Analytic Geometry – Solutions to Pattern Finding Activity

<table>
<tr><th>Question</th><th colspan="2">Solutions (Including Equations that Describe the Relationships)</th></tr>
<tr>
<td>1. How many regions are there in the fourteenth diagram?

1 2 3</td>
<td>How is the number of regions (r) related to the diagram number (d)?
<table><tr><th>d</th><th>$r=d+1$</th></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>3</td></tr><tr><td>3</td><td>4</td></tr><tr><td>.
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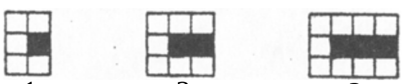
.</td></tr><tr><td>14</td><td>15</td></tr></table></td>
<td>From the table we can see that the number of regions is always one more than the diagram number.
This relationship can be described by the following equation:
$$r=d+1$$</td>
</tr>
<tr>
<td>2. How many shaded squares are there in the eighth diagram? How many unshaded squares are there in the eighth diagram?

1 2 3</td>
<td>How is the number of shaded squares (s) related to the diagram number (d)? How is the number of unshaded squares (u) related to the diagram number (d)?
<table><tr><th>d</th><th>$s=d$</th><th>$u=2d+3$</th></tr><tr><td>1</td><td>1</td><td>5</td></tr><tr><td>2</td><td>2</td><td>7</td></tr><tr><td>3</td><td>3</td><td>9</td></tr><tr><td>.
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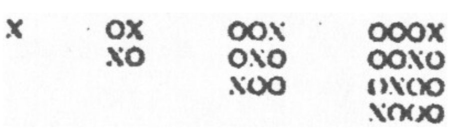
.</td></tr><tr><td>8</td><td>8</td><td>19</td></tr></table></td>
<td>From the table we can see that the number of shaded squares and the number of unshaded squares are related to the diagram number according to the following equations:
$$s=d$$
(The # of shaded squares is equal to the diagram #.)
$$u=2d+3$$</td>
</tr>
<tr>
<td>3. How many “X’s” are in the twentieth diagram? How many “O’s” are there in the twentieth diagram?

1 2 3 4</td>
<td>How is the number of “X’s” (X) related to the diagram number (d)? How is the number of “O’s” (O) related to the diagram number (d)?
<table><tr><th>d</th><th>$X=d$</th><th>$O=d^2-d$</th></tr><tr><td>1</td><td>1</td><td>0</td></tr><tr><td>2</td><td>2</td><td>2</td></tr><tr><td>3</td><td>3</td><td>6</td></tr><tr><td>4</td><td>4</td><td>12</td></tr><tr><td>.
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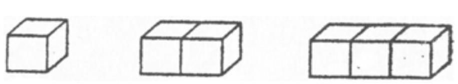
.</td></tr><tr><td>20</td><td>20</td><td>380</td></tr></table></td>
<td>From the table we can see that the number of “X’s” and the number of “O’s” are related to the diagram number according to the following equations:
$$X=d$$
$$O=d^2-d$$
The second equation can also be written as
$$O=d(d-1).$$</td>
</tr>
<tr>
<td>4. How many faces are visible in the twentieth diagram?

1 2 3</td>
<td>How is the number of visible faces (f) related to the diagram number (d)?
<table><tr><th>d</th><th>$f=3d+2$</th></tr><tr><td>1</td><td>5</td></tr><tr><td>2</td><td>8</td></tr><tr><td>3</td><td>11</td></tr><tr><td>.
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.</td><td>.
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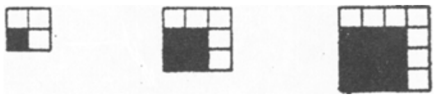
.</td></tr><tr><td>20</td><td>62</td></tr></table></td>
<td>From the table we can see that the number of visible faces is always two more than triple the diagram number. This relationship can be described by the following equation:
$$f=3d+2$$</td>
</tr>
</table>

5. How many shaded squares are there in the twelfth diagram? How many unshaded squares are there in the twelfth diagram?



1 2 3

How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	$s=d^2$	$u=2d+1$
1	1	3
2	4	5
3	9	7
.	.	.
12	144	25

From the table we can see that the number of shaded squares and the number of unshaded squares are ***related to*** the diagram number according to the following equations:

$$s=d^2$$

$$u=2d+1$$

6. A cow is milked twice a day. Each time she gives 11 kg of milk. Calculate the total milk production after
(i) 16 days (ii) 49 days

How is the total milk production (m) ***related to*** the time in days (t)?

t	$m=22t$
1	22
2	44
3	66
.	.
16	352
49	1078

From the table we can see that the total milk production is 22 times the number of days. This ***relationship*** can be described by the following equation:

$$m=22t$$

7. The sum of the interior angles of each polygon is shown. What is the sum of the interior angles of a decagon (a polygon with 10 sides).

180° 360° 540°

3 4 5

How is the sum of the interior angles (s) ***related to*** the number of sides (n)?

n	$s=180(n-2)$
3	180
4	360
5	540
.	.
10	1440

From the table we can see that the sum of the interior angles is the product of 180 and two less than the number of sides. This ***relationship*** can be described by the following equation:

$$s=180(n-2)$$

8. How many "X's" are in the tenth diagram? How many "O's" are there in the tenth diagram?

1 2 3

How is the number of "X's" (X) ***related to*** the diagram number (d)? How is the number of "O's" (O) ***related to*** the diagram number (d)?

d	$X=d+1$	$O=\frac{d(d+1)}{2}$
1	2	1
2	3	3
3	4	6
.	.	.
10	11	55

From the table we can see that the number of "X's" and the number of "O's" are ***related to*** the diagram number according to the following equations:

$$X=d+1$$

$$O=\frac{d(d+1)}{2}=\frac{d^2+d}{2}$$

9. The cubes along one diagonal of each cube of a face are coloured (including the faces that can't be seen). How many cubes are coloured on the fifth diagram?

1 2 3 4

How is the number of coloured cubes (c) ***related to*** the diagram number (d)?

d	$c=6(d-2)+4, d\neq 1$
1	1
2	4
3	10
4	16
5	22

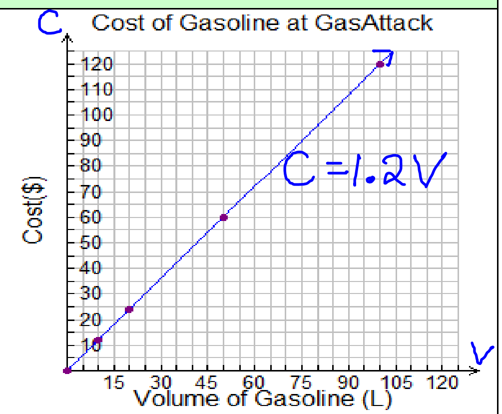
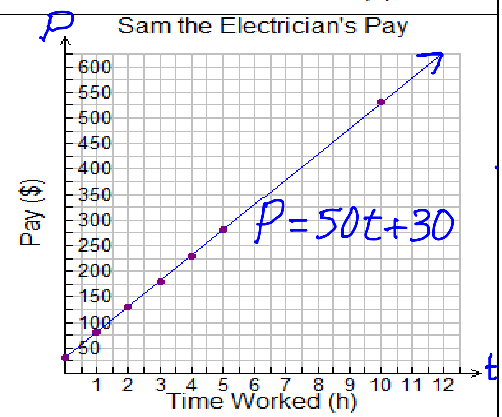
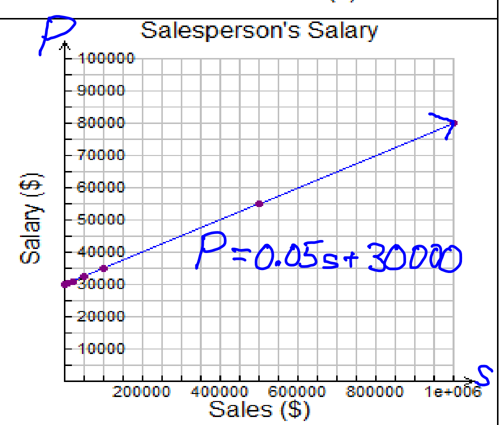
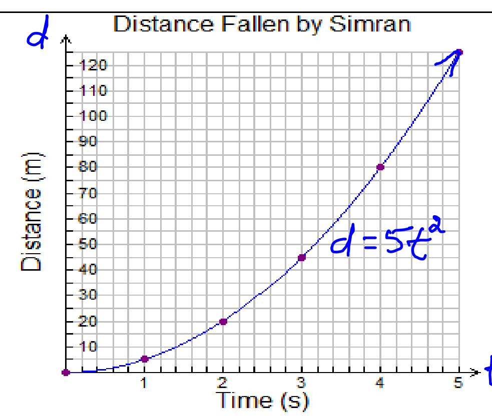
From the table we can see that the number of coloured cubes is six times, 2 less than the diagram number, all increased by four. This ***relationship*** can be described by the following equation:

$$c=6(d-2)+4, d\neq 1.$$

By simplifying, the equation can be written $c=6d-8, d\neq 1$. (The equation does not hold for $d=1$.)

