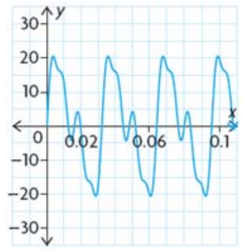
UNIT 4 – COMBINATIONS OF FUNCTIONS

Operations on Functions

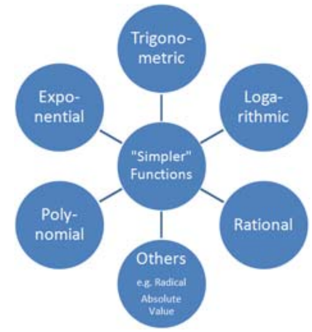
Introduction – Building more Complex Functions from Simpler Functions

The sound produced when a person strums a guitar chord represents the combination of sounds made by several different strings. The sound made by each string can be represented by a sine function. The period of each function is based on the frequency of the sound, whereas the loudness of the individual sounds varies and is related to the amplitude of each function. These sine functions are literally added together to produce the desired sound.
The sound of a G chord played on a six-string acoustic guitar can be approximated by the following combination of sine functions:
$y = 16\sin 196x + 9\sin 392x + 4\sin 784x$



This example is taken from our textbook. It is *not correct* to suggest that the given equation models a G chord or that it models six notes played simultaneously. In reality, this equation models three different G notes played simultaneously, the lowest and highest of which are separated in pitch by two octaves. Note that the frequency of G just below middle C is 196 Hz (196 Hertz or 196 cycles per second) and that doubling the frequency raises the pitch by one octave. In addition, the textbook does not indicate what *x* and *y* represent. It is most likely that *x* represents time and that *y* represents the change in air pressure that occurs at a given point as the sound wave propagates past that point.

Just as complex molecules are built from atoms and simpler molecules, mathematical operations can be used to build more complicated functions from simpler ones. If we think of the functions that we have studied in this course as the ***elements*** or ***basic building blocks*** of functions, we can create functions with a richer variety of features just by combining the "elementary functions" functions examined in this course.
The following is a list of operations that can be performed on functions to produce new functions. This list is by no means exhaustive.

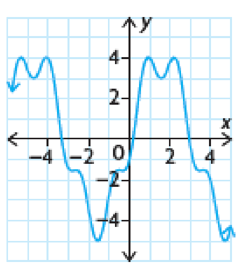


- **Sum of Functions**
  Let *f* and *g* represent any two functions. Then, we can define a new function $h = f + g$ that is the ***sum*** of *f* and *g*:
  $h(x) = (f+g)(x) = f(x) + g(x)$
- **Difference of Functions**
  Let *f* and *g* represent any two functions. Then, we can define a new function $h = f - g$ that is the ***difference*** of the functions *f* and *g*:
  $h(x) = (f-g)(x) = f(x) - g(x)$
- **Product of Functions**
  Let *f* and *g* represent any two functions. Then, we can define a new function $h = fg$ that is the ***product*** of *f* and *g*:
  $h(x) = (fg)(x) = f(x)g(x)$
- **Quotient of Functions**
  Let *f* and *g* represent any two functions. Then, we can define a new function $h = \dfrac{f}{g}$ that is the ***quotient*** of *f* and *g*: $h(x) = \left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$
- **Composition of Functions**
  Let *f* and *g* represent any two functions. Then, we can define a new function $h = f \circ g$ that is called the ***composition*** of *f* with *g*: $h(x) = (f \circ g)(x) = f(g(x))$
  (We can also say that "*f* is composed with *g*.")

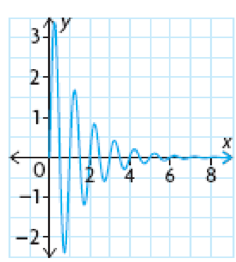
Activity

Shown below is a series of graphs of functions that are "built from" the simpler functions that we have studied. Match the graphs with the equations given below.

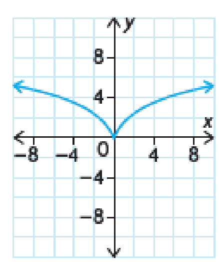
Graph 1



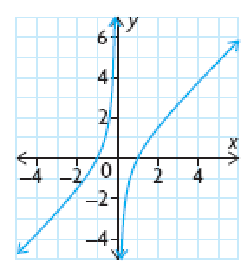
Graph 2



Graph 3



Graph 4



(a) $y = x\sqrt{x-1}$ **(b)** $y = 4\sin x - \cos 4x$ **(c)** $y = x - \dfrac{1}{x}$ **(d)** $y = 5\log\left(|x|+1\right)$

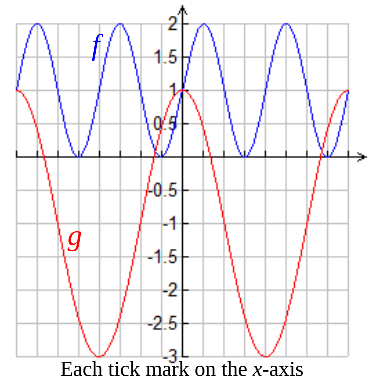
(e) $y = x^2 \sin x$ **(f)** $y = \begin{cases} -0.5(x-2)^2 + 2, & x < 0 \\ 0.5(x-2)^2 - 2, & x \ge 0 \end{cases}$ **(g)** $y = 0.5^x\left[4\sin(2\pi x)\right]$ **(h)** $y = \dfrac{x^3}{x+1}$

Example 1

Shown at the right are the graphs of the functions $f(x)=\sin 2x+1$ (blue) and $g(x)=2\cos x-1$ (red). Various combinations of these functions are shown in the following table. Note the following important features of *f* and *g*:

$$\text{Domain of } f=\mathbb{R}\text{, Range of } f=\{y\in\mathbb{R}:0<y<2\}$$
$$\text{Domain of } g=\mathbb{R}\text{, Range of } g=\{y\in\mathbb{R}:-3<y<1\}$$

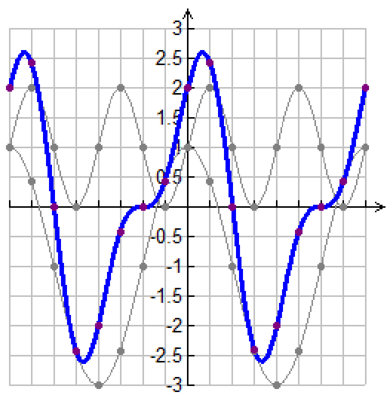
Once we have the graphs of *f* and *g*, we can sketch the graphs of combinations of *f* and *g* by using the *y*-co-ordinates of certain key points on the graphs of *f* and *g*.



