means that he is moving to the left (toward the motion sensor) with a speed of about 6.28 m/s. Notice that the answer –6.28 m/s agrees with the graph shown above. The tangent line at $t = 2$ leans to the left, which means that its slope should be negative. In addition, a quick, rough calculation of $\dfrac{\text{rise}}{\text{run}}$ yields $\dfrac{4-0}{1.7-2.35} \doteq -6.15$, which is in close agreement with the answer obtained above.

Now use the following graphs to confirm the answers given above for the instantaneous quantities.







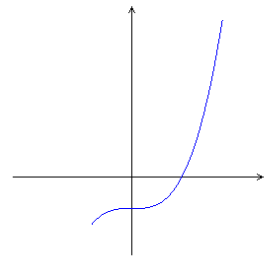
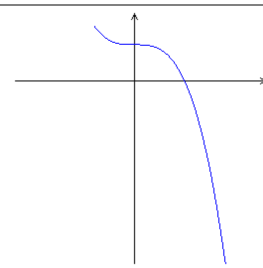
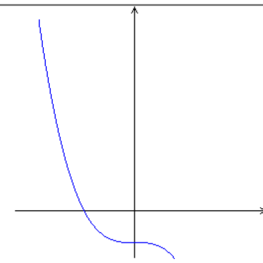
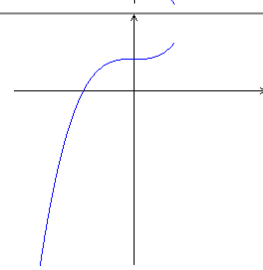
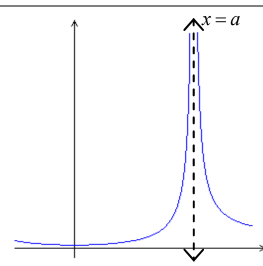
Homework
pp. 369 – 373: 2, 3, 6, 8, 9, 10, 11, 12, 13, 14

End Behaviours and other Tendencies of Trigonometric Functions

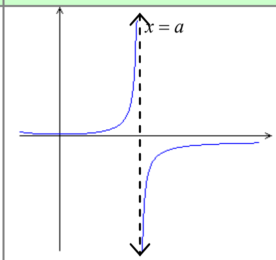
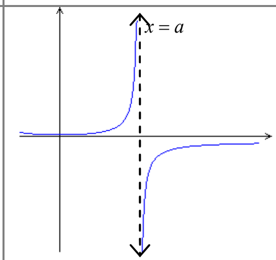
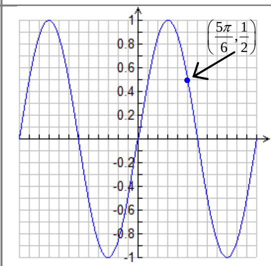
Review of Notation

Examples are given in the following table of notation that is used to describe the behaviour of some function *f* as *x* undergoes some change such as tending toward a particular value or getting larger and larger without bound.

- Note that $x \to \infty$ can also be written $x \to +\infty$.
- "Arbitrarily far from" means "as far as desired from."
- "Arbitrarily close to" means "as close as desired to."

Notation used in this Course	*Calculus Notation**	*Meaning*	*What it Looks Like*
As $x \to \infty$, $f(x) \to \infty$	$\lim\limits_{x\to\infty} f(x) = \infty$	**Read:** As *x* approaches (positive) infinity, $f(x)$ approaches (positive) infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***positive direction*** by making *x* far enough from the origin in the ***positive direction***.	
As $x \to \infty$, $f(x) \to -\infty$	$\lim\limits_{x\to\infty} f(x) = -\infty$	**Read:** As *x* approaches (positive) infinity, $f(x)$ approaches negative infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***negative direction*** by making *x* far enough from the origin in the ***positive direction***.	
As $x \to -\infty$, $f(x) \to \infty$	$\lim\limits_{x\to-\infty} f(x) = \infty$	**Read:** As *x* approaches negative infinity, $f(x)$ approaches (positive) infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***positive direction*** by making *x* far enough from the origin in the ***negative direction***.	
As $x \to -\infty$ $f(x) \to -\infty$	$\lim\limits_{x\to-\infty} f(x) = -\infty$	**Read:** As *x* approaches negative infinity, $f(x)$ approaches negative infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***negative direction*** by making *x* far enough from the origin in the ***negative direction***.	
As $x \to a$ $f(x) \to \infty$	$\lim\limits_{x\to a} f(x) = \infty$	**Read:** As *x* approaches *a*, $f(x)$ approaches (positive) infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***positive direction*** by making *x* close enough but not equal to *a* (from both the left and right sides).	