Direct Variation, Partial Variation or Neither?

Complete the following table.

Situation	*Type of Variation (Circle One)*	*Table of Values*	*Initial Value (b) and Constant of Variation (m)*	*Graph and Equation*
Gasoline at GasAttack costs $1.20/L. How does the ***cost*** of gasoline vary with the ***volume*** of gasoline purchased?	Graph passes through origin Partial / Direct / Neither Starting value of C is 0.	V (L) | C ($) 0 | 0 10 | 12 20 | 24 50 | 60 100 | 120	$b =$ 0 $m =$ 1.2	
Sam the electrician charges a base fee of $30 plus $50/h. How does Sam's ***pay*** vary with the ***time*** worked?	Graph doesn't pass through the origin Partial / Direct / Neither Starting value of P is 30	t (h) | P ($) 0 | 30 1 | 80 2 | 130 3 | 180 4 | 230 5 | 280 10 | 530	$b =$ 30 $m =$ 50	
Abdul the salesperson is paid a base salary of $30,000 plus 5% of sales. How does Abdul's ***pay*** vary with the amount of ***sales***?	5% = 0.05 Partial / Direct / Neither e.g. s = 50000 P = 0.05(50000) + 30000 = 32500	s ($) | P ($) 0 | 30000 10000 | 30500 20000 | 31000 50000 | 52500 100000 | 55000 500000 | 55000 1000000 | 80000	$b =$ 30000 $m =$ 0.05	
Simran likes bungee jumping. Whenever she jumps, her speed increases at a rate of 10 m/s. How does the ***distance*** fallen vary with ***time***?	See next page for explanation Partial / Direct / Neither Simran must be attached to a very long BUNGEE cord!	t (s) | d (m) 0 | 0 1 | 5 2 | 20 3 | 45 4 | 80 5 | 125	$b =$ 0 $m =$ N/A because the relation is not linear	

Explanation of Simran's Bungee Jumping Adventure

Simran's speed increases at a constant rate of 10 m/s. This means that at any given time, her speed is 10 m/s faster than it was exactly one second earlier. This is shown in the following table for the first nine seconds.

Time (s)	0	1	2	3	4	5	6	7	8	9
Speed (m/s) at the Given Time	0	10	20	30	40	50	60	70	80	90

The next table shows Simran's average speed in each of the first nine one-second time intervals. Since Simran's speed increases at a constant rate, the average speed during any time interval is simply the average of the initial and final speeds. For example, the average speed in the first second (0 s to 1 s) is calculated as follows:

$$\text{Average speed in the first second} = \frac{\text{speed at 0 s} + \text{speed at 1 s}}{2} = \frac{0+10}{2} = 5$$

Then the distance fallen during the given time interval is simply the average speed multiplied by the elapsed time. For example, the distance fallen in the first second is calculated as follows:

$$\text{Distance fallen in the first second} = (\text{average speed}) \times (\text{time elapsed}) = (5 \text{ m/s}) \times (1 \text{ s}) = 5 \text{ m}$$

Time Interval	0 s to 1 s	1 s to 2 s	2 s to 3 s	3 s to 4 s	4 s to 5 s	5 s to 6 s	6 s to 7 s	7 s to 8 s	8 s to 9 s
Average Speed (m/s) during Time Interval	5	15	25	35	45	55	65	75	85
Distance Fallen during Time Interval	5	15	25	35	45	55	65	75	

Finally, the distance fallen after a given amount of time has elapsed is calculated by adding the distance fallen up to exactly one second earlier to the distance fallen in the last second. For example, the distance fallen after 5 s is calculated as follows:

Distance fallen after 5 s have elapsed = (Distance fallen after 4 s) + (Distance fallen from 4 s to 5 s) =80 m+45 m = 125 m

Time (s)	0	1	2	3	4	5	6	7	8	9
Distance Fallen (m) by the Given Time	0	5	$5+15$ $=20$	$20+25$ $=45$	$45+35$ $=80$	$80+45$ $=125$	$125+55$ $=180$	$180+65$ $=245$	$245+75$ $=320$	$320+85$ $=405$

DEEPER ANALYSIS OF RELATIONS

Victim: Mr. Solutions

Question	*Solutions*	*Questions*

1.

Question: 1. How many regions are there in the fourteenth diagram?

1 2 3

Solutions: How is the number of regions (r) ***related to*** the diagram number (d)?

d	$r = d + 1$
1	2
2	3
3	4
.	.
.	.
.	.
14	15

Questions:

Equation of Relation	$r = d + 1$
Independent Variable	d
Dependent Variable	r
Linear or Non-Linear?	linear
If Linear, Partial or Direct Variation	Partial
If Linear, Constant of Variation	1
Initial Value	1

2.

Question: 2. How many shaded squares are there in the eighth diagram? How many unshaded squares are there in the eighth diagram?

1 2 3

Solutions: How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

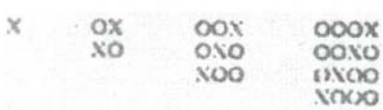
d	$s = d$	$u = 2d + 3$
1	1	5
2	2	7
3	3	9
.	.	.
.	.	.
.	.	.
8	8	19

Questions:

Equation of Relation	$s = d$	$u = 2d + 3$
Independent Variable	d	d
Dependent Variable	s	u
Linear or Non-Linear?	Linear	Linear
If Linear, Partial or Direct Variation	Direct	Partial
If Linear, Constant of Variation	1	2
Initial Value	0	3

3.

Question: 3. How many "X's" are in the twentieth diagram? How many "O's" are there in the twentieth diagram?



1 2 3 4

Solutions: How is the number of "X's" (X) ***related to*** the diagram number (d)? How is the number of "O's" (O) ***related to*** the diagram number (d)?

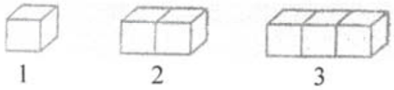
d	$X = d$	$O = d^2 - d$
1	1	0
2	2	2
3	3	6
4	4	12
.	.	.
.	.	.
.	.	.
20	20	380

Questions:

Equation of Relation	$X = d$	$O = d^2 - d$
Independent Variable	d	d
Dependent Variable	X	O
Linear or Non-Linear?	linear	non-linear
If Linear, Partial or Direct Variation	direct	N/A
If Linear, Constant of Variation	1	N/A
Initial Value	0	0

4.

Question: 4. How many faces are visible in the twentieth diagram?



1 2 3

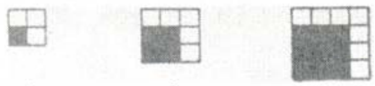
Solutions: How is the number of visible faces (f) ***related to*** the diagram number (d)?

d	$f = 3d + 2$
1	5
2	8
3	11
.	.
.	.
.	.
20	62

Questions:

Equation of Relation	$f = 3d + 2$
Independent Variable	d
Dependent Variable	f
Linear or Non-Linear?	Linear
If Linear, Partial or Direct Variation	Partial
If Linear, Constant of Variation	3
Initial Value	2

5. How many shaded squares are there in the twelfth diagram? How many unshaded squares are there in the twelfth diagram?



1 2 3

How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	$s=d^2$	$u=2d+1$
1	1	3
2	4	5
3	9	7
⋮	⋮	⋮
12	144	25

Equation of Relation	$s=d^2$	$u=2d+1$
Independent Variable	d	d
Dependent Variable	s	u
Linear or Non-Linear?	non-linear	linear
If Linear, Partial or Direct Variation	N/A	Partial
If Linear, Constant of Variation	N/A	2
Initial Value	0	1

6. A cow is milked twice a day. Each time she gives 11 kg of milk. Calculate the total milk production after
(i) 16 days (ii) 49 days

How is the total milk production (m) ***related to*** the time in days (t)?

t	$m=22t$
1	22
2	44
3	66
⋮	⋮
16	352
49	1078

Equation of Relation	$m=22t$
Independent Variable	t
Dependent Variable	m
Linear or Non-Linear?	Linear
If Linear, Partial or Direct Variation	Direct
If Linear, Constant of Variation	22
Initial Value	0

7. The sum of the interior angles of each polygon is shown. What is the sum of the interior angles of a decagon (a polygon with 10 sides).