Each tick mark on the *x*-axis represents $\frac{\pi}{4}$ radians.

<table>
<tr><th>Equation of Combination of f and g</th><th>Graph of Combination of f and g</th><th>How Points were Obtained</th><th>Domain and Range of Combination of f and g</th></tr>
<tr>
<td>

$(f+g)(x)=\sin 2x+2\cos x$

The *y*-co-ordinate of any point on the graph of $f+g$ is found by ***adding*** the *y*-co-ordinates of the corresponding points on *f* and *g* (i.e. the points having the same *x*-co-ordinate).
</td>
<td>


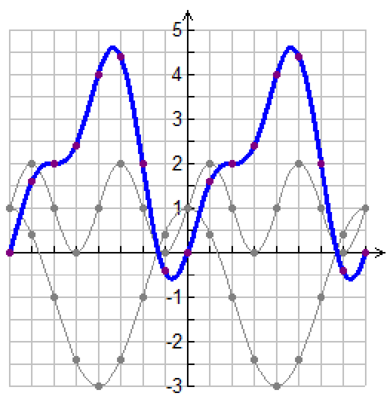
</td>
<td>
<table>
<tr><th>x</th><th>f(x)</th><th>g(x)</th><th>(f+g)(x)</th></tr>
<tr><td>-2π</td><td>1</td><td>1</td><td>2</td></tr>
<tr><td>$-\frac{3\pi}{2}$</td><td>1</td><td>-1</td><td>0</td></tr>
<tr><td>$-\pi$</td><td>1</td><td>-3</td><td>-2</td></tr>
<tr><td>$-\frac{\pi}{2}$</td><td>1</td><td>-1</td><td>0</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>2</td></tr>
<tr><td>$\frac{\pi}{2}$</td><td>1</td><td>-1</td><td>0</td></tr>
<tr><td>π</td><td>1</td><td>-3</td><td>-2</td></tr>
<tr><td>$\frac{3\pi}{2}$</td><td>1</td><td>-1</td><td>0</td></tr>
<tr><td>2π</td><td>1</td><td>1</td><td>2</td></tr>
</table>
</td>
<td>

$D=\mathbb{R}$

$R=\left\{y\in\mathbb{R}:-\frac{3\sqrt{3}}{2}\le y\le\frac{3\sqrt{3}}{2}\right\}$

The upper and lower limits of $f+g$ were found by using calculus.
</td>
</tr>
<tr>
<td>

$(f-g)(x)=\sin 2x-2\cos x+2$

The *y*-co-ordinate of any point on the graph of $f-g$ is found by ***subtracting*** the *y*-co-ordinates of the corresponding points on *f* and *g* (i.e. the points having the same *x*-co-ordinate).
</td>
<td>


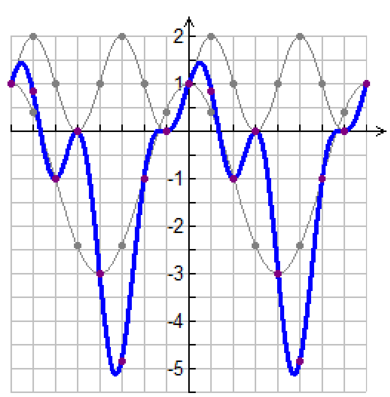
</td>
<td>
<table>
<tr><th>x</th><th>f(x)</th><th>g(x)</th><th>(f−g)(x)</th></tr>
<tr><td>-2π</td><td>1</td><td>1</td><td>0</td></tr>
<tr><td>$-\frac{3\pi}{2}$</td><td>1</td><td>-1</td><td>2</td></tr>
<tr><td>$-\pi$</td><td>1</td><td>-3</td><td>4</td></tr>
<tr><td>$-\frac{\pi}{2}$</td><td>1</td><td>-1</td><td>2</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>0</td></tr>
<tr><td>$\frac{\pi}{2}$</td><td>1</td><td>-1</td><td>2</td></tr>
<tr><td>π</td><td>1</td><td>-3</td><td>-2</td></tr>
<tr><td>$\frac{3\pi}{2}$</td><td>1</td><td>-1</td><td>0</td></tr>
<tr><td>2π</td><td>1</td><td>1</td><td>2</td></tr>
</table>
</td>
<td>

$D=\mathbb{R}$

$R=\left\{y\in\mathbb{R}:\frac{4-3\sqrt{3}}{2}\le y\le\frac{4+3\sqrt{3}}{2}\right\}$

The upper and lower bounds of $f-g$ were found by using calculus.
</td>
</tr>
<tr>
<td>

$(fg)(x)=(\sin 2x+1)(2\cos x-1)$

The *y*-co-ordinate of any point on the graph of fg is found by ***multiplying*** the *y*-co-ordinates of the corresponding points on *f* and *g* (i.e. the points having the same *x*-co-ordinate).
</td>
<td>


</td>
<td>
<table>
<tr><th>x</th><th>f(x)</th><th>g(x)</th><th>(fg)(x)</th></tr>
<tr><td>-2π</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>$-\frac{3\pi}{2}$</td><td>1</td><td>-1</td><td>-1</td></tr>
<tr><td>$-\pi$</td><td>1</td><td>-3</td><td>-3</td></tr>
<tr><td>$-\frac{\pi}{2}$</td><td>1</td><td>-1</td><td>-1</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>$\frac{\pi}{2}$</td><td>1</td><td>-1</td><td>-1</td></tr>
<tr><td>π</td><td>1</td><td>-3</td><td>-3</td></tr>
<tr><td>$\frac{3\pi}{2}$</td><td>1</td><td>-1</td><td>-1</td></tr>
<tr><td>2π</td><td>1</td><td>1</td><td>1</td></tr>
</table>
</td>
<td>

$D=\mathbb{R}$

$R=\{y\in\mathbb{R}:-5.12\le y\le 1.45\}$

The upper and lower bounds of fg are ***approximations*** and were found by using TI-Interactive.
</td>
</tr>
</table>

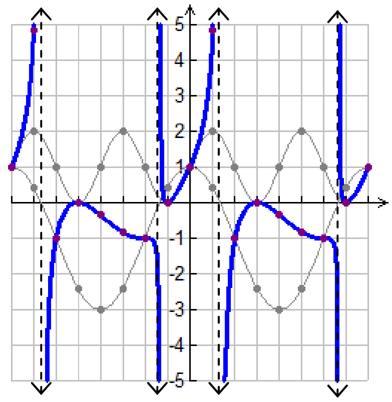
Equation of Combination of f and g	Graph	How Points were Obtained	Domain and Range of Combination of f and g

Equation of Combination of f and g

$$\left(\frac{f}{g}\right)(x)=\frac{\sin 2x+1}{2\cos x-1}$$

The y-co-ordinate of any point on the graph of $\frac{f}{g}$ is found by ***dividing*** the y-co-ordinates of the corresponding points on f and g (i.e. the points having the same x-co-ordinate).