* This is just a preview of calculus. You are not required to use calculus notation in this course.

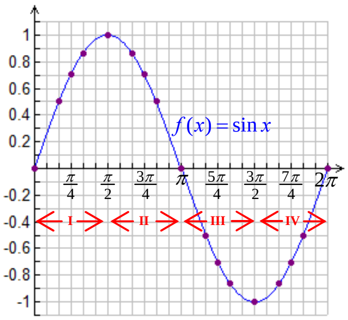
Notation used in this Course	Calculus Notation*	Meaning	What it Looks Like
As $x \to a^{+}$, $f(x) \to -\infty$	$\lim\limits_{x \to a^{+}} f(x) = -\infty$	**Read:** As x approaches a ***from the right***, $f(x)$ approaches negative infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***negative direction*** by making x close enough to a from the right side but not equal to a.	
As $x \to a^{-}$, $f(x) \to \infty$	$\lim\limits_{x \to a^{-}} f(x) = \infty$	**Read:** As x approaches a ***from the left***, $f(x)$ approaches (positive) infinity. **Meaning:** We can make $f(x)$ arbitrarily far from the origin in the ***positive direction*** by making x close enough to a from the left side but not equal to a.	
As $x \to a$ $f(x) \to L$	$\lim\limits_{x \to a} f(x) = L$	**Read:** As x approaches a, $f(x)$ approaches L. **Meaning:** We can make $f(x)$ arbitrarily close to L by making x close enough but not equal to a.	

In the example given in the previous row, as $x \to \frac{5\pi}{6}$, $f(x) \to \frac{1}{2}$. This means that we can make $f(x) = \sin x$ as close as we desire to $\frac{1}{2}$ by making x close enough but not equal to $\frac{5\pi}{6}$. In the language of calculus, this is written $\lim\limits_{x \to \frac{5\pi}{6}} f(x) = \frac{1}{2}$.

* This is just a preview of calculus. You are not required to use calculus notation in this course.

Exercises

Determine the tendency of $f(x)$.