180° 360° 540°

3 4 5

How is the sum of the interior angles (s) ***related to*** the number of sides (n)?

n	$s=180(n-2)$
3	180
4	360
5	540
⋮	⋮
10	1440

Equation of Relation	$s=180(n-2)$
Independent Variable	n
Dependent Variable	s
Linear or Non-Linear?	Linear
If Linear, Partial or Direct Variation	Partial
If Linear, Constant of Variation	180
Initial Value	-360

8. How many "X's" are in the tenth diagram? How many "O's" are there in the tenth diagram?

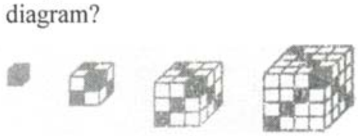
1 2 3

How is the number of "X's" (X) ***related to*** the diagram number (d)? How is the number of "O's" (O) ***related to*** the diagram number (d)?

d	$X=d+1$	$O=\frac{d(d+1)}{2}$
1	2	1
2	3	3
3	4	6
⋮	⋮	⋮
10	11	55

Equation of Relation	$X=d+1$	$O=\frac{d(d+1)}{2}$
Independent Variable	d	d
Dependent Variable	X	O
Linear or Non-Linear?	Linear	Non-linear
If Linear, Partial or Direct Variation	Partial	N/A
If Linear, Constant of Variation	1	N/A
Initial Value	1	0

9. The cubes along one diagonal of each cube of a face are coloured (including the faces that can't be seen). How many cubes are coloured on the fifth diagram?



1 2 3 4

How is the number of coloured cubes (c) ***related to*** the diagram number (d)?

d	$c=6(d-2)+4, d\neq 1$
1	1
2	4
3	10
4	16
5	22

Equation of Relation	$c=6(d-2)+4, d\neq 1$
Independent Variable	d
Dependent Variable	c
Linear or Non-Linear?	Linear
If Linear, Partial or Direct Variation	Partial
If Linear, Constant of Variation	6
Initial Value	-8

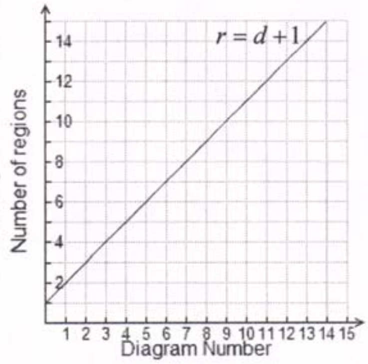
Opening Activity Revisited

Question	Table of Values	Graph and its Properties

1. How many regions are there in the fourteenth diagram?

How is the number of regions (r) ***related to*** the diagram number (d)?

d	$r = d + 1$
1	2
2	3
3	4
.	.
.	.
.	.
14	15



Slope: 1

Vertical Intercept: 1

Independent Variable: d

Dependent Variable: r

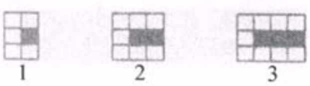
Linear or Non-Linear? L

Direct or Partial Variation? Partial

Constant of Variation 1

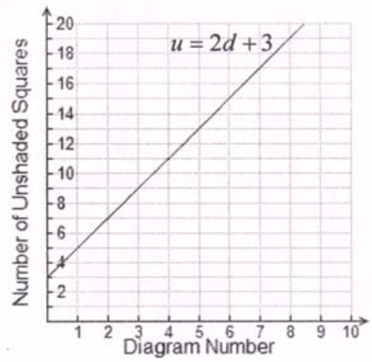
Initial Value 1

2. How many shaded squares are there in the eighth diagram? How many unshaded squares are there in the eighth diagram?



How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	$s = d$	$u = 2d + 3$
1	1	5
2	2	7
3	3	9
.	.	.
.	.	.
.	.	.
8	8	19



Slope: 2

Vertical Intercept: 3

Independent Variable: d

Dependent Variable: u

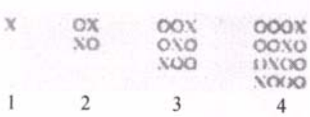
Linear or Non-Linear? L

Direct or Partial Variation? Partial

Constant of Variation 2

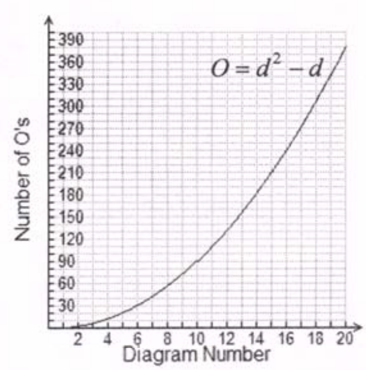
Initial Value 3

3. How many "X's" are in the twentieth diagram? How many "O's" are there in the twentieth diagram?



How is the number of "X's" (X) ***related to*** the diagram number (d)? How is the number of "O's" (O) ***related to*** the diagram number (d)?

d	$X = d$	$O = d^2 - d$
1	1	0
2	2	2
3	3	6
4	4	12
.	.	.
.	.	.
.	.	.
20	20	380



Slope: N/A

Vertical Intercept: 0

Independent Variable: d

Dependent Variable: O

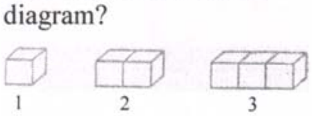
Linear or Non-Linear? N

Direct or Partial Variation? N/A

Constant of Variation N/A

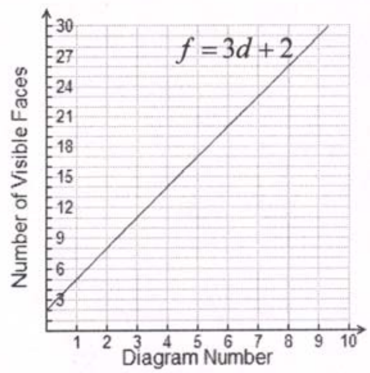
Initial Value 0

4. How many faces are visible in the twentieth diagram?



How is the number of visible faces (f) ***related to*** the diagram number (d)?

d	$f = 3d + 2$
1	5
2	8
3	11
.	.
.	.
.	.
20	62



Slope: 3

Vertical Intercept: 2

Independent Variable: d

Dependent Variable: f

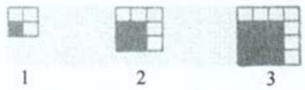
Linear or Non-Linear? L

Direct or Partial Variation? Partial

Constant of Variation 3

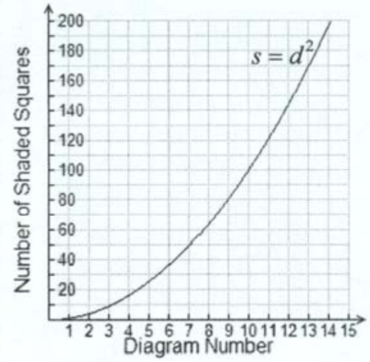
Initial Value 2

5. How many shaded squares are there in the twelfth diagram? How many unshaded squares are there in the twelfth diagram?



How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	$s=d^2$	$u=2d+1$
1	1	3
2	4	5
.	.	.
.	.	.
.	.	.
12	144	25



Slope: N/A

Vertical Intercept: 0

Independent Variable: d

Dependent Variable: s

Linear or Non-Linear? N

Direct or Partial Variation? N/A

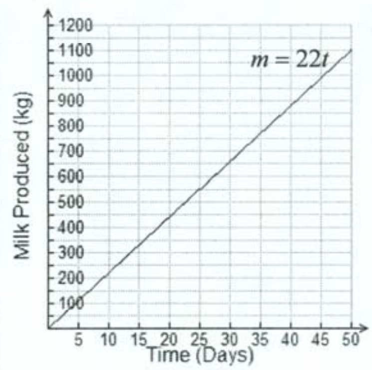
Constant of Variation N/A

Initial Value 0

6. A cow is milked twice a day. Each time she gives 11 kg of milk. Calculate the total milk production after (i) 16 days (ii) 49 days

How is the total milk production (m) ***related to*** the time in days (t)?

t	$m=22t$
1	22
2	44
.	.
.	.
.	.
16	352
49	1078



Slope: 22

Vertical Intercept: 0

Independent Variable: t

Dependent Variable: m

Linear or Non-Linear? L

Direct or Partial Variation? Direct

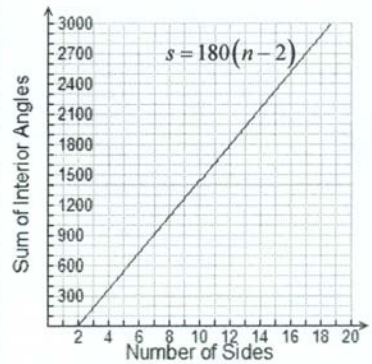
Constant of Variation 22

Initial Value 0

7. The sum of the interior angles of each polygon is shown. What is the sum of the interior angles of a decagon (a polygon with 10 sides).