Graph



How Points were Obtained

x	f(x)	g(x)	(f /g)(x)
-2π	1	1	1
$-\frac{3\pi}{2}$	1	−1	−1
$-\pi$	1	−3	−1/3
$-\frac{\pi}{2}$	1	−1	−1
0	1	1	1
$\frac{\pi}{2}$	1	−1	−1
π	1	−3	−1/3
$\frac{3\pi}{2}$	1	−1	−1
2π	1	1	1

Domain and Range of Combination of f and g

$D=\left\{x\in\mathbb{R}: x\neq\frac{(6n+1)\pi}{3}, x\neq\frac{(6n+5)\pi}{3}, n\in\mathbb{Z}\right\}$

$R=\mathbb{R}$

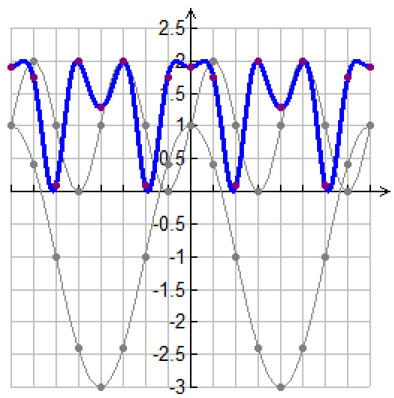
The vertical asymptotes are located wherever $2\cos x-1=0$. In the interval $[0,2\pi]$, this equation has two solutions, $x=\frac{\pi}{3}$ and $x=\frac{5\pi}{3}$. All other solutions are obtained by adding or subtracting any multiple of 2π.

Equation of Combination of f and g

$$(f\circ g)(x)=f(g(x))$$
$$=\sin(2(2\cos x-1))+1$$
$$=\sin(4\cos x-2)+1$$

The y-co-ordinate of any point on the graph of $f\circ g$ is found by ***applying the function f*** to the y-co-ordinate of the corresponding point on g (i.e. the point having the same x-co-ordinate).

Graph



How Points were Obtained

x	f(x)	g(x)	f(g(x))
-2π	1	1	f(1)
$-\frac{3\pi}{2}$	1	−1	f(−1)
$-\pi$	1	−3	f(−3)
$-\frac{\pi}{2}$	1	−1	f(−1)
0	1	1	f(1)
$\frac{\pi}{2}$	1	−1	f(−1)
π	1	−3	f(−3)
$\frac{3\pi}{2}$	1	−1	f(−1)
2π	1	1	f(1)

Domain and Range of Combination of f and g

$D=\mathbb{R}$

$R=\{y\in\mathbb{R}: 0\le y\le 2\}$

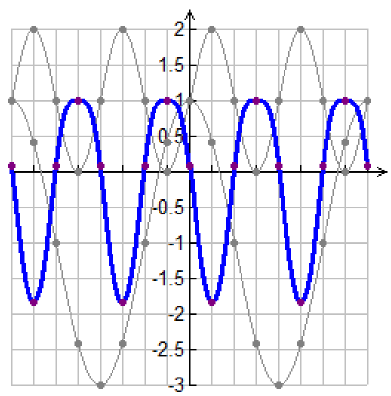
The upper and lower bounds of $f\circ g$ were found graphically.

Equation of Combination of f and g

$$(g\circ f)(x)=g(f(x))$$
$$=2\cos(\sin 2x+1)-1$$

The y-co-ordinate of any point on the graph of $g\circ f$ is found by ***applying the function g*** to the y-co-ordinate of the corresponding point on f (i.e. the point having the same x-co-ordinate).

Graph



How Points were Obtained

x	f(x)	g(x)	g(f(x))
-2π	1	1	g(1)
$-\frac{3\pi}{2}$	1	−1	g(1)
$-\pi$	1	−3	g(1)
$-\frac{\pi}{2}$	1	−1	g(1)
0	1	1	g(1)
$\frac{\pi}{2}$	1	−1	g(1)
π	1	−3	g(1)
$\frac{3\pi}{2}$	1	−1	g(1)
2π	1	1	g(1)

Domain and Range of Combination of f and g

$D=\mathbb{R}$

$R=\{y\in\mathbb{R}: 2\cos 2-1\le y\le 1\}$

The upper and lower bounds of $g\circ f$ were found graphically.

Example 2

Given $f(x)=2^x$ and $g(x)=\log_2(x+5)$, find the equations of $f+g$, $f-g$, fg, $\frac{f}{g}$, $f\circ g$ and $g\circ f$.

(a) $(f+g)(x)=f(x)+g(x)$
$=2^x+\log_2(x+5)$

(b) $(f-g)(x)=f(x)-g(x)$
$=2^x-\log_2(x+5)$

(c) $(fg)(x)=f(x)g(x)$
$=2^x\log_2(x+5)$

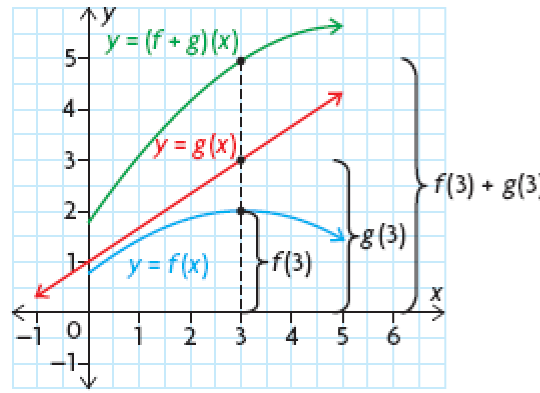
(d) $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}=\frac{2^x}{\log_2(x+5)}$

(e) $(f\circ g)(x)=f(g(x))$
$=2^{\log_2(x+5)}$
$=x+5$

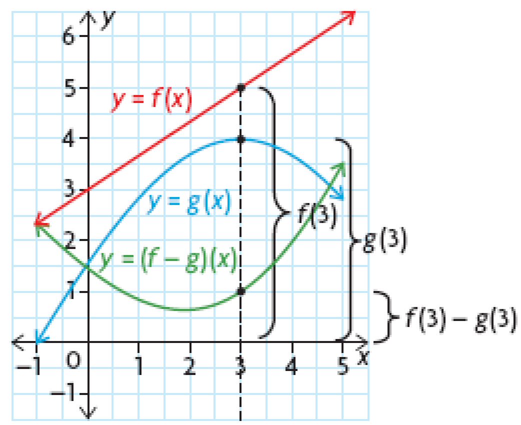
(f) $(g\circ f)(x)=g(f(x))$
$=\log_2(2^x+5)$

Key Ideas

- When two functions $f(x)$ and $g(x)$ are combined to form the function $(f+g)(x)$, the new function is called the sum of f and g. For any given value of x, the value of the function is represented by $f(x)+g(x)$. The graph of $f+g$ can be obtained from the graphs of functions f and g by adding corresponding y-coordinates.