As $x \to 0^+$, $f(x) \to$ _____

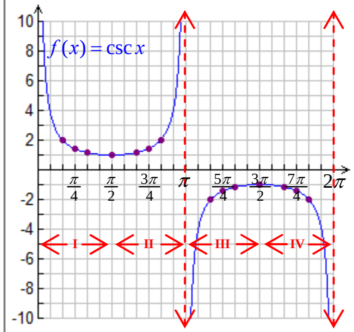
As $x \to 0^-$, $f(x) \to$ _____

As $x \to 0$, $f(x) \to$ _____

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ _____

As $x \to \frac{\pi}{2}^+$, $f(x) \to$ _____

As $x \to \frac{\pi}{2}$, $f(x) \to$ _____



As $x \to 0^+$, $f(x) \to$ _____

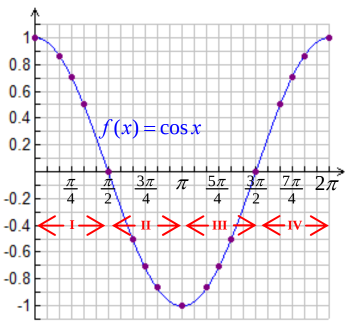
As $x \to 0^-$, $f(x) \to$ _____

As $x \to \frac{\pi}{2}$, $f(x) \to$ _____

As $x \to \frac{3\pi}{2}$, $f(x) \to$ _____

As $x \to \pi^-$, $f(x) \to$ _____

As $x \to \pi^+$, $f(x) \to$ _____



As $x \to 0^+$, $f(x) \to$ _____

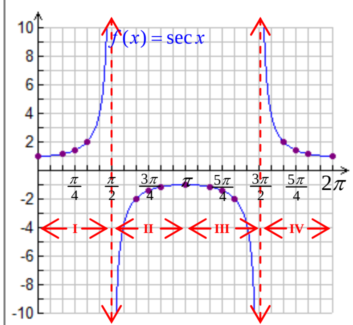
As $x \to 0^-$, $f(x) \to$ _____

As $x \to 0$, $f(x) \to$ _____

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ _____

As $x \to \frac{\pi}{2}^+$, $f(x) \to$ _____

As $x \to \frac{\pi}{2}$, $f(x) \to$ _____



As $x \to 0$, $f(x) \to$ _____

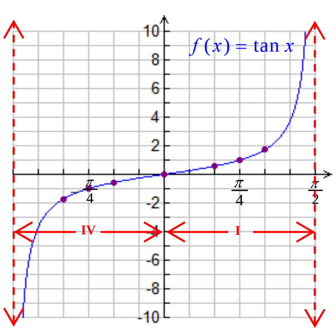
As $x \to \pi$, $f(x) \to$ _____

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ _____

As $x \to \frac{\pi}{2}^+$, $f(x) \to$ _____

As $x \to \frac{3\pi}{2}^-$, $f(x) \to$ _____

As $x \to \frac{3\pi}{2}^+$, $f(x) \to$ _____



As $x \to 0$, $f(x) \to$ _____

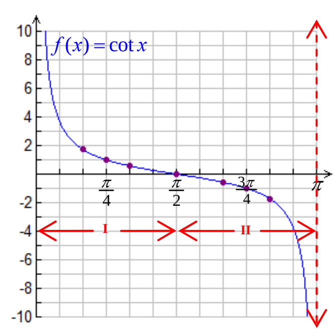
As $x \to \frac{\pi}{4}$, $f(x) \to$ _____

As $x \to \frac{\pi}{6}$, $f(x) \to$ _____

As $x \to -\frac{\pi}{2}^+$, $f(x) \to$ _____

As $x \to \frac{\pi}{2}^-$, $f(x) \to$ _____

As $x \to \frac{3\pi}{2}^+$, $f(x) \to$ _____



As $x \to 0^+$, $f(x) \to$ _____

As $x \to \frac{\pi}{4}$, $f(x) \to$ _____

As $x \to \frac{\pi}{6}$, $f(x) \to$ _____

As $x \to \frac{\pi}{2}$, $f(x) \to$ _____

As $x \to \pi^-$, $f(x) \to$ _____

As $x \to \pi^+$, $f(x) \to$ _____

B. TRIGONOMETRIC FUNCTIONS

OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of the meaning and application of radian measure;
2. make connections between trigonometric ratios and the graphical and algebraic representations of the corresponding trigonometric functions and between trigonometric functions and their reciprocals, and use these connections to solve problems;
3. solve problems involving trigonometric equations and prove trigonometric identities.

SPECIFIC EXPECTATIONS

1. Understanding and Applying Radian Measure

By the end of this course, students will:

1.1 recognize the radian as an alternative unit to the degree for angle measurement, define the radian measure of an angle as the length of the arc that subtends this angle at the centre of a unit circle, and develop and apply the relationship between radian and degree measure

1.2 represent radian measure in terms of π (e.g., $\frac{\pi}{3}$ radians, 2π radians) and as a rational number (e.g., 1.05 radians, 6.28 radians)

1.3 determine, with technology, the primary trigonometric ratios (i.e., sine, cosine, tangent) and the reciprocal trigonometric ratios (i.e., cosecant, secant, cotangent) of angles expressed in radian measure

1.4 determine, without technology, the exact values of the primary trigonometric ratios and the reciprocal trigonometric ratios for the special angles $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, and their multiples less than or equal to 2π

2. Connecting Graphs and Equations of Trigonometric Functions

By the end of this course, students will:

2.1 sketch the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ for angle measures expressed in radians, and determine and describe some key properties (e.g., period of 2π, amplitude of 1) in terms of radians