How is the sum of the interior angles (s) ***related to*** the number of sides (n)?

n	$s=180(n-2)$
3	180
4	360
.	.
.	.
.	.
10	1440



Slope: 180

Vertical Intercept: -360

Independent Variable: n

Dependent Variable: s

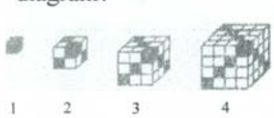
Linear or Non-Linear? L

Direct or Partial Variation? Partial

Constant of Variation 180

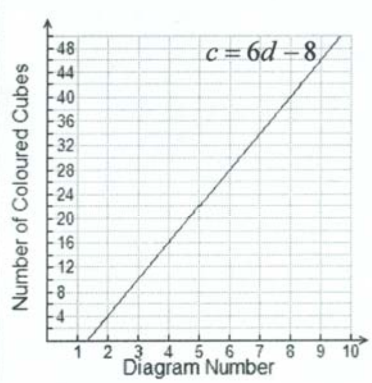
Initial Value -360

8. The cubes along one diagonal of each cube of a face are coloured (including the faces that can't be seen). How many cubes are coloured on the fifth diagram?



How is the number of coloured cubes (c) ***related to*** the diagram number (d)?

d	$c=6(d-2)+4, d\neq 1$
1	1
2	4
3	10
4	16



Slope: 6

Vertical Intercept: -8

Independent Variable: d

Dependent Variable: c

Linear or Non-Linear? L

Direct or Partial Variation? Partial

Constant of Variation 6

Initial Value -8

m = slope = 2, b = vertical int. = 3

Drawing Conclusions

Complete the following table.

Equation	*Linear or Non-Linear*	*Slope (If Linear)*	*Vertical Intercept*	*Partial or Direct Variation (If Linear)*	*Constant of Variation (If Linear)*	*Initial Value*
1. $r = d + 1$	L	1	1	P	1	1
2. $u = 2d + 3$	L	2	3	P	2	3
3. $O = d^2 - d$	N	NA	0	NA	NA	0
4. $f = 3d + 2$	L	3	2	P	3	2
5. $s = d^2$	N	NA	0	NA	NA	0
6. $m = 22t$	L	22	0	D	22	0
7. $s = 180n - 360$	L	180	-360	P	180	-360
8. $c = 6d - 8$	L	6	-8	P	6	-8
9. $O = \frac{1}{2}d(d+1) = \frac{1}{2}d^2 + \frac{1}{2}d$	N	NA	0	NA	NA	0

m = 180, b = -360

Observations

1. Describe how you can use the *equation of a relation* to determine

(a) whether it is *linear* or *non-linear*

The equation will take the form

$y = mx + b$ (slope, vertical intercept)

$x \to$ ind., $y \to$ dep., $m, b \to$ const.

(b) the *slope*, if the relation is linear

m = slope

(c) the *vertical intercept* (initial value)

b = vertical intercept

2. Equations 7 and 8 were originally written as $s = 180(n-2)$ and $c = 6(d-2)+4$. Show how each of these were simplified to produce the equivalent forms $s = 180n - 360$ and $c = 6d - 8$.

Use distributive property!

3. What is the connection between slope and the constant of variation?

m = slope = constant of variation

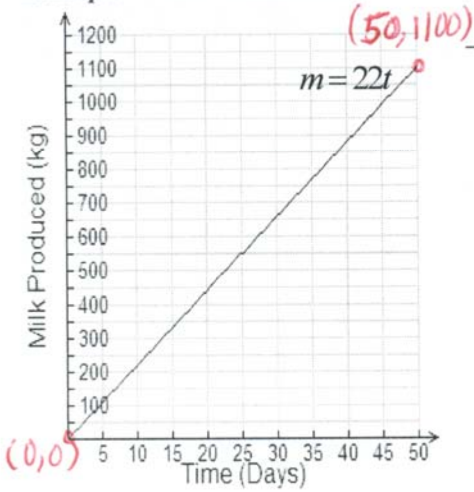
4. What is the connection between the vertical intercept and the initial value?

first or starting

b = vertical intercept = y-intercept = initial value

Slope as a Rate of Change

Example – Milk Production



t	$m=22t$
0	0
1	22
2	44
3	66
4	88
5	110
6	132
7	154
8	176
9	198
10	220

Slope = Constant of Variation =22

Explain how you can determine this using…

(a) …the graph choose two points for which the exact co-ordinates are known

$\therefore$ slope $= m = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = \frac{1100-0}{50-0} = \frac{1100}{50} = 22$

(b) …the equation → slope (coefficient of variable term)

$m = 22t$ is in the form $y = mx + b$ slope y-int

(c) …the table

Choose any two points (i.e. rows), then calculate $\frac{\Delta y}{\Delta x}$ e.g. (2, 44), (10, 220)

Observation

Every day, the cow produces 22 kg of milk. We can express this as a ***rate***, that is, the cow produces 22 kg/day (i.e. 22 kg per day or 22 kg every day). This example suggests that ***slope can also be interpreted as a rate of change***!

Rate of Change Definition

Let x represent an independent variable and y represent a variable whose value depends on x.
By the ***rate of change of y with respect to x*** we mean ***how fast*** y changes as the value of x changes.

Examples of Rate of Change

Name	*Independent Variable*	*Dependent Variable*	*Verbal Description*	*Example*
Speed	Time (t)	Distance (d)	***Speed*** is the rate of change of d with respect to t. That is, speed is a measure of how fast distance changes over time. (Units must be distance/time.)	A car travels at a speed of 120 km/h.
Hourly Wage	Time (t)	Money (M)	An ***hourly wage*** is the rate of change of M with respect to t. That is, hourly wage measures how fast money is earned over time. (Units must be money/time.)	Selene earns $25/h.
Fuel Efficiency	Distance (d)	Fuel Used (f)	***Fuel efficiency*** is the rate of change of f with respect to d. That is, fuel efficiency measures how fast fuel is used over distance travelled. (Units must be volume/distance.)	The Toyota Prius has a fuel efficiency of 4.3 L/100 km.

Summary

$$m = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in independent variable}}{\text{change in dependent variable}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \text{slope} = \text{constant of variation} = \begin{cases} y/x, & \text{direct variation} \\ (y-b)/x, & \text{partial variation} \end{cases},$$

b = initial value = vertical intercept = y-intercept

m = slope = the rate of change of y with respect to x = how fast y changes as x changes

Is the Relation Linear? Analytic Geometry Activity

Background

There is a very simple way to tell whether a relation is linear. The key to understanding this is to realize the following:

Linear Relation

↕

***Slope* is Constant**

↕

***Rate of Change* is Constant**

↕

**If the Independent Variable (i.e. the x-value) Changes by a Constant Amount,
Then the Dependent Variable (i.e. the y-value) Must Also Change by a Constant Amount**

↕

If Δx is constant, Δy must also be constant!

Example and Exercises

From the above, we can conclude that a relation is linear if Δy is constant whenever Δx is constant. The Δy values are called ***first differences***. Therefore, a relation is linear if the first differences are constant whenever Δx is constant.

- From the table, we can see that $\Delta x = 1$ and $\Delta y = -2$.
- Since both Δx and Δy are constant, the relation must be linear.
- slope $= m = \dfrac{\Delta y}{\Delta x} = \dfrac{-2}{1} = -2$
- y-intercept $= b = 4$ because the point $(0,4)$ belongs to the relation.
- The equation of the relation must be $y = -2x + 4$

x	y	Δy
-3	10	–
-2	8	$8-10=-2$
-1	6	$6-8=-2$
0	4	$4-6=-2$
1	2	$2-4=-2$
2	0	$0-2=-2$
3	-2	$-2-0=-2$

- From the table, we can see that $\Delta x =$ 2 and $\Delta y =$ 4.
- Since both Δx and Δy are constant, the relation must be linear.
- slope $= m = \dfrac{\Delta y}{\Delta x} = \dfrac{4}{2} = 2$
- y-intercept $= b =$ 3 because the point (0,3) belongs to the relation.
- The equation of the relation must be $y = 2x+3$

x	y	Δy
0	3	–
2	7	4
4	11	4
6	15	4
8	19	4
10	23	4
12	27	4

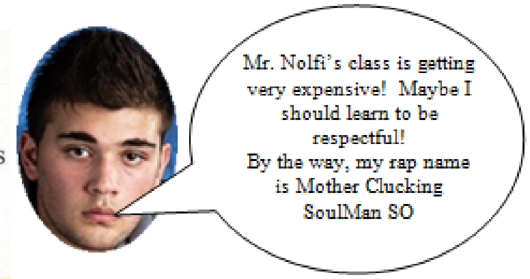
- From the table, we can see that $\Delta x =$ 1 and $\Delta y =$ 1.
- Since both Δx and Δy are constant, the relation must be linear.
- slope $= m = \dfrac{\Delta y}{\Delta x} = \dfrac{1}{1} = 1$
- y-intercept $= b =$ 3 because the point (0,3) belongs to the relation.
- The equation of the relation must be $y = x+3$

