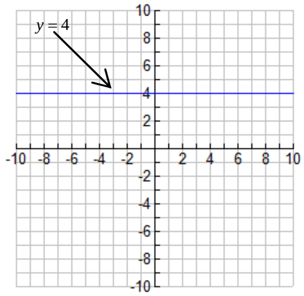
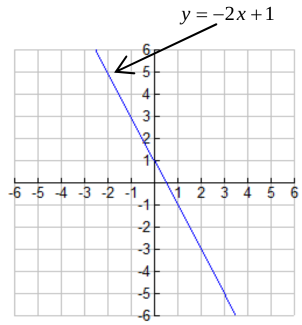
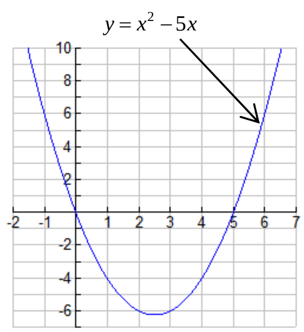
Unit 3 – Polynomial and Rational Functions

Introduction to Polynomial Functions

Polynomials that you Already Know and Love

Class of Polynomial Function	*General Equation*	*Example Graph*	*Features*
Constant Polynomial Functions	$f(x)=C,\ \ C\in\mathbb{R}$		• slope = rate of change of y with respect to x = 0
Linear Polynomial Functions	$f(x)=mx+b$		• m = slope = steepness of line = vertical stretch factor = rate of change of y with respect to x • b = y-intercept • $-\dfrac{b}{m}$ = x-intercept
Quadratic Polynomial Functions	$f(x)=ax^2+bx+c$ (Standard Form) $f(x)=a(x-h)^2+k$ (Vertex Form)		• a = vertical stretch factor • $a>0\rightarrow$parabola opens upward • $a<0\rightarrow$parabola opens downward • $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ = x-intercept(s) = zeros = roots If $b^2-4ac>0$, there are ***two*** x-intercepts If $b^2-4ac=0$, there is ***one*** x-intercept If $b^2-4ac<0$, there are ***no*** x-intercepts • c = y-intercept • co-ordinates of vertex: $\left(-\dfrac{b}{2a},\dfrac{4ac-b^2}{4a}\right)$

Examples

These are polynomial expressions.
$3x^2-5x+3$
$-4x+5x^7-3x^4+2$
$\frac{2}{5}x^3-3x^5+4$
$\sqrt{4}x^3-\frac{\sqrt{5}}{3}x^2+2x-\frac{1}{4}$
$3x-5$
-7
$-4x$
$(2x-3)(x+1)^2$

These are not polynomial expressions.
$\sqrt{x}+5x^3$
$\dfrac{1}{2x+5}$
$6x^3+5x^2-3x+2+4x^{-1}$
$\dfrac{3x^2+5x-1}{2x^2+x-3}$
4^x+5
$\sin(x-30)$
$x^2y+3x-4y^{-2}$
$3x^3+4x^{2.5}$

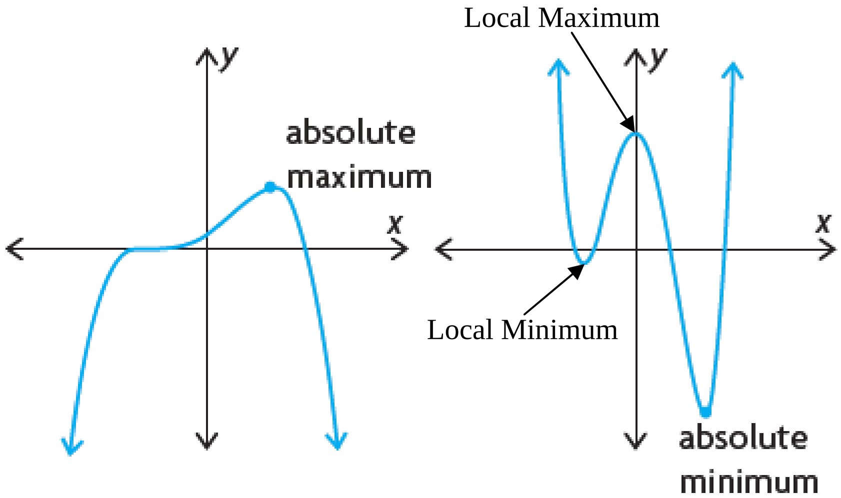
General Form of a Polynomial Function

- Let n be any whole number and $a_0,\ a_1,\ a_2, \ldots a_{n-1},\ a_n$ be real numbers such that $a_n \neq 0$. Then, the function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is called ***a polynomial of degree n***.
- The ***degree n*** of the polynomial is equal to the ***exponent*** of the highest power x^n.
- The numbers $a_0,\ a_1,\ a_2, \ldots, a_{n-1},\ a_n$ are called the ***numerical coefficients*** or the ***coefficients*** of the polynomial.
- The coefficient a_n of the highest power x^n is called the ***leading coefficient*** of the polynomial.

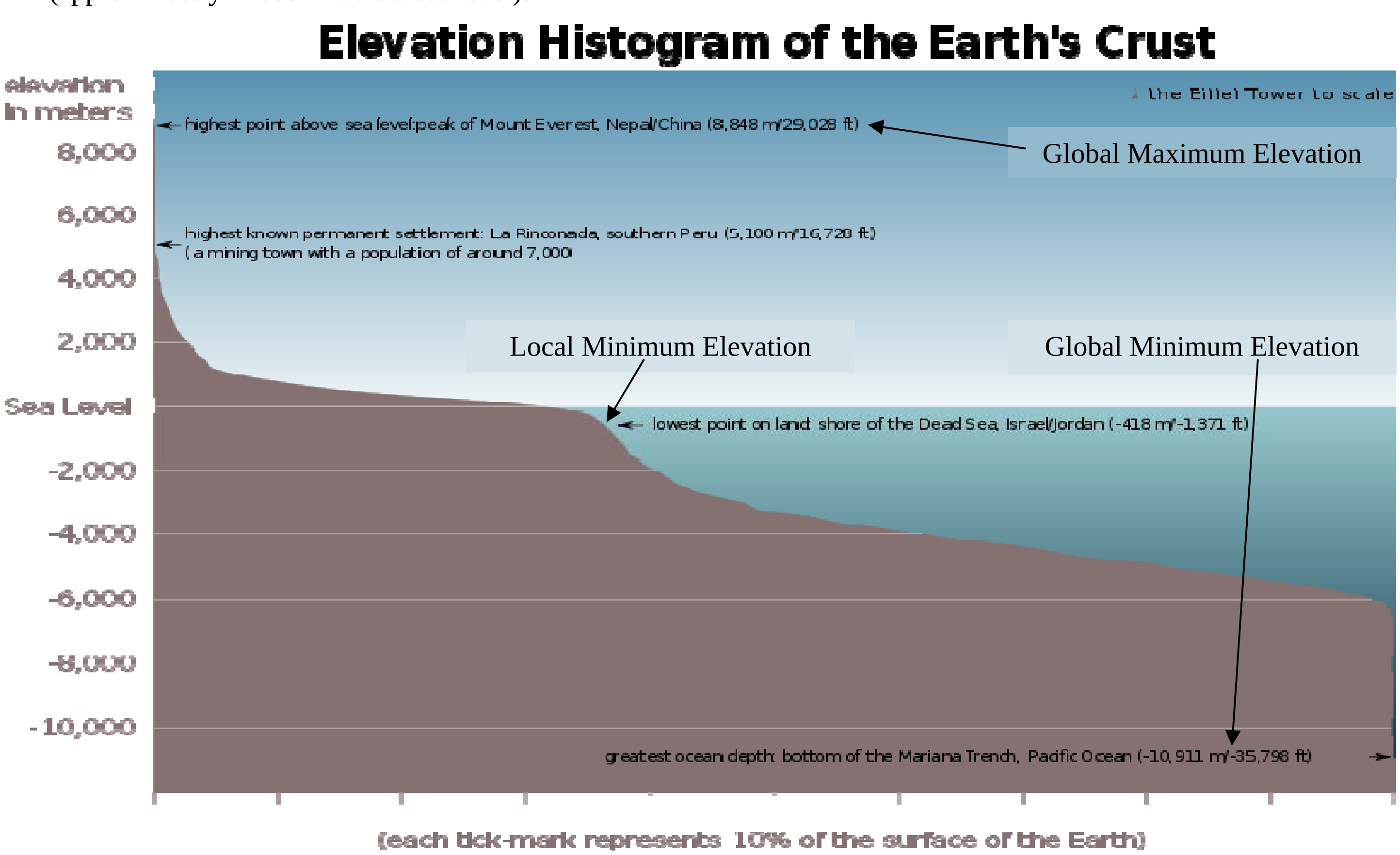
Extreme (Turning) Points

In general, an ***extreme point*** or a ***turning point*** of a function is any point at which the function ***changes direction***.

- If a point on a graph has a y-co-ordinate that is greater than or equal to the y-co-ordinate of any other point on the graph, it is called an ***absolute*** or ***global maximum point***.
- If a point on a graph has a y-co-ordinate that is less than or equal to the y-co-ordinate of any other point on the graph, it is called an ***absolute*** or ***global minimum point***.
- Collectively, maximum and minimum points are called ***extreme points***.

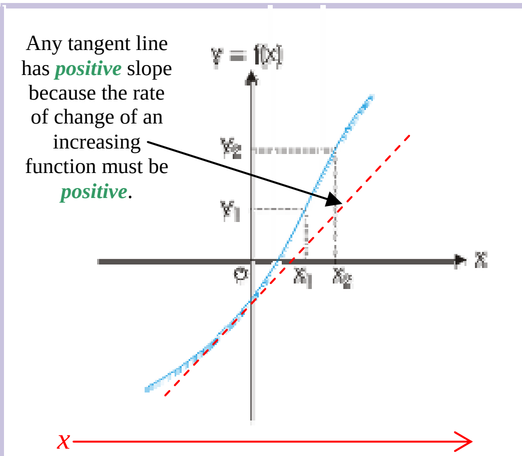


- By contrast, ***local extreme*** points are maximum or minimum points in a restricted region of a function.
- The global maximum and minimum points of the Earth's crust are, respectively, the peak of Mount Everest (approximately 8848 m above sea level) and the deepest part of the Mariana Trench in the Pacific Ocean (approximately 11033 m below sea level).

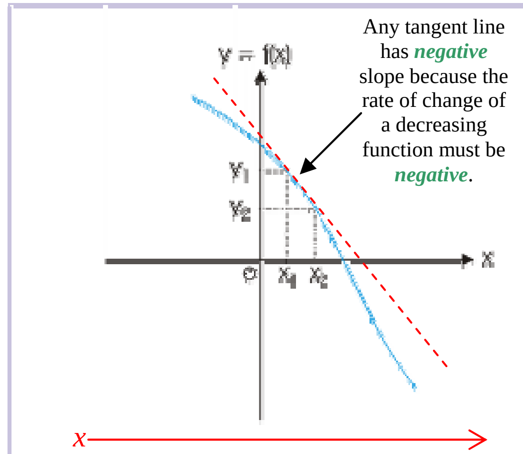


Increasing and Decreasing Functions

- The function f is said to be ***(strictly) increasing*** if for all choices of x_1 and x_2 such that $x_1 < x_2$, $f(x_1) < f(x_2)$.
  - That is, f is said to be ***(strictly) increasing*** if $y = f(x)$ increases as x increases.
  - If you imagine walking along the graph of an increasing function, you would always be walking ***uphill*** as you move from left to right.
- The function f is said to be ***(strictly) decreasing*** if for all choices of x_1 and x_2 such that $x_1 < x_2$, $f(x_1) > f(x_2)$.
  - That is, f is said to be ***(strictly) decreasing*** if $y = f(x)$ decreases as x increases.
  - If you imagine walking along the graph of a decreasing function, you would always be walking ***downhill*** as you move from left to right.



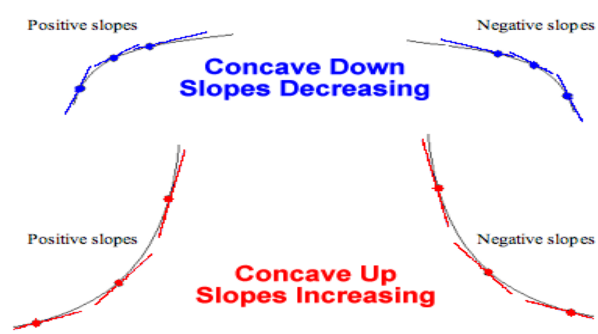
A Strictly Increasing Function
As x increases, so does $y = f(x)$.



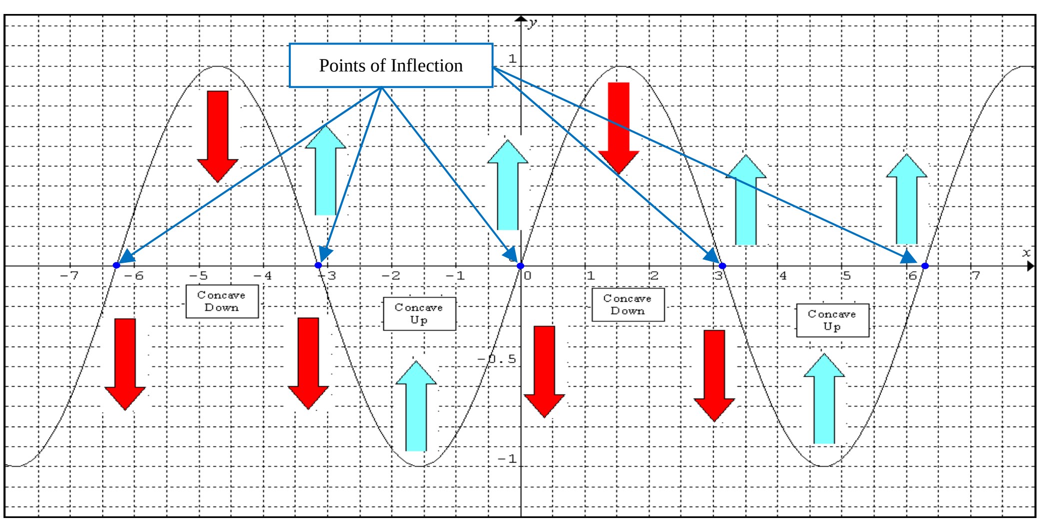
A Strictly Decreasing Function
As x increases, $y = f(x)$ decreases.