2.2 make connections between the tangent ratio and the tangent function by using technology to graph the relationship between angles in radians and their tangent ratios and defining this relationship as the function $f(x) = \tan x$, and describe key properties of the tangent function

2.3 graph, with technology and using the primary trigonometric functions, the reciprocal trigonometric functions (i.e., cosecant, secant, cotangent) for angle measures expressed in radians, determine and describe key properties of the reciprocal functions (e.g., state the domain, range, and period, and identify and explain the occurrence of asymptotes), and recognize notations used to represent the reciprocal functions [e.g., the reciprocal of $f(x) = \sin x$ can be represented using $\csc x$, $\frac{1}{f(x)}$, or $\frac{1}{\sin x}$, but not using $f^{-1}(x)$ or $\sin^{-1} x$, which represent the inverse function]

2.4 determine the amplitude, period, and phase shift of sinusoidal functions whose equations are given in the form $f(x) = a\sin(k(x - d)) + c$ or $f(x) = a\cos(k(x - d)) + c$, with angles expressed in radians

2.5 sketch graphs of $y = a\sin(k(x - d)) + c$ and $y = a\cos(k(x - d)) + c$ by applying transformations to the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ with angles expressed in radians, and state the period, amplitude, and phase shift of the transformed functions

Sample problem: Transform the graph of $f(x) = \cos x$ to sketch $g(x) = 3\cos(2x) - 1$, and state the period, amplitude, and phase shift of each function.

2.6 represent a sinusoidal function with an equation, given its graph or its properties, with angles expressed in radians

Sample problem: A sinusoidal function has an amplitude of 2 units, a period of π, and a maximum at (0, 3). Represent the function with an equation in two different ways.

2.7 pose problems based on applications involving a trigonometric function with domain expressed in radians (e.g., seasonal changes in temperature, heights of tides, hours of daylight, displacements for oscillating springs), and solve these and other such problems by using a given graph or a graph generated with or without technology from a table of values or from its equation

Sample problem: The population size, P, of owls (predators) in a certain region can be modelled by the function $P(t) = 1000 + 100\sin\left(\frac{\pi t}{12}\right)$, where t represents the time in months. The population size, p, of mice (prey) in the same region is given by $p(t) = 20\,000 + 4000\cos\left(\frac{\pi t}{12}\right)$. Sketch the graphs of these functions, and pose and solve problems involving the relationships between the two populations over time.

3. Solving Trigonometric Equations

By the end of this course, students will:

3.1 recognize equivalent trigonometric expressions [e.g., by using the angles in a right triangle to recognize that $\sin x$ and $\cos\left(\frac{\pi}{2} - x\right)$ are equivalent; by using transformations to recognize that $\cos\left(x + \frac{\pi}{2}\right)$ and $-\sin x$ are equivalent], and verify equivalence using graphing technology

3.2 explore the algebraic development of the compound angle formulas (e.g., verify the formulas in numerical examples, using technology; follow a demonstration of the algebraic development [student reproduction of the development of the general case is not required]), and use the formulas to determine exact values of trigonometric ratios [e.g., determining the exact value of $\sin\left(\frac{\pi}{12}\right)$ by first rewriting it in terms of special angles as $\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$]

3.3 recognize that trigonometric identities are equations that are true for every value in the domain (i.e., a counter-example can be used to show that an equation is not an identity), prove trigonometric identities through the application of reasoning skills, using a variety of relationships (e.g., $\tan x = \frac{\sin x}{\cos x}$; $\sin^2 x + \cos^2 x = 1$; the reciprocal identities; the compound angle formulas), and verify identities using technology

Sample problem: Use the compound angle formulas to prove the double angle formulas.

3.4 solve linear and quadratic trigonometric equations, with and without graphing technology, for the domain of real values from 0 to 2π, and solve related problems

Sample problem: Solve the following trigonometric equations for $0 \le x \le 2\pi$, and verify by graphing with technology: $2\sin x + 1 = 0$; $2\sin^2 x + \sin x - 1 = 0$; $\sin x = \cos 2x$; $\cos 2x = \frac{1}{2}$.