$\Delta x = 1$

$\Delta x = 2$

x	y	Δy
-3	0	–
-2	1	1
-1	2	1
0	3	1
2	5	2
4	7	2
6	9	2

Problem 1 – Solimon's Dilemma – A Linear Relation



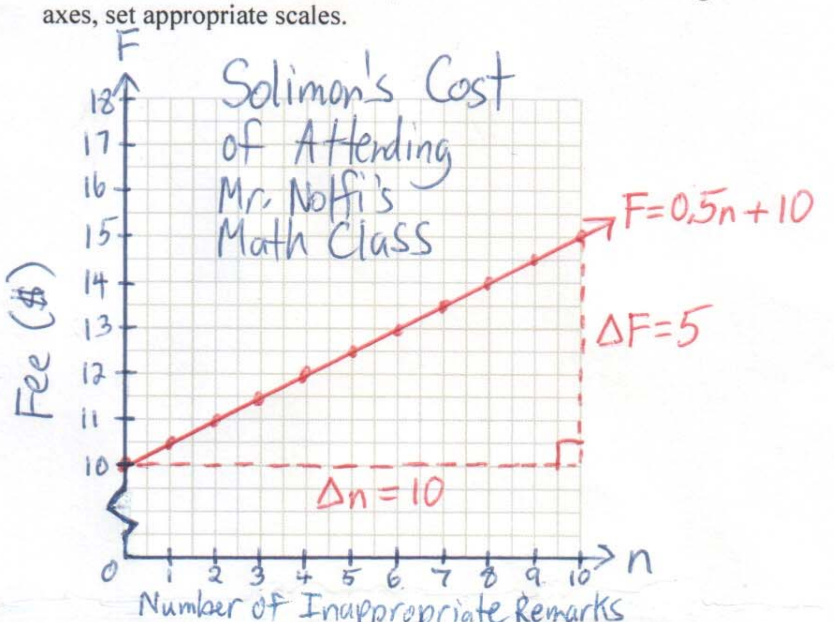
Mr. Nolfi believes very strongly in the importance of showing respect to others. Unfortunately, this view was not shared by one of his former students, the infamous Solimon. He often blurted out inappropriate remarks such as referring to his classmates as "retards" or "idiots."

After unsuccessfully having tried several strategies to teach Solimon the value of respect, Mr. Nolfi was forced to resort to a monetary tactic. He decided to charge Solimon a base fee of \$10.00 ***plus*** \$0.50 per inappropriate comment.

(a) Complete the following table of values.
$n \rightarrow$ number of inappropriate comments
$F \rightarrow$ fee Solimon pays in dollars
$\Delta F \rightarrow$ change in the fee (first difference)

n	F	ΔF
0	10	–
1	10.50	0.50
2	11.00	0.50
3	11.50	0.50
4	12.00	0.50
5	12.50	0.50
6	13.00	0.50
7	13.50	0.50
8	14.00	0.50
9	14.50	0.50
10	15.00	0.50

(b) Graph the relation between the number of inappropriate comments made and the fee that has to be paid. In addition to labelling the axes, set appropriate scales.



(c) The independent variable is n.

The dependent variable is F.

(d) Write an equation that relates the dependent variable to the independent variable. $F=0.5n+10$

(e) Explain in ***three different ways*** why the relation between F and n must be ***linear***.

(i) first differences are constant (ΔF is constant when Δn is constant)

(ii) graph is a straight line

(iii) the equation contains only terms of degree at most 1.

(f) Calculate the ***slope*** of the line that you sketched in part (b). What is the ***meaning*** of the slope? Don't forget the units! How could you determine the slope without using the graph?

(i) Slope $= \frac{\Delta F}{\Delta n} = \frac{5}{10} = 0.5$

(ii) meaning: \$0.50 per inappropriate remark

(iii) slope could be calculated using table of values <u>OR</u> can be seen from the equation (coefficient of n)

(g) Determine the vertical intercept (i.e. y-intercept or initial value). What is the ***meaning*** of the vertical intercept?

vertical intercept = 10

Meaning: the base fee is \$10.00.

(h) How much would Solimon have to pay if he made one inappropriate comment every minute in a single math period?

\# inappropriate remarks $= n = 76$

$F = 0.5(76)+10 = 38+10 = 48$

Solimon would have to pay \$48.00.

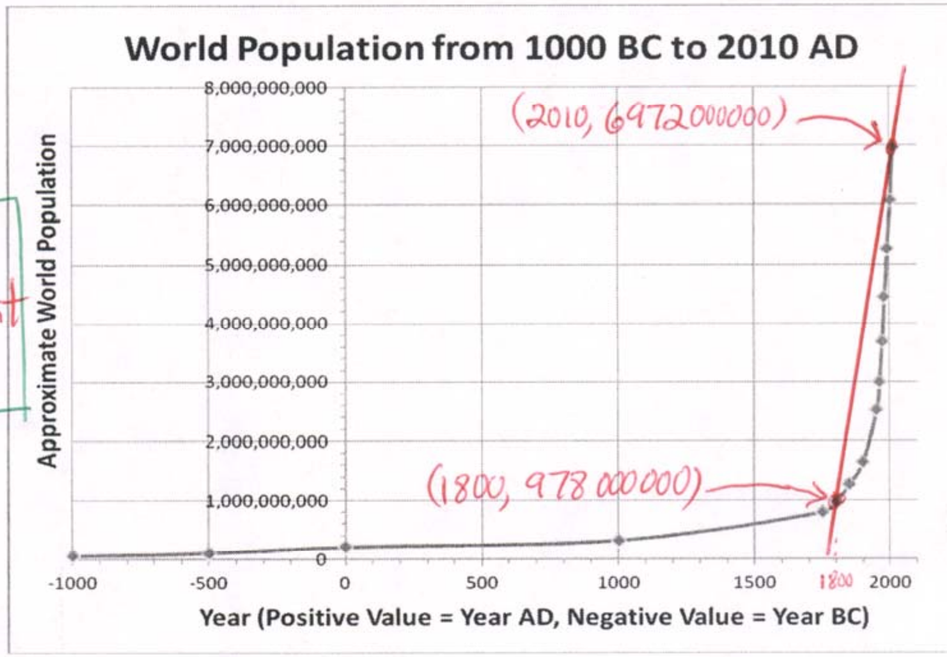
Problem 2 – World Population – A Non-Linear Relation

The table and graph given below show how the world population has changed over the last 3 millennia (3000 years). From both the table and the graph, one can clearly see that the relation between global population and time is ***non-linear***.

Year	*Population*
−1000	50000000
−500	100000000
1	200000000
1000	310000000
1750	791000000
1800	978000000
1850	1262000000
1900	1650000000
1950	2519000000
1960	2982000000
1970	3692000000
1980	4435000000
1990	5263000000
2000	6070000000
2010	6972000000

constant Δt

ΔP not constant



Note

- For graphing convenience, the dividing line between the BC and AD eras is shown as year 0. However, there was no "year 0"in reality. The BC era ended with year 1, which was immediately followed by year 1 in the AD era.
- Some authors refer to the BC ("before Christ") era as BCE ("before the current/common era") and to the AD (*anno domini* or "In the year of the Lord") era as CE ("current/common era").

(a) Calculate $\frac{\Delta P}{\Delta t} = \frac{\text{change in population}}{\text{change in time}}$ from 1800 to 2010.

$$\frac{\Delta P}{\Delta t} = \frac{6972000000 - 978000000}{2010-1800}$$

the change in population

$$= \frac{5994000000}{210} \doteq 28500000$$

the change in time

(b) Interpret your answer from question (a) as a rate of change.

From 1800 to 2010, the population increased at an average rate of about 28500000 people/year.

(c) On the grid given above, sketch a line whose slope equals the value you calculated in question (a). What conclusion(s) can you draw?

The average rate of change of population with respect to time from 1800–2010 is about 28500000 people/year. (See graph for the line)

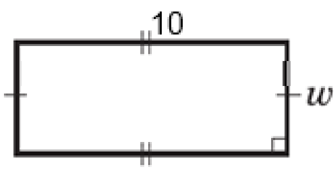
(d) Use the values given in the table to explain why the relation between world population and time must be non-linear. (Be careful! Remember that both Δx and Δy must be constant for a relation to be linear. To show that a relation is non-linear, you must show that for some part of the relation, Δy is ***not constant*** when Δx ***is*** constant.)

From 1750 to 1950, Δt is constant between consecutive t-values. However, the first differences, ΔP, are not constant.

Activity: Complete the following table. The first row is done for you.