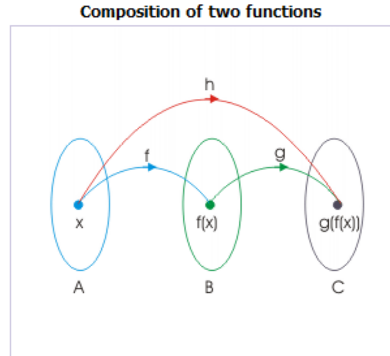
- Similarly, the difference of two functions, $f-g$, is $(f-g)(x)=f(x)-g(x)$. The graph of $f-g$ can be obtained by subtracting the y-coordinate of g from the y-coordinate of f for every pair of corresponding x-values.



- When two functions, $f(x)$ and $g(x)$, are combined to form the function $(f\times g)(x)$, the new function is called the product of f and g. For any given value of x, the function value is represented by $f(x)\times g(x)$. The graph of $f\times g$ can be obtained from the graphs of functions f and g by multiplying each y-coordinate of f by the corresponding y-coordinate of g.

- When two functions, $f(x)$ and $g(x)$, are combined to form the function $(f\div g)(x)$, the new function is called the quotient of f and g. For any given value of x, the value of the function is represented by $f(x)\div g(x)$. The graph of $f\div g$ can be obtained from the graphs of functions f and g by dividing each y-coordinate of f by the corresponding y-coordinate of g.

Composition of two functions



- Two functions, f and g, can be combined using a process called composition, which can be represented by $f(g(x))$. The output for the inner function g is used as the input for the outer function f. The function $f(g(x))$ can be denoted by $(f\circ g)(x)$.

$$(f\circ g)(x)=f(\underbrace{g(x)}_{\text{inner function}})$$

outer function

When f is composed with g, take the output for the inner function g and use it as the input for the outer function f.

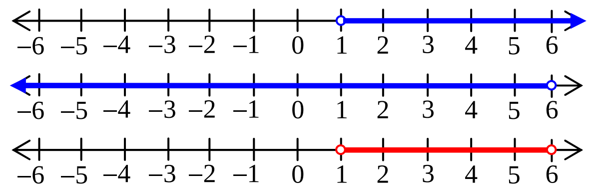
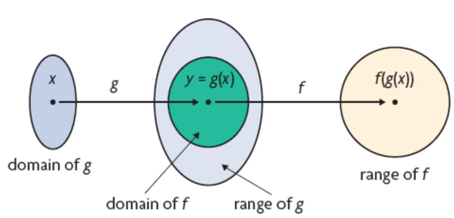
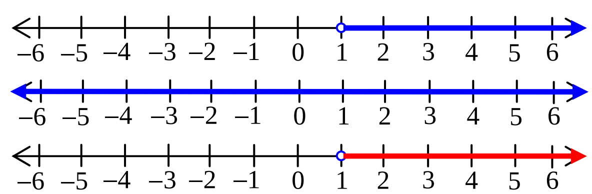
$$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$

Homework

p. 529: 8, 9 | p. 537: 1cdef, 2, 4 | p. 542: 1, 2 | p.552: 1, 6

Determining Characteristics of Combinations of Functions

Domain

Combination of f and g	Domain	Example
$f+g$ $f-g$ fg	The ***sum***, ***difference*** and ***product*** of the functions *f* and *g* are defined wherever ***both*** *f* and *g* are defined. Thus, the domain of the combined function is the ***intersection*** of the domain of *f* and the domain of *g*. That is, a value *x* is in the domain of $f+g$, $f-g$ or fg if and only if *x* is in the domain of *f* ***and*** *x* is in the domain of *g*. **intersection** a set that contains the elements that are common to both sets; the symbol for intersection is $\cap$ $A \cap B = \{x : x \in A \text{ and } x \in B\}$	Let $f(x)=\log_5(x-1)$ and $g(x)=\log_5(-(x-6))$. • domain of $f=\{x\in\mathbb{R}: x>1\}$ • domain of $g=\{x\in\mathbb{R}: x<6\}$ Therefore, $\text{domain}(f+g)=\{x\in\mathbb{R}: x>1\}\cap\{x\in\mathbb{R}: x<6\}$ $=\{x\in\mathbb{R}: x>1 \text{ and } x<6\}=(1,6)$ 
$\dfrac{f}{g}$	The ***quotient*** of the functions *f* and *g* is defined • wherever ***both*** *f* and *g* are defined ***AND*** • wherever $g(x)\neq 0$ (to avoid division by zero) Thus, the domain of the function $\dfrac{f}{g}$ is the ***intersection*** of the domain of *f* and the domain of *g*, ***not including*** the values in the domain of *g* for which $g(x)=0$. That is, a value *x* is in the domain of $\dfrac{f}{g}$ if and only if *x* is in the domain of *f*, *x* is in the domain of *g* ***and*** $g(x)\neq 0$.	Let $f(x)=\sin x$ and $g(x)=\cos 2x$. • domain of $f=\mathbb{R}$ • domain of $g=\mathbb{R}$ Therefore, This symbol means that the elements in the set should be ***excluded***. $\text{domain}\left(\dfrac{f}{g}\right)=\mathbb{R}\cap\mathbb{R}-\{x\in\mathbb{R}: g(x)=0\}$ $=\mathbb{R}-\{x\in\mathbb{R}: \cos 2x=0\}$ $=\{x\in\mathbb{R}: \cos 2x\neq 0\}$ $=\left\{x\in\mathbb{R}: x\neq\dfrac{(2n+1)\pi}{4}, n\in\mathbb{Z}\right\}$
$f\circ g$	Recall that $(f\circ g)(x)=f(g(x))$. That is, the ***output*** of *g* is the ***input*** of *f*. This means that the set of inputs to *f* belong to the ***range*** of *g*. Therefore, the domain of $f\circ g$ consists of all values in the ***range of g*** that are also in the domain of *f*. In other words, the domain of $f\circ g$ is equal to the intersection of the domain of *f* and the range of *g*. 	Let $f(x)=\log_5(x-1)$ and $g(x)=\log_5(-(x-6))$. • domain of $f=\{x\in\mathbb{R}: x>1\}$ • ***range*** of $g=\mathbb{R}$ Therefore, $\text{domain}(f\circ g)=\{x\in\mathbb{R}: x>1\}\cap\mathbb{R}$ $=\{x\in\mathbb{R}: x>1\}=(1,\infty)$ 

Range

Suppose that *f* represents a function that is a combination of simpler functions. If the domain of each of the constituent functions is known, it is generally a simple matter to determine the domain of *f*. Unfortunately, the same cannot be said of determining the range of *f*. In general, it is not as simple a matter to predict the range of *f* given only the range of each of the constituent functions. Often, the best approach is to determine the range directly from the equation of *f*.