Concavity and Points of Inflection (Optional Topic)

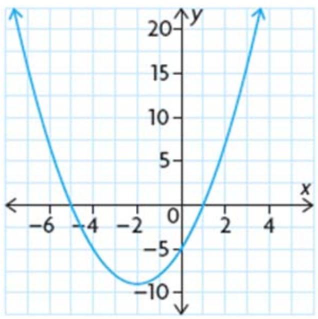
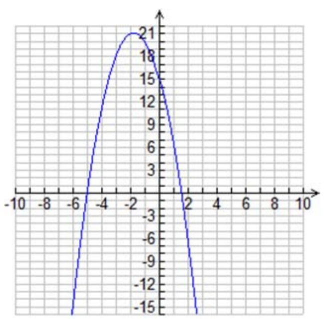
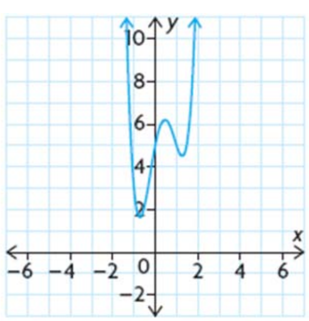
- ***Concave Down*** shape ("frowny")
  $\rightarrow$ slope decreases
  $\rightarrow$ rate of change decreases
- ***Concave Up*** shape ("smiley")
  $\rightarrow$ slope increases
  $\rightarrow$ rate of change increases

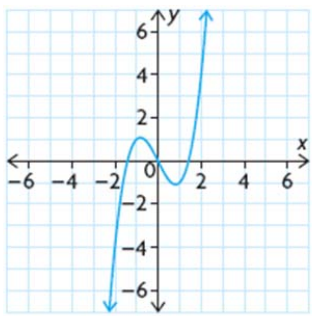
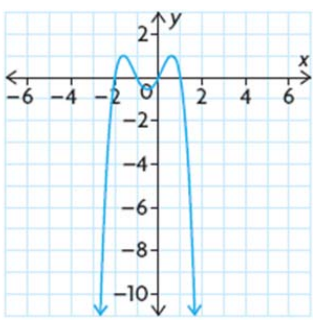
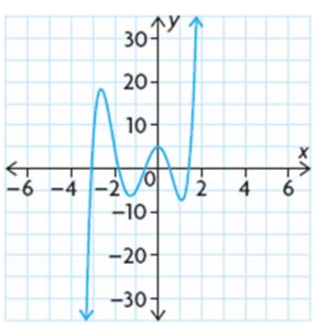


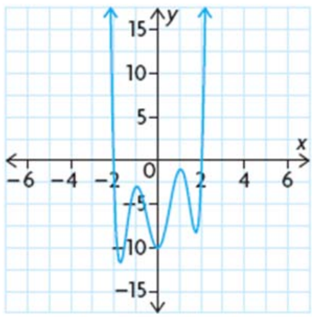
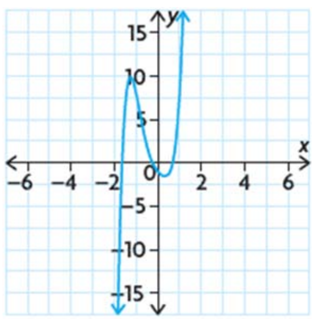
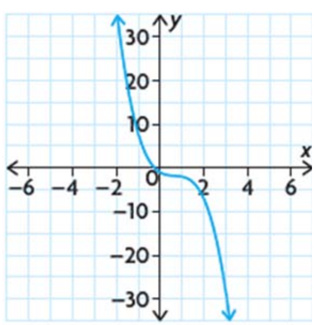
- A ***point of inflection*** is a point at which concavity changes.

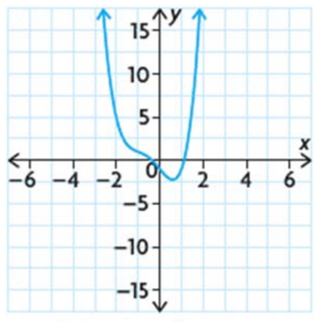


Behaviour of Polynomial Functions

Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviours		# of Turning Points	# of Points of Inflection	Intervals of Decrease	Intervals of Increase	Intervals where f is Concave Down	Intervals where f is Concave Up
				$x \to -\infty$	$x \to +\infty$						
 $f(x) = x^2 + 4x - 5$	2	even	+1	$y \to +\infty$	$y \to +\infty$	1	0	$(-\infty, -2)$	$(-2, \infty)$	nowhere	$(-\infty, \infty)$
 $f(x) = -2x^2 - 7x + 15$	2	even	–2	$y \to -\infty$	$y \to -\infty$	1	0	$\left(-\frac{7}{4}, \infty\right)$	$\left(-\infty, -\frac{7}{4}\right)$	$(-\infty, \infty)$	nowhere
 $f(x) = 3x^4 - 4x^3 - 4x^2 + 5x + 5$	4	even	+3	$y \to +\infty$	$y \to +\infty$	3	2	approximate values given here $(-\infty, -0.9)$ $(0.4, 1.2)$	approximate values given here $(-0.9, 0.4)$ $(1.2, \infty)$	approximate values given here $(0, 0.9)$	approximate values given here $(-\infty, 0)$ $(0.9, \infty)$

Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviours		# of Turning Points	# of Points of Inflection	Intervals of Decrease	Intervals of Increase	Intervals where f is Concave Down	Intervals where f is Concave Up
				$x \to -\infty$	$x \to +\infty$						
 $f(x) = x^3 - 2x$											
 $f(x) = -x^4 - 2x^3 + x^2 + 2x$											
 $f(x) = 2x^5 + 7x^4 - 3x^3 - 18x^2 + 5$											

Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviours		# of Turning Points	# of Points of Inflection	Intervals of Decrease	Intervals of Increase	Intervals where *f* is Concave Down	Intervals where *f* is Concave Up
				$x \to -\infty$	$x \to +\infty$						
 $f(x) = 2x^6 - 12x^4 + 18x^2 + x - 10$											
 $f(x) = 5x^5 + 5x^4 - 2x^3 + 4x^2 - 3x$											
 $f(x) = -2x^3 + 4x^2 - 3x - 1$											

Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviours		# of Turning Points	# of Points of Inflection	Intervals of Decrease	Intervals of Increase	Intervals where f is Concave Down	Intervals where f is Concave Up
				$x \to -\infty$	$x \to +\infty$						
 $f(x) = x^4 + 2x^3 - 3x - 1$											

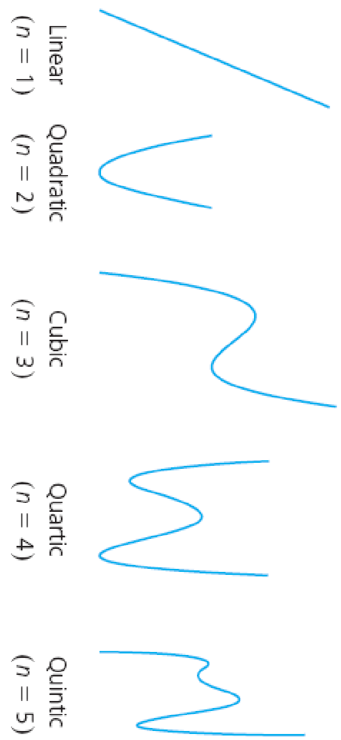
In Summary

Key Idea

- A polynomial in one variable is an expression of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number. The expression contains only one variable, with the powers arranged in descending order. For example, $2x + 5$, $3x^2 + 2x - 1$, and $5x^4 + 3x^3 - 6x^2 + 5x - 8$.

Need to Know

- In any polynomial expression, the exponents on the variable must be whole numbers.
- A polynomial function is any function that contains a polynomial expression in one variable. The degree of the function is the highest exponent in the expression. For example, $f(x) = 6x^3 - 3x^2 + 4x - 9$ has a degree of 3.
- The nth finite differences of a polynomial function of degree n are constant.
- The domain of a polynomial function is the set of real numbers, $\{x \in \mathbf{R}\}$.
- The range of a polynomial function may be all real numbers, or it may have a lower bound or an upper bound (but not both).
- The graphs of polynomial functions do not have horizontal or vertical asymptotes.
- The graphs of polynomial functions of degree zero are horizontal lines. The shape of other graphs depends on the degree of the function. Five typical shapes are shown for various degrees:



Determining the Behaviour of Polynomial Functions

In Summary

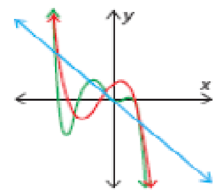
Key Ideas

- Polynomial functions of the same degree have similar characteristics.
- The degree and the leading coefficient in the equation of a polynomial function indicate the end behaviours of the graph.
- The degree of a polynomial function provides information about the shape, turning points, and zeros of the graph.

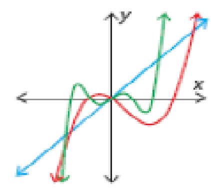
Need to Know

End Behaviours

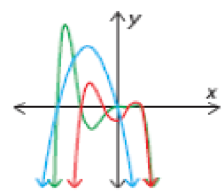
- An odd-degree polynomial function has opposite end behaviours.
  - If the leading coefficient is negative, then the function extends from the second quadrant to the fourth quadrant; that is, as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$.



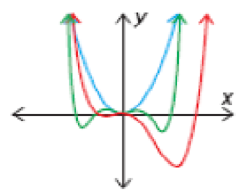
  - If the leading coefficient is positive, then the function extends from the third quadrant to the first quadrant; that is, as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$.



- An even-degree polynomial function has the same end behaviours.
  - If the leading coefficient is negative, then the function extends from the third quadrant to the fourth quadrant; that is, as $x \to \pm\infty$, $y \to -\infty$.



  - If the leading coefficient is positive, then the function extends from the second quadrant to the first quadrant; that is, as $x \to \pm\infty$, $y \to \infty$.



Turning Points

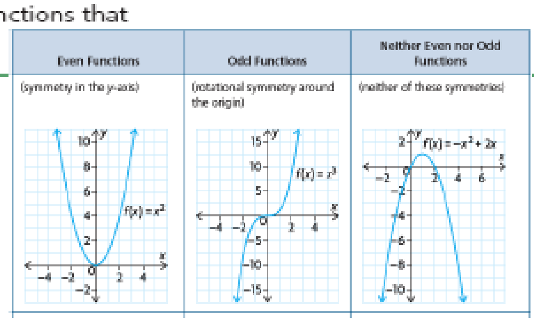
- A polynomial function of degree n has at most $n - 1$ turning points.
- An even degree polynomial has an odd number of turning points. An odd degree polynomial has an even number of turning points.

Number of Zeros

- A polynomial function of degree n may have up to n distinct zeros.
- A polynomial function of odd degree must have at least one zero.
- A polynomial function of even degree may have no zeros.

Symmetry

- Some polynomial functions are symmetrical in the y-axis. These are even functions, where $f(-x) = f(x)$.
- Some polynomial functions have rotational symmetry about the origin. These are odd functions, where $f(-x) = -f(x)$.
- Most polynomial functions have no symmetrical properties. These are functions that are neither even nor odd, with no relationship between $f(-x)$ and $f(x)$.



Number of Points of Inflection

- A polynomial function of degree n has at most $n - 2$ points of inflection.
- A polynomial of odd degree ≥ 3 must have at least one point of inflection.
- The number of points of inflection of a polynomial function can exceed the number of turning points. (e.g. $f(x) = x^3$ has no turning points and one point of inflection)

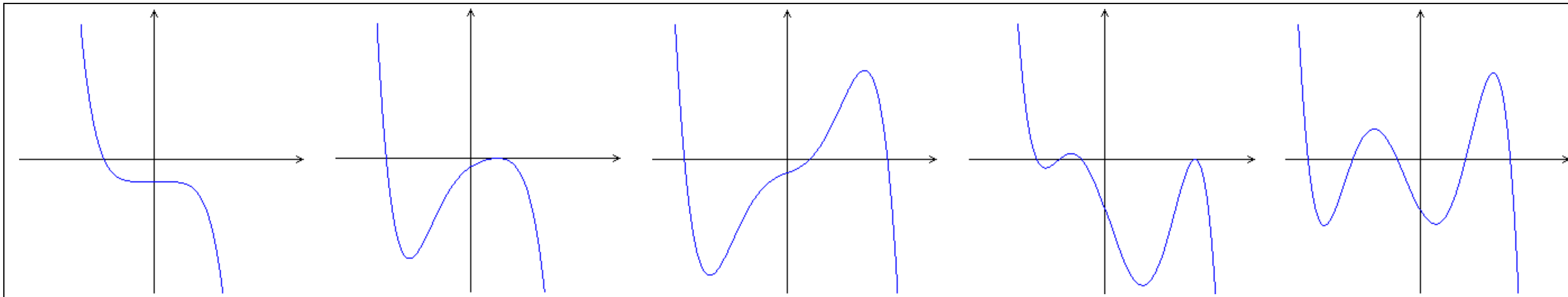
Example

Equation $f(x)=-3x^5+4x^3-8x^2+7x-5$		General Comments	End Behaviours		# of Zeros Possible	# of Turning Points Possible	# of Points of Inflection Possible	Absolute Max, Min or Neither?
			$x\to-\infty$	$x\to+\infty$				
Degree	5	Since the degree of this polynomial is odd, it has ***opposite*** end behaviours. The leading coefficient is negative, which means that the graph must extend from quadrant II to quadrant IV.	$y\to+\infty$ When x is a very large ***negative*** number such as –1000, $-3x^5$ has an extremely large ***positive*** value and has a greater effect on the value of the function than the other terms. Therefore, as $x\to-\infty$, $y\to+\infty$.	$y\to-\infty$ When x is a very large ***positive*** number such as 1000, $-3x^5$ has an extremely large ***negative*** value and has a greater effect on the value of the function than the other terms. Therefore, as $x\to+\infty$, $y\to-\infty$.	The degree is odd, which means that the polynomial function must have at least one zero. Since the degree is 5, there can be no more than 5 zeros.	The degree of *f* is odd, which means that it has opposite end behaviours. Hence, it must have an ***even*** number of turning points. As a result, *f* can have 0, 2 or 4 turning points.	For a polynomial function, the number of points of inflection is at most two less than the degree of the polynomial. Hence, there are either 1. 2 or 3 points of inflection.	Since the degree of *f* is odd, it cannot have any absolute extreme points.
Even or Odd Degree?	Odd							
Leading Coefficient	–3							

Summary

- *f* has opposite end behaviours: As $x\to-\infty$, $f(x)\to\infty$ ($\lim\limits_{x\to-\infty} f(x)=\infty$). As $x\to\infty$, $f(x)\to-\infty$ ($\lim\limits_{x\to\infty} f(x)=-\infty$).
- Since $f(x)=-3x^5+4x^3-8x^2+7x-5$, then $f(0)=-5$, which means that the *y*-intercept must be –5.
- *f* has 1 to 5 zeroes
- *f* has 0, 2 or 4 turning points but cannot have any absolute (global) turning points
- *f* has 1, 2 or 3 points of inflection

Possible graphs of $f(x)$



One Zero
Zero Turning Points
One Point of Inflection

Two Zeros
Two Turning Points
One Point of Inflection

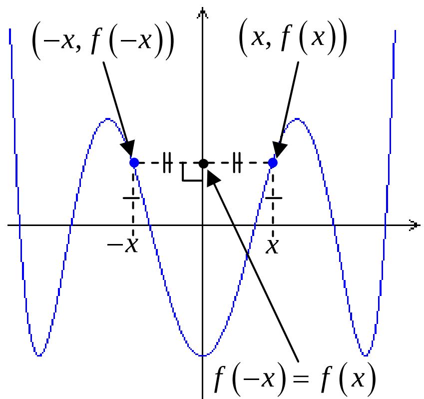
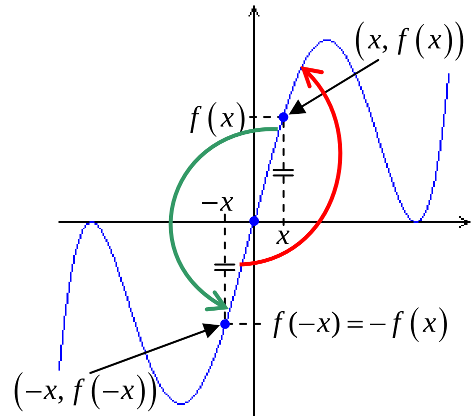
Three Zeros
Two Turning Points
Two Points of Inflection

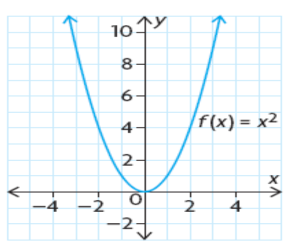
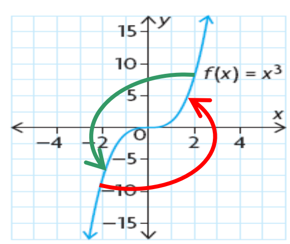
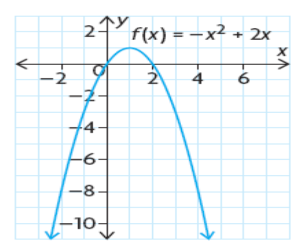
Four Zeros
Four Turning Points
Three Points of Inflection

Five Zeros
Four Turning Points
Three Points of Inflection

Symmetry – Even and Odd Functions

Certain functions can be classified according to symmetry that they exhibit. Two important categories of symmetries are shown in the diagrams below.