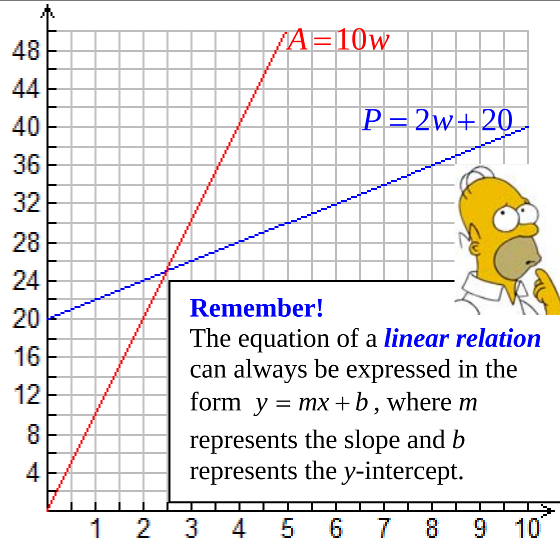
Shape	Perimeter and Area for Given Values	Plot Graphs, State Equations, Answer Questions

Row 1



w	P
2	24
4	28
6	32
8	36
10	40

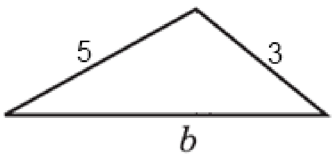
w	A
1	10
2	20
3	30
4	40
5	50



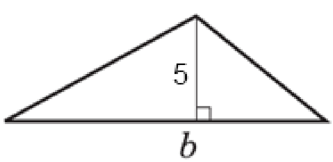
Remember!
The equation of a ***linear relation*** can always be expressed in the form $y = mx + b$, where m represents the slope and b represents the y-intercept.

Slopes: P: 2, A: 10
Vertical Intercepts: P: 20, A: 0
Independent Variable: w
Dependent Variables: P, A
Linear or Non-Linear? Both linear
Direct or Partial Variation? P: partial, A: direct
Constants of Variation: P: 2, A: 10
Initial Values: P: 20, A: 0

Row 2



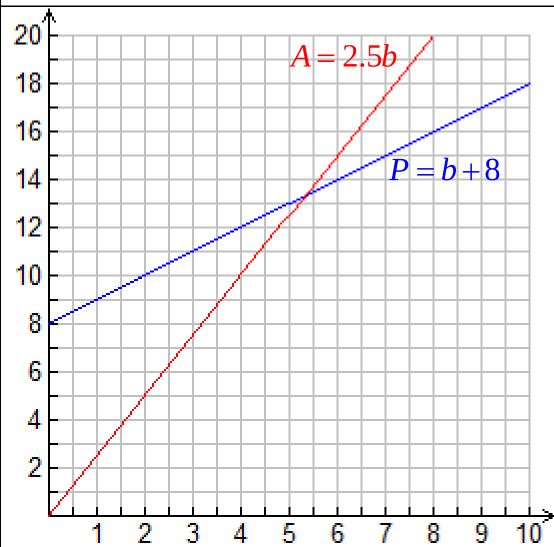
Use this triangle for perimeter calculations.



Use this triangle for area calculations.

b	P
3	11
4	12
5	13
6	14
7	15

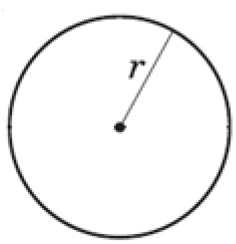
b	A
2	5
4	10
6	15
8	20
10	25



Slopes: P: 1, A: 2.5
Vertical Intercepts: P: 8, A: 0
Independent Variable: b
Dependent Variables: P, A
Linear or Non-Linear? Both linear
Direct or Partial Variation? P: partial, A: direct
Constants of Variation: P: 1, A: 2.5
Initial Values: P: 8, A: 0

Row 3

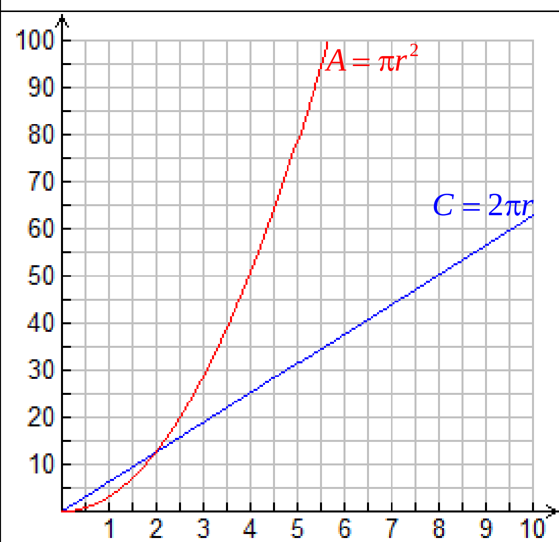
Value of π used: $\pi \doteq 3.14$



Values rounded to one decimal place

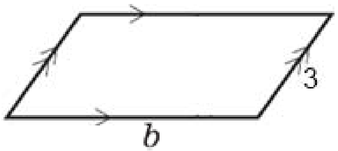
r	C
2	12.6
4	25.1
6	37.7
8	50.2
10	62.8

r	A
2	12.6
4	50.2
6	113.0
8	201.0
10	314.0

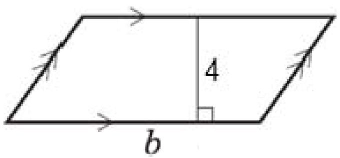


Slopes: P: N/A, A: 2π
Vertical Intercepts: P: 0, A: 0
Independent Variable: r
Dependent Variables: P, A
Linear or Non-Linear? P: linear, A: non-linear
Direct or Partial Variation? P: direct, A: N/A
Constants of Variation: P: N/A, A: 2π
Initial Values: P: 0, A: 0

Shape	Perimeter and Area for Given Values	Plot Graphs, State Equations, Answer Questions



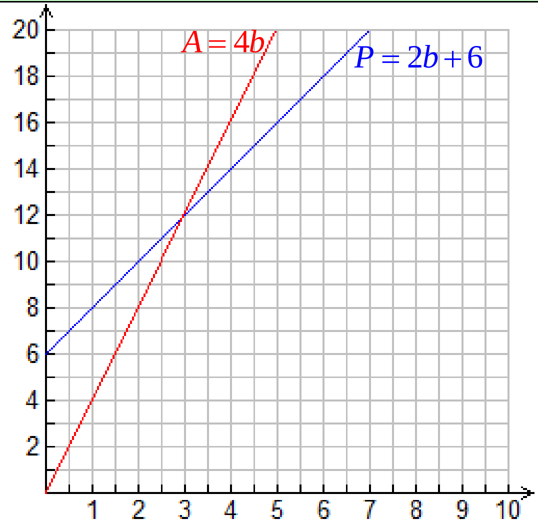
Use this parallelogram for perimeter calculations.



Use this parallelogram for area calculations.

b	P
2	10
4	14
6	18
8	22
10	26

b	A
2	8
4	16
6	24
8	32
10	40



Slopes: P: 2, A: 4

Vertical Intercepts: P: 6, A: 0

Independent Variable: b

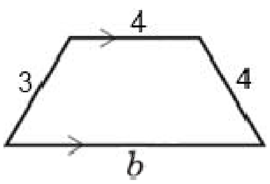
Dependent Variables: P, A

Linear or Non-Linear? Both linear

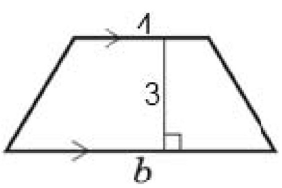
Direct or Partial Variation? P partial, A direct

Constants of Variation: P: 2, A: 4

Initial Values: P: 6, A: 0



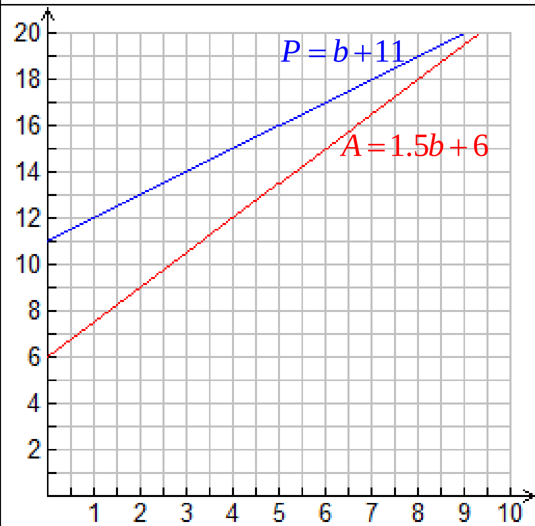
Use this trapezoid for perimeter calculations.



Use this parallelogram for area calculations.

b	P
2	13
4	15
6	17
8	19
10	21

b	A
2	9
4	12
6	15
8	18
10	21



Slopes: P: 1, A: 1.5

Vertical Intercepts: P: 11, A: 6

Independent Variable: b

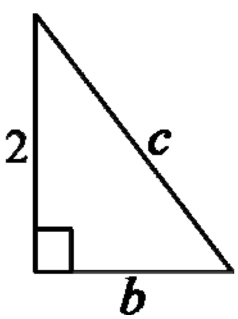
Dependent Variables: P, A

Linear or Non-Linear? Both linear

Direct or Partial Variation? P partial, A partial

Constants of Variation: P: 1, A: 2.5

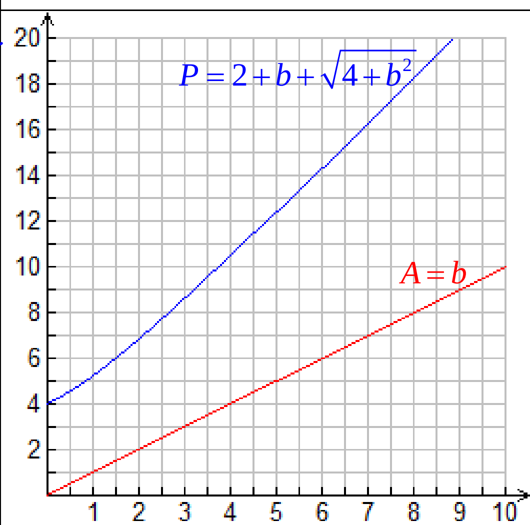
Initial Values: P: 11, A: 6



Values rounded to one decimal place

b	P
2	6.8
4	10.5
6	14.3
8	18.2
10	22.2

b	A
2	2
4	4
6	6
8	8
10	10



Slopes: P: N/A, A: 1

Vertical Intercepts: P: 4, A: 0

Independent Variable: b

Dependent Variables: P, A

Linear or Non-Linear? P: linear, A: non-linear

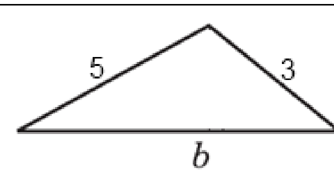
Direct or Partial Variation? P: N/A, A: direct

Constants of Variation: P: N/A, A: 1

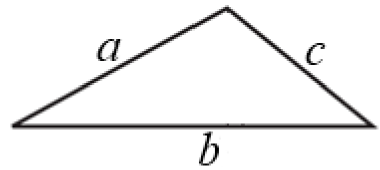
Initial Values: P: 4, A: 0

Question

For the triangle shown at the right, explain why the value of b must be greater than 2 and less than 8. (See answer on next page.)



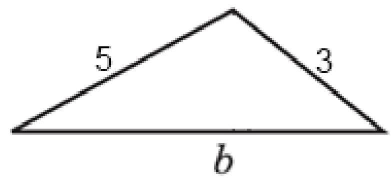
As shown in the diagram, let *a*, *b* and *c* represent the side lengths of any triangle.
Then, it must be the case that



$$a + c > b$$
$$a + b > c$$
and $b + c > a$.

This follows directly from the fact that the shortest path between two points is a straight line.

Then for the given triangle, it must be true that



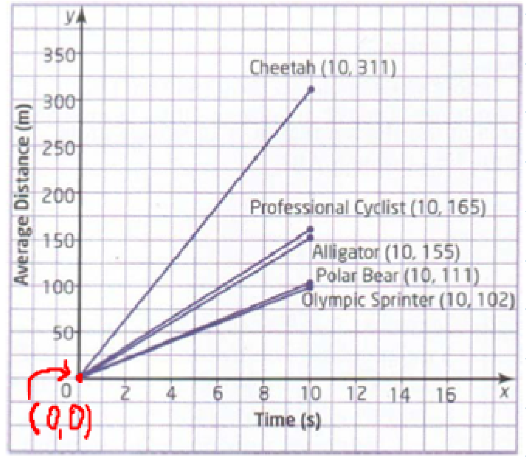
$$5 + 3 > b$$
and $b + 3 > 5$.

Therefore, $8 > b$ and $b > 2$, which means that the value of *b* must be greater than 2 and less than 8.

ANALYTIC GEOMETRY: REVIEW PROBLEMS

1. Consider the graphs shown at the right. Each graph gives a typical example of how ***average distance*** varies over time for a ten-second sprint performed by various animals, an Olympic sprinter and a professional cyclist.

(a) ***Using only the graphs, estimate the slope*** of each line segment. (Do not use the given co-ordinates to obtain your estimate.) Show how you arrived at your estimate. In addition, state the ***average speed*** in each case.



	Cheetah	Cyclist	Alligator	Polar Bear	Sprinter
Estimated Slope	Answers will VARY				
Estimated Average Speed					

(b) Now ***calculate the exact slope*** of each line segment as well as the ***exact average speed***. Show all calculations.

		Cheetah	Cyclist	Alligator	Polar Bear	Sprinter
$\frac{\Delta y}{\Delta x}$	Exact Slope	$\frac{311-0}{10-0}=31.1$	$\frac{165-0}{10-0}=16.5$	$\frac{155-0}{10-0}=15.5$	$\frac{111-0}{10-0}=11.1$	$\frac{102-0}{10-0}=10.2$
$\frac{d}{t}$	Exact Average Speed	$\frac{311\text{ m}}{10\text{ s}}=31.1\text{ m/s}$	$\frac{16.5\text{ m}}{10\text{ s}}=16.5\text{ m/s}$	$\frac{155\text{ m}}{10\text{ s}}=15.5\text{ m/s}$	$\frac{111\text{ m}}{10\text{ s}}=11.1\text{ m/s}$	$\frac{102\text{ m}}{10\text{ s}}=10.2\text{ m/s}$

(c) Let d represent average distance in metres and t represent time in seconds. Write an equation of each line.

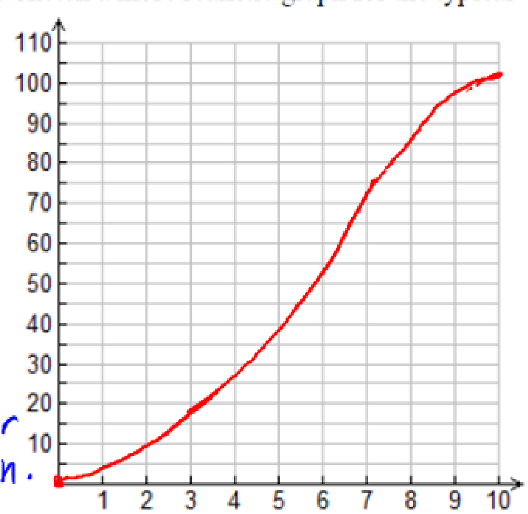
	Cheetah	Cyclist	Alligator	Polar Bear	Sprinter
Equation	$d=31.1t$	$d=16.5t$	$d=15.5t$	$d=11.1t$	$d=10.2t$

(d) Being straight lines, each of the given graphs is of a linear relation. This might suggest to some that the ***speed is constant*** (not average speed) in each case. (Of course, average speed over an interval of time ***must be constant***.)

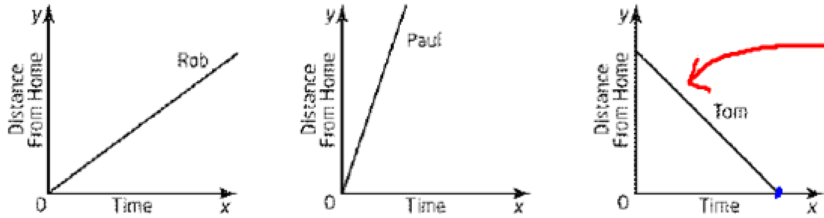
(i) Explain why it is not realistic for the speed to be constant.

It is very difficult to maintain a constant sprinting speed. In the case of the Olympic sprinter for instance, it takes time to accelerate to top speed. As the sprint progresses, the runner tires and gradually slows down.

(ii) Sketch a more realistic graph for the typical Olympic sprinter.



2. Consider the graphs below. Whose graph slopes downward? What does this indicate?



Negative Slope

∴ rate of change of distance from home over time is negative

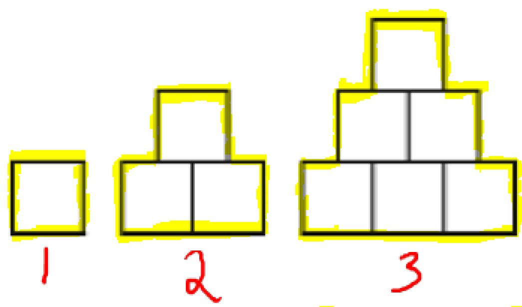
(a) Tom's graph slopes downward. This indicates that his distance from home is decreasing.
(b) Paul's graph slopes downward. This indicates that his distance from home is decreasing.
(c) Rob's graph slopes downward. This indicates that his distance from home is increasing.
(d) Paul's graph slopes downward. This indicates that his distance from home is increasing.

∴ distance from home is decreasing over time

3. Which statement is true?

(a) The constant of variation of a linear relation equals its slope. ✓
(b) $\frac{\Delta y}{\Delta x}$ is equal to the slope of a line as well as the rate of change of the dependent variable y with respect to x. ✓
(c) A linear relation has an equation of the form $y = mx + b$. ✓
(d) All of the above.

4. Consider the pattern shown at the right. Each square has a side length of 1 cm.

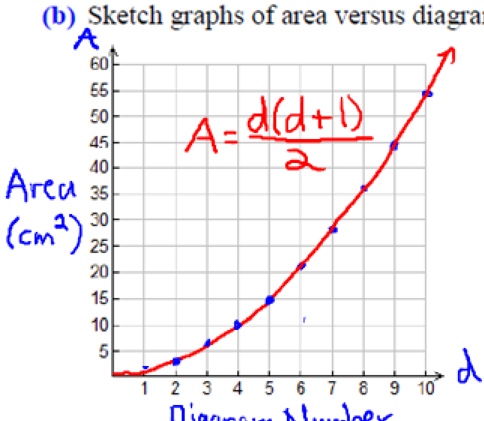


(a) Create a table comparing the diagram number with the area and perimeter for that diagram.

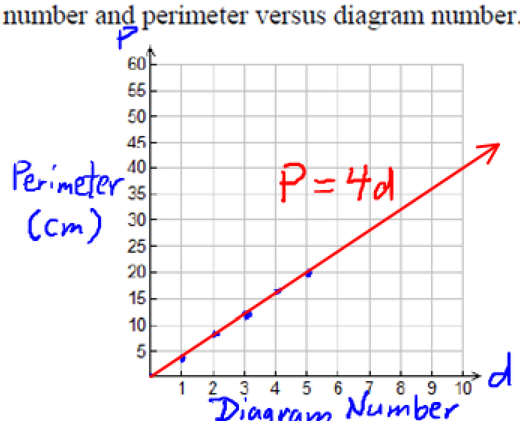
Diagram Number (d)	Area (A)	Perimeter (P)
1	1	4
2	3 ($+2$)	8 ($+4$)
3	6 ($+3$)	12 ($+4$)

- Area increases non-linearly because the first differences are not constant
- Perimeter increases linearly because the first differences are constant.

(b) Sketch graphs of area versus diagram number and perimeter versus diagram number. Is either relation linear?



$A = \frac{d(d+1)}{2}$



$P = 4d$

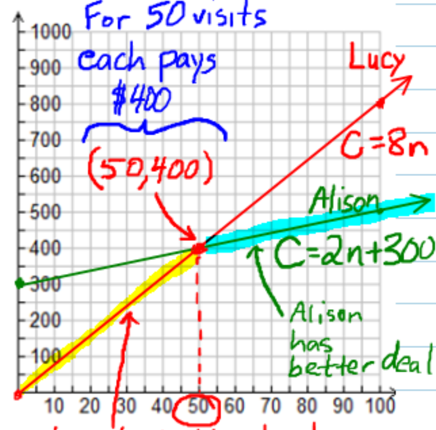
slope
$= \frac{12-8}{3-2}$
$= \frac{4}{1}$
$= 4$

(c) Write equations for both A and P, the dependent variables, in terms of d, the independent variable.

5. Alison and Lucy belong to different fitness clubs. Alison has a membership that cost her \$300 and she pays \$2 each time she visits the club. Lucy has a pay-as-you-go membership and she pays \$8 each time she visits her club.

(a) Let n represent the number of visits to the fitness club and let C represent the total cost in dollars. Write equations for C in terms of n for both Lucy and Alison. In addition, sketch the graph of each relation on a single grid.

	Equation for C in terms of n
Lucy	$C = 8n$
Alison	$C = 2n + 300$



(b) Use your graphs to *estimate* the values of n for which Lucy has a better deal and the values of n for which Alison has a better deal.

Lucy has a better deal up to 50 visits.
Alison has a better deal for more than 50 visits.

(c) Now solve an equation to determine the exact value of n at which Lucy and Alison pay exactly the same amount. Use your solution to determine the values of n for which Lucy has a better deal and the values of n for which Alison has a better deal.

What Lucy pays = What Alison pays

$8n = 2n + 300$

$\therefore 8n - 2n = 2n + 300 - 2n$

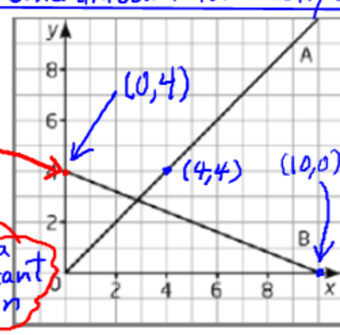
$\therefore 6n = 300$

$\therefore \frac{6n}{6} = \frac{300}{6}$

$\therefore n = 50$

Lucy has a better deal for fewer than 50 visits. Alison has a better deal for more than 50 visits. They pay the same amount for exactly 50 visits.

6. The graph shows two relations, A and B, one direct and one partial variation.



(a) Identify the partial variation. B

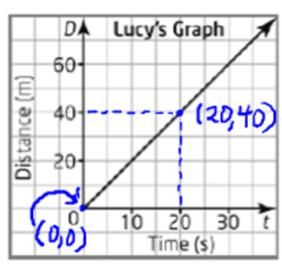
(b) Give the fixed, or initial, value for the partial variation. 4

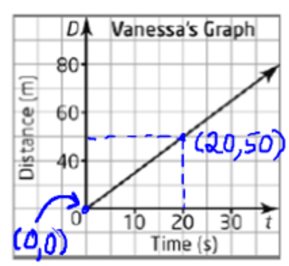
(c) Which relation has a greater constant of variation? same as slope

A: slope $= \frac{4-0}{4-0} = 1$ B: slope $= \frac{0-4}{10-0} = \frac{-4}{10} = -\frac{2}{5}$

Line A has a greater constant of variation

7. Lucy and Vanessa are walking home from school.





(a) How far did each person walk in 20 s?

Lucy: 40 m Vanessa: 50 m

(b) What is the slope of each graph?

Lucy: slope $= \frac{40-0}{20-0} = 2$ Vanessa: slope $= \frac{50-0}{20-0} = 2.5$

(c) Who walked faster? Explain.

Vanessa walked faster (2.5 m/s versus 2 m/s)

(d) Whose graph looks steeper?

Lucy's graph looks steeper. However, as calculated in (b), the slope of Vanessa's graph is larger. This is caused by the different scales on the y-axes. If the scales were the same, Vanessa's graph would look steeper.

8. The length of a trip *varies directly* with the amount of gasoline used. Yael's car used 16 L for the first 145 km of his trip from Toronto to Montreal.

(a) How much gasoline, rounded to the nearest litre, should he expect to use in the remaining 400 km of his trip?

F → amount of fuel used (L)
d → distance travelled (km)
$\frac{16\,L}{145\,km}$ = constant of variation = slope

$\therefore F = \frac{16}{145}d$
If $d = 400$ then
$F = \frac{16}{145}\left(\frac{400}{1}\right) \doteq 44$

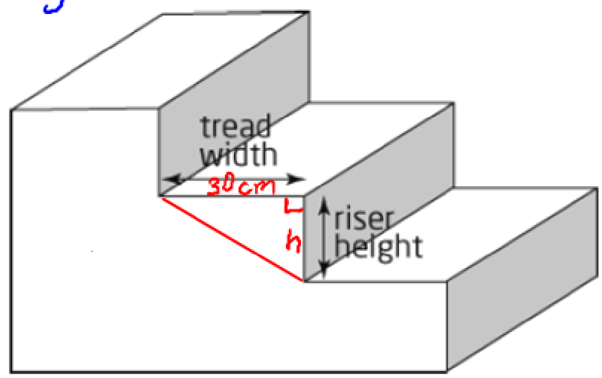
Yael should expect to use about 44 L of fuel.

(b) If gasoline costs \$1.13/L, can he complete the trip with a budget of \$70?

Cost of fuel $\doteq \underbrace{(44+16)}_{\text{Total fuel used}}(\$1.13) = 60(\$1.13) = \$67.80$

Yael can complete the trip with a budget of \$70.00.

9. Jorgen is designing a set of steps from his deck to the garden 2 m below. He knows that a comfortable slope for steps is about 0.6. In addition, he wants the tread width to be 30 cm.



(a) What should the height of each riser be?

Let h represent the riser height in cm.
$\because$ slope $= 0.6$
$\therefore \frac{h}{30} = 0.6$
$\therefore \frac{30}{1}\left(\frac{h}{30}\right) = 30(0.6)$
$\therefore h = 18$

The riser height should be 18 cm.

(b) How many steps will the staircase have? Be sure to give an integer answer and explain the effects of your choice.

$$\# \text{ steps} = \frac{\text{total height}}{\text{riser height}} = \frac{200\text{ cm}}{18\text{ cm}} \doteq 11.1$$

There should be 11 steps. In addition, the riser height should be increased slightly to ensure that that the staircase spans the entire 2 m distance.

10. **Modified True/False**

Indicate whether each statement is true or false. If the statement is false, change the underlined part(s) to make the statement true.

F Partial variation occurs when the ratio of the dependent variable to the independent variable is constant. Change: Direct Variation

F Any linear relation has an equation of the form $y = mx + b$, where m represents the fixed, or initial value of y, and b represents the constant of variation. Change: m represents slope, b represents initial or fixed value

F The vertical intercept, constant of variation and rate of change all represent the same concept for a linear relation. Change: slope

F The following are all units of change: Change: rate of change

kilometres per hour, dollars per kilogram, litres per 100 km, breaths per minute