For a good illustration of this, see Example 1 on page 3. Notice that in each case, the domains of *f* and *g* are closely related to the domain of the combination of *f* and *g*. On the other hand, with the exception of $f \circ g$, the range of each combination of *f* and *g* bears very little resemblance to the range of either *f* or *g*. (The results are summarized below.)

$f(x) = \sin 2x + 1$	$g(x) = 2\cos x - 1$
Domain of $f = \mathbb{R}$	Domain of $g = \mathbb{R}$
Range of $f = \{y \in \mathbb{R} : 0 \le y \le 2\}$	Range of $g = \{y \in \mathbb{R} : -3 \le y \le 1\}$

Function	*Domain*	*Range*
$(f+g)(x) = \sin 2x + 2\cos x$	$D = \mathbb{R}$	$R = \left\{ y \in \mathbb{R} : -\frac{3\sqrt{3}}{2} \le y \le \frac{3\sqrt{3}}{2} \right\}$
$(f-g)(x) = \sin 2x - 2\cos x + 2$	$D = \mathbb{R}$	$R = \left\{ y \in \mathbb{R} : \frac{4-3\sqrt{3}}{2} \le y \le \frac{4+3\sqrt{3}}{2} \right\}$
$(fg)(x) = (\sin 2x + 1)(2\cos x - 1)$	$D = \mathbb{R}$	$R = \{y \in \mathbb{R} : -5.12 \le y \le 1.45\}$ (Values are approximate here)
$\left(\frac{f}{g}\right)(x) = \frac{\sin 2x + 1}{2\cos x - 1}$	$D = \mathbb{R}$	$R = \mathbb{R}$
$(f \circ g)(x) = f(g(x)) = \sin(2(2\cos x - 1)) + 1$	$D = \mathbb{R}$	$R = \{y \in \mathbb{R} : 0 \le y \le 2\}$
$(g \circ f)(x) = g(f(x)) = 2\cos(\sin 2x + 1) - 1$	$D = \mathbb{R}$	$R = \{y \in \mathbb{R} : 2\cos 2 - 1 \le y \le 1\}$

Example

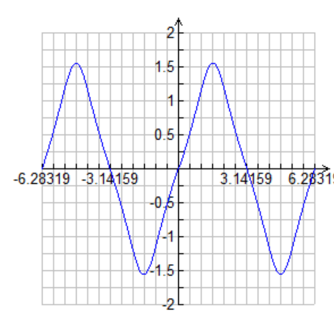
Find the domain and range of $y = \tan(\sin x)$.

Solution

If $f(x) = \tan x$ and $g(x) = \sin x$, then $y = \tan(\sin x) = f(g(x))$.

$$\text{domain}(f) = \left\{ x \in \mathbb{R} : x \ne \frac{(2n+1)\pi}{2}, n \in \mathbb{Z} \right\}, \qquad \text{range}(f) = \mathbb{R}$$

$$\text{domain}(g) = \mathbb{R}, \qquad \text{range}(g) = \{y \in \mathbb{R} : -1 \le y \le 1\}$$

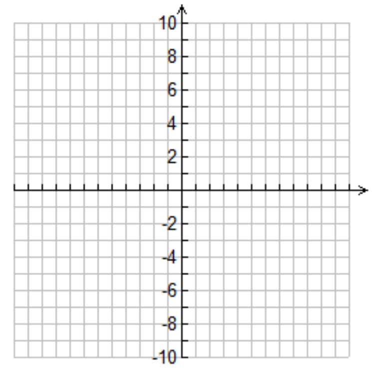
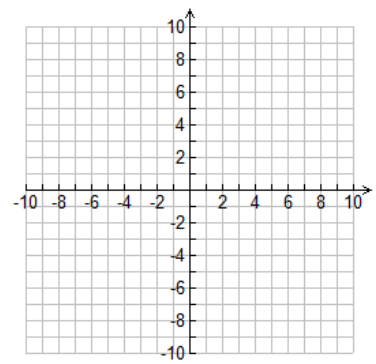
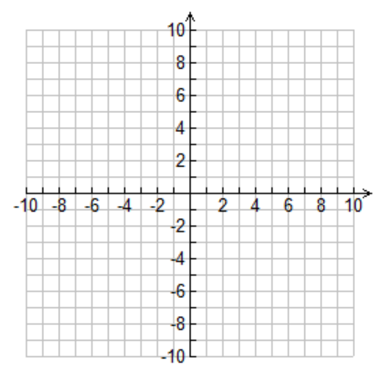
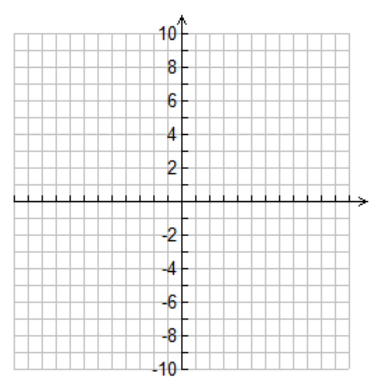


The maximum value of $y = \tan(\sin x)$ is $\tan 1 \doteq 1.5574$ and the minimum value is $\tan(-1) \doteq -1.5574$

The graph of $y = (f \circ g)(x) = \tan(\sin x)$ is shown at the right. Clearly, the domain of $f \circ g$ is $\mathbb{R}$. This occurs because the value of $\sin x$ must lie between –1 and 1 and $f(x) = \tan x$ is defined for all values in the interval $[-1,1]$. (Note that the tan function is undefined at $x = \pm\frac{\pi}{2}$ but it ***is*** defined for all values in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which contains the interval $[-1,1]$.) Since $f(x) = \tan x$ is a strictly increasing function, then its maximum value on $[-1,1]$ is $\tan 1$ and its minimum value on $[-1,1]$ is $\tan(-1)$. Therefore, $\text{range}(f) = \{y \in \mathbb{R} : \tan(-1) \le y \le \tan 1\}$.

Activity

Complete the following table.

Equation of Function	Characteristics of Function		Graph
$f(x)=\csc(\cos x)$ Domain: Range:	Zeros: ______________ y-intercept: __________ Domain: ____________ Range: _____________ Asymptote(s): ________ ___________________	As $x\to\frac{\pi}{2}^-$, $f(x)\to$ ______ As $x\to\frac{\pi}{2}^+$, $f(x)\to$ ______ As $x\to-\frac{\pi}{2}^-$, $f(x)\to$ _____ As $x\to-\frac{\pi}{2}^+$, $f(x)\to$ _____ As $x\to 0$, $f(x)\to$ ______	
$f(x)=\log_2(x^2+1)$ Domain: Range:	Zeros: ______________ y-intercept: __________ Domain: ____________ Range: _____________ Asymptote(s): ________ ___________________	As $x\to\infty$, $f(x)\to$ ______ As $x\to-\infty$, $f(x)\to$ ______ As $x\to 0$, $f(x)\to$ ______ Intervals of Increase: ______ ______________________ Intervals of Decrease: ______ ______________________	
$f(x)=\log_2(x^2-1)$ Domain: Range:	Zeros: ______________ y-intercept: __________ Domain: ____________ Range: _____________ Asymptote(s): ________ ___________________	As $x\to\infty$, $f(x)\to$ ______ As $x\to-\infty$, $f(x)\to$ ______ As $x\to 1^+$, $f(x)\to$ ______ As $x\to 1^-$, $f(x)\to$ ______ Intervals of Increase: ______ ______________________ Intervals of Decrease: ______ ______________________	
$f(x)=\csc x+\tan x$ Domain: Range:	Zeros: ______________ y-intercept: __________ Domain: ____________ Range: _____________ Asymptote(s): ________ ___________________	As $x\to\frac{\pi}{2}^-$, $f(x)\to$ ______ As $x\to\frac{\pi}{2}^+$, $f(x)\to$ ______ As $x\to\pi^-$, $f(x)\to$ ______ As $x\to\pi^+$, $f(x)\to$ ______ Intervals of Increase: ______ ______________________ Intervals of Decrease: ______ ______________________	

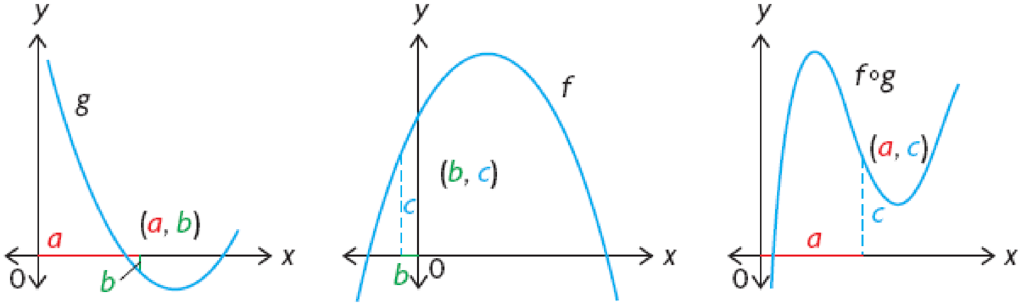
The Composition of a Function with its Inverse

Complete the following table.

$f(x)$	$f^{-1}(x)$	$(f \circ f^{-1})(x) = f(f^{-1}(x))$	$(f^{-1} \circ f)(x) = f^{-1}(f(x))$
$f(x) = x^2$			
$f(x) = 2x + 7$			
$f(x) = 10^x$			
$f(x) = \log_3(x)$			
$f(x) = \frac{1}{x-3}$			

Given the results in the table, make a conjecture about the result of composing a function with its inverse. Explain why this result ***should not*** be surprising.

- Algebraically, the composition of f with g is denoted by $(f \circ g)(x)$, whereas the composition of g with f is denoted by $(g \circ f)(x)$. In most cases, $(f \circ g)(x) \neq (g \circ f)(x)$ because the order in which the functions are composed matters.
- Let $(a, b) \in g$ and $(b, c) \in f$. Then $(a, c) \in f \circ g$. A point in $f \circ g$ exists where an element in the range of g is also in the domain of f. The function $f \circ g$ exists only when the range of g overlaps the domain of f.



- The domain of $(f \circ g)(x)$ is a subset of the domain of g. It is the set of values, x, in the domain of g for which $g(x)$ is in the domain of f.
- If both f and f^{-1} are functions, then $(f^{-1} \circ f)(x) = x$ for all x in the domain of f, and $(f \circ f^{-1})(x) = x$ for all x in the domain of f^{-1}.

Homework
p. 530: 10, 12, 13, 15, 16   p. 538-539: 5, 6, 7, 9, 12, 15, 16   p. 542: 3   p. 552-554: 2, 3, 4, 7, 8, 12, 13, 14, 15

Solving Equations that are Difficult or Impossible to Solve Algebraically

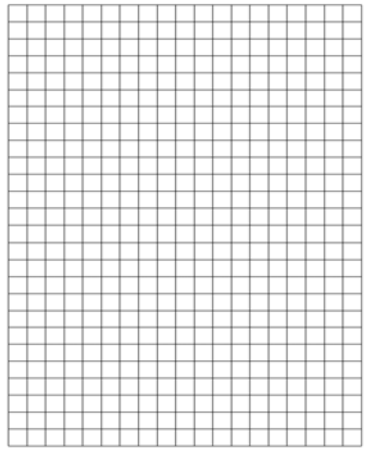
Mathematical Models

Activity – Under Pressure

Part A – Forming a Hypothesis

A tire is inflated to 400 kilopascals (kPa). Because of a slow leak, over the course of a few hours the tire deflates until it is flat. The following data were collected over the first 45 minutes.

Time, t, (min)	*Pressure, P, (kPa)*
0	400
5	335
10	295
15	255
20	225
25	195
30	170
35	150
40	135
45	115



Create a scatter plot for P against time t. Sketch the curve of best fit for tire pressure.

Part B – Testing Your Hypothesis and Choosing a "Best Fit Model"

The data were plotted using *The Geometer's Sketchpad*® and saved in a file called "Under Pressure.gsp." Open this file and follow the instructions that appear on the screen.

Enter your best fit equations and number of hits in the table below.