Even Functions	*Odd Functions*
 $f(x) = x^2(x-2)^2(x+2.5)^2 - 5$ ***An EVEN Polynomial Function*** For all ***even*** functions *f*, $f(-x) = f(x)$ and *f* is symmetric in the *y*-axis.	 $f(x) = x(x-2)^2(x+2.5)^2$ ***An ODD Polynomial Function*** For all ***odd*** functions *f*, $f(-x) = -f(x)$ and *f* has rotational symmetry about the origin.

Even Functions	Odd Functions	Neither Even nor Odd Functions
(symmetry in the *y*-axis)	(rotational symmetry around the origin)	(neither of these symmetries)
		

Important Questions

1. Complete the following table. (Don't forget to investigate the graph of each function!)

Function	*Odd or Even?*	*Explanation*	*Function*	*Odd or Even?*	*Explanation*
$f(x) = 2^x$	Neither	$f(-x) = 2^{-x} = \frac{1}{2^x} \neq f(x)$ $f(-x) = \frac{1}{2^x} \neq -2^x = -f(x)$	$f(x) = \tan x$		
$f(x) = \log_2 x$			$f(x) = \csc x$		
$f(x) = \sin x$			$f(x) = \sec x$		
$f(x) = \cos x$			$f(x) = \cot x$		

2. Is it possible for a function to have symmetry in the *x*-axis? Explain.

3. Besides symmetry in the *y*-axis and rotational symmetry about the origin, are there any other symmetries that a function can have?

4. Complete the following tables.

<table>
<tr><th colspan="2">Equation
$g(x)=2x^4+x^2+2$</th><th rowspan="2">General Comments (Including any Symmetry)</th><th colspan="2">End Behaviours</th><th rowspan="2"># of Zeros Possible</th><th rowspan="2"># of Turning Points Possible</th><th rowspan="2"># of Points of Inflection Possible</th><th rowspan="2">Absolute Max, Min or Neither?</th></tr>
<tr><th colspan="2"></th><th>$x \to -\infty$</th><th>$x \to +\infty$</th></tr>
<tr><td>Degree</td><td></td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td><td rowspan="3"></td></tr>
<tr><td>Even or Odd Degree?</td><td></td></tr>
<tr><td>Leading Coefficient</td><td></td></tr>
</table>

Possible graphs of $g(x)$

Homework
pp. 136 – 138: 3, 4, 5, 7, 8, 9, 12, 13, 14, 16

The Advantages of Writing Polynomial Expressions in Factored Form

Factoring – The "F" Word of Math

To the disappointment of many students, a great deal of time is spent developing factoring skills in high school mathematics. While factoring in and of itself is often tedious and sometimes may even appear to be purposeless, its importance in understanding polynomial functions cannot be underestimated. For instance, consider the polynomial function $f(x)=x^4-4x^3-7x^2+22x+24$. The table given below illustrates how much more convenient and informative the factored form of the equation can be.

Information that can be obtained easily when the Polynomial Equation is written in Standard Form	*Information that can be obtained easily when the Polynomial Equation is written in Factored Form*
$f(x)=x^4-4x^3-7x^2+22x+24$ From this form of the equation, we can only determine the following. • The highest power is x^4. Therefore, the end behaviours are the same (as $x\to\pm\infty$, $y\to\infty$). • The *y*-intercept is $f(0)=24$.	$f(x)=(x+1)(x+2)(x-3)(x-4)$ From this form of the equation, we can easily determine the much more. • The highest power is $x(x)(x)(x)=x^4$. Therefore, the end behaviours are the same (as $x\to\pm\infty$, $y\to\infty$). • The *y*-intercept is $1(2)(-3)(-4)=24$. • The zeros of the function are –1, –2, 3, 4 • There must be 3 turning points

Example 1

Sketch a possible graph of $f(x)=-(x+2)(x-1)(x-3)^2$

Solution

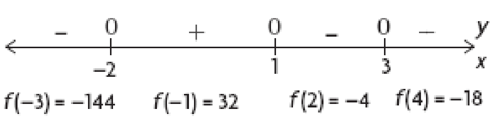
$$f(0) = -(0+2)(0-1)(0-3)^2$$
$$= -(2)(-1)(-3)^2$$
$$= 18$$

Calculate the *y*-intercept.

$$0 = -(x+2)(x-1)(x-3)^2$$
$$x=-2,\ x=1,\ \text{or } x=3$$

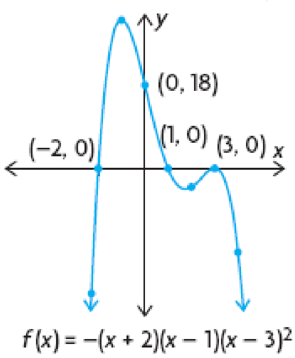
Determine the *x*-intercepts by letting $f(x)=0$. Use the factors to solve the resulting equation for *x*.

Use values of *x* that fall between the *x*-intercepts as test values to determine the location of the function above or below the *x*-axis.



Since the function lies below the *x*-axis on both sides of $x = 3$, the graph must just touch the *x*-axis and not cross over at this point. The order of 2 on the factor $(x-3)^2$ confirms the parabolic shape near $x = 3$.

Determine the end behaviours of the function.



Because the degree is even and the leading coefficient is negative, the graph extends from third quadrant to the fourth quadrant; that is, as $x\to\pm\infty$, $y\to-\infty$.

This is a possible graph of $f(x)$ estimating the locations of the turning points.

Order (Multiplicity) of Zeros

Let *r* represent a zero of a polynomial function $f(x)$. The ***order*** or ***multiplicity*** of *r* is equal to the "number of times" that *r* appears as a root of the polynomial equation $f(x)=0$. This can be stated more precisely as follows:

Let $f(x)$ represent a polynomial function and let *r* represent one of its zeros. We say that the zero *r* has ***order*** or ***multiplicity*** *k* if *k* is the largest possible value such that $(x-r)^k$ ***is*** a factor of $f(x)$. (i.e. $(x-r)^k$ ***is*** a factor of $f(x)$ but $(x-r)^m$ ***is not*** a factor of $f(x)$ for $m>k$.)

Example

$f(x)=x^2(x-5)^4(x-1)^3(x+4)^5(x+2)$

Zero	Order (Multiplicity)
–4	5
–2	1
0	2
1	3
5	4

Example 2

Write the equation of a cubic function with x-intercepts –2, 3 and $\frac{2}{5}$ and y-intercept 6.

Solution

Let f represent the cubic polynomial function. Note that the x-intercepts of f are the same as the zeros of f. Therefore, the equation of f must take the form $f(x) = a(x+2)(x-3)(5x-2)$ where $a \in \mathbb{R}$. Then,

$$f(x) = a(x+2)(x-3)(5x-2)$$

Use the zeros of the function to create factors for the correct family of polynomials. Since this function has three zeros and it is cubic, the order of each factor must be 1.

$$6 = a(0+2)(0-3)(5(0)-2)$$
$$6 = a(2)(-3)(-2)$$
$$6 = 12a$$
$$a = \frac{1}{2}$$

Use the y-intercept to calculate the value of a.

Substitute $x = 0$ and $y = 6$ into the equation, and solve for a.

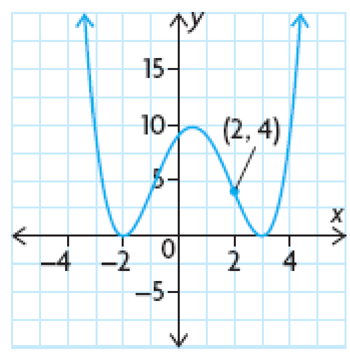
$$f(x) = \frac{1}{2}(x+2)(x-3)(5x-2)$$

Write the equation in factored form.

Therefore, the equation of f in factored form is $f(x) = \frac{1}{2}(x+2)(x-3)(5x-2)$.

Exercise 1

Write an equation of the graph shown at the right. In addition, state its domain and range.



Solution

Exercise 2

Sketch the graph of $f(x) = x^4 + 2x^3$.

Solution

In Summary

Key Idea

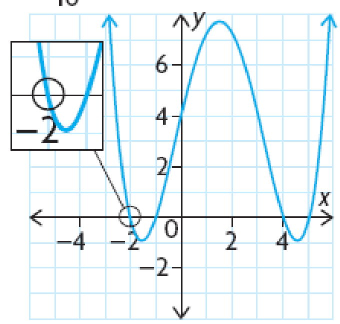
- The zeros of the polynomial function $y = f(x)$ are the same as the roots of the related polynomial equation, $f(x) = 0$.

Need to Know

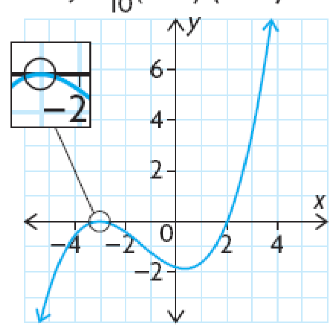
- To determine the equation of a polynomial function in factored form, follow these steps:
  - Substitute the zeros $(x_1, x_2, \ldots, x_n)$ into the general equation of the appropriate family of polynomial functions of the form $y = a(x - x_1)(x - x_2)\ldots(x - x_n)$.
  - Substitute the coordinates of an additional point for x and y, and solve for a to determine the equation.
- If any of the factors of a polynomial function are linear, then the corresponding x-intercept is a point where the curve passes through the x-axis. The graph has a linear shape near this x-intercept.

Zeros of order (multiplicity) 1

$y = \frac{1}{10}(x + 2)(x + 1)(x - 4)(x - 5)$



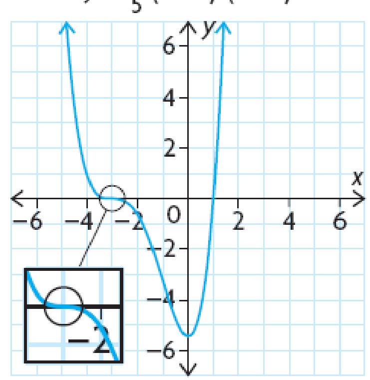
$y = \frac{1}{10}(x + 3)^2(x - 2)$



- If any of the factors of a polynomial function are squared, then the corresponding x-intercepts are turning points of the curve and the x-axis is tangent to the curve at these points. The graph has a parabolic shape near these x-intercepts.

Zeros of order (multiplicity) 2

$y = \frac{1}{5}(x + 3)^3(x - 1)$



- If any of the factors of a polynomial function are cubed, then the corresponding x-intercepts are points where the x-axis is tangent to the curve and also passes through the x-axis. The graph has a cubic shape near these x-intercepts.

Zeros of order (multiplicity) 3

Homework
pp. 146 – 148: 1, 2, 4, 5, 6ef, 7, 8, 9bd, 12, 14, 16

Using Transformations to Sketch the Graphs of Cubic and Quartic Functions

Review

Recall that we can sketch the graph of $y=g(x)=af\left(b(x-h)\right)+k$ by applying stretches/compressions and translations to the graph of $y=f(x)$.

$$y=g(x)=af(b(x-h))+k$$

$$x \rightarrow -h \rightarrow \times b \rightarrow b(x-h) \rightarrow \boxed{f} \rightarrow f\left(b(x-h)\right) \rightarrow \times a \rightarrow +k \rightarrow af\left(b(x-h)\right)+k$$

$$x \rightarrow \boxed{g} \rightarrow af\left(b(x-h)\right)+k$$

Algebraic Form	Transformations expressed in Words	Mapping Notation
Base Function for all Polynomials of Degree n $f(x)=x^n$ ***General Equation of g*** $g(x)=af\left(b(x-h)\right)+k$ $=a\left(b(x-h)\right)^n+k$ $=ab^n(x-h)^n+k$	***Horizontal*** **1.** Stretch/compress by a factor of $1/b=b^{-1}$ depending on whether $0<b<1$ or $b>1$. If b is negative, there is also a reflection in the y-axis. **2.** Shift h units right if $h>0$ or h units left if $h<0$. ***Vertical*** **1.** Stretch/compress by a factor of a depending on whether $a>1$ or $0<a<1$. If a is negative, there is also a reflection in the x-axis. **2.** Shift k units up/down depending on whether k is positive or negative.	(x,y) $\downarrow$ $\left(\frac{1}{b}x+h, ay+k\right)$

Example 1

Sketch the graph of $g(x)=-4\left(\frac{1}{3}x+1\right)^3+2$ by applying transformations to the graph of $f(x)=x^3$.

Solution

Method 1

First, write the equation of g to conform with the general form $g(x)=af\left(b(x-h)\right)+k$:

$$g(x)=-4\left(\frac{1}{3}x+1\right)^3+2=-4\left(\frac{1}{3}(x+3)\right)^3+2$$

Then decide what the transformations should be. By now, you should be able to glance at the equation and immediately write the transformation using mapping notation:

$$(x,y)\rightarrow(3x-3,-4y+2)$$

Method 2

First, simplify the equation fully:

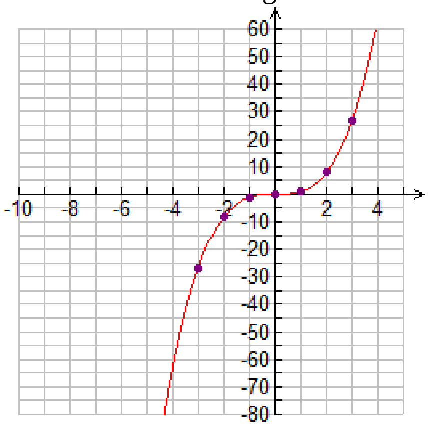
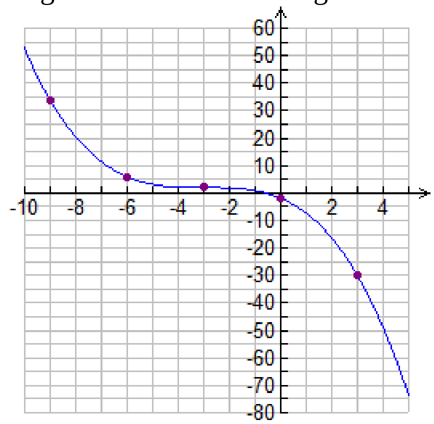
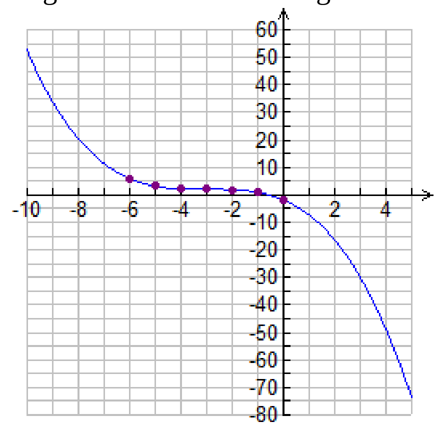
$$g(x)=-4\left(\frac{1}{3}x+1\right)^3+2$$
$$=-4\left(\frac{1}{3}(x+3)\right)^3+2$$
$$=-4\left(\frac{1}{3}\right)^3(x+3)^3+2$$
$$=-\frac{4}{27}(x+3)^3+2$$

Then write the transformation using mapping notation:

$$(x,y)\rightarrow\left(x-3,-\frac{4}{27}y+2\right)$$

Although this answer is not the same as that obtained using method 1, it is equivalent to it when applied to the base function $f(x)=x^3$

Pre-image Points on Graph of $f(x)=x^3$	Image Points on Graph of $g(x)=-4\left((1/3)x+1\right)^3+2$	
(x,y)	Method 1: $(3x-3,-4y+2)$	Method 2: $\left(x-3,(-4/27)y+2\right)$
$(0,0)$	$(-3,2)$	$(-3,2)$
$(1,1)$	$(0,-2)$	$\left(-2,\frac{50}{27}\right)$
$(-1,-1)$	$(-6,6)$	$\left(-4,\frac{58}{27}\right)$
$(2,8)$	$(3,-30)$	$\left(-1,\frac{22}{27}\right)$
$(-2,-8)$	$(-9,34)$	$\left(-5,\frac{86}{27}\right)$
$(3,27)$	$(6,-106)$	$(0,-2)$
$(-3,-27)$	$(-12,110)$	$(-6,6)$

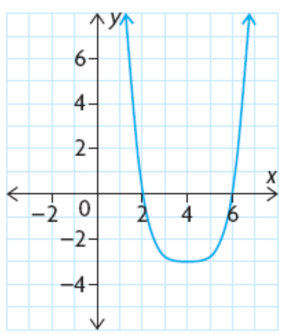
Base Graph: $y = x^3$	***Image under the Transformation given Above***	
Note the Pre-image Points	Image Points obtained using Method 1	Image Points obtained using Method 2
		

Example 2

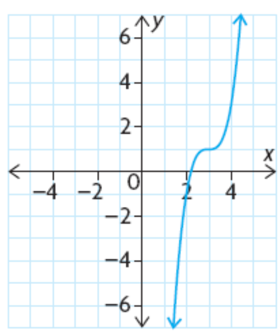
Match each function with the most suitable graph. Explain your reasoning.

a) $y = 2(x-3)^3 + 1$ **b)** $y = -\frac{1}{3}(x+1)^3 - 1$ **c)** $y = 0.2(x-4)^4 - 3$ **d)** $y = -1.5(x+3)^4 + 4$

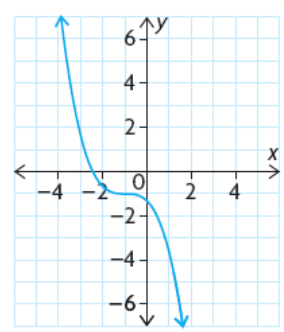
A



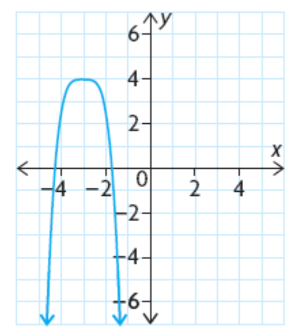
B



C



D



Solution

Because they are equations of polynomials of odd degree, a) and b) must describe functions with opposite end behaviours. Therefore, they can only match with graphs B and C. Since equation a) describes a cubic polynomial that has a vertical shift of one unit upward with respect to the base function $y = x^3$, it can only be matched with graph B. Similarly, equation c) can only be matched with graph C. Since graph A has same end behaviours and as $x \to \pm\infty$, $y \to \infty$, it can only match with equation c). Finally, by a process of elimination, graph D must match with equation d).

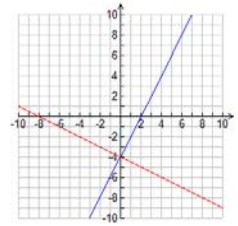
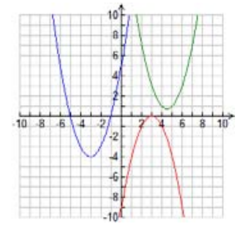
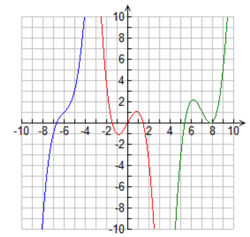
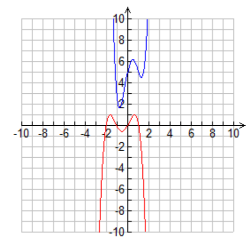
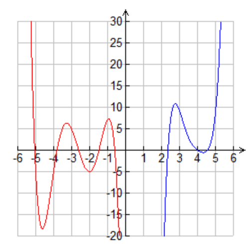
a) $\to$ B, b) $\to$ C, c) $\to$ A, d) $\to$ D

Homework

pp. 157 – 158: 6, 7, 8, 9cf, 10, 11, 13, 15

Solving Polynomial Equations of Degree 3 or Higher

Introduction

Polynomial Function	*Degree*	*Corresponding Polynomial Equation*	*# Zeros of Function (=# of Roots of Equation)*	*General Solution of Polynomial Equation in Terms of Coefficients*	*Typical Graphs*
$f(x)=a_1x+a_0$	1	$a_1x+a_0=0$	exactly 1	$x=-\dfrac{a_0}{a_1}$	
$f(x)=a_2x^2+a_1x+a_0$	2	$a_2x^2+a_1x+a_0=0$	0, 1 or 2	$x=\dfrac{-a_1\pm\sqrt{a_1^2-4a_2a_0}}{2a_2}$	
$f(x)=a_3x^3+a_2x^2+a_1x+a_0$	3	$a_3x^3+a_2x^2+a_1x+a_0=0$	1, 2 or 3	The roots of $a_3x^3+a_2x^2+a_1x+a_0=0$ can be expressed in terms of the coefficients a_3, a_2, a_1 and a_0. However, doing so involves complicated algebraic manipulations. See Cubic Function for more information.	
$f(x)=a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$	4	$a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$	0, 1, 2, 3 or 4	The roots of $a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$ can be expressed in terms of the coefficients a_4, a_3, a_2, a_1 and a_0. However, doing so involves complicated algebraic manipulations. See Quartic Equation for more information.	
$f(x)=a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$	5	$a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$	1, 2, 3, 4 or 5	For any polynomial equation of degree 5 or higher, it is ***not possible*** to express the roots in terms of the coefficients of the polynomial. This was proved in 1824 by the brilliant young Norwegian mathematician Niels Henrik Abel (1802 – 1829).	

Long Division and the Remainder Theorem – A Limited Approach to Solving Polynomial Equations

As summarized above, linear and quadratic polynomial equations are the ***only ones*** for which it is always easy to find exact solutions! Any cubic or quartic polynomial equation can also be solved exactly but the process is somewhat long and tedious. Exact solutions for polynomial equations of degree five or higher can be found only in certain special cases. In most cases, it is ***impossible*** to solve polynomial equations of degree five or higher using algebraic methods.

- Only a small class of equations can be solved by using algebraic methods. Therefore, methods of approximation are used. Such methods must be executed by a computer because they involve copious calculations.
- For our purposes, we shall solve certain polynomial equations of ***degree three*** or higher by ***guessing*** one of the roots. This method is based on a theorem known as the ***Remainder Theorem***.

The Remainder Theorem

Suppose that *p* is a polynomial function of degree *n* and that $a \in \mathbb{R}$. Then, *p* can be expressed uniquely in the form

$$p(x) = (x-a)q(x) + R,$$

where *q* is a polynomial of degree *n* – 1 and $R \in \mathbb{R}$. The value *R* is called the ***remainder***.

It follows immediately that $p(a) = R$, that is, the remainder can be calculated simply by substituting *a* into the polynomial. That is, when $p(x)$ is divided by $x-a$, the remainder is $p(a)$.

The remainder theorem gives us a quick way to calculate the remainder obtained when a polynomial is divided by a linear binomial. It states that a polynomial can be divided by a linear binomial to obtain a quotient $q(x)$, which is a polynomial of degree *n* – 1, and a remainder *R*, which is a real number.

Examples

$$
\begin{array}{rl}
 & x^2+2x+5 \\
x-2\,\big) & x^3+0x^2+x+1 \\
 & x^3-2x^2 \\ \hline
 & 2x^2+x \\
 & 2x^2-4x \\ \hline
 & 5x+1 \\
 & 5x-10 \\ \hline
 & 11
\end{array}
$$

$$\therefore p(x) = x^3+x+1 = (x-2)(x^2+2x+5)+11$$

Notice that

$$
\begin{aligned}
p(2) &= (2-2)(2^2+2(2)+5)+11 \\
&= 0(2^2+2(2)+5)+11 \\
&= 11
\end{aligned}
$$

$$
\begin{array}{rl}
 & x^2+2x+5 \\
x-2\,\big) & x^3+0x^2+x-10 \\
 & x^3-2x^2 \\ \hline
 & 2x^2+x \\
 & 2x^2-4x \\ \hline
 & 5x-10 \\
 & 5x-10 \\ \hline
 & 0
\end{array}
$$

$$\therefore p(x) = x^3+x-10 = (x-2)(x^2+2x+5)$$

Notice that

$$
\begin{aligned}
p(2) &= (2-2)(2^2+2(2)+5) \\
&= 0(2^2+2(2)+5) \\
&= 0
\end{aligned}
$$

The remainder theorem gives us a method to solve certain polynomial equations of degree 3 or higher. As we can see from the examples at the left, $x-a$ divides into a polynomial $p(x)$ whenever the remainder *R* is zero. By the remainder theorem, $p(a) = R$. Therefore, $x-a$ divides into a polynomial $p(x)$ whenever $p(a) = 0$. Another way of stating this is that $x-a$ divides into $p(x)$ if *a* is a root of the polynomial equation $p(x) = 0$. The example below shows how we can exploit this to solve a cubic equation.

How to find a Value "a" such that $p(a) = 0$

To apply the remainder theorem, we need to find a value *a* such that $p(a) = 0$. How do we go about finding such a value? A simple observation can help us narrow down the possibilities.

Let $p(x) = bx^3 + cx^2 + dx + f$.

If $p(a) = 0$, then by the remainder theorem, $p(x) = (x-a)(mx^2+nx+s) = mx^3 + (n-am)x^2 - anx - as$.

Comparing the two different forms of $p(x)$, we can conclude that $b = m$, $c = n - am$, $d = -an$ and $f = -as$

Since $f = -as$, we can conclude that *a* must divide into *f*.

If $p(x) = bx^3 + cx^2 + dx + f$ and $p(a) = 0$, then *a* must divide into *f*.

A Corollary of the Remainder Theorem – The Factor Theorem

> ***The Factor Theorem***
> Let p represent any polynomial function. Then, $x-a$ is a factor of $p(x)$ ***if and only if*** $p(a)=0$.

Note

The phrase "if and only if" in the above statement expresses the ***logical equivalence*** of the statements "$x-a$ is a factor of $p(x)$" and "$p(a)=0$." That is, the use of "if and only if" means that ***both*** of the following statements are true.

- If $x-a$ is a factor of $p(x)$, then $p(a)=0$. (This statement is true.)
- If $p(a)=0$, then $x-a$ is a factor of $p(x)$. (The ***converse*** of the above statement is ***also*** true.)

In general, consider the following statements:

- If P is true then Q is true. (This is called a ***conditional statement***.)
- If Q is true then P is true. (This statement is called the ***converse*** of the above ***conditional statement***.)

If ***both*** of the above statements are true, then we can write

- P is true ***if and only*** if Q is true. (This statement is called a ***biconditional statement*** or a ***logical equivalence***.)

Note that ***not all statements*** are biconditional. Consider the following:

1. If I do all my homework each and every day, then Mr. Nolfi is extremely happy with my effort. (The statement)
2. If Mr. Nolfi is extremely happy with my effort, then I do all my homework each and every day. (The converse.)

3. If it is raining, then the roads are wet. (The statement.)
4. If the roads are wet, then it is raining. (The converse.)

Statements 1 and 3 are true in all possible cases and hence, we call them "true." However, the converses of the statements are false in some cases, and so, we call them false. It is possible that Mr. Nolfi is extremely happy with a student's efforts even if the student did not complete all of his/her homework. Similarly, the roads can be wet even if it is not raining.

Example of using the Factor Theorem to Solve a Cubic Equation

Solve $2r^3+45r^2-2052=0$

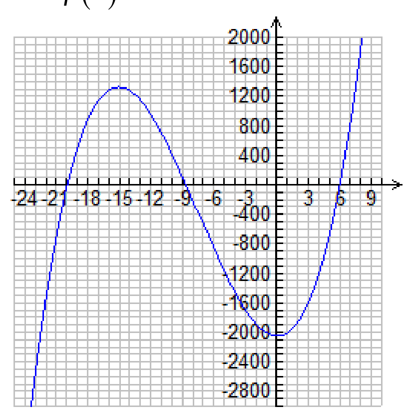
Solution 1

Let $f(r)=2r^3+45r^2-2052$.

Using the result on the previous page, we know that if $f(a)=0$, then a must divide into 2052. Therefore, it is only necessary to try factors of 2052 when searching for values of a such that $f(a)=0$.

a (Factors of 2052)	$f(a)$	Conclusion
±1	$f(-1)=-2009$ $f(1)=-2005$	$(r-(-1))=(r+1)$ is not a factor of $f(r)$ $(r-1)$ is not a factor of $f(r)$
±2	$f(-2)=-1888$ $f(2)=-1856$	$(r-(-2))=(r+2)$ is not a factor of $f(r)$ $(r-2)$ is not a factor of $f(r)$
±3	$f(-3)=-1701$ $f(3)=-1593$	$(r-(-3))=(r+3)$ is not a factor of $f(r)$ $(r-3)$ is not a factor of $f(r)$
±4	$f(-4)=-1460$ $f(4)=-1204$	$(r-(-4))=(r+4)$ is not a factor of $f(r)$ $(r-4)$ is not a factor of $f(r)$
±6	$f(-6)=-864$ $f(6)=0$	$(r-(-6))=(r+6)$ is not a factor of $f(r)$ $(r-6)$ ***is*** a factor of $f(r)$

The Graph of $f(r)=2r^3+45r^2-2052$



Since the degree of f is odd, f must have opposite end behaviours.

As shown above, by trial and error we find that $f(6)=0$. Therefore, $r-6$ divides into $f(r)$ with remainder zero. This means that $r-6$ is a factor of $f(r)$.

By ***long division***, we find that $f(r)=(r-6)(2r^2+57r+342)$.

$$\begin{array}{r}
2r^2+57r+342 \\
r-6\,\overline{)\,2r^3+45r^2+0r-2052} \\
\underline{2r^3-12r^2} \\
57r^2+0r \\
\underline{57r^2-342r} \\
342r-2052 \\
\underline{342r-2052} \\
0
\end{array}$$

$$\begin{aligned}
&2r^3+45r^2-2052=0 \\
&\therefore (r-6)(2r^2+57r+342)=0 \\
&\therefore r-6=0 \text{ or } 2r^2+57r+342=0 \\
&\therefore r=6 \text{ or } r=\frac{-57\pm\sqrt{513}}{4}
\end{aligned}$$

Solution 2

We know that $r-6$ is a factor of $f(r)$. Therefore, there exist real numbers *a*, *b* and *c* such that

$$\begin{aligned}
&(r-6)(ar^2+br+c)=2r^3+45r^2+0r-2052 \\
&\therefore ar^3+br^2+cr-6ar^2-6br-6c=2r^3+45r^2+0r-2052 \\
&\therefore ar^3+(b-6a)r^2+(c-6b)r-6c=2r^3+45r^2+0r-2052 \\
&\therefore a=2,\ b-6a=45,\ c-6b=0,\ -6c=-2052 \\
&\therefore a=2,\ b-12=45,\ c-6b=0,\ c=342 \\
&\therefore a=2,\ b=57,\ c=342 \\
&\therefore 2r^3+45r^2-2052=(r-6)(2r^2+57r+342)
\end{aligned}$$

Homework

pp. 204 – 206: 5, 6, 7def, 9ad, 10, 12, 14, 15, 16, 18

FACTORING POLYNOMIALS

Review – Common Factoring and Factoring Quadratic Polynomials

An ***expression*** is ***factored*** if it is written as a ***product***.

Common Factoring	*Factor Simple Trinomial*	*Factor Complex Trinomial*	*Difference of Squares*
Example $-42m^3n^2+13mn^2p-39m^4n^3q$ $=-13mn^2(4m^2-p+3m^3nq)$	***Example*** $n^2-20n+91$ $=(n-7)(n-13)$ ***Rough Work*** $(-7)(-13)=91$ $-7+(-13)=-20$	***Example*** $10x^2-x-21$ $=(10x^2-15x)+(14x-21)$ $=5x(2x-3)+7(2x-3)$ $=(2x-3)(5x+7)$ ***Rough Work*** $(10)(-21)=-210$, $(-15)(14)=-210$ $-15+14=-1$	***Example*** $98x^2-50y^2$ $=2(49x^2-25y^2)$ $=2((7x)^2-(5y)^2)$ $=2(7x-5y)(7x+5y)$

Factoring Polynomials of Degree Three or Higher

Factor each of the following polynomials.

$$x^4-6x^3+2x^2-12x$$
$$=\left(x^4-6x^3\right)+\left(2x^2-12x\right)$$
$$=x^3(x-6)+2x(x-6)$$
$$=(x-6)\left(x^3+2x\right)$$
$$=x(x-6)\left(x^2+2\right)$$

This method is called ***factoring by grouping***. It can be used in certain special cases but does not work very well in general.

In general, the factor theorem is more useful in factoring polynomials of degree three or higher.

$$4x^4+6x^3-6x^2-4x$$
$$=x\left(4x^3+6x^2-6x-4\right)$$
$$=x(x-1)\left(4x^2+10x+4\right)$$
$$=2x(x-1)\left(2x^2+5x+2\right)$$
$$=2x(x-1)(2x+1)(x+2)$$

Let $f(x)=4x^3+6x^2-6x-4$.

Then,

$$f(1)=4\left(1^3\right)+6\left(1^2\right)-6(1)-4=0$$

Therefore, by the factor theorem, $x-1$ is a factor of $f(x)$.

$$\begin{array}{r|l} & 4x^2+10x+4 \\ \hline x-1 & 4x^3+6x^2-6x-4 \\ & \underline{4x^3-4x^2} \\ & 10x^2-6x \\ & \underline{10x^2-10x} \\ & 4x-4 \\ & \underline{4x-4} \\ & 0 \end{array}$$

$$x^3-64$$
$$=(x-4)\left(x^2+4x+16\right)$$

This is an example of how a ***difference of cubes*** is factored.
Notice that the quadratic factor cannot be factored further.

$$b^2-4ac$$
$$=4^2-4(1)(16)$$
$$=16-64$$
$$=-48$$
$$<0$$

Let $f(x)=x^3-64$.

Then,

$$f(4)=4^3-64=0$$

Therefore, by the factor theorem, $x-4$ is a factor of $f(x)$.

$$\begin{array}{r|l} & x^2+4x+16 \\ \hline x-4 & x^3+0x^2+0x-64 \\ & \underline{x^3-4x^2} \\ & 4x^2+0x \\ & \underline{4x^2-16x} \\ & 16x-64 \\ & \underline{16x-64} \\ & 0 \end{array}$$

$$x^3+27$$
$$=(x+3)\left(x^2-3x+9\right)$$

This is an example of how a ***sum of cubes*** is factored.
Notice that the quadratic factor cannot be factored further.

$$b^2-4ac$$
$$=(-3)^2-4(1)(9)$$
$$=9-36$$
$$=-27$$
$$<0$$

Let $f(x)=x^3+27$. Then,

$$f(-3)=(-3)^3+27=0$$

Therefore, by the factor theorem, $x+3$ is a factor of $f(x)$.

$$\begin{array}{r|l} & x^2-3x+9 \\ \hline x+3 & x^3+0x^2+0x+27 \\ & \underline{x^3+3x^2} \\ & -3x^2+0x \\ & \underline{-3x^2-9x} \\ & 9x+27 \\ & \underline{9x+27} \\ & 0 \end{array}$$

Factoring Sums and Differences of Cubes

The last two examples on the previous page suggest a general method for factoring sums and differences of cubes.

Difference of Cubes

Let $f(x)=x^3-y^3$, where y represents some constant.

Since $f(y)=y^3-y^3=0$, by the factor theorem, $x-y$ must be a factor of $f(x)$.

$$\begin{array}{r|l} & x^2+yx+y^2 \\ \hline x-y & x^3+0x^2+0x-y^3 \\ & \underline{x^3-yx^2} \\ & \qquad yx^2+0x \\ & \qquad \underline{yx^2-y^2x} \\ & \qquad\qquad y^2x-y^3 \\ & \qquad\qquad \underline{y^2x-y^3} \\ & \qquad\qquad\qquad 0 \end{array}$$

Therefore, $f(x)=x^3-y^3=(x-y)(x^2+yx+y^2)$

Sum of Cubes

Let, $f(x)=x^3+y^3$, where y represents some constant.

Since $f(-y)=(-y)^3+y^3=0$, by the factor theorem, $x-(-y)=x+y$ must be a factor of $f(x)$.

$$\begin{array}{r|l} & x^2-yx+y^2 \\ \hline x+y & x^3+0x^2+0x+y^3 \\ & \underline{x^3+yx^2} \\ & \qquad -yx^2+0x \\ & \qquad \underline{-yx^2-y^2x} \\ & \qquad\qquad y^2x+y^3 \\ & \qquad\qquad \underline{y^2x+y^3} \\ & \qquad\qquad\qquad 0 \end{array}$$

Therefore, $f(x)=x^3+y^3=(x+y)(x^2-yx+y^2)$

Summary

cubes

$$x^3-y^3=(x-y)(x^2+xy+y^2)$$

difference

cubes

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$

sum

Examples

Factor each of the following:

(a) $8x^3+27$

$$=(2x)^3+3^3$$
$$=(2x+3)\left((2x)^2-3(2x)+3^2\right)$$
$$=(2x+3)(4x^2-6x+9)$$

(b) $64a^3-8b^3$

$$=8(8a^3-b^3)$$
$$=8\left((2a)^3-b^3\right)$$
$$=8(2a-b)\left((2a)^2+(2a)(b)+b^2\right)$$
$$=8(2a-b)(4a^2+2ab+b^2)$$

(c) $7m^4-448m$

$$=7m(m^3-64)$$
$$=7m(m^3-4^3)$$
$$=7m(m-4)(m^2+4m+16)$$

(d) $125x^9+512$

$$=(5x^3)^3+8^3$$
$$=(5x^3+8)\left((5x^3)^2-8(5x^3)+8^2\right)$$
$$=(5x^3+8)(125x^6-40x^3+64)$$

Important Question

Calculate the discriminant (b^2-4ac) of the quadratic polynomials in x obtained in the factorizations of x^3-y^3 and x^3+y^3. What do you notice? What conclusions can you draw?

Homework

p. 177: 4, 5, 6ef, 7ef, 9, 10, 11, 13, 15, 16, 17

p. 182: 4, 5, 6, 7, 8, 9, 10

Solving Polynomial Inequalities

Introductory Problem

Preetika has a monthly budget of \$1800. Each month she must pay \$950 for rent, \$375 for transportation, \$250 for food and \$100 in miscellaneous expenses. Assuming that there are no other expenses, determine the number of times per month that Preetika can go to the movies. (Assume an average price of admission of \$10.00 per movie.)

Solution

Although you might be tempted to solve this problem by using an equation, technically it would be incorrect. To see this, consider the following restatement of the problem:

total monthly expenses ***must be less than or equal to*** \$1750.00

$\therefore$ \$950 + \$375 + \$250 + \$100 + cost of movies $\leq$ \$1750

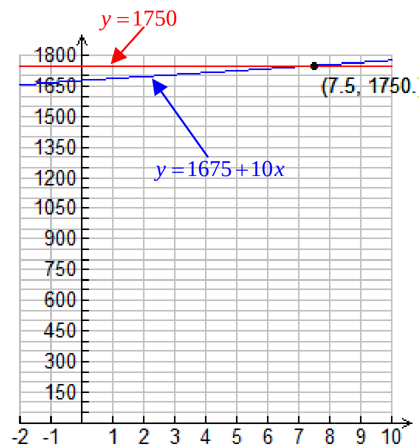
$\therefore$ \$1675 + cost of movies $\leq$ \$1750

Now if we let *x* represent the number of times that Preetika goes to the movies in one month, then we can represent this problem using the following inequality:

$$1675+10x \leq 1750$$

We can solve this inequality as follows:

$$\begin{aligned} 1675+10x &\leq 1750 \\ \therefore 10x &\leq 1750-1675 \\ \therefore 10x &\leq 75 \\ \therefore \frac{10x}{10} &\leq \frac{75}{10} \\ \therefore x &\leq 7.5 \end{aligned}$$



The solution set of the inequality is $\{x \in \mathbb{R} : x \leq 7.5\}$, or in interval notation, $(-\infty, 7.5]$.

The solution informs us that Preetika can stay within her monthly budget if she goes to the movies 7.5 or fewer times per month. Obviously, the number of times that Preetika goes to the movies must be a whole number. Therefore, Preetika can afford to go to the movies no more than 7 times per month.

Represent the solution on a number line.
A solid dot is placed on 7.5 since this number is included in the solution set.

Investigation

Consider the following series of inequalities. In which cases is the reasoning invalid? Can you draw any conclusions?

$$\begin{aligned} &\because 2<5 \\ &\therefore 2+6<5+6 \\ &\therefore 8<11 \\ &\therefore 8(2)<11(2) \\ &\therefore 16<22 \end{aligned}$$

?

$$\begin{aligned} &\because -3<-1 \\ &\therefore -3+4<-1+4 \\ &\therefore 1<3 \\ &\therefore \frac{1}{5}<\frac{3}{5} \end{aligned}$$

?

$$\begin{aligned} &\because 2>1 \\ &\therefore 2-6>1-6 \\ &\therefore -4>-5 \\ &\therefore 3(-4)>3(-5) \\ &\therefore -12>-15 \end{aligned}$$

?

$$\begin{aligned} &\because 2>1 \\ &\therefore 2+3>1+3 \\ &\therefore 5>4 \\ &\therefore -3(5)>-3(4) \\ &\therefore -15>-12 \end{aligned}$$

?

$$\begin{aligned} &-3<-1 \\ &\therefore -3+2<-1+2 \\ &\therefore -1<1 \\ &\therefore \frac{-1}{-5}<\frac{1}{-5} \\ &\therefore \frac{1}{5}<-\frac{1}{5} \end{aligned}$$

?

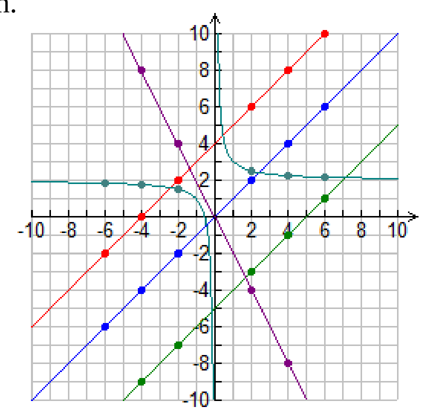
Working with Equations and Inequalities – Similarities and Differences

1. Whether solving an equation or an inequality, whatever operation is performed to one side must also be performed to the other side. Another way of stating this is that ***whatever function is applied to one side must also be applied to the other side***.
2. Whether solving an equation or an inequality, the ***general approach is to apply inverse operations to both sides in the order OPPOSITE the order of operations***.
3. When the same function is applied to both sides of an ***equation, equality is always preserved***.
4. However, when the same function is applied to both sides of an ***inequality, the inequality is NOT always preserved***.
   - Whenever a strictly increasing function is applied to both sides, the inequality is preserved.
   - However, when other functions are applied to both sides, the inequality may not be preserved.
   - In particular, if both sides of an inequality are ***multiplied or divided by a negative number***, the inequality ***must be reversed***. This happens because multiplying or dividing a set of numbers by a negative value causes the order of the numbers in the set to be inverted.

Understanding why an Inequality sometimes needs to be Reversed

Consider the values in the following table. Note that in the first three columns, the values are in ascending order (i.e. written from smallest to largest). Another way of interpreting this is that the operations of adding and subtracting ***preserved the order of the values***. In the fourth column, however, the values are in descending order (i.e. written from largest to smallest). The operation of multiplying by –2 caused the ***order of the values to be reversed***. A slightly more extreme example is given in the fifth column. When the function $y=\frac{1}{x}+2$ is applied, the numbers are in descending order up to a point, then briefly ascend only to descend once again.

x	$y=x+4$	$y=x-5$	$y=-2x$	$y=\frac{1}{x}+2$
-6	-2	-11	12	$11/6$
-4	0	-9	8	$7/4$
-2	2	-7	4	$3/2$
2	6	-3	-4	$5/2$
4	8	-1	-8	$9/4$
6	10	1	-12	$13/6$



To the left is a graphical view of the table. Certain operations (functions) preserve order, some reverse order and others produce mixed results.

Valid Deductions involving Inequalities		
$\because -6<-4$ $\therefore -6+4<-4+4$ $\therefore -2<0$	$\because -6<-4$ $\therefore -6-5<-4-5$ $\therefore -11<-9$	$\because -2<2$ $\therefore -2-5<2-5$ $\therefore -7<-3$
$\because -2<4$ $\therefore -2(-2)>2(-2)$ $\therefore 4>-4$	$\because -6<6$ $\therefore -2(-6)>-2(6)$ $\therefore 12>-12$	$\because 6>-4$ $\therefore -2(6)<-2(-4)$ $\therefore -12<8$

Invalid Deductions involving Inequalities	
$\because -6<6$ $\therefore -2(-6)<-2(6)$ $\therefore 12<-12$	$\because 4>2$ $\therefore 1/4>1/2$
$\because 4>2$ $\therefore 1/4>1/2$ $\therefore 1/4+2>1/2+2$ $\therefore 9/4>5/2$	$\because 4>2$ $\therefore 4^2>2^2$ $\therefore 1/4^2>1/2^2$ $\therefore 1/16>1/4$

Examples

Solve each of the following inequalities. Note that a good way to visualize the solution is to use a number line.

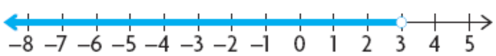
1. $-3x+1>-8$

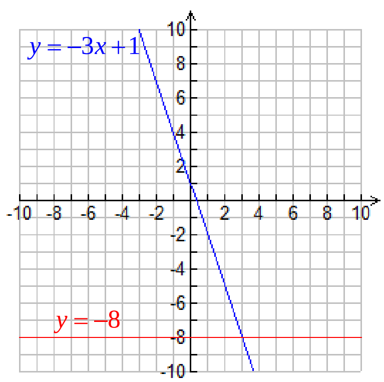
Solution

$$-3x+1>-8$$
$$\therefore -3x+1-1>-8-1$$
$$\therefore -3x>-9$$
$$\therefore \frac{-3x}{-3}<\frac{-9}{-3}$$
$$\therefore x<3$$

Remember to reverse the inequality when multiplying or dividing by a negative number.

The solution set of the inequality is $\{x\in\mathbb{R}: x<3\}$, or in interval notation, $(-\infty,3)$.



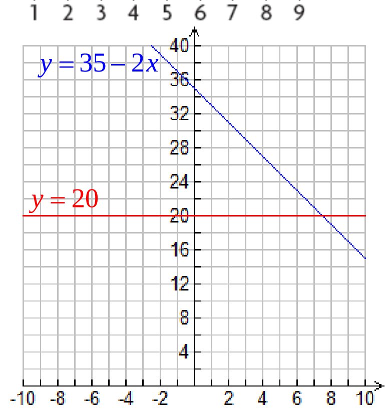


2. $35-2x\ge 20$

Solution

$$35-2x\ge 20$$
$$\therefore 35-2x-35\ge 20-35$$
$$\therefore -2x\ge -15$$
$$\therefore \frac{-2x}{-2}\le\frac{-15}{-2}$$
$$\therefore x\le\frac{15}{2}$$

The solution set of the inequality is $\left\{x\in\mathbb{R}: x\le\frac{15}{2}\right\}$, or in interval notation, $\left(-\infty,\frac{15}{2}\right]$.

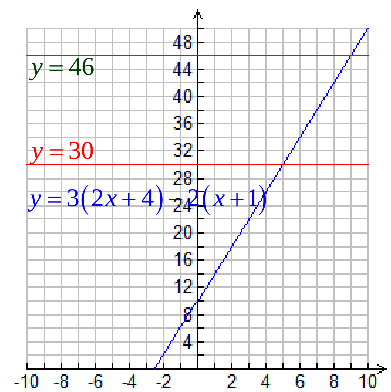


3. $30\le 3(2x+4)-2(x+1)\le 46$

Solution

$$30\le 3(2x+4)-2(x+1)\le 46$$
$$\therefore 30\le 6x+12-2x-2\le 46$$
$$\therefore 30\le 4x+10\le 46$$
$$\therefore 30-10\le 4x+10-10\le 46-10$$
$$\therefore 20\le 4x\le 36$$
$$\therefore \frac{20}{4}\le\frac{4x}{4}\le\frac{36}{4}$$
$$\therefore 5\le x\le 9$$

The solution set of the inequality is $\{x\in\mathbb{R}: 5\le x\le 9\}$, or in interval notation, $[5,9]$.



4. $x^2-5x+6>0$

Solution

$$x^2-5x+6>0$$
$$\therefore (x-2)(x-3)>0$$

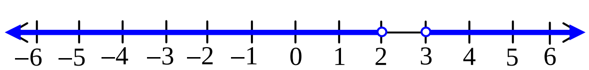
If the product $(x-2)(x-3)$ is positive, then either both factors are positive OR both factors are negative.

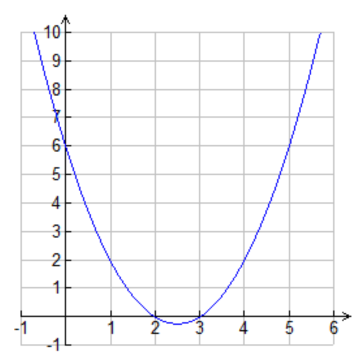
$\therefore x-2>0$ and $x-3>0$ ***or*** $x-2<0$ and $x-3<0$

$\therefore x>2$ and $x>3$ ***or*** $x<2$ and $x<3$

$\therefore x>3$ ***or*** $x<2$

The solution set of the inequality is $\{x\in\mathbb{R}: x<2 \text{ or } x>3\}$, or in interval notation, $(-\infty,2)\cup(3,\infty)$.





Shown above is the graph of $y=x^2-5x+6$. Notice that the parabola dips below the *x*-axis between *x* = 2 and *x* = 3 and that it is above the *x*-axis wherever *x* < 2 or *x* > 3. Remember that wherever a graph is ***above*** the *x*-axis, all *y*-values of points on that part of the graph are ***positive***. Similarly, wherever a graph is ***below*** the *x*-axis, all *y*-values of points on that part of the graph are ***negative***.

5. $2x^3+3x^2-17x+12>0$

Solution

$$2x^3+3x^2-17x+12>0$$

$$\therefore (x-1)\left(2x^2+5x-12\right)>0$$

$$\therefore (x-1)(2x-3)(x+4)>0$$

$$\therefore (x-1)(x+4)(2x-3)>0$$

As can be seen from the factorization of $f(x)=2x^3+3x^2-17x+12$, its zeros are –4, 1 and $\dfrac{3}{2}$.

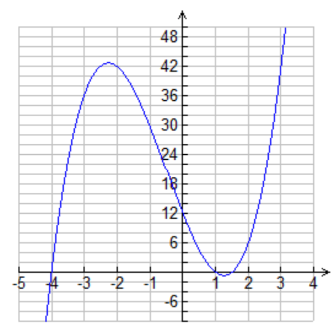
From the graph of $f(x)$ shown at the right, it's obvious that $f(x)>0$ (i.e. the graph is ***above*** the x-axis) if x is between –4 and 1 or if x is greater than $\dfrac{3}{2}$. Thus, the solution set of the inequality is

$$\left\{x\in\mathbb{R}: -4<x<1 \text{ or } x>\frac{3}{2}\right\}.$$

To verify this solution set, consider the table shown below.

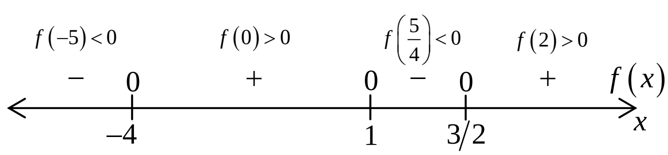
Let $f(x)=2x^3+3x^2-17x+12$. Since, $f(1)=0$ by the factor theorem, $x-1$ must be a factor of $f(x)$.

$$\begin{array}{r|l} & 2x^2+5x-12 \\ \hline x-1 & 2x^3+3x^2-17x+12 \\ & \underline{2x^3-2x^2} \\ & \qquad 5x^2-17x \\ & \qquad \underline{5x^2-5x} \\ & \qquad\qquad -12x+12 \\ & \qquad\qquad \underline{-12x+12} \\ & \qquad\qquad\qquad 0 \end{array}$$

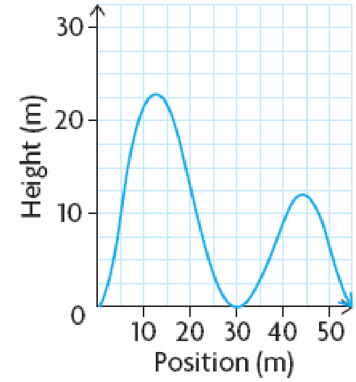


	$x<-4$	$-4<x<1$	$1<x<\frac{3}{2}$	$x>\frac{3}{2}$
$(x-1)$	–	–	+	+
$(x+4)$	–	+	+	+
$\left(x-\frac{3}{2}\right)$	–	–	–	+
Their Product	–	+	–	+

We can also use a number line to determine the sign of $f(x)$ over each interval.



6. The height of one section of a roller coaster can be modelled by the polynomial function $h(x)=\frac{1}{40000000}x^2(x-30)^2(x-55)^2$, where $h(x)$ is the height above the ground in metres, measured at the position x metres along the ground from the start. At what points will the roller coaster car be more than 9 metres above the ground?



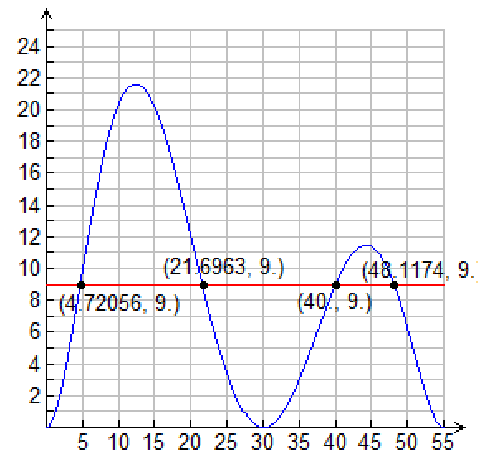
Solution

This problem is equivalent to solving the inequality $\frac{1}{40000000}x^2(x-30)^2(x-55)^2>9$.

Expanding and simplifying produces a polynomial inequality that is hopelessly complicated. Therefore, it is best to use a graphical approach in this case.

By using graphing software such as TI-Interactive or a graphing calculator, sketch the graphs of $y=\frac{1}{40000000}x^2(x-30)^2(x-55)^2$ and $y=9$. Then find the points of intersection (using for example the "Intersection…" option in the "Calculate" menu in TI-Interactive). Once the points of intersection are found, it's easy to see the approximate solution set of the inequality. The roller coaster is more than 9 m above the ground wherever the graph of

$y=\frac{1}{40000000}x^2(x-30)^2(x-55)^2$ lies above the graph of $y=9$.



Therefore, the roller coaster will be more than 9 m above the ground approximately between 4.7 m and 21.7 m from the starting point and approximately between 40 m and 48.1 m from the starting point.

Homework
pp. 213-215: 1f, 3, 5bef, 7f, 9, 10, 11, 13, 16, 18 pp. 225-228: 1d, 2, 5, 6bdf, 7ace, 8, 10, 11, 13, 14, 15, 18

Investigating Rational Functions

What is a Rational Function?

Just as a ***rational number*** is the ***ratio of two integers***, a ***rational function*** is the ***ratio of two polynomial functions***.

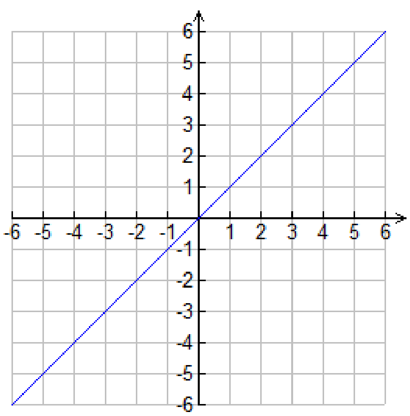
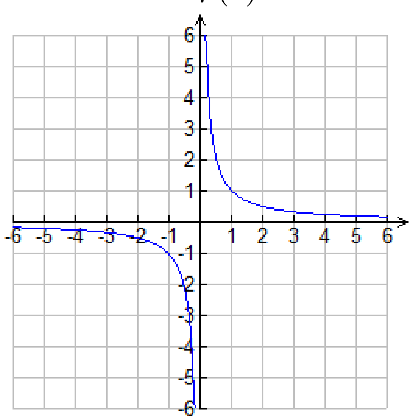
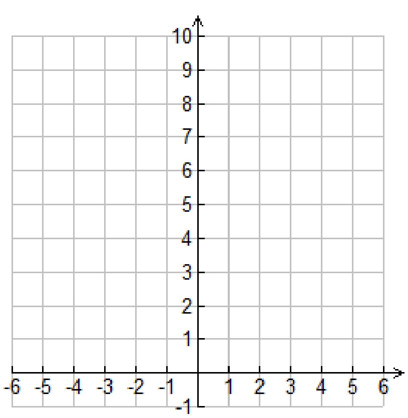
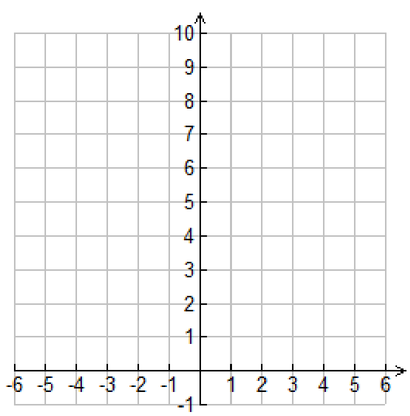
- ***Rational Numbers*** have the form $r = \frac{a}{b}$ where $a \in \mathbb{Z},\ b \in \mathbb{Z},\ \text{b} \neq 0$ (*a* and *b* are integers, *b* must be nonzero)
- ***Rational Functions*** have the form $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions such that $q(x) \neq 0$.

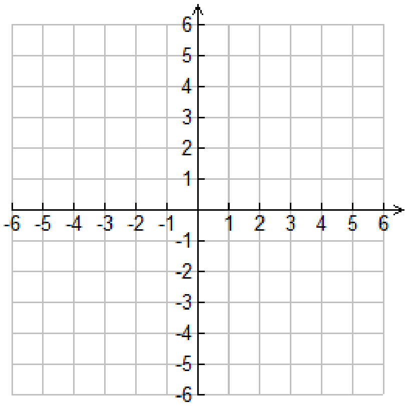
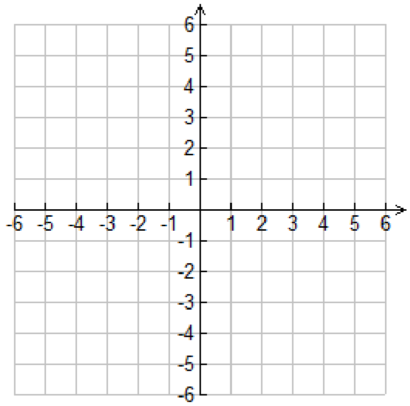
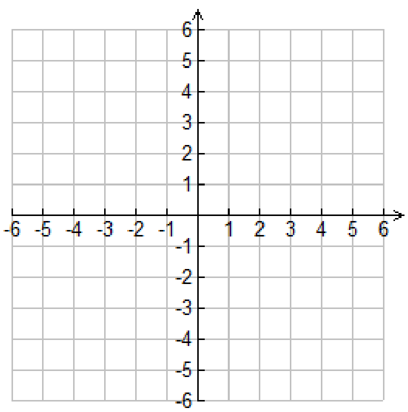
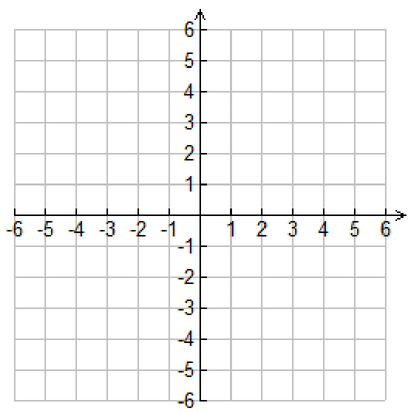
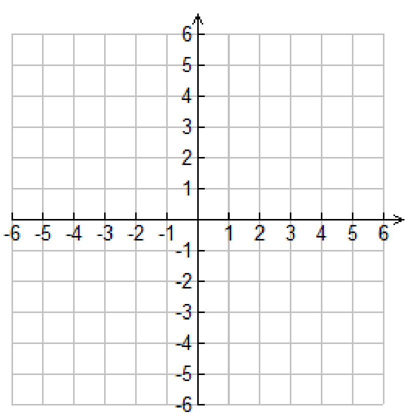
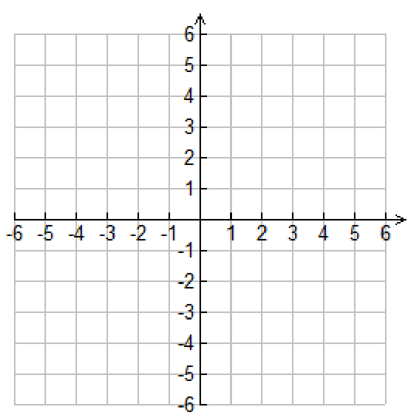
Graphs of the Simplest Rational Functions – Reciprocals of Polynomial Functions

Use a graphing calculator or graphing software to complete the following table.

Legend

VA→Vertical Asymptote(s) HA→Horizontal Asymptote(s) IP→Intervals on which the Function is Positive
IN→Intervals on which the Function is Negative II→Intervals on which the Function is Increasing
ID→Intervals on which the Function is Decreasing ICU→Intervals on which the Function is Concave Up
ICD→Intervals on which the Function is Concave Down PON→Points at which the Function is 1 or –1

Graph of Function	***Graph of its Reciprocal***	***Characteristics of Function***	***Characteristics of the Reciprocal of the Function***
$f(x) = x$ 	$g(x) = \frac{1}{f(x)} = \frac{1}{x}$ 	Zeros: 1 (at $x = 0$) VA: none HA: none IP: $(0, \infty)$ IN: $(-\infty, 0)$ II: $(-\infty, \infty)$ ID: none ICU: none ICD: none PON: $(-1,-1)$, $(1,1)$	Zeros: none VA: $x = 0$ HA: $y = 0$ IP: $(0, \infty)$ IN: $(-\infty, 0)$ II: none ID: $(-\infty,0),(0,\infty)$ ICU: $(0, \infty)$ ICD: $(-\infty, 0)$ PON: $(-1,-1)$, $(1,1)$
$f(x) = (x-1)^2$ 	$g(x) = \frac{1}{f(x)} = \frac{1}{(x-1)^2}$ 	Zeros: ____________ VA: ____________ HA: ____________ IP: ____________ IN: ____________ II: ____________ ID: ____________ ICU: ____________ ICD: ____________ PON: ____________	Zeros: ____________ VA: ____________ HA: ____________ IP: ____________ IN: ____________ II: ____________ ID: ____________ ICU: ____________ ICD: ____________ PON: ____________

Graph of Function	*Graph of its Reciprocal*	*Characteristics of Function*	*Characteristics of the Reciprocal of the Function*
$f(x)=x^2-4$ 	$g(x)=\dfrac{1}{f(x)}=\dfrac{1}{x^2-4}$ 	Zeros: ________ VA: ________ HA: ________ IP: ________ IN: ________ II: ________ ID: ________ ICU: ________ ICD: ________ PON: ________	Zeros: ________ VA: ________ HA: ________ IP: ________ IN: ________ II: ________ ID: ________ ICU: ________ ICD: ________ PON: ________
$f(x)=2x-3$ 	$g(x)=\dfrac{1}{f(x)}=\dfrac{1}{2x-3}$ 	Zeros: ________ VA: ________ HA: ________ IP: ________ IN: ________ II: ________ ID: ________ ICU: ________ ICD: ________ PON: ________	Zeros: ________ VA: ________ HA: ________ IP: ________ IN: ________ II: ________ ID: ________ ICU: ________ ICD: ________ PON: ________
$f(x)=(x-2)(x+3)$ 	$g(x)=\dfrac{1}{f(x)}=\dfrac{1}{(x-2)(x+3)}$ 	Zeros: ________ VA: ________ HA: ________ IP: ________ IN: ________ II: ________ ID: ________ ICU: ________ ICD: ________ PON: ________	Zeros: ________ VA: ________ HA: ________ IP: ________ IN: ________ II: ________ ID: ________ ICU: ________ ICD: ________ PON: ________

Graphs of Rational Functions of the Form $f(x)=\dfrac{ax+b}{cx+d}$ *(Quotients of Linear Polynomials)*

Complete the following table.

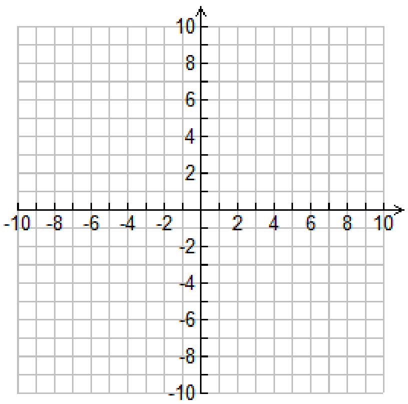
Legend

The following symbols are used in addition to the abbreviations given on page 30.

D→Domain   R→Range   X→x-intercept(s)   Y→y-intercept

Graph and Characteristics of Rational Function	*Graph and Characteristics of Rational Function*

$f(x)=\dfrac{x+1}{x-1}$



As $x\to\infty$, $f(x)\to$ ________

As $x\to-\infty$, $f(x)\to$ ________

As $x\to1^-$, $f(x)\to$ ________

As $x\to1^+$, $f(x)\to$ ________

Zeros: ________

VA: ________

HA: ________

IP: ________

IN: ________

II: ________

ID: ________

ICU: ________

ICD: ________

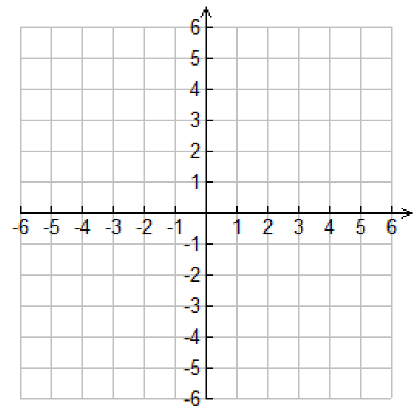
D: ________

R: ________

X: ________

Y: ________

$f(x)=\dfrac{x}{x-3}$



As $x\to\infty$, $f(x)\to$ ________

As $x\to-\infty$, $f(x)\to$ ________

As $x\to3^-$, $f(x)\to$ ________

As $x\to3^+$, $f(x)\to$ ________

Zeros: ________

VA: ________

HA: ________

IP: ________

IN: ________

II: ________

ID: ________

ICU: ________

ICD: ________

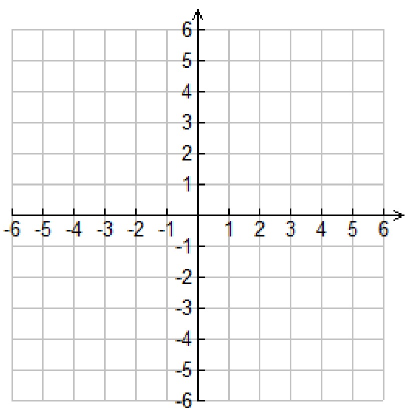
D: ________

R: ________

X: ________

Y: ________

$f(x)=\dfrac{x-2}{3x+4}$



As $x\to\infty$, $f(x)\to$ ________

As $x\to-\infty$, $f(x)\to$ ________

As $x\to-\frac{4}{3}^-$, $f(x)\to$ ________

As $x\to-\frac{4}{3}^+$, $f(x)\to$ ________

Zeros: ________

VA: ________

HA: ________

IP: ________

IN: ________

II: ________

ID: ________

ICU: ________

ICD: ________

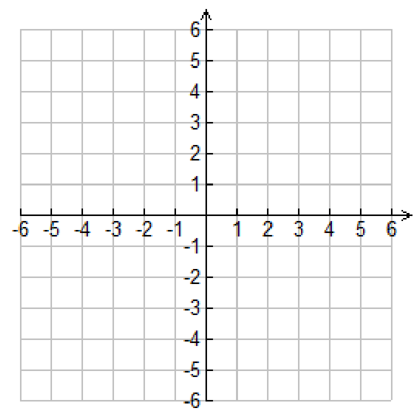
D: ________

R: ________

X: ________

Y: ________

$f(x)=\dfrac{x-3}{2x-6}$



As $x\to\infty$, $f(x)\to$ ________

As $x\to-\infty$, $f(x)\to$ ________

As $x\to3^-$, $f(x)\to$ ________

As $x\to3^+$, $f(x)\to$ ________

Zeros: ________

VA: ________

HA: ________

IP: ________

IN: ________

II: ________

ID: ________

ICU: ________

ICD: ________

D: ________

R: ________

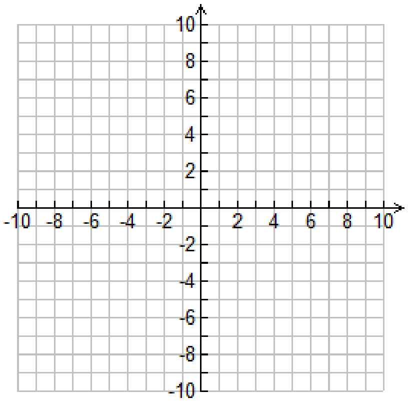
X: ________

Y: ________

Graphs of other Rational Functions

Graph and Characteristics of Rational Function

$$f(x)=\frac{9x}{1+x^2}$$



As $x\to\infty$, $f(x)\to$ ______

As $x\to-\infty$, $f(x)\to$ ______

As $x\to1^-$, $f(x)\to$ ______

As $x\to1^+$, $f(x)\to$ ______

Zeros: ______

VA: ______

HA: ______

IP: ______

IN: ______

II: ______

ID: ______

ICU: ______

ICD: ______

D: ______

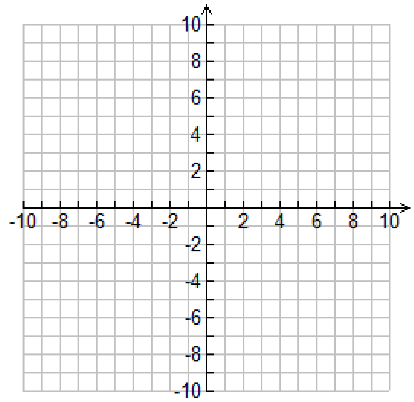
R: ______

X: ______

Y: ______

Graph and Characteristics of Rational Function

$$f(x)=\frac{x+1}{x^2-2x-3}$$



As $x\to\infty$, $f(x)\to$ ______

As $x\to-\infty$, $f(x)\to$ ______

As $x\to3^-$, $f(x)\to$ ______

As $x\to3^+$, $f(x)\to$ ______

Zeros: ______

VA: ______

HA: ______

IP: ______

IN: ______

II: ______

ID: ______

ICU: ______

ICD: ______

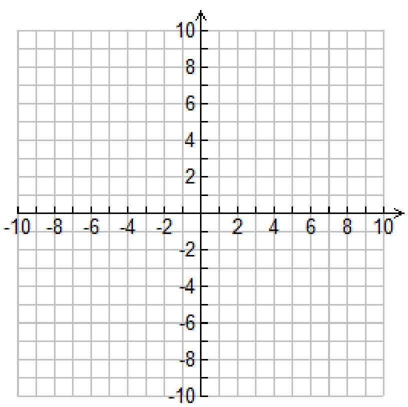
D: ______

R: ______

X: ______

Y: ______

$$f(x)=\frac{x^2-1}{x-1}$$



As $x\to\infty$, $f(x)\to$ ______

As $x\to-\infty$, $f(x)\to$ ______

As $x\to1^-$, $f(x)\to$ ______

As $x\to1^+$, $f(x)\to$ ______

Zeros: ______

VA: ______

HA: ______

IP: ______

IN: ______

II: ______

ID: ______

ICU: ______

ICD: ______

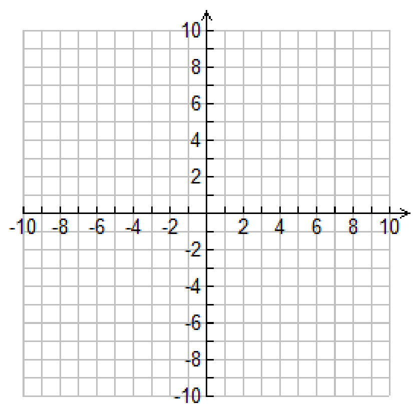
D: ______

R: ______

X: ______

Y: ______

$$f(x)=\frac{0.5x^2+1}{x-1}$$



As $x\to\infty$, $f(x)\to$ ______

As $x\to-\infty$, $f(x)\to$ ______

As $x\to1^-$, $f(x)\to$ ______

As $x\to1^+$, $f(x)\to$ ______

Zeros: ______

VA: ______

HA: ______

IP: ______

IN: ______

II: ______

ID: ______

ICU: ______

ICD: ______

D: ______

R: ______

X: ______

Y: ______

Summary – General Characteristics of Rational Functions

Reciprocals of Polynomials	*Rational Functions of the Form* $f(x)=\dfrac{ax+b}{cx+d}$	*General Rational Functions*
Let f represent a polynomial function of degree one or greater. • Wherever $y=f(x)$ has a ***zero***, $y=\dfrac{1}{f(x)}$ has a ***vertical asymptote***. • The x-axis (i.e. the line $y=0$) is always a ***horizontal asymptote*** of $y=\dfrac{1}{f(x)}$. • Wherever $y=f(x)$ ***increases***, $y=\dfrac{1}{f(x)}$ ***decreases*** (and vice versa) • Since $y=f(x)$ and $y=\dfrac{1}{f(x)}$ have the same sign, if $y=f(x)$ lies above the x-axis so does $y=\dfrac{1}{f(x)}$ (and vice versa). • A point whose y-co-ordinate is ±1 is ***invariant***. That is, such a point lies on both $y=f(x)$ and $y=\dfrac{1}{f(x)}$.	• $y=f(x)$ has either a ***vertical asymptote*** or a ***hole*** at $x=-\dfrac{d}{c}$. • A hole will occur if $ax+b$ and $cx+d$ have a common factor that "divides out" in such a way that only a constant remains. More precisely, a "hole" occurs at $x=-\dfrac{d}{c}$ if $ax+b=k(cx+d)$ for some non-zero real number k. In this case, $f(x)=\dfrac{ax+b}{cx+d}$ $=\dfrac{k(cx+d)}{cx+d}=k,\ k\neq-\dfrac{d}{c}$ • As $x\to\pm\infty$, $f(x)\to\dfrac{a}{c}$. Therefore, $y=f(x)$ has a ***horizontal asymptote*** at $x=\dfrac{a}{c}$.	Let f and g represent polynomial functions and define the rational function q as $q(x)=\dfrac{f(x)}{g(x)}$. • Wherever $g(x)=0$, $q(x)=\dfrac{f(x)}{g(x)}$ either has a ***vertical asymptote*** or a ***hole***. • If $q(x)\to k$ as $x\to\pm\infty$, where $k\in\mathbb{R}$, then $q(x)$ has a ***horizontal asymptote*** at $y=k$. Note that this can occur only if the degree of $f(x)$ is less than or equal to the degree of $g(x)$. • If the degree of $f(x)$ is exactly one greater than the degree of $g(x)$, then $q(x)$ has an ***oblique asymptote***.

Homework

pp. 254-257: 1, 4, 5gh, 6, 10, 11, 12, 13, 16

p. 262: 1, 2, 3

pp. 271-274: 1, 3, 5, 7, 11, 13, 14

Rational Equations and Inequalities

Example 1

When eating together, it takes Peter and Homer ***two hours*** to eat a certain number of hamburgers. By himself, Homer can eat the same number of hamburgers in ***fifteen fewer minutes*** than it takes Peter to do the same. How long does it take Peter to eat the hamburgers by himself?

Solution

Let t represent the amount of time, in minutes, that it takes Peter to eat all the burgers. Then $t - 15$ represents the time that it takes Homer to do the same. In addition,

- $\frac{1}{t}$ represents the ***fraction*** of burgers eaten by Peter in ***one minute*** when eating alone
- $\frac{1}{t-15}$ represents the ***fraction*** of burgers eaten by Homer in ***one minute*** when eating alone
- $\frac{1}{120}$ represents the ***fraction*** of burgers eaten by Peter and Homer in ***one minute*** when eating together

To understand this, consider an example. Suppose that it takes Homer 180 minutes to eat all the burgers by himself. This means that in one minute, he eats $\frac{1}{180}$ of the burgers.

Also, $t-15>0$ because it takes Homer more than zero minutes to eat all the burgers. Therefore, $t>15$.

(Fraction Peter eats in one minute) + (Fraction Homer eats in one minute) = fraction they eat together in one minute

Therefore, $\frac{1}{t}+\frac{1}{t-15}=\frac{1}{120}$.

$$\therefore 120t(t-15)\left(\frac{1}{t}+\frac{1}{t-15}\right)=120t(t-15)\left(\frac{1}{120}\right)$$

$$\therefore \left(\frac{120\overset{1}{\cancel{t}}(t-15)}{1}\right)\left(\frac{1}{\underset{1}{\cancel{t}}}\right)+\left(\frac{120t\overset{1}{\cancel{(t-15)}}}{1}\right)\left(\frac{1}{\underset{1}{\cancel{t-15}}}\right)=\left(\frac{\overset{1}{\cancel{120}}t(t-15)}{1}\right)\left(\frac{1}{\underset{1}{\cancel{120}}}\right)$$

$$\therefore 120(t-15)+120t=t(t-15)$$

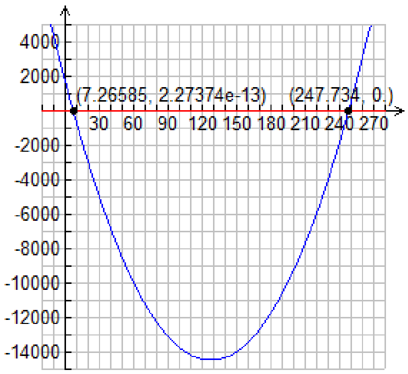
$$\therefore 120t+120t-1800=t^2-15t$$

$$\therefore t^2-255t+1800=0$$

$$\therefore t=\frac{-(-255)\pm\sqrt{(-255)^2-4(1)(1800)}}{2(1)}$$

$$\therefore t=\frac{255\pm\sqrt{57825}}{2}=\frac{255\pm15\sqrt{257}}{2}$$

$$\therefore t=\frac{255-15\sqrt{257}}{2}\doteq 7.27 \text{ or } t=\frac{255+15\sqrt{257}}{2}\doteq 247.73$$



Clearly, 7.27 minutes ***cannot be*** the correct answer. First of all, we observed above that $t>15$. Furthermore, if it takes Homer and Peter ***two hours together*** to eat all the burgers, Peter ***cannot possibly*** eat all of them in only 7.27 minutes. Therefore, it must take Peter about 247.73 minutes (4 hours, 7 minutes, 44 seconds) to eat all the burgers.

Example 2

Solve $x-2<\frac{8}{x}$.

Solution

Since the value of x could be negative, multiplying both sides by x might require that the inequality be reversed. A solution that involves multiplying both sides by x requires two cases, one for $x>0$ and another for $x<0$. Instead of dividing the solution into two cases, it is easier to use the following approach.

$$x-2<\frac{8}{x},\ x\neq 0$$

$$\therefore x-2-\frac{8}{x}<0$$

$$\therefore \frac{x^2}{x}-\frac{2x}{x}-\frac{8}{x}<0 \text{ (Write each term with a common denominator)}$$

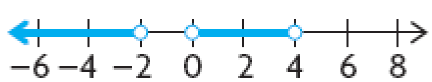
$$\therefore \frac{x^2-2x-8}{x}<0$$

$$\therefore \frac{(x-4)(x+2)}{x}<0$$

This final inequality states that the expression on the left side must be negative. A chart such as the following is an organized method of determining the intervals on which $\frac{(x-4)(x+2)}{x}$ is negative.

	$x<-2$	$-2<x<0$	$0<x<4$	$x>4$
$x-4$	$-$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$\frac{(x-4)(x+2)}{x}$	$\frac{(-)(-)}{-}=-$	$\frac{(-)(+)}{-}=+$	$\frac{(-)(+)}{+}=-$	$\frac{(+)(+)}{+}=+$

The expression $\frac{(x-4)(x+2)}{x}$ is negative if $x<-2$ or if $0<x<4$. Hence, the solution set of the inequality is $\{x\in\mathbb{R}: x<-2 \text{ or } 0<x<4\}$. In interval notation this can be written $(-\infty,-2)\cup(0,4)$.



Homework

pp. 285-287: 6, 7cf, 11, 12, 13, 15, 16

pp. 295-297: 1, 3, 4ef, 5f, 11, 14, 15

C. POLYNOMIAL AND RATIONAL FUNCTIONS

OVERALL EXPECTATIONS

By the end of this course, students will:

1. identify and describe some key features of polynomial functions, and make connections between the numeric, graphical, and algebraic representations of polynomial functions;
2. identify and describe some key features of the graphs of rational functions, and represent rational functions graphically;
3. solve problems involving polynomial and simple rational equations graphically and algebraically;
4. demonstrate an understanding of solving polynomial and simple rational inequalities.

SPECIFIC EXPECTATIONS

1. Connecting Graphs and Equations of Polynomial Functions

By the end of this course, students will:

1.1 recognize a polynomial expression (i.e., a series of terms where each term is the product of a constant and a power of x with a non-negative integral exponent, such as $x^3 - 5x^2 + 2x - 1$); recognize the equation of a polynomial function, give reasons why it is a function, and identify linear and quadratic functions as examples of polynomial functions

1.2 compare, through investigation using graphing technology, the numeric, graphical, and algebraic representations of polynomial (i.e., linear, quadratic, cubic, quartic) functions (e.g., compare finite differences in tables of values; investigate the effect of the degree of a polynomial function on the shape of its graph and the maximum number of x-intercepts; investigate the effect of varying the sign of the leading coefficient on the end behaviour of the function for very large positive or negative x-values)

Sample problem: Investigate the maximum number of x-intercepts for linear, quadratic, cubic, and quartic functions using graphing technology.

1.3 describe key features of the graphs of polynomial functions (e.g., the domain and range the shape of the graphs, the end behaviour o: the functions for very large positive or negative x-values)

Sample problem: Describe and compare the key features of the graphs of the functions $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^3 + x^2$, and $f(x) = x^3 + x$.

1.4 distinguish polynomial functions from sinusoidal and exponential functions [e.g., $f(x) = \sin x$, $g(x) = 2^x$], and compare and contrast the graphs of various polynomial functions with the graphs of other types of functions

1.5 make connections, through investigation using graphing technology (e.g., dynamic geometry software), between a polynomial function given in factored form [e.g., $f(x) = 2(x-3)(x+2)(x-1)$] and the x-intercepts of its graph, and sketch the graph of a polynomial function given in factored form using its key features (e.g., by determining intercepts and end behaviour; by locating positive and negative regions using test values between and on either side of the x-intercepts)

Sample problem: Investigate, using graphing technology, the x-intercepts and the shapes of the graphs of polynomial functions with one or more repeated factors, for example, $f(x) = (x-2)(x-3)$, $f(x) = (x-2)(x-2)(x-3)$, $f(x) = (x-2)(x-2)(x-2)(x-3)$, and $f(x) = (x+2)(x+2)(x-2)(x-2)(x-3)$, by considering whether the factor is repeated an even or an odd number of times. Use your conclusions to sketch $f(x) = (x+1)(x+1)(x-3)(x-3)$, and verify using technology.

1.6 determine, through investigation using technology, the roles of the parameters a, k, d, and c in functions of the form $y = af(k(x-d)) + c$, and describe these roles in terms of transformations on the graphs of $f(x) = x^3$ and $f(x) = x^4$ (i.e., vertical and horizontal translations; reflections in the axes; vertical and horizontal stretches and compressions to and from the x- and y-axes)

Sample problem: Investigate, using technology, the graph of $f(x) = 2(x-d)^3 + c$ for various values of d and c, and describe the effects of changing d and c in terms of transformations.

1.7 determine an equation of a polynomial function that satisfies a given set of conditions (e.g., degree of the polynomial, intercepts, points on the function), using methods appropriate to the situation (e.g., using the x-intercepts of the function; using a trial-and-error process with a graphing calculator or graphing software; using finite differences), and recognize that there may be more than one polynomial function that can satisfy a given set of conditions (e.g., an infinite number of polynomial functions satisfy the condition that they have three given x-intercepts)

Sample problem: Determine an equation for a fifth-degree polynomial function that intersects the x-axis at only 5, 1, and −5, and sketch the graph of the function.

1.8 determine the equation of the family of polynomial functions with a given set of zeros and of the member of the family that passes through another given point [e.g., a family of polynomial functions of degree 3 with zeros 5, −3, and −2 is defined by the equation $f(x) = k(x-5)(x+3)(x+2)$, where k is a real number, $k \neq 0$; the member of the family that passes through $(-1, 24)$ is $f(x) = -2(x-5)(x+3)(x+2)$]

Sample problem: Investigate, using graphing technology, and determine a polynomial function that can be used to model the function $f(x) = \sin x$ over the interval $0 \leq x \leq 2\pi$.

1.9 determine, through investigation, and compare the properties of even and odd polynomial functions [e.g., symmetry about the y-axis or the origin; the power of each term; the number of x-intercepts; $f(x) = f(-x)$ or $f(-x) = -f(x)$], and determine whether a given polynomial function is even, odd, or neither

Sample problem: Investigate numerically, graphically, and algebraically, with and without technology, the conditions under which an even function has an even number of x-intercepts.

2. Connecting Graphs and Equations of Rational Functions

By the end of this course, students will:

2.1 determine, through investigation with and without technology, key features (i.e., vertical and horizontal asymptotes, domain and range, intercepts, positive/negative intervals, increasing/decreasing intervals) of the graphs of rational functions that are the reciprocals of linear and quadratic functions, and make connections between the algebraic and graphical representations of these rational functions [e.g., make connections between $f(x) = \dfrac{1}{x^2 - 4}$ and its graph by using graphing technology and by reasoning that there are vertical asymptotes at $x = 2$ and $x = -2$ and a horizontal asymptote at $y = 0$ and that the function maintains the same sign as $f(x) = x^2 - 4$]

Sample problem: Investigate, with technology, the key features of the graphs of families of rational functions of the form $f(x) = \dfrac{1}{x+n}$ and $f(x) = \dfrac{1}{x^2+n}$, where n is an integer, and make connections between the equations and key features of the graphs.

2.2 determine, through investigation with and without technology, key features (i.e., vertical and horizontal asymptotes, domain and range, intercepts, positive/negative intervals, increasing/decreasing intervals) of the graphs of rational functions that have linear expressions in the numerator and denominator [e.g., $f(x) = \dfrac{2x}{x-3}$, $h(x) = \dfrac{x-2}{3x+4}$], and make connections between the algebraic and graphical representations of these rational functions

Sample problem: Investigate, using graphing technology, key features of the graphs of the family of rational functions of the form $f(x) = \frac{8x}{nx+1}$ for $n = 1, 2, 4,$ and 8, and make connections between the equations and the asymptotes.

2.3 sketch the graph of a simple rational function using its key features, given the algebraic representation of the function

3. Solving Polynomial and Rational Equations

By the end of this course, students will:

3.1 make connections, through investigation using technology (e.g., computer algebra systems), between the polynomial function $f(x)$, the divisor $x - a$, the remainder from the division $\frac{f(x)}{x-a}$, and $f(a)$ to verify the remainder theorem and the factor theorem

Sample problem: Divide $f(x) = x^4 + 4x^3 - x^2 - 16x - 14$ by $x - a$ for various integral values of a using a computer algebra system. Compare the remainder from each division with $f(a)$.

3.2 factor polynomial expressions in one variable, of degree no higher than four, by selecting and applying strategies (i.e., common factoring, difference of squares, trinomial factoring, factoring by grouping, remainder theorem, factor theorem)

Sample problem: Factor: $x^3 + 2x^2 - x - 2$; $x^4 - 6x^3 + 4x^2 + 6x - 5$.

3.3 determine, through investigation using technology (e.g., graphing calculator, computer algebra systems), the connection between the real roots of a polynomial equation and the x-intercepts of the graph of the corresponding polynomial function, and describe this connection [e.g., the real roots of the equation $x^4 - 13x^2 + 36 = 0$ are the x-intercepts of the graph of $f(x) = x^4 - 13x^2 + 36$]

Sample problem: Describe the relationship between the x-intercepts of the graphs of linear and quadratic functions and the real roots of the corresponding equations. Investigate, using technology, whether this relationship exists for polynomial functions of higher degree.

3.4 solve polynomial equations in one variable, of degree no higher than four (e.g., $2x^3 - 3x^2 + 8x - 12 = 0$), by selecting and applying strategies (i.e., common factoring, difference of squares, trinomial factoring, factoring by grouping, remainder theorem, factor theorem), and verify solutions using technology (e.g., using computer algebra systems to determine the roots; using graphing technology to determine the x-intercepts of the graph of the corresponding polynomial function)

3.5 determine, through investigation using technology (e.g., graphing calculator, computer algebra systems), the connection between the real roots of a rational equation and the x-intercepts of the graph of the corresponding rational function, and describe this connection [e.g., the real root of the equation $\frac{x-2}{x-3} = 0$ is 2, which is the x-intercept of the function $f(x) = \frac{x-2}{x-3}$; the equation $\frac{1}{x-3} = 0$ has no real roots, and the function $f(x) = \frac{1}{x-3}$ does not intersect the x-axis]

3.6 solve simple rational equations in one variable algebraically, and verify solutions using technology (e.g., using computer algebra systems to determine the roots; using graphing technology to determine the x-intercepts of the graph of the corresponding rational function)

3.7 solve problems involving applications of polynomial and simple rational functions and equations [e.g., problems involving the factor theorem or remainder theorem, such as determining the values of k for which the function $f(x) = x^3 + 6x^2 + kx - 4$ gives the same remainder when divided by $x - 1$ and $x + 2$]

Sample problem: Use long division to express the given function $f(x) = \frac{x^2 + 3x - 5}{x - 1}$ as the sum of a polynomial function and a rational function of the form $\frac{A}{x-1}$ (where A is a constant), make a conjecture about the relationship between the given function and the polynomial function for very large positive and negative x-values, and verify your conjecture using graphing technology.

4. Solving Inequalities

By the end of this course, students will:

4.1 explain, for polynomial and simple rational functions, the difference between the solution to an equation in one variable and the solution to an inequality in one variable, and demonstrate that given solutions satisfy an inequality (e.g., demonstrate numerically and graphically that the solution to $\frac{1}{x+1} < 5$ is $x < -1$ or $x > -\frac{4}{5}$);

4.2 determine solutions to polynomial inequalities in one variable [e.g., solve $f(x) \geq 0$, where $f(x) = x^3 - x^2 + 3x - 9$] and to simple rational inequalities in one variable by graphing the corresponding functions, using graphing technology, and identifying intervals for which x satisfies the inequalities

4.3 solve linear inequalities and factorable polynomial inequalities in one variable (e.g., $x^3 + x^2 > 0$) in a variety of ways (e.g., by determining intervals using x-intercepts and evaluating the corresponding function for a single x-value within each interval; by factoring the polynomial and identifying the conditions for which the product satisfies the inequality), and represent the solutions on a number line or algebraically (e.g., for the inequality $x^4 - 5x^2 + 4 < 0$, the solution represented algebraically is $-2 < x < -1$ or $1 < x < 2$)