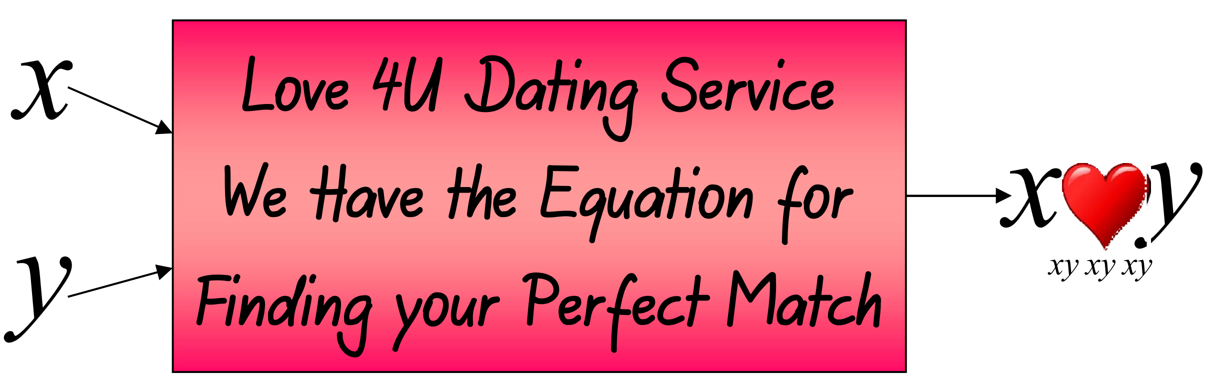
Unit 3 – Relations and Analytic Geometry

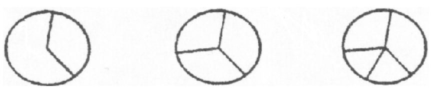
Introduction – Math is like a Dating Service…

Math is Like a Dating Service.

It's all about Relationships!



Analytic Geometry – Opening Activity – Find the Patterns

<table>
<tr><th>Question</th><th>Pattern?</th><th>Explanation</th></tr>
<tr>
<td>1. How many regions are there in the fourteenth diagram?

1 2 3</td>
<td>How is the number of regions (r) related to the diagram number (d)?
<table><tr><th>d</th><th>r</th></tr><tr><td></td><td></td></tr></table></td>
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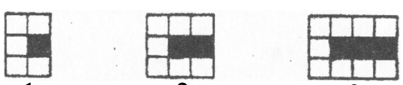
<td>2. How many shaded squares are there in the eighth diagram? How many unshaded squares are there in the eighth diagram?

1 2 3</td>
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<table><tr><th>d</th><th>s</th><th>u</th></tr><tr><td></td><td></td><td></td></tr></table></td>
<td></td>
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<tr>
<td>3. How many “X’s” are in the twentieth diagram? How many “O’s” are there in the twentieth diagram?
<table><tr><td>X</td><td>OX
XO</td><td>OOX
OXO
XOO</td><td>OOOX
OOXO
OXOO
XOOO</td></tr><tr><td>1</td><td>2</td><td>3</td><td>4</td></tr></table></td>
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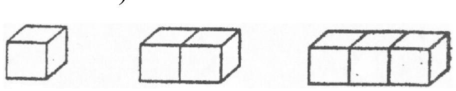
<tr>
<td>4. How many faces are visible in the twentieth diagram? (Include the back and sides.)

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<td>How is the number of visible faces (f) related to the diagram number (d)?
<table><tr><th>d</th><th>f</th></tr><tr><td></td><td></td></tr></table></td>
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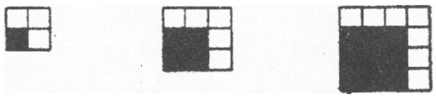
</table>

5. How many shaded squares are there in the twelfth diagram? How many unshaded squares are there in the twelfth diagram?



1 2 3

How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	s	u

6. A cow is milked twice a day. Each time she gives 11 kg of milk. Calculate the total milk production after
(i) 16 days (ii) 49 days

How is the total milk production (m) ***related to*** the time in days (t)?

t	m

7. The sum of the interior angles of each polygon is shown. What is the sum of the interior angles of a decagon (a polygon with 10 sides).

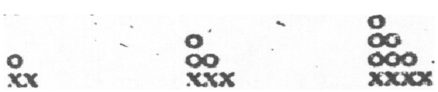
180° 360° 540°

3 4 5

How is the sum of the interior angles (s) ***related to*** the number of sides (n)?

n	s

8. How many "X's" are in the tenth diagram? How many "O's" are there in the tenth diagram?

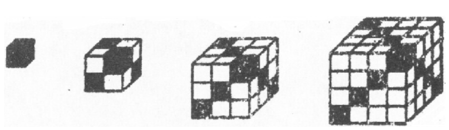


1 2 3

How is the number of "X's" (X) ***related to*** the diagram number (d)? How is the number of "O's" (O) ***related to*** the diagram number (d)?

d	X	O

9. The cubes along one diagonal of each cube of a face are coloured (including the faces that can't be seen). How many cubes are coloured on the fifth diagram?



1 2 3 4

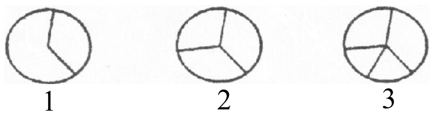
How is the number of coloured cubes (c) ***related to*** the diagram number (d)?

d	c

Analytic Geometry – Solutions to Pattern Finding Activity

Question	***Solutions (Including Equations that Describe the Relationships)***

1. How many regions are there in the fourteenth diagram?



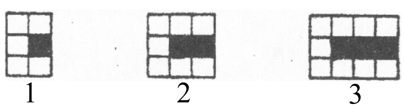
How is the number of regions (r) ***related to*** the diagram number (d)?

d	$r = d + 1$
1	2
2	3
3	4
.	.
.	.
.	.
14	15

From the table we can see that the number of regions is always ***one more than the diagram number***. This ***relationship*** can be described by the following equation:

$$r = d + 1$$

2. How many shaded squares are there in the eighth diagram? How many unshaded squares are there in the eighth diagram?



How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	$s = d$	$u = 2d + 3$
1	1	5
2	2	7
3	3	9
.	.	.
.	.	.
.	.	.
8	8	19

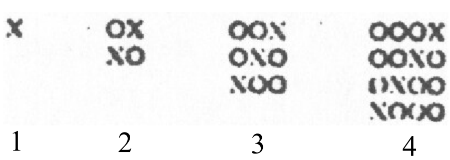
From the table we can see that the number of shaded squares and the number of unshaded squares are ***related to*** the diagram number according to the following equations:

$$s = d$$

(The # of shaded squares is equal to the diagram #.)

$$u = 2d + 3$$

3. How many "X's" are in the twentieth diagram? How many "O's" are there in the twentieth diagram?



How is the number of "X's" (X) ***related to*** the diagram number (d)? How is the number of "O's" (O) ***related to*** the diagram number (d)?

d	$X = d$	$O = d^2 - d$
1	1	0
2	2	2
3	3	6
4	4	12
.	.	.
.	.	.
.	.	.
20	20	380

From the table we can see that the number of "X's" and the number of "O's" are ***related to*** the diagram number according to the following equations:

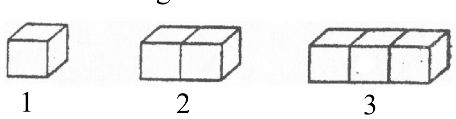
$$X = d$$

$$O = d^2 - d$$

The second equation can also be written as

$$O = d(d-1).$$

4. How many faces are visible in the twentieth diagram?



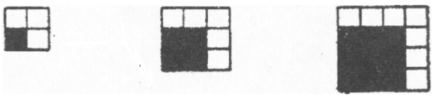
How is the number of visible faces (f) ***related to*** the diagram number (d)?

d	$f = 3d + 2$
1	5
2	8
3	11
.	.
.	.
.	.
20	62

From the table we can see that the number of visible faces is always two more than triple the diagram number. This ***relationship*** can be described by the following equation:

$$f = 3d + 2$$

5. How many shaded squares are there in the twelfth diagram? How many unshaded squares are there in the twelfth diagram?



1 2 3

How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	$s = d^2$	$u = 2d + 1$
1	1	3
2	4	5
3	9	7
⋮	⋮	⋮
12	144	25

From the table we can see that the number of shaded squares and the number of unshaded squares are ***related to*** the diagram number according to the following equations:

$$s = d^2$$

$$u = 2d + 1$$

6. A cow is milked twice a day. Each time she gives 11 kg of milk. Calculate the total milk production after
(i) 16 days (ii) 49 days

How is the total milk production (m) ***related to*** the time in days (t)?

t	$m = 22t$
1	22
2	44
3	66
⋮	⋮
16	352
49	1078

From the table we can see that the total milk production is 22 times the number of days. This ***relationship*** can be described by the following equation:

$$m = 22t$$

7. The sum of the interior angles of each polygon is shown. What is the sum of the interior angles of a decagon (a polygon with 10 sides).

180° 360° 540°

3 4 5

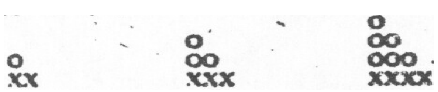
How is the sum of the interior angles (s) ***related to*** the number of sides (n)?

n	$s = 180(n-2)$
3	180
4	360
5	540
⋮	⋮
10	1440

From the table we can see that the sum of the interior angles is the product of 180 and two less than the number of sides. This ***relationship*** can be described by the following equation:

$$s = 180(n-2)$$

8. How many "X's" are in the tenth diagram? How many "O's" are there in the tenth diagram?



1 2 3

How is the number of "X's" (X) ***related to*** the diagram number (d)? How is the number of "O's" (O) ***related to*** the diagram number (d)?

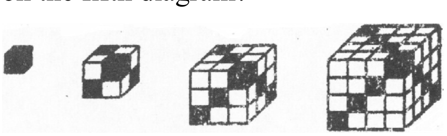
d	$X = d + 1$	$O = \frac{d(d+1)}{2}$
1	2	1
2	3	3
3	4	6
⋮	⋮	⋮
10	11	55

From the table we can see that the number of "X's" and the number of "O's" are ***related to*** the diagram number according to the following equations:

$$X = d + 1$$

$$O = \frac{d(d+1)}{2} = \frac{d^2 + d}{2}$$

9. The cubes along one diagonal of each cube of a face are coloured (including the faces that can't be seen). How many cubes are coloured on the fifth diagram?



1 2 3 4

How is the number of coloured cubes (c) ***related to*** the diagram number (d)?

d	$c = 6(d-2) + 4, d \neq 1$
1	1
2	4
3	10
4	16
5	22

From the table we can see that the number of coloured cubes is six times, 2 less than the diagram number, all increased by four. This ***relationship*** can be described by the following equation:

$$c = 6(d-2) + 4, d \neq 1.$$

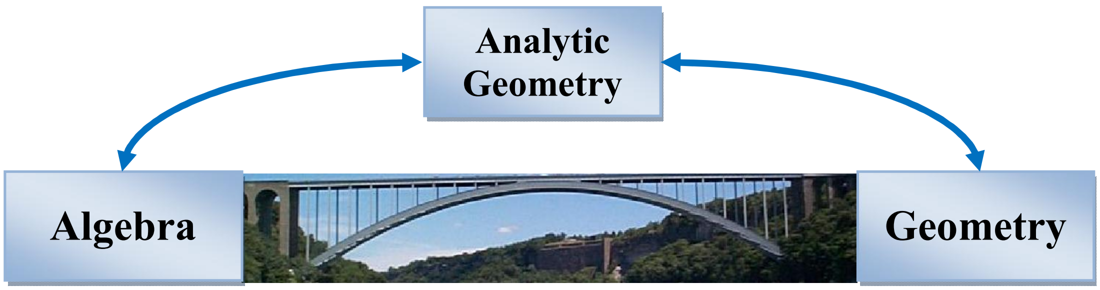
By simplifying, the equation can be written $c = 6d - 8, d \neq 1$. (The equation does not hold for $d = 1$.)

INTRODUCTION TO ANALYTIC GEOMETRY

What is Analytic Geometry?

Analytic Geometry (also known as ***Co-ordinate Geometry*** or ***Cartesian Geometry***) is a branch of mathematics that allows us to bridge the gap between algebra and geometry. As you already know, algebra deals mainly with symbols while geometry deals mainly with pictures. Until the French philosopher René Descartes developed analytic geometry, these two subjects were seen largely as separate mathematical disciplines, islands unto themselves. Thanks in large part to Descartes' brilliant insight, we now have a powerful tool that allows us to draw pictures of algebraic equations as well as better understand the connections between algebraic and geometric ideas.

René Descartes, (31 March 1596 – 11 February 1650), also known as *Renatus Cartesius* (Latinized form), was a French philosopher, mathematician, physicist and writer who spent most of his adult life in the Dutch Republic. He has been dubbed the "Father of Modern Philosophy," and much of subsequent Western philosophy is a response to his writings, which continue to be studied closely to this day. In particular, his *Meditations on First Philosophy* continues to be a standard text at most university philosophy departments. Descartes' influence in mathematics is also apparent, the Cartesian co-ordinate system allowing geometric shapes to be expressed in algebraic equations. As such, he is also recognized as the father of analytic geometry. Descartes was also one of the key figures in the Scientific Revolution.



Analytic Geometry – Key Ideas

Independent Variable

- An ***independent variable*** is a variable whose value can be chosen freely.
- The value of an independent variable never depends on the value of any other variable.

Dependent Variable

- A ***dependent variable*** is a variable whose value ***depends*** on the value of another variable.
- The ***choice*** of the value of the independent variable ***determines*** the value of the dependent variable.

d	$r = d + 1$
1	2
2	3
3	4
⋮	⋮
14	15

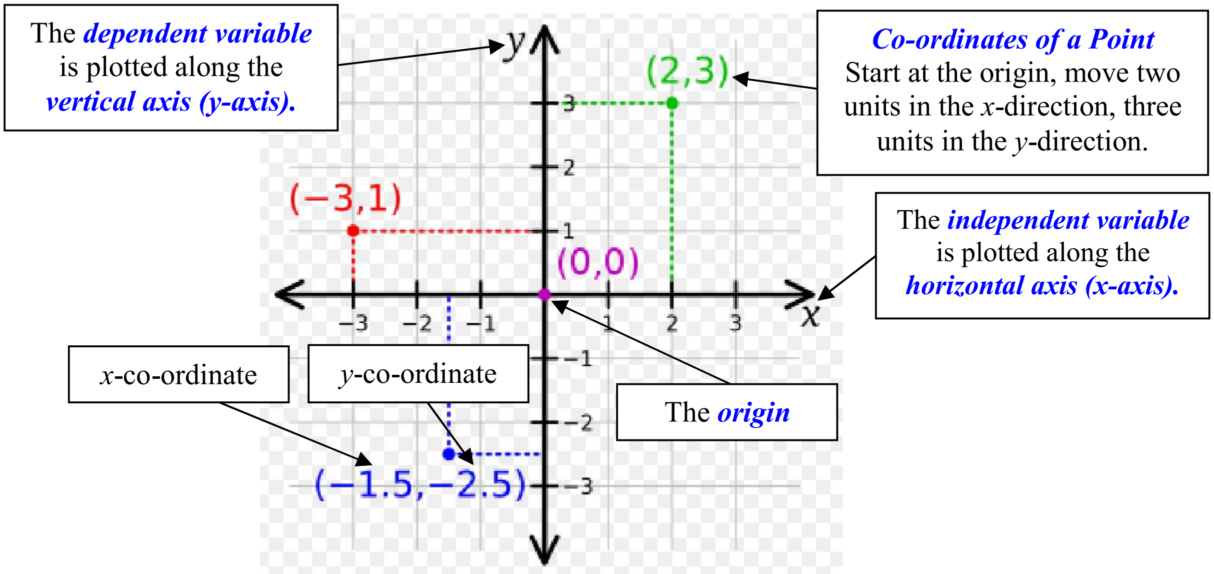
- d Represents the "diagram number"
  It is the ***independent variable*** because the diagram # does not depend on any other value
- r Represents the "number of regions"
  It is the ***dependent variable*** because its value depends on the diagram number.
- $r = d + 1$ This is the equation that describes how r and d are related to each other. The number of regions is equal to one more than the diagram number.

Relation

Any mathematical relationship is called a ***relation***. Often, relations can be described by equations.

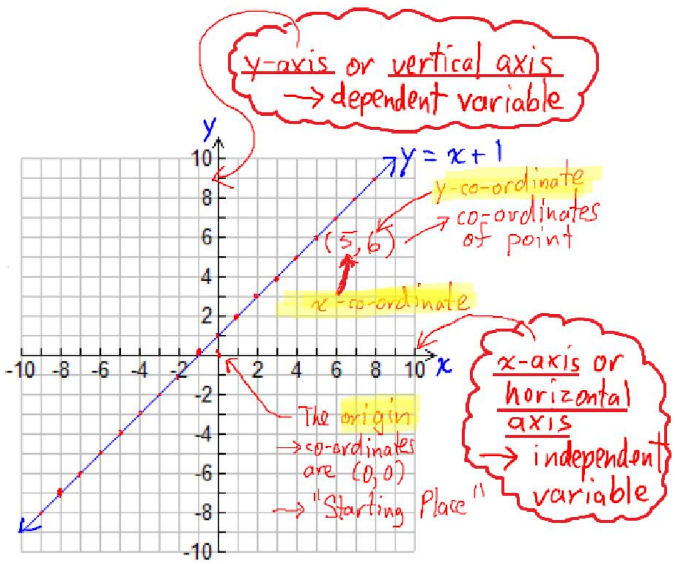
Graphs of Relations – Important Terminology

- x-axis (horizontal axis) (for plotting values of the ***independent variable***)
- y-axis (vertical axis) (for plotting values of the ***dependent variable***)
- Point, ordered pair, co-ordinates of a point, x-co-ordinate, y-co-ordinate
- The origin
- The equation of the relation



Example – Graph of a Linear Relation

- x The independent variable
- y The dependent variable
- $y = x + 1$ The equation that describes how x and y are related.
- The relation is called ***linear*** because its graph is a ***line***.



The independent variable

The dependent variable

x	$y=x+1$	(x,y)
-10	-9	(-10,-9)
-9	-8	(-9,-8)
-8	-7	(-8,-7)
-7	-6	
-6	-5	
-5	-4	
-4	-3	
-3	-2	
-2	-1	
-1	0	
0	1	
1	2	
2	3	
3	4	
4	5	
5	6	
6	7	
7	8	
8	9	
9	10	(9,10)

co-ordinates of points (ordered pairs)

Properties of Linear Relations – Direct and Partial Variation

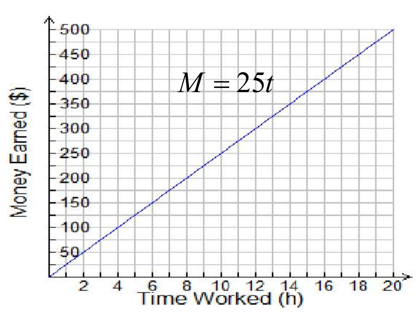
Linear relations come in two varieties, those in which the dependent variable varies ***directly*** with the independent variable and those in which the dependent variable varies ***partially*** with the independent variable.

Example of Direct Variation

Naomi is paid \$25.00/h.

Let t represent time worked in hours and M represent money earned in dollars.

t	M
0	0
5	125
10	250
15	375
20	500
25	625
30	750



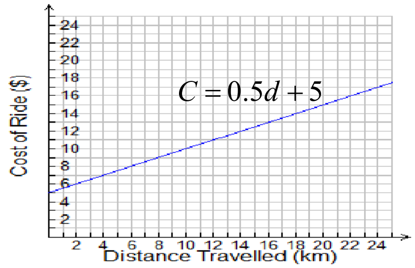
- We say that M varies directly with t.
- For direct variation, the graph passes through the origin.
- For direct variation, $y/x = m$, where m is a constant called the ***constant of variation***.
- In this example, the constant of variation is 25: $M/t = 25$.

Example of Partial Variation

A taxi ride costs \$5.00 (the "flat fee") plus \$0.50 per kilometre.

Let d represent distance in km and C represent cost, in dollars.

d	C
0	5.00
5	7.50
10	10.00
15	12.50
20	15.00
25	17.50



- We say that C ***varies partially*** with d.
- For partial variation, the graph ***does not*** pass through the origin.
- For partial variation, $(y-b)/x = m$, where m is the ***constant of variation*** and b is the ***initial value***.
- In this example, the constant of variation is 0.5 and the initial value is 5: $(C-5)/d = 0.5$.

Note: For a direct variation, the initial value is 0 (i.e. $b = 0$).

Practice – Direct Variation

1. Find the constant of variation for each direct variation.

a) The cost for a long-distance telephone call varies directly with time. A 12-min phone call cost \$0.96.

b) The total mass of magazines varies directly with the number of magazines. The mass of 8 magazines is 3.6 kg.

c) The distance travelled varies directly with time. In 3 h, Alex drove 195 km.

2. The cost, C, in dollars, of wood required to frame a sandbox varies directly with the perimeter, P, in metres, of the sandbox.

a) A sandbox has perimeter 9 m. The wood cost \$20.70. Find the constant of variation for this relationship. What does this represent?

b) Write an equation relating C and P.

c) Use the equation to find the cost of wood for a sandbox with perimeter 15 m.

3. The cost, C, in dollars, to park in a downtown parking lot varies directly with the time, t, in hours. The table shows the cost for different times.

t **(h)**	***C*** **(\$)**
0	0
0.5	1.50
1	3.00
1.5	4.50
2	6.00
2.5	7.50

a) Graph the data in the table.

b) Write the constant of variation for this relationship. What does it represent?

c) Write an equation relating C and t.

4. The distance, d, in kilometres, Kim travels varies directly with the time, t, in hours, she drives. Kim is travelling at 80 km/h.

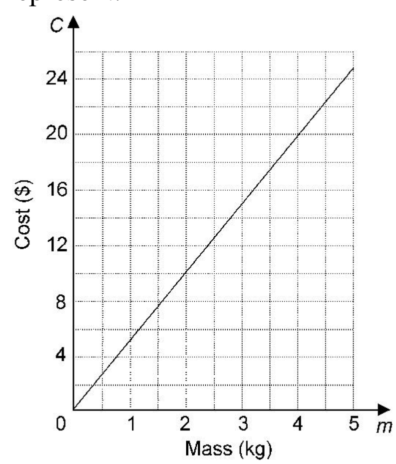
a) Assign letters for variables. Make a table of values to show the distance Kim travelled after 0 h, 1 h, 2 h, and 3 h.

b) Graph the relationship.

c) What is the constant of variation for this relationship?

d) Write an equation in the form $y = kx$.

5. a) Describe a situation this graph could represent.



b) Write an equation for this relationship.

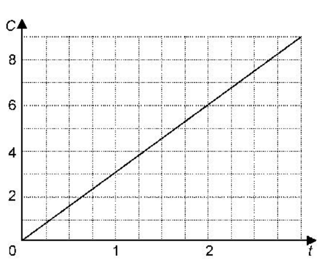
Answers

1. a) 0.08 **b)** 0.45 **c)** 65

2. a) 2.30; the cost per metre, in dollars, of wood

b) $C = 2.3P$ **c)** \$34.50

3. a)



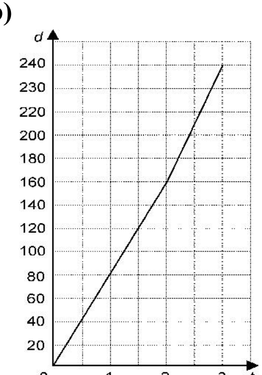
3. b) 3.00; the cost per hour to park in this parking lot

c) $C = 3.00t$

4. a)

t **(h)**	***d*** **(km)**
0	0
1	80
2	160
3	240

4. b)



4. c) 80 **d)** $d = 80t$

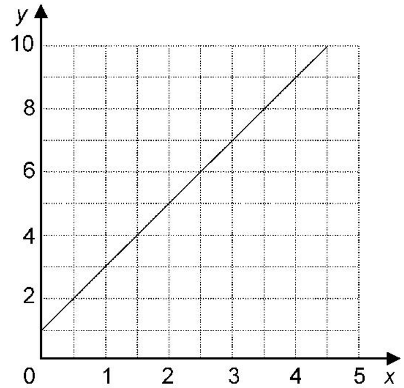
5. a) Tomatoes cost \$5.00 per kg.

b) $C = 5m$

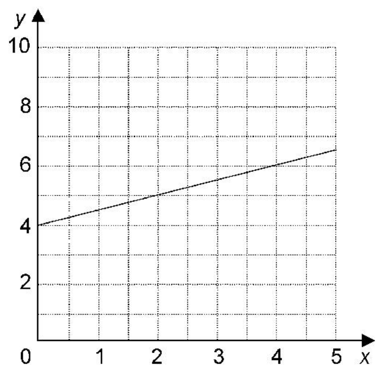
Practice – Partial Variation

1. Identify each relation as a direct variation or a partial variation.

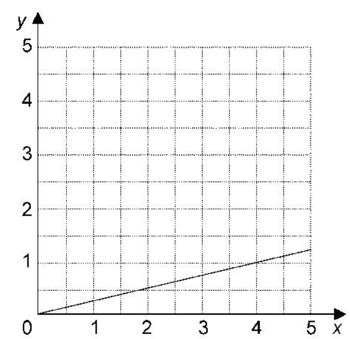
   a)



   b)



   c)



2. Identify each relation as a direct variation or a partial variation.

   a) $y = 3x + 2$ **b)** $y = 2x$
   c) $C = 0.65n$ **d)** $h = 5t + 2$

3. The relationship in the table is a partial variation.

x	*y*
0	3
1	4
2	5
3	6
4	7

   a) Use the table to identify the initial value of *y* and the constant of variation.
   b) Write an equation in the form $y = mx + b$.
   c) Graph the relation. Describe the graph.

4. Latoya is a sales representative. She earns a weekly salary of $240 plus 15% commission on her sales.

   a) Copy and complete the table of values.

Sales ($)	Earnings ($)
0	
100	
200	
300	
400	
500	

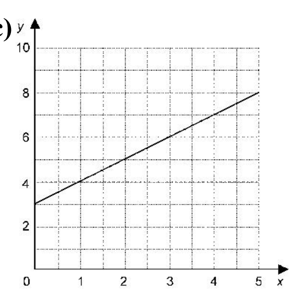
   b) Identify the initial value and the constant of variation.
   c) Write an equation relating Latoya's earnings, *E*, and her sales, *S*.
   d) Graph the relation.

Answers

1. **a)** partial variation
   b) partial variation
   c) direct variation

2. **a)** partial variation
   b) direct variation
   c) direct variation
   d) partial variation

3. **a)** 3; 1
   b) $y = x + 3$
   c)



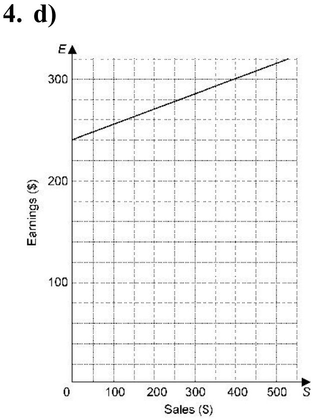
   The graph intersects the *y*-axis at (0, 3). As the *x*-values increase by 1, the *y*-values also increase by 1.

4. **a)**

Sales ($)	Earnings ($)
0	240
100	255
200	270
300	285
400	300
500	315

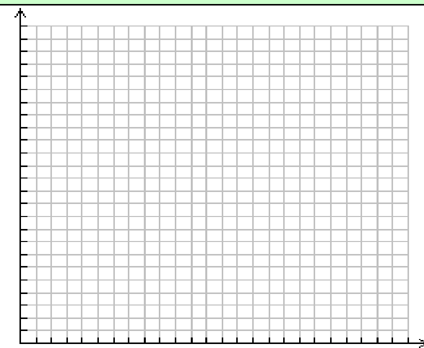
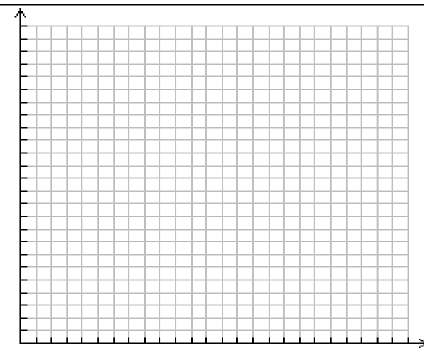
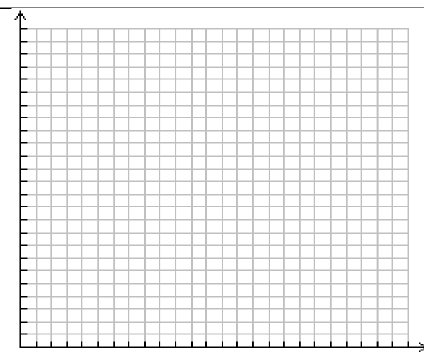
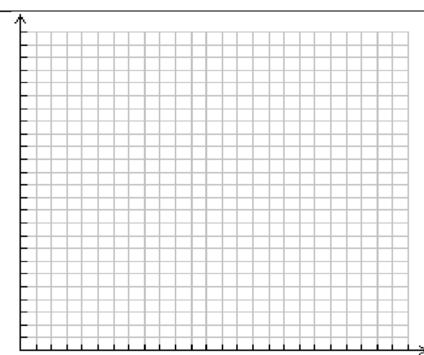
   b) 240; 0.15
   c) $E = 0.15S + 240$

4. **d)**



DIRECT VARIATION, PARTIAL VARIATION OR NEITHER?

Complete the following table.

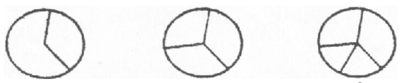
Situation	***Type of Variation (Circle One)***	***Table of Values***	***Initial Value (b) and Constant of Variation (m)***	***Graph and Equation***
Gasoline at GasAttack costs \$1.20/L. How does the ***cost*** of gasoline vary with the ***volume*** of gasoline purchased?	Partial / Direct / Neither	V (L) | C (\$)	b = ________ m = ________	
Sam the electrician charges a base fee of \$30 plus \$50/h. How does Sam's ***pay*** vary with the ***time*** worked?	Partial / Direct / Neither	t (h) | P (\$)	b = ________ m = ________	
Abdul the salesperson is paid a base salary of \$30,000 plus 5% of sales. How does Abdul's ***pay*** vary with the amount of ***sales***?	Partial / Direct / Neither	s (\$) | P (\$)	b = ________ m = ________	
Simran likes bungee jumping. Whenever she jumps, her speed increases at a rate of 10 m/s. How does the ***distance*** fallen vary with ***time***?	Partial / Direct / Neither	t (s) | d (m)	b = ________ m = ________	

DEEPER ANALYSIS OF RELATIONS

Victim: ______________________________

Question / *Solutions* / *Questions*

1. How many regions are there in the fourteenth diagram?



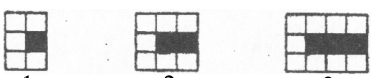
1 2 3

How is the number of regions (r) ***related to*** the diagram number (d)?

d	$r = d + 1$
1	2
2	3
3	4
.	.
.	.
.	.
14	15

Equation of Relation	$r = d + 1$
Independent Variable	d
Dependent Variable	r
Linear or Non-Linear?	linear
If Linear, Partial or Direct Variation	Partial
If Linear, Constant of Variation	1
Initial Value	1

2. How many shaded squares are there in the eighth diagram? How many unshaded squares are there in the eighth diagram?



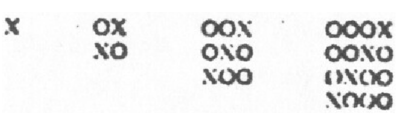
1 2 3

How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	$s = d$	$u = 2d + 3$
1	1	5
2	2	7
3	3	9
.	.	.
.	.	.
.	.	.
8	8	19

Equation of Relation	$s = d$	$u = 2d + 3$
Independent Variable		
Dependent Variable		
Linear or Non-Linear?		
If Linear, Partial or Direct Variation		
If Linear, Constant of Variation		
Initial Value		

3. How many "X's" are in the twentieth diagram? How many "O's" are there in the twentieth diagram?



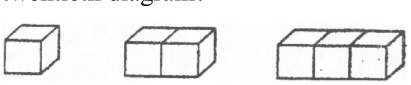
1 2 3 4

How is the number of "X's" (X) ***related to*** the diagram number (d)? How is the number of "O's" (O) ***related to*** the diagram number (d)?

d	$X = d$	$O = d^2 - d$
1	1	0
2	2	2
3	3	6
4	4	12
.	.	.
.	.	.
.	.	.
20	20	380

Equation of Relation	$X = d$	$O = d^2 - d$
Independent Variable		
Dependent Variable		
Linear or Non-Linear?		
If Linear, Partial or Direct Variation		
If Linear, Constant of Variation		
Initial Value		

4. How many faces are visible in the twentieth diagram?



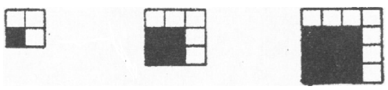
1 2 3

How is the number of visible faces (f) ***related to*** the diagram number (d)?

d	$f = 3d + 2$
1	5
2	8
3	11
.	.
.	.
.	.
20	62

Equation of Relation	$f = 3d + 2$
Independent Variable	
Dependent Variable	
Linear or Non-Linear?	
If Linear, Partial or Direct Variation	
If Linear, Constant of Variation	
Initial Value	

5. How many shaded squares are there in the twelfth diagram? How many unshaded squares are there in the twelfth diagram?



1 2 3

How is the number of shaded squares (s) ***related to*** the diagram number (d)?
How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	$s=d^2$	$u=2d+1$
1	1	3
2	4	5
3	9	7
.	.	.
.	.	.
.	.	.
12	144	25

Equation of Relation	$s=d^2$	$u=2d+1$
Independent Variable		
Dependent Variable		
Linear or Non-Linear?		
If Linear, Partial or Direct Variation		
If Linear, Constant of Variation		
Initial Value		

6. A cow is milked twice a day. Each time she gives 11 kg of milk. Calculate the total milk production after
(i) 16 days (ii) 49 days

How is the total milk production (m) ***related to*** the time in days (t)?

t	$m=22t$
1	22
2	44
3	66
.	.
.	.
.	.
16	352
49	1078

Equation of Relation	$m=22t$
Independent Variable	
Dependent Variable	
Linear or Non-Linear?	
If Linear, Partial or Direct Variation	
If Linear, Constant of Variation	
Initial Value	

7. The sum of the interior angles of each polygon is shown. What is the sum of the interior angles of a decagon (a polygon with 10 sides).

180° 360° 540°

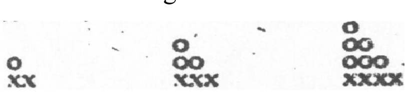
3 4 5

How is the sum of the interior angles (s) ***related to*** the number of sides (n)?

n	$s=180(n-2)$
3	180
4	360
5	540
.	.
.	.
.	.
10	1440

Equation of Relation	$s=180(n-2)$
Independent Variable	
Dependent Variable	
Linear or Non-Linear?	
If Linear, Partial or Direct Variation	
If Linear, Constant of Variation	
Initial Value	

8. How many "X's" are in the tenth diagram? How many "O's" are there in the tenth diagram?



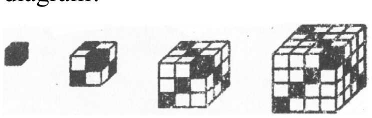
1 2 3

How is the number of "X's" (X) ***related to*** the diagram number (d)?
How is the number of "O's" (O) ***related to*** the diagram number (d)?

d	$X=d+1$	$O=\frac{d(d+1)}{2}$
1	2	1
2	3	3
3	4	6
.	.	.
.	.	.
.	.	.
10	11	55

Equation of Relation	$X=d+1$	$O=\frac{d(d+1)}{2}$
Independent Variable		
Dependent Variable		
Linear or Non-Linear?		
If Linear, Partial or Direct Variation		
If Linear, Constant of Variation		
Initial Value		

9. The cubes along one diagonal of each cube of a face are coloured (including the faces that can't be seen). How many cubes are coloured on the fifth diagram?



1 2 3 4

How is the number of coloured cubes (c) ***related to*** the diagram number (d)?

d	$c=6(d-2)+4, d\neq 1$
1	1
2	4
3	10
4	16
5	22

Equation of Relation	$c=6(d-2)+4, d\neq 1$
Independent Variable	
Dependent Variable	
Linear or Non-Linear?	
If Linear, Partial or Direct Variation	
If Linear, Constant of Variation	
Initial Value	

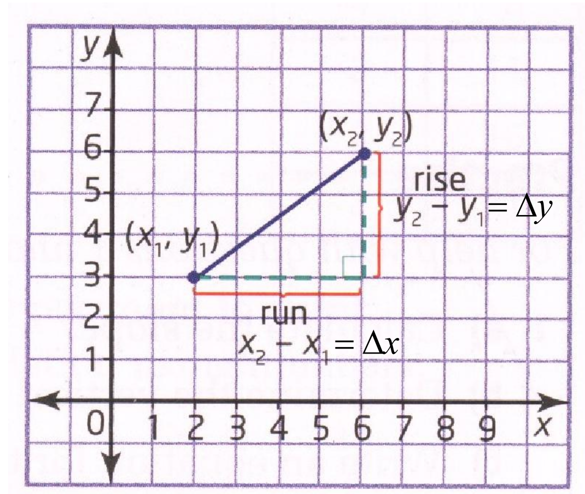
UNDERSTANDING SLOPE

Meaning and Calculation of Slope

- ***Slope*** measures the ***steepness*** of a line (i.e. how "slanted" it is). The steeper a line is, the greater its slope.
- ***Slope*** is the ***ratio*** of the ***rise*** to the ***run***.

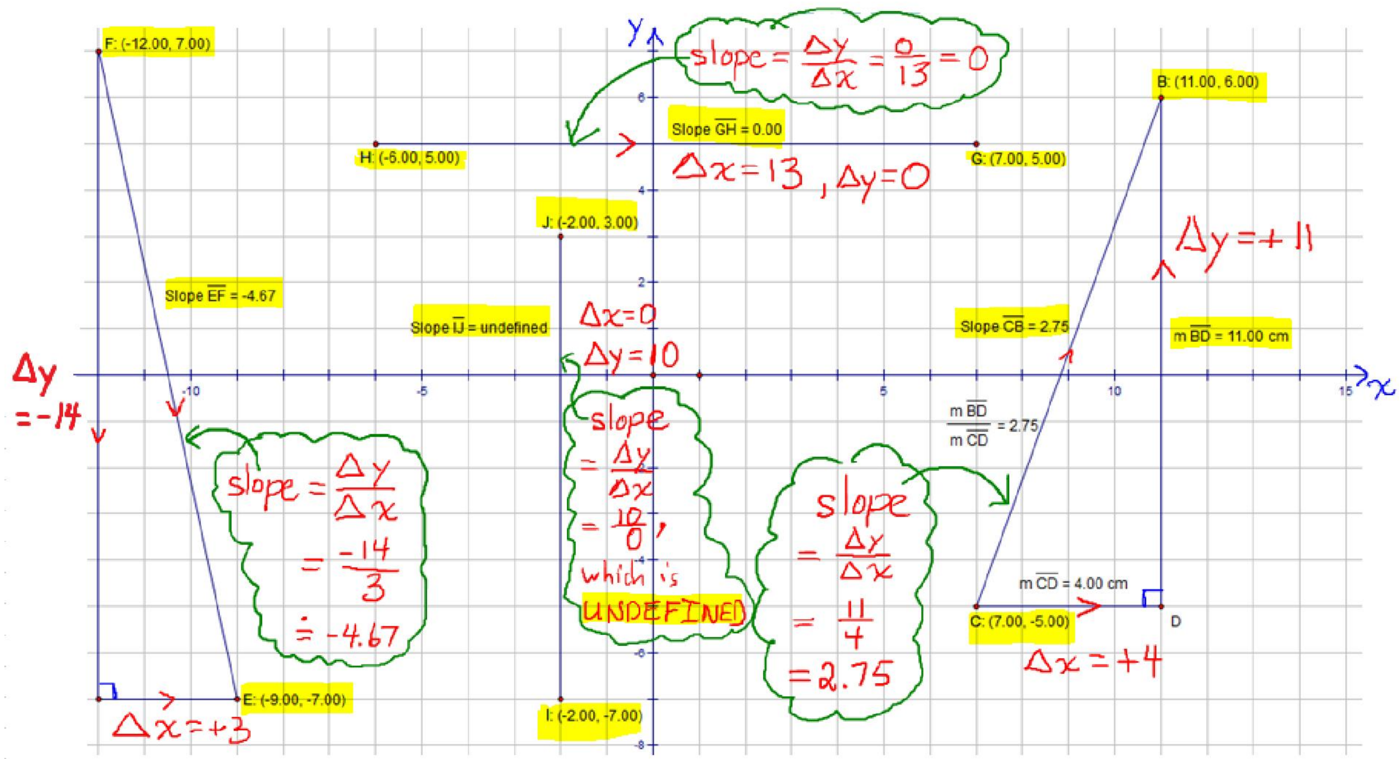
$$\text{slope} = \frac{\text{rise}}{\text{run}}$$
$$= \frac{\text{change in dependent variable}}{\text{change in independent variable}}$$
$$= \frac{\Delta y}{\Delta x}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{\text{difference of the } y\text{-co-ordinates}}{\text{difference of the } x\text{-co-ordinates}}$$

> Δ → The Greek letter "delta" (uppercase)
> → Used in math and science to represent "the change in"



- Slope can be positive, negative, zero or undefined (see below).

Examples



Summary

Horizontal Line

- ***Zero*** Slope (like flat terrain or flat roof → no slope)
- The dependent variable remains constant as the independent variable increases.

Vertical Line

- ***Undefined*** Slope (like vertical climb → infinitely steep)
- The dependent variable takes on all possible values while the independent variable remains constant.

Line Leaning to the Right (Goes Upward to the Right)

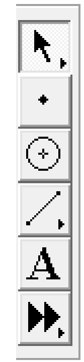
- ***Positive*** Slope (Δx and Δy have ***the same*** sign)
- The dependent variable increases as the independent variable increases.

Line Leaning to the Left (Goes Downward to the Right)

- ***Negative*** Slope (Δx and Δy have ***opposite*** sign)
- The dependent variable decreases as the independent variable increases.

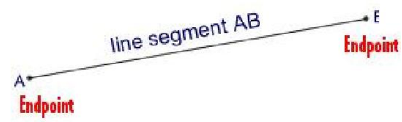
Investigating Slopes

As described in class, open "The Geometer's Sketchpad" (GSP)

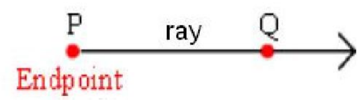


- The ***Toolbox*** appears on the left of the screen when you start Sketchpad, and includes six tools.
- The ***Selection Arrow*** tools: Use this tool to select and drag objects in your sketch. The three variations of the tool allow you to drag-translate (move), drag-rotate (turn), and drag-dilate (shrink or grow) objects.
- The ***Point Tool***: Use this tool to construct points.
- The ***Compass*** Tool: Use this tool to construct circles.
- The ***Straightedge Tool***: Use this tool to construct straight objects. The three variations of the tool allow you to construct ***segments***, ***rays***, and ***lines***.
- The ***Text Tool***: Use this tool to create and edit text and labels.
- The ***Custom Tools*** icon: Use this icon to define, use, and manage custom tools

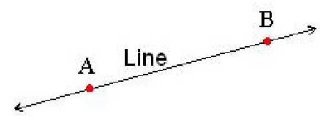
Instructions



1. Open the "Graph" menu and select "Show Grid."
2. Open the "Graph" menu and select "Snap Points."
3. Open the "Edit" menu, select "Preferences" and then the "Text" tab. ***Uncheck*** "For all new points."
4. Label the red points on the x-axis. Label the point at the origin A and the other B.



5. Open the "Graph" menu and choose "Plot Points…"
6. Plot a point with x-co-ordinate 1 and y-co-ordinate 6. (The point should be automatically labelled C)



7. Points A and C should be highlighted. (Only these two points should be highlighted!!!) Open the "Construct" menu and select segment.
   <u>*You have constructed a line segment with endpoints A and C!*</u>
8. Open the "Measure" menu and choose slope. (You can only measure the slope of a line that is highlighted!)

   Slope of AC is ______________________

Now experiment with several different lines. Complete the following table:

Co-ordinates of Endpoints of Line Segment	***Slope as Measured by GSP (Correct to 6 Decimal Places)***	***Slope as Measured by You (Expressed as a Fraction)***

Carefully observe your results. What is the relationship between the slope and the co-ordinates of the endpoints of the lines? Can you draw any conclusions?

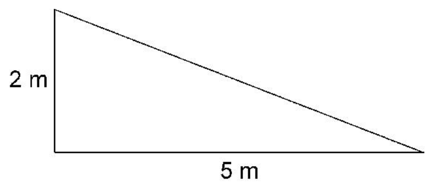
What to Hand In

Hand in your work. Save your file in your personal folder on the "I:" drive. Also save a copy in your "G:" drive.

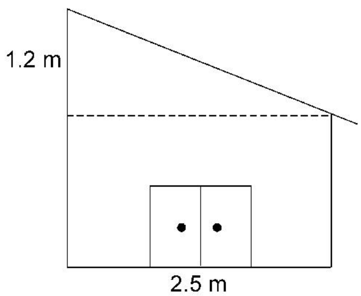
Practice – Slope

1. Find the slope of the "slanted" line segment in each object.

a)

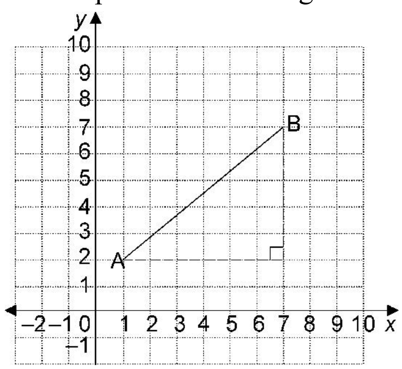


b)

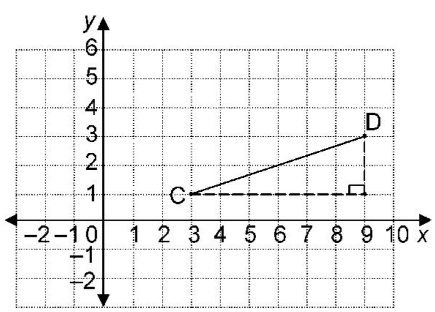


2. Find the slope of each line segment.

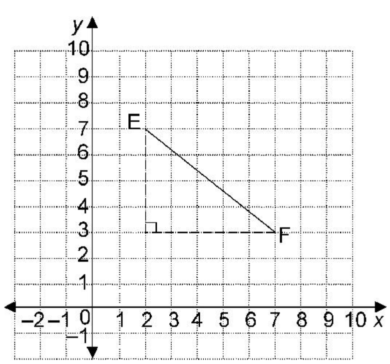
a)



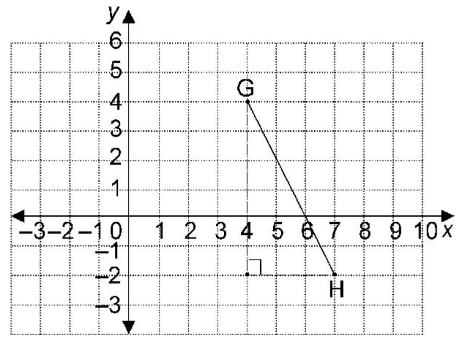
b)



c)

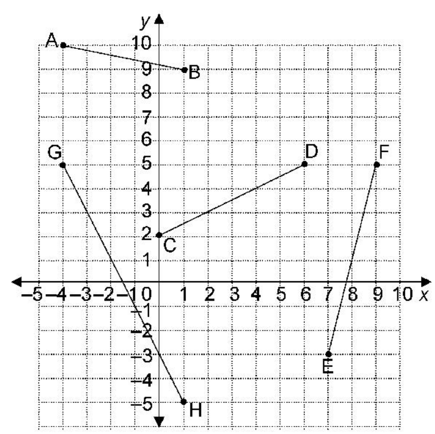


d)



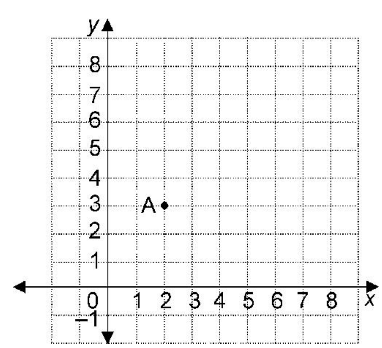
3. For safety, the slope of a staircase must be greater than 0.58 and less than 0.70. A staircase has a vertical rise of 2.4 m over a horizontal run of 3.5 m.
   a) Find the slope of the staircase.
   b) Is the staircase safe?

4. Find the slope of each line segment.



a) AB **b)** CD **c)** EF **d)** GH

5. Point $A(2, 3)$ is plotted on the grid. Draw a line segment AB with slope $-\frac{1}{2}$. What are possible coordinates of B?



Answers

1. **a)** $-\frac{2}{5}$ **b)** -0.48 **2.** **a)** $\frac{5}{6}$ **b)** $\frac{1}{3}$ **c)** $-\frac{4}{5}$ **d)** -2 **4.** **a)** $-\frac{1}{5}$ **b)** $\frac{1}{2}$ **c)** 4 **d)** -2

3. **a)** 0.69 **b)** Yes **5.** Answers will vary. Possible answer: $B(6, 1)$

Opening Activity Revisited

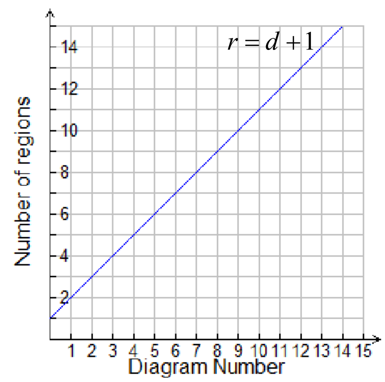
Question	*Table of Values*	*Graph and its Properties*

1. How many regions are there in the fourteenth diagram?

1 2 3

How is the number of regions (r) ***related to*** the diagram number (d)?

d	$r = d + 1$
1	2
2	3
3	4
.	.
.	.
.	.
14	15



Slope: ________________

Vertical Intercept: _______

Independent Variable: ____

Dependent Variable: _____

Linear or Non-Linear? ____

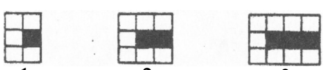
Direct or Partial Variation?

____

Constant of Variation _____

Initial Value ___________

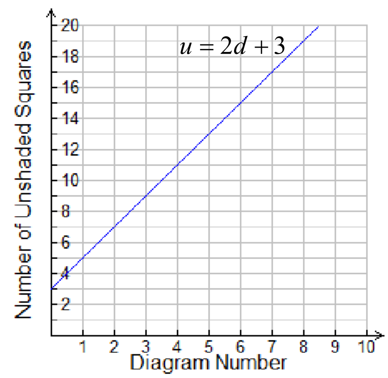
2. How many shaded squares are there in the eighth diagram? How many unshaded squares are there in the eighth diagram?



1 2 3

How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	$s = d$	$u = 2d + 3$
1	1	5
2	2	7
3	3	9
.	.	.
.	.	.
.	.	.
8	8	19



Slope: ________________

Vertical Intercept: _______

Independent Variable: ____

Dependent Variable: _____

Linear or Non-Linear? ____

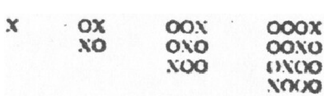
Direct or Partial Variation?

____

Constant of Variation _____

Initial Value ___________

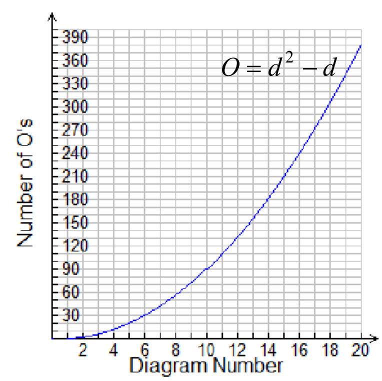
3. How many "X's" are in the twentieth diagram? How many "O's" are there in the twentieth diagram?



1 2 3 4

How is the number of "X's" (X) ***related to*** the diagram number (d)? How is the number of "O's" (O) ***related to*** the diagram number (d)?

d	$X = d$	$O = d^2 - d$
1	1	0
2	2	2
3	3	6
4	4	12
.	.	.
.	.	.
.	.	.
20	20	380



Slope: ________________

Vertical Intercept: _______

Independent Variable: ____

Dependent Variable: _____

Linear or Non-Linear? ____

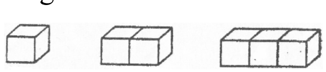
Direct or Partial Variation?

____

Constant of Variation _____

Initial Value ___________

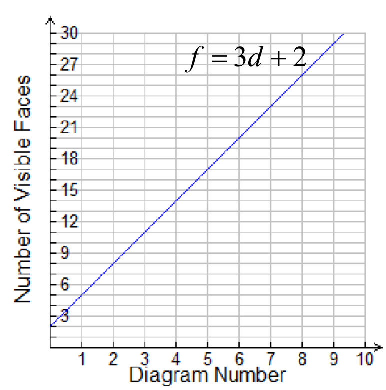
4. How many faces are visible in the twentieth diagram?



1 2 3

How is the number of visible faces (f) ***related to*** the diagram number (d)?

d	$f = 3d + 2$
1	5
2	8
3	11
.	.
.	.
.	.
20	62



Slope: ________________

Vertical Intercept: _______

Independent Variable: ____

Dependent Variable: _____

Linear or Non-Linear? ____

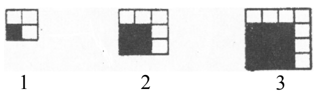
Direct or Partial Variation?

____

Constant of Variation _____

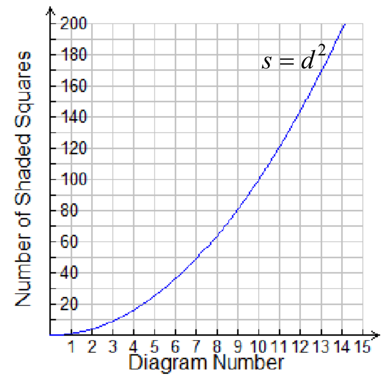
Initial Value ___________

5. How many shaded squares are there in the twelfth diagram? How many unshaded squares are there in the twelfth diagram?



How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	$s=d^2$	$u=2d+1$
1	1	3
2	4	5
.	.	.
.	.	.
.	.	.
12	144	25



Slope: ____________

Vertical Intercept: ________

Independent Variable: _____

Dependent Variable: ______

Linear or Non-Linear? _____

Direct or Partial Variation?

____

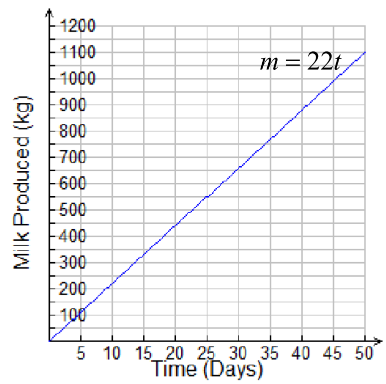
Constant of Variation ______

Initial Value ____________

6. A cow is milked twice a day. Each time she gives 11 kg of milk. Calculate the total milk production after (i) 16 days (ii) 49 days

How is the total milk production (m) ***related to*** the time in days (t)?

t	$m=22t$
1	22
2	44
.	.
.	.
.	.
16	352
49	1078



Slope: ____________

Vertical Intercept: ________

Independent Variable: _____

Dependent Variable: ______

Linear or Non-Linear? _____

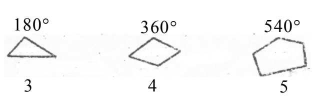
Direct or Partial Variation?

____

Constant of Variation ______

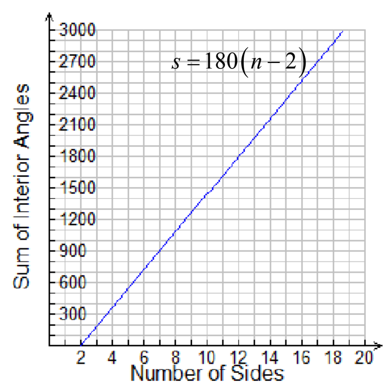
Initial Value ____________

7. The sum of the interior angles of each polygon is shown. What is the sum of the interior angles of a decagon (a polygon with 10 sides).



How is the sum of the interior angles (s) ***related to*** the number of sides (n)?

n	$s=180(n-2)$
3	180
4	360
.	.
.	.
.	.
10	1440



Slope: ____________

Vertical Intercept: ________

Independent Variable: _____

Dependent Variable: ______

Linear or Non-Linear? _____

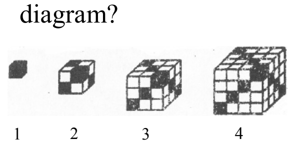
Direct or Partial Variation?

____

Constant of Variation ______

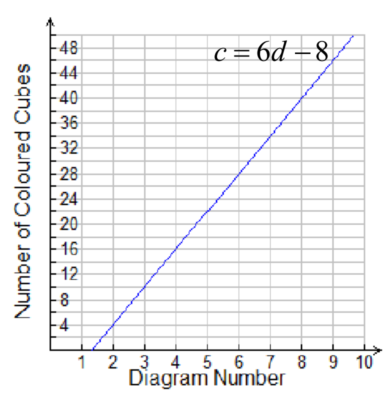
Initial Value ____________

8. The cubes along one diagonal of each cube of a face are coloured (including the faces that can't be seen). How many cubes are coloured on the fifth diagram?



How is the number of coloured cubes (c) ***related to*** the diagram number (d)?

d	$c=6(d-2)+4, d\neq 1$
1	1
2	4
3	10
4	16



Slope: ____________

Vertical Intercept: ________

Independent Variable: _____

Dependent Variable: ______

Linear or Non-Linear? _____

Direct or Partial Variation?

____

Constant of Variation ______

Initial Value ____________

Drawing Conclusions

Complete the following table.

Equation	*Linear or Non-Linear*	*Slope (If Linear)*	*Vertical Intercept (y-intercept)*	*Partial or Direct Variation (If Linear)*	*Constant of Variation (If Linear)*	*Initial Value*
1. $r = d + 1$						
2. $u = 2d + 3$						
3. $O = d^2 - d$						
4. $f = 3d + 2$						
5. $s = d^2$						
6. $m = 22t$						
7. $s = 180n - 360$						
8. $c = 6d - 8$						
9. $O = \frac{1}{2}d(d+1) = \frac{1}{2}d^2 + \frac{1}{2}d$						

Observations

1. Describe how you can use the ***equation of a relation*** to determine

(a) whether it is ***linear*** or ***non-linear***

(b) the ***slope***, if the relation is linear

(c) the ***vertical intercept*** (initial value)

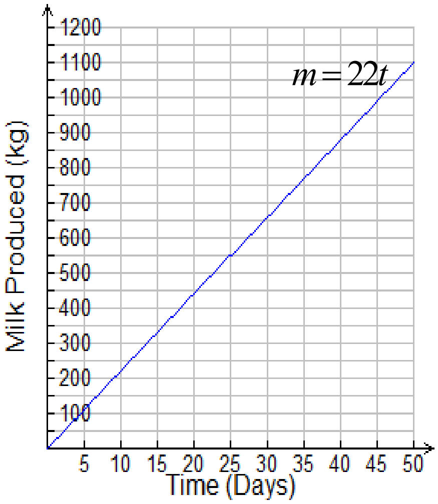
2. Equations 7 and 8 were originally written as $s = 180(n-2)$ and $c = 6(d-2)+4$. Show how each of these were simplified to produce the equivalent forms $s = 180n - 360$ and $c = 6d - 8$.

3. What is the connection between slope and the constant of variation?

4. What is the connection between the vertical intercept (y-intercept) and the initial value?

SLOPE AS A RATE OF CHANGE

Example – Milk Production



t	$m = 22t$
0	0
1	22
2	44
3	66
4	88
5	110
6	132
7	154
8	176
9	198
10	220

Slope = Constant of Variation =22

Explain how you can determine this using…

(a) …the graph

(b) …the equation

(c) …the table

Observation

Every day, the cow produces 22 kg of milk. We can express this as a ***rate***, that is, the cow produces 22 kg/day (i.e. 22 kg per day or 22 kg every day). This example suggests that ***slope can also be interpreted as a rate of change***!

Rate of Change Definition

Let x represent an independent variable and y represent a variable whose value depends on x.
By the ***rate of change of y with respect to x*** we mean ***how fast*** y changes as the value of x changes.

Examples of Rate of Change

Name	*Independent Variable*	*Dependent Variable*	*Verbal Description*	*Example*
Speed	Time (t)	Distance (d)	***Speed*** is the rate of change of d with respect to t. That is, speed is a measure of how fast distance changes over time. (Units must be distance/time.)	A car travels at a speed of 120 km/h.
Hourly Wage	Time (t)	Money (M)	An ***hourly wage*** is the rate of change of M with respect to t. That is, hourly wage measures how fast money is earned over time. (Units must be money/time.)	Selene earns $25/h.
Fuel Efficiency	Distance (d)	Fuel Used (f)	***Fuel efficiency*** is the rate of change of f with respect to d. That is, fuel efficiency measures how fast fuel is used over distance travelled. (Units must be volume/distance.)	The Toyota Prius has a fuel efficiency of 4.3 L/100 km.

Summary

$$m = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in dependent variable}}{\text{change in independent variable}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \text{slope} = \text{constant of variation} = \begin{cases} y/x, & \text{direct variation} \\ (y-b)/x, & \text{partial variation} \end{cases}, \qquad b = \text{initial value} = \text{vertical intercept} = y\text{-intercept}$$

m = slope = the rate of change of y with respect to x = how fast y changes as x changes

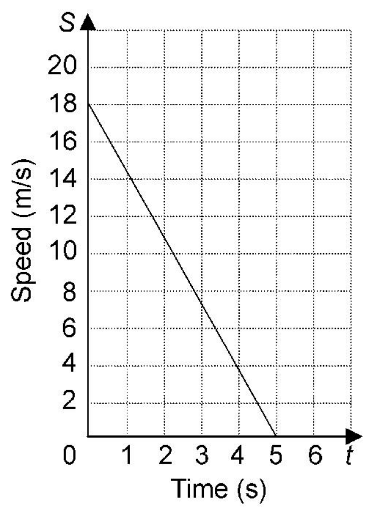
Practice – Slope as a Rate of Change

1. At rest, Vicky takes 62 breaths every 5 min. What is Vicky's rate of change of number of breaths?

2. When he is sleeping, Jeffrey's heart beats 768 times in 12 min. What is the rate of change of number of heartbeats?

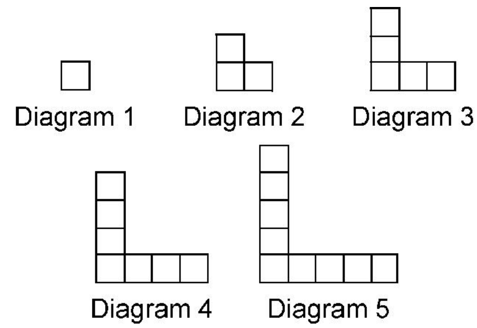
3. A race car driver completed a 500-km closed course in 2.8 h. What is the rate of change of distance with respect to time (i.e. the speed)?

4. The graph shows the speed of the cars on a roller coaster once the brakes are applied.



   a) Find the slope of the graph.
   b) Interpret the slope as a rate of change.

5. **a)** Make a table relating the number of squares to the diagram number. Graph the data in the table.

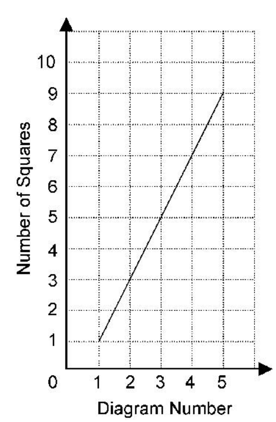


   b) Find the slope of the graph.
   c) Interpret the slope as a rate of change.

Answers

1. 12.4 breaths/min
2. 64 beats/min
3. 179 km/h
4. **a)** –3.6
   b) Once the brakes are applied, the speed of the cars decreases at a rate of 3.6 m/s each second (i.e. 3.6 (m/s)/s or 3.6 m/s^2).
5. **a)**

Diagram Number	Number of Squares
1	1
2	3
3	5
4	7
5	9



   b) 2
   c) Each diagram has two more squares than the previous diagram.

IS THE RELATION LINEAR?

Background

There is a very simple way to tell whether a relation is linear. The key to understanding this is to realize the following:

Linear Relation

$\updownarrow$

***Slope* is Constant**

$\updownarrow$

***Rate of Change* is Constant**

$\updownarrow$

If the Independent Variable (i.e. the x-value) Changes by a Constant Amount,
Then the Dependent Variable (i.e. the y-value) Must Also Change by a Constant Amount

$\updownarrow$

If Δx is constant, Δy must also be constant!

Example and Exercises

From the above, we can conclude that a relation is linear if Δy is constant whenever Δx is constant. The Δy values are called ***first differences***. ***First differences*** are the differences between consecutive y-values in a table of values when the differences in the x-values are constant. Therefore, a relation is linear if the first differences are constant.

- From the table, we can see that $\Delta x = 1$ and $\Delta y = -2$.
- Since both Δx and Δy are constant, the relation must be linear.
- slope $= m = \dfrac{\Delta y}{\Delta x} = \dfrac{-2}{1} = -2$
- y-intercept $= b = 4$ because the point $(0,4)$ belongs to the relation.
- The equation of the relation must be $y = -2x + 4$

x	y	Δy (1st differences)
-3	10	–
-2	8	$8 - 10 = -2$
-1	6	$6 - 8 = -2$
0	4	$4 - 6 = -2$
1	2	$2 - 4 = -2$
2	0	$0 - 2 = -2$
3	-2	$-2 - 0 = -2$

- From the table, we can see that $\Delta x =$ ______ and $\Delta y =$ ______.
- Since both Δx and Δy are ___________ , the relation must be _______________.
- slope $= m = \dfrac{\Delta y}{\Delta x} = \underline{\qquad} =$
- y-intercept $= b =$ ________ because the point __________ belongs to the relation.
- The equation of the relation must be ________________

x	y	Δy (1st differences)
0	3	–
2	7	
4	11	
6	15	
8	19	
10	23	
12	27	

- From the table, we can see that $\Delta x =$ ______ and $\Delta y =$ ______.
- Since both Δx and Δy are ___________ , the relation must be _______________.
- slope $= m = \dfrac{\Delta y}{\Delta x} = \underline{\qquad} =$
- y-intercept $= b =$ ________ because the point __________ belongs to the relation.
- The equation of the relation must be ________________

x	y	Δy (1st differences)
-3	0	–
-2	1	
-1	2	
0	3	
2	5	
4	7	
6	9	

Problem 1 – Solimon's Dilemma – A Linear Relation

Mr. Nolfi believes very strongly in the importance of showing respect to others. Unfortunately, this view was not shared by one of his former students, the infamous Solimon. He often blurted out inappropriate remarks such as referring to his classmates as "retards" or "idiots."

After unsuccessfully having tried several strategies to teach Solimon the value of respect, Mr. Nolfi was forced to resort to a monetary tactic. He decided to charge Solimon a **base fee** of \$10.00 ***plus*** \$0.50 per inappropriate comment.

(a) Complete the following table of values.
$n \rightarrow$ number of inappropriate comments
$F \rightarrow$ fee Solimon pays in dollars
$\Delta F \rightarrow$ change in the fee (first differences)

n	F	ΔF (1st differences)
0		–
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

Notice that $\Delta n = 1$, which is constant.

(b) Graph the relation between the number of inappropriate comments made and the fee that has to be paid. In addition to labelling the axes, set appropriate scales.

(c) The independent variable is __________.

The dependent variable is __________.

(d) Write an equation that relates the dependent variable to the independent variable.

(e) Explain in ***three different ways*** why the relation between F and n must be ***linear***.

(f) Calculate the ***slope*** of the line that you sketched in part (b). What is the ***meaning*** of the slope? Don't forget the units! How could you determine the slope without using the graph?

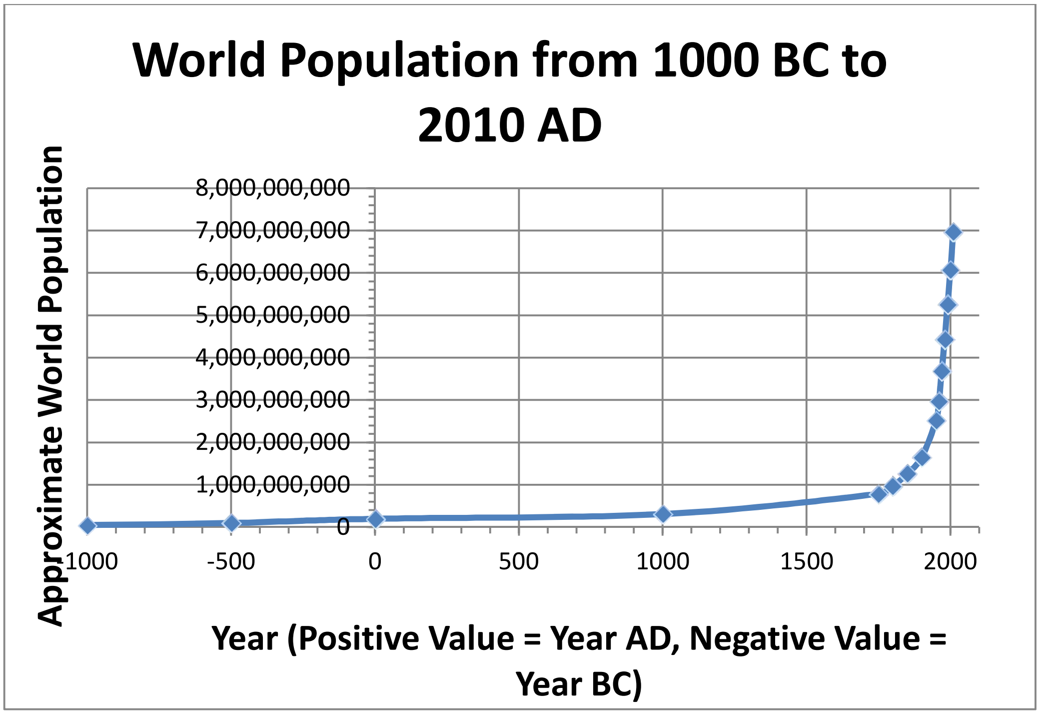
(g) Determine the vertical intercept (i.e. y-intercept or initial value). What is the ***meaning*** of the vertical intercept?

(h) How much would Solimon have to pay if he made one inappropriate comment every minute in a single period?

Problem 2 – World Population – A Non-Linear Relation

The table and graph given below show how the world population has changed over the last 3 millennia (3000 years). From both the table and the graph, one can clearly see that the relation between global population and time is ***non-linear***.

Year	*Population*
−1000	50000000
−500	100000000
1	200000000
1000	310000000
1750	791000000
1800	978000000
1850	1262000000
1900	1650000000
1950	2519000000
1960	2982000000
1970	3692000000
1980	4435000000
1990	5263000000
2000	6070000000
2010	6972000000



Note

- For graphing convenience, the dividing line between the BC and AD eras is shown as year 0. However, there was no "year 0"in reality. The BC era ended with year 1, which was immediately followed by year 1 in the AD era.
- Some authors refer to the BC ("before Christ") era as BCE ("before the current/common era") and to the AD (*anno domini* or "In the year of the Lord") era as CE ("current/common era").

(a) Calculate $\dfrac{\Delta P}{\Delta t} = \dfrac{\text{change in population}}{\text{change in time}}$ from 1800 to 2010.

(b) Interpret your answer from question (a) as a rate of change.

(c) On the grid given above, sketch a line whose slope equals the value you calculated in question (a). What conclusion(s) can you draw?

(d) Use the values given in the table to explain why the relation between world population and time must be non-linear. (Be careful! Remember that both Δx and Δy must be constant for a relation to be linear. To show that a relation is non-linear, you must show that for some part of the relation, Δy is ***not constant*** when Δx ***is*** constant.)

Practice – First Differences

1. Consider the relation $y = 2x - 3$.

a) Make a table of values for x-values from 0 to 5.
b) Graph the relation.
c) Classify the relation as linear or non-linear.
d) Add a third column to the table in part a). Label the column "First Differences." Find the differences between consecutive y-values and record them in this column.

2. Consider the relation $y = \frac{1}{2}x^2$.

a) Make a table of values for x-values from 0 to 5.
b) Graph the relation.
c) Classify the relation as linear or non-linear.
d) Find the differences between consecutive y-values. Add a column to your table in part a) to record the first differences.

3. Refer to your answers to questions 1 and 2. How can you use first differences to tell if a relation is linear or non-linear?

4. Copy and complete each table. State whether each relation is linear or non-linear.

a)

x	y	**First Differences**
0	8	
1	10	
2	13	
3	17	

b)

x	y	**First Differences**
0	2	
1	6	
2	10	
3	14	

5. Copy and complete the table for each equation. Identify each relation as linear or non-linear.

x	y	**First Differences**
1		
2		
3		
4		

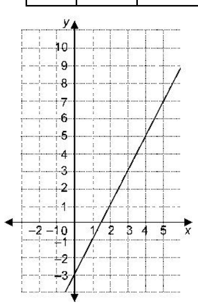
a) $y = 2^x$ **b)** $y = -3x$ **c)** $y = x + 1$ **d)** $y = x^2 + 1$

Answers

1. a, d)

x	y	**1st Differences**
0	−3	–
1	−1	2
2	1	2
3	3	2
4	5	2
5	7	2

b)

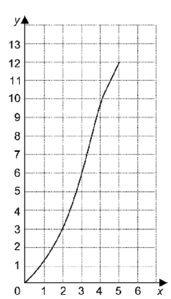


c) linear

2. a, d)

x	y	**1st Differences**
0	0	–
1	0.5	0.5
2	2	1.5
3	4.5	2.5
4	8	3.5
5	12.5	4.5

b)



c) non-linear

3. If the first differences are equal, the relation is linear. If the first differences are not equal, the relation is non-linear.

4. a)

x	y	**1st Differences**
0	8	–
1	10	2
2	13	3
3	17	4

non-linear

5. a)

x	y	**1st Differences**
1	2	–
2	4	2
3	8	4
4	16	8

non-linear

b)

x	y	**1st Differences**
1	−3	–
2	−6	−3
3	−9	−3
4	−12	−3

linear

c)

x	y	**1st Differences**
1	2	–
2	3	1
3	4	1
4	5	1

linear

d)

x	y	**1st Differences**
1	2	–
2	5	3
3	10	5
4	17	7

non-linear

Summary of Analytic Geometry

Relation

Independent Variable: The variable whose value can be chosen freely; its value does not depend on any other value

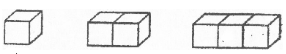
Dependent Variable: The variable whose value is determined entirely by the value of the independent variable

Equation: Describes the ***relationship*** between the independent and dependent variables

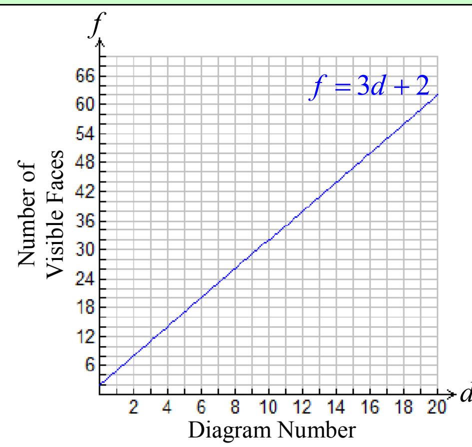
Relation: Any mathematical relationship; that is, any correspondence between two (or more) variables

First Differences: The differences between ***consecutive y-values*** in a table of values when the ***differences in the x-values are constant***.

Example

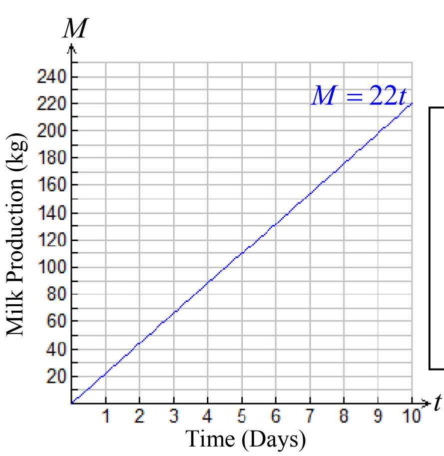
<table>
<tr><th>Situation</th><th>Description of the Relation using a Table</th><th>Description of the Relation using Words</th><th rowspan="1">Description of the Relation using a Graph or Diagram</th></tr>
<tr><td rowspan="3">How many faces are visible in the twentieth diagram?


Ind. Var. – d (diagram #)
Dep. Var. – f (# vis. faces)</td><td rowspan="3">How is the number of visible faces (f) related to the diagram number (d)?

d | $f = 3d + 2$
1 | 5
2 | 8
3 | 11
. | .
. | .
. | .
20 | 62</td><td>The number of visible faces is equal to two more than triple the diagram number.</td><td rowspan="3">
</td></tr>
<tr><td>Description of the Relation using an Equation</td></tr>
<tr><td>$f = 3d + 2$</td></tr>
</table>

Linear Relations (x = independent variable, y = dependent variable)

- **Graph of a Linear Relation:** a ***line***.
- **Direct Variation:** The graph of a linear relation passes through the origin; ***y varies directly with x***; $y/x = m$, $y = mx$
- **Partial Variation:** The graph of a linear relation ***does not*** pass through the origin; ***y varies partially with x***; $(y-b)/x = m$, $y = mx + b$
- ***y changes by a constant amount*** whenever ***x changes by a constant amount***; i.e. the ***first differences*** are constant
- **Slope of a Linear Relation:** is a measure of the ***steepness of a line***
- **For all Linear Relations:** slope = constant of variation = rate of change of y with respect to x
- **Rate of Change of *y* with Respect to *x*:** means "how fast y changes as x changes"
- ***y*-intercept = initial value = vertical intercept:** point at which line crosses the y-axis = "starting value"
- **Equation of Every Linear Relation**: can be written in the form $y = mx + b$, where m = slope, b = y-intercept



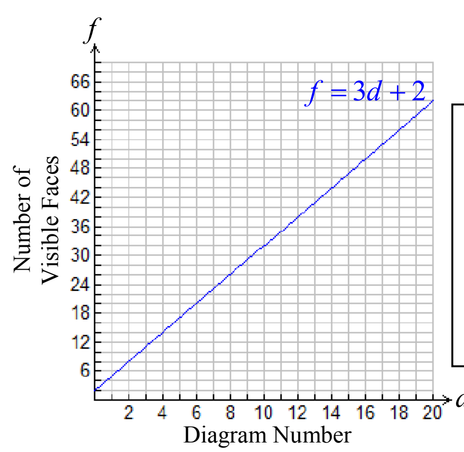
Direct Variation

M varies directly with t

Slope = Constant of Variation =22

Rate of Change = 22 kg/day

Initial Value = vertical intercept = 0



Partial Variation

f varies partially with d

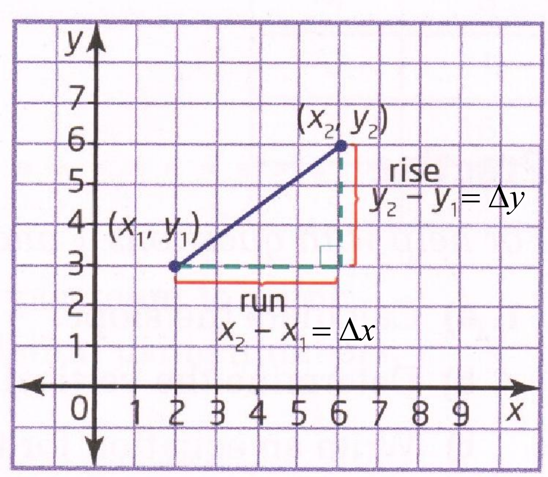
Slope = Constant of Variation = 3

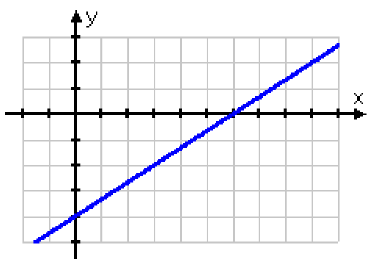
Rate of Change = 3 faces/diagram number

Initial Value = vertical intercept = 2

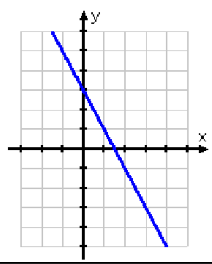
Slope Details

- $$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in dependent variable}}{\text{change in independent variable}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$
- A line that ***leans to the right*** (goes upward to the right) has ***positive slope*** (i.e. a positive rate of change) because both the rise and the run must have the same sign.
- A line that ***leans to the left*** (goes downward to the right) has ***negative slope*** (i.e. a negative rate of change) because the rise and the run must be of opposite sign.
- A ***horizontal line*** has ***zero slope*** because the rise must be zero and the run can have any value.
- A ***vertical line*** has ***undefined slope*** because the rise can have any value but the run must be zero. (Division by zero is undefined.)

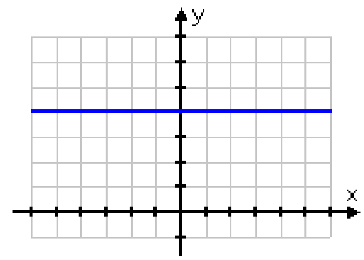




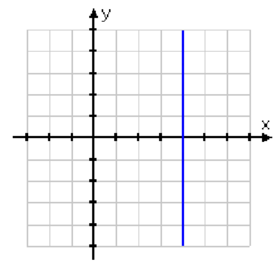
Positive Slope
Positive Rate of Change
As x increases, y increases.



Negative Slope
Negative Rate of Change
As x increases, y decreases.



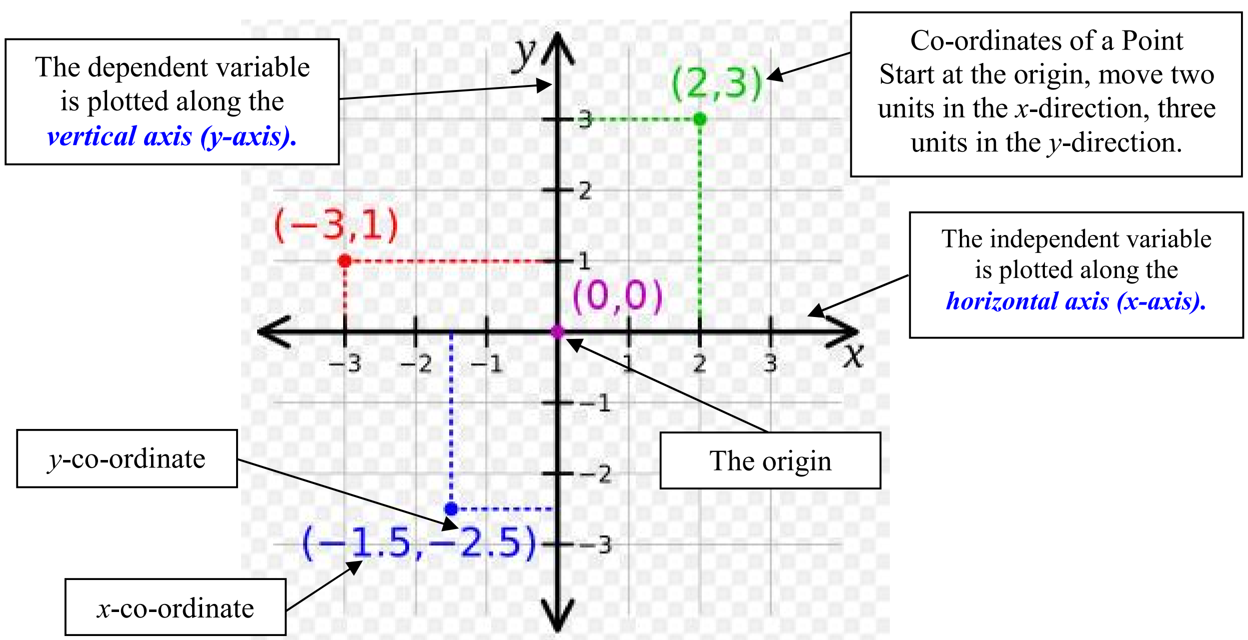
Zero Slope
Zero Rate of Change
As x increases, y is constant.



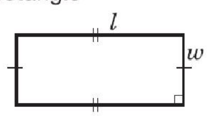
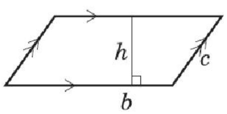
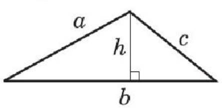
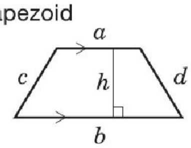
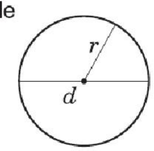
Undefined Slope
Undefined Rate of Change
x is constant, y varies freely.

Analytic Geometry Background Information

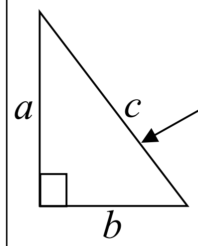
- Also known as ***co-ordinate geometry*** or ***Cartesian geometry***
- Developed by René Descartes, a French philosopher, physicist, mathematician and writer
- René Descartes' Latinized name was *Renatus Cartesius*, which explains why analytic geometry is also known as ***Cartesian geometry***
- Analytic geometry unifies algebra with geometry, providing us with a systematic link between these two extremely important branches of mathematics. This gives us a tool for drawing pictures (i.e. graphs) of equations.



Perimeter and Area Equations

Geometric Figure	Perimeter	Area
Rectangle 	$P = l + l + w + w$ or $P = 2(l + w)$	$A = lw$
Parallelogram 	$P = b + b + c + c$ or $P = 2(b + c)$	$A = bh$
Triangle 	$P = a + b + c$	$A = \frac{bh}{2}$ or $A = \frac{1}{2}bh$
Trapezoid 	$P = a + b + c + d$	$A = \frac{(a+b)h}{2}$ or $A = \frac{1}{2}(a+b)h$
Circle 	$C = \pi d$ or $C = 2\pi r$	$A = \pi r^2$

Pythagorean Theorem



The ***hypotenuse*** is the ***longest side*** of a right triangle. It is always found ***opposite*** the right angle.

In ***any*** right triangle, **the square of the hypotenuse** is equal to **the sum of the squares of the other two sides**. That is,

$$c^2 = a^2 + b^2$$

By using your knowledge of rearranging equations, you can rewrite this equation as follows:

$$b^2 = c^2 - a^2$$

and

$$a^2 = c^2 - b^2$$

The Meaning of π

In ***any*** circle, the ***ratio*** of the ***circumference*** to the ***diameter*** is equal to a ***constant*** value that we call π. That is,

$$C : d = \pi .$$

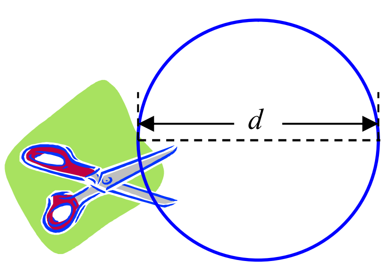
Alternatively, this may be written as

$$\frac{C}{d} = \pi$$

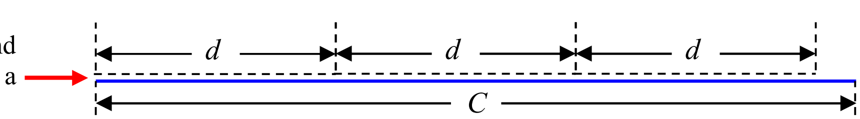
or, by multiplying both sides by d, in the more familiar form

$$C = \pi d .$$

If we recall that $d = 2r$, then we finally arrive at the most common form of this ***relationship***: $C = 2\pi r$.

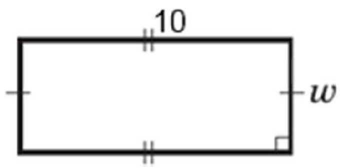


Snip the circle and stretch it out into a straight line.



As you can see, the length of the circumference is slightly more than three diameters. The exact length of C, of course, is π diameters or πd.

Activity: Complete the following table. The first row is done for you.

<table>
<tr><th>Shape</th><th>Perimeter and Area for Given Values</th><th colspan="2">Plot Graphs, State Equations, Answer Questions</th></tr>
<tr>
<td>
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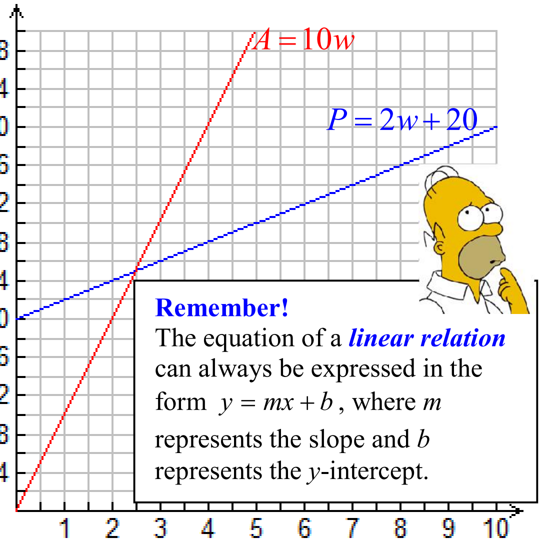
<td>

Remember!
The equation of a linear relation can always be expressed in the form $y=mx+b$, where m represents the slope and b represents the y-intercept.</td>
<td>Slopes: P: 2, A:10
Vertical Intercepts: P: 20, A: 0
Independent Variable: w
Dependent Variables: P, A
Linear or Non-Linear? Both linear
Direct or Partial Variation? P partial, A direct
Constants of Variation: P: 2, A: 10
Initial Values: P: 20, A: 0</td>
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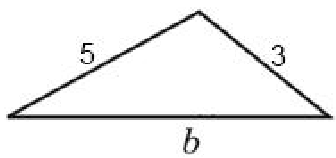
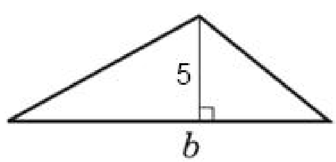
<td>

Use this triangle for perimeter calculations.



Use this triangle for area calculations.</td>
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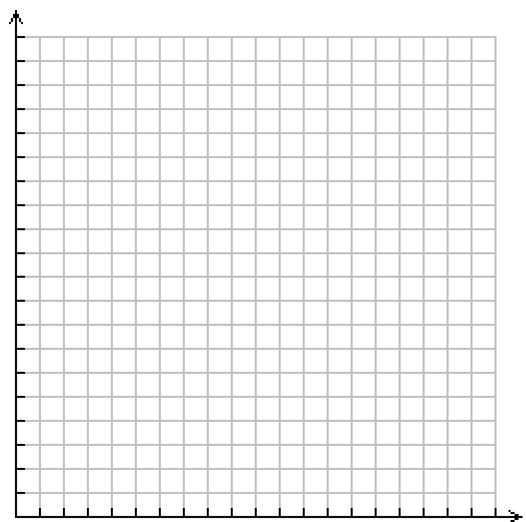
<td></td>
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Independent Variable: _________
Dependent Variables:
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Linear or Non-Linear? _________
Direct or Partial Variation?
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Constants of Variation: ________
Initial Values: _____</td>
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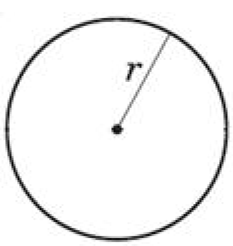
<td>
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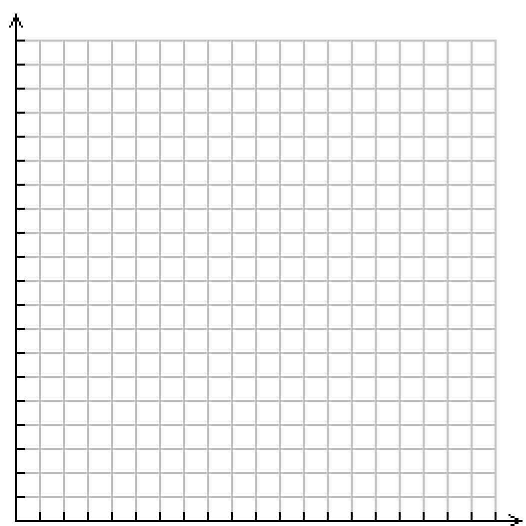
<td></td>
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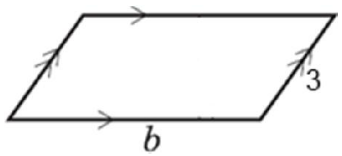
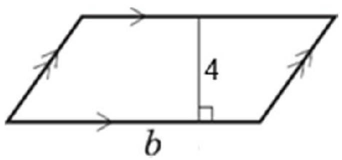


Use this parallelogram for perimeter calculations.


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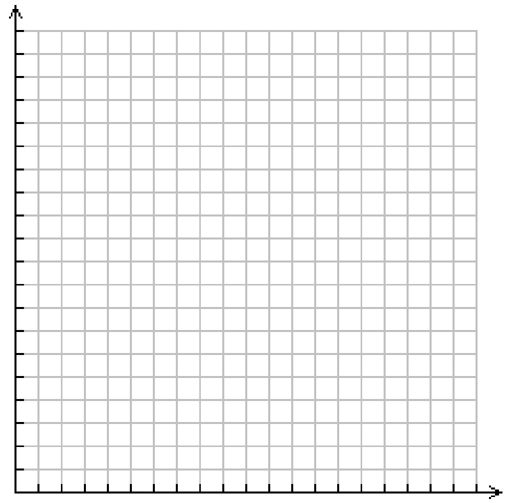
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Independent Variable: __________
Dependent Variables: ____________
Linear or Non-Linear? __________
Direct or Partial Variation? ____________
Constants of Variation: ________
Initial Values: _____
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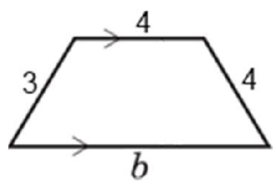
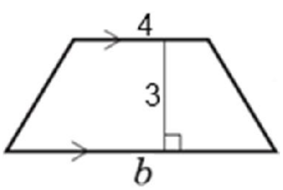


Use this trapezoid for perimeter calculations.


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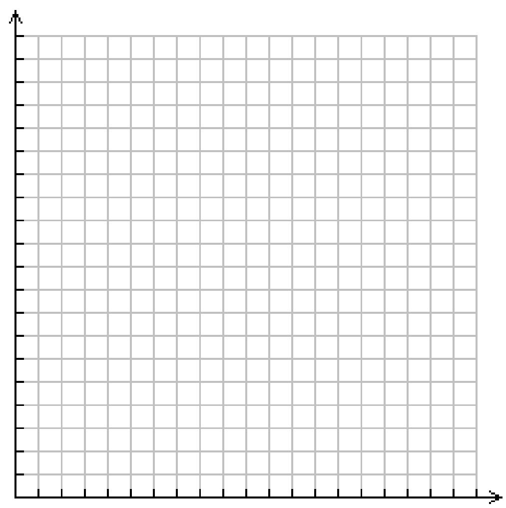
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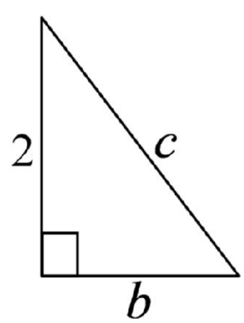

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