	Linear Model $f(x) = mx + b$	*Quadratic Model* $f(x) = a(x-h)^2 + k$	*Exponential Model* $f(x) = a \cdot b^{x-h} + k$
Your Best Fit Equations			
Number of Hits			

Equation of the best fit model: →

Part C – Evaluating Your Model

1. Is the quadratic model a valid choice if you consider the entire domain of the quadratic function and the long term trend of the data in this context? Explain why or why not.

2. Using each of the three "best fit" models, predict the pressure remaining in the tire after one hour. How do your predictions compare? Which of the 3 models gives the most reasonable prediction? Justify your answer.

3. Using each of the three "best fit" models, determine how long it will take before the tire pressure drops below 23 kPA?

4. Justify, in detail, why you think the model you obtained is the best model for the data in this scenario. Consider more than the number of hits in your answer.

Part D: Pumped Up

Johanna is pumping up her bicycle tire and monitoring the pressure every 5 pumps of the air pump. Her data are shown below. Determine the algebraic model that best represents these data and use your model to determine how many pumps it will take to inflate the tire to the recommended pressure of 65 psi (pounds per square inch).

Number of Pumps	*Tire Pressure (psi)*
0	14
5	30
10	36
15	41
20	46
25	49

$1 \text{ psi} \doteq 6.89 \text{ kPa}$

In Summary

Key Ideas

- A mathematical model is just that—a model. It will not be a perfect description of a real-life situation; but if it is a good model, then you will be able to use it to describe the real-life situation and make predictions.
- Increasing the amount of data you have for creating a mathematical model improves the accuracy of the model.
- A scatter plot gives you a visual representation of the data. Examining the scatter plot may give you an idea of what kind of function could be used to model the data. Graphing your mathematical model on the scatter plot is a visual way to confirm that it is a good fit.

Need to Know

- If you have to choose between a simple function and a complicated function, and if both fit the data equally well, the simple function is generally preferred.
- The function you choose should make sense in the context of the problem; for the growth of a population, you may want to consider an exponential model or a logistic model.
- One way to compare mathematical models created using regression analysis is to examine the value of R^2. This is the fraction of the variation in the response variable (y), which is explained by the mathematical model based on the predictor variable (x).
- Mathematical models are useful for **interpolating**. They are not necessarily useful for **extrapolating** because they assume that the trend in the data will continue. Many factors can affect the relationship between the independent variable and the dependent variable and change the trend.
- It is often necessary to restrict the domain of a mathematical model to represent a realistic situation.

APPENDIX – ONTARIO MINISTRY OF EDUCATION GUIDELINES

Note: Many of the expectations listed below have already been addressed in other units. This unit focuses on combinations of functions and on selecting appropriate mathematical models.

D. CHARACTERISTICS OF FUNCTIONS

OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of average and instantaneous rate of change, and determine, numerically and graphically, and interpret the average rate of change of a function over a given interval and the instantaneous rate of change of a function at a given point;
2. determine functions that result from the addition, subtraction, multiplication, and division of two functions and from the composition of two functions, describe some properties of the resulting functions, and solve related problems;
3. compare the characteristics of functions, and solve problems by modelling and reasoning with functions, including problems with solutions that are not accessible by standard algebraic techniques.

SPECIFIC EXPECTATIONS

1. Understanding Rates of Change

By the end of this course, students will:

1.1 gather, interpret, and describe information about real-world applications of rates of change, and recognize different ways of representing rates of change (e.g., in words, numerically, graphically, algebraically)

1.2 recognize that the rate of change for a function is a comparison of changes in the dependent variable to changes in the independent variable, and distinguish situations in which the rate of change is zero, constant, or changing by examining applications, including those arising from real-world situations (e.g., rate of change of the area of a circle as the radius increases, inflation rates, the rising trend in graduation rates among Aboriginal youth, speed of a cruising aircraft, speed of a cyclist climbing a hill, infection rates)

Sample problem: The population of bacteria in a sample is 250 000 at 1:00 p.m., 500 000 at 3:00 p.m., and 1 000 000 at 5:00 p.m. Compare methods used to calculate the change in the population and the rate of change in the population between 1:00 p.m. to 5:00 p.m. Is the rate of change constant? Explain your reasoning.

1.3 sketch a graph that represents a relationship involving rate of change, as described in words, and verify with technology (e.g., motion sensor) when possible

Sample problem: John rides his bicycle at a constant cruising speed along a flat road. He then decelerates (i.e., decreases speed) as he climbs a hill. At the top, he accelerates (i.e., increases speed) on a flat road back to his constant cruising speed, and he then accelerates down a hill. Finally, he comes to another hill and glides to a stop as he starts to climb. Sketch a graph of John's speed versus time and a graph of his distance travelled versus time.

1.4 calculate and interpret average rates of change of functions (e.g., linear, quadratic, exponential, sinusoidal) arising from real-world applications (e.g., in the natural, physical, and social sciences), given various representations of the functions (e.g., tables of values, graphs, equations)

Sample problem: Fluorine-20 is a radioactive substance that decays over time. At time 0, the mass of a sample of the substance is 20 g. The mass decreases to 10 g after 11 s, to 5 g after 22 s, and to 2.5 g after 33 s. Compare the average rate of change over the 33-s interval with the average rate of change over consecutive 11-s intervals.

1.5 recognize examples of instantaneous rates of change arising from real-world situations, and make connections between instantaneous rates of change and average rates of change (e.g., an average rate of change can be used to approximate an instantaneous rate of change)

Sample problem: In general, does the speedometer of a car measure instantaneous rate of change (i.e., instantaneous speed) or average rate of change (i.e., average speed)? Describe situations in which the instantaneous speed and the average speed would be the same.

1.6 determine, through investigation using various representations of relationships (e.g., tables of values, graphs, equations), approximate instantaneous rates of change arising from real-world applications (e.g., in the natural, physical, and social sciences) by using average rates of change and reducing the interval over which the average rate of change is determined

Sample problem: The distance, d metres, travelled by a falling object in t seconds is represented by $d = 5t^2$. When $t = 3$, the instantaneous speed of the object is 30 m/s. Compare the average speeds over different time intervals starting at $t = 3$ with the instantaneous speed when $t = 3$. Use your observations to select an interval that can be used to provide a good approximation of the instantaneous speed at $t = 3$.

1.7 make connections, through investigation, between the slope of a secant on the graph of a function (e.g., quadratic, exponential, sinusoidal) and the average rate of change of the function over an interval, and between the slope of the tangent to a point on the graph of a function and the instantaneous rate of change of the function at that point

Sample problem: Use tangents to investigate the behaviour of a function when the instantaneous rate of change is zero, positive, or negative.

1.8 determine, through investigation using a variety of tools and strategies (e.g., using a table of values to calculate slopes of secants or graphing secants and measuring their slopes with technology), the approximate slope of the tangent to a given point on the graph of a function (e.g., quadratic, exponential, sinusoidal) by using the slopes of secants through the given point (e.g., investigating the slopes of secants that approach the tangent at that point more and more closely), and make connections to average and instantaneous rates of change

1.9 solve problems involving average and instantaneous rates of change, including problems arising from real-world applications, by using numerical and graphical methods (e.g., by using graphing technology to graph a tangent and measure its slope)

Sample problem: The height, h metres, of a ball above the ground can be modelled by the function $h(t) = -5t^2 + 20t$, where t is the time in seconds. Use average speeds to determine the approximate instantaneous speed at $t = 3$.

2. Combining Functions

By the end of this course, students will:

2.1 determine, through investigation using graphing technology, key features (e.g., domain, range, maximum/minimum points, number of zeros) of the graphs of functions created by adding, subtracting, multiplying, or dividing functions [e.g., $f(x) = 2^{-x} \sin 4x$, $g(x) = x^2 + 2^x$, $h(x) = \dfrac{\sin x}{\cos x}$], and describe factors that affect these properties

Sample problem: Investigate the effect of the behaviours of $f(x) = \sin x$, $f(x) = \sin 2x$, and $f(x) = \sin 4x$ on the shape of $f(x) = \sin x + \sin 2x + \sin 4x$.

2.2 recognize real-world applications of combinations of functions (e.g., the motion of a damped pendulum can be represented by a function that is the product of a trigonometric function and an exponential function; the frequencies of tones associated with the numbers on a telephone involve the addition of two trigonometric functions), and solve related problems graphically

Sample problem: The rate at which a contaminant leaves a storm sewer and enters a lake depends on two factors: the concentration of the contaminant in the water from the sewer and the rate at which the water leaves the sewer. Both of these factors vary with time. The concentration of the contaminant, in kilograms per cubic metre of water, is given by $c(t) = t^2$, where t is in seconds. The rate at which water leaves the sewer, in cubic metres per second, is given by $w(t) = \dfrac{1}{t^4 + 10}$. Determine the time at which the contaminant leaves the sewer and enters the lake at the maximum rate.

2.3 determine, through investigation, and explain some properties (i.e., odd, even, or neither; increasing/decreasing behaviours) of functions formed by adding, subtracting, multiplying, and dividing general functions [e.g., $f(x) + g(x)$, $f(x)g(x)$]

Sample problem: Investigate algebraically, and verify numerically and graphically, whether the product of two functions is even or odd if the two functions are both even or both odd, or if one function is even and the other is odd.

2.4 determine the composition of two functions [i.e., $f(g(x))$] numerically (i.e., by using a table of values) and graphically, with technology, for functions represented in a variety of ways (e.g., function machines, graphs, equations), and interpret the composition of two functions in real-world applications

Sample problem: For a car travelling at a constant speed, the distance driven, d kilometres, is represented by $d(t) = 80t$, where t is the time in hours. The cost of gasoline, in dollars, for the drive is represented by $C(d) = 0.09d$. Determine numerically and interpret $C(d(5))$, and describe the relationship represented by $C(d(t))$.

2.5 determine algebraically the composition of two functions [i.e., $f(g(x))$], verify that $f(g(x))$ is not always equal to $g(f(x))$ [e.g., by determining $f(g(x))$ and $g(f(x))$, given $f(x) = x + 1$ and $g(x) = 2x$], and state the domain [i.e., by defining $f(g(x))$ for those x-values for which $g(x)$ is defined and for which it is included in the domain of $f(x)$] and the range of the composition of two functions

Sample problem: Determine $f(g(x))$ and $g(f(x))$ given $f(x) = \cos x$ and $g(x) = 2x + 1$, state the domain and range of $f(g(x))$ and $g(f(x))$, compare $f(g(x))$ with $g(f(x))$ algebraically, and verify numerically and graphically with technology.

2.6 solve problems involving the composition of two functions, including problems arising from real-world applications

Sample problem: The speed of a car, v kilometres per hour, at a time of t hours is represented by $v(t) = 40 + 3t + t^2$. The rate of gasoline consumption of the car, c litres per kilometre, at a speed of v kilometres per hour is represented by $c(v) = \left(\frac{v}{500} - 0.1\right)^2 + 0.15$. Determine algebraically $c(v(t))$, the rate of gasoline consumption as a function of time. Determine, using technology, the time when the car is running most economically during a four-hour trip.

2.7 demonstrate, by giving examples for functions represented in a variety of ways (e.g., function machines, graphs, equations), the property that the composition of a function and its inverse function maps a number onto itself [i.e., $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ demonstrate that the inverse function is the reverse process of the original function and that it undoes what the function does]

2.8 make connections, through investigation using technology, between transformations (i.e., vertical and horizontal translations; reflections in the axes; vertical and horizontal stretches and compressions to and from the x- and y-axes) of simple functions $f(x)$ [e.g., $f(x) = x^3 + 20$, $f(x) = \sin x$, $f(x) = \log x$] and the composition of these functions with a linear function of the form $g(x) = A(x + B)$

Sample problem: Compare the graph of $f(x) = x^2$ with the graphs of $f(g(x))$ and $g(f(x))$, where $g(x) = 2(x - d)$, for various values of d. Describe the effects of d in terms of transformations of $f(x)$.

3. Using Function Models to Solve Problems

By the end of this course, students will:

3.1 compare, through investigation using a variety of tools and strategies (e.g., graphing with technology; comparing algebraic representations; comparing finite differences in tables of values) the characteristics (e.g., key features of the graphs, forms of the equations) of various functions (i.e., polynomial, rational, trigonometric, exponential, logarithmic)

3.2 solve graphically and numerically equations and inequalities whose solutions are not accessible by standard algebraic techniques

Sample problem: Solve: $2x^2 < 2^x$; $\cos x = x$, with x in radians.

3.3 solve problems, using a variety of tools and strategies, including problems arising from real-world applications, by reasoning with functions and by applying concepts and procedures involving functions (e.g., by constructing a function model from data, using the model to determine mathematical results, and interpreting and communicating the results within the context of the problem)

Sample problem: The pressure of a car tire with a slow leak is given in the following table of values:

Time, t (min)	Pressure, P (kPa)
0	400
5	335
10	295
15	255
20	225
25	195
30	170

Use technology to investigate linear, quadratic, and exponential models for the relationship of the tire pressure and time, and describe how well each model fits the data. Use each model to predict the pressure after 60 min. Which model gives the most realistic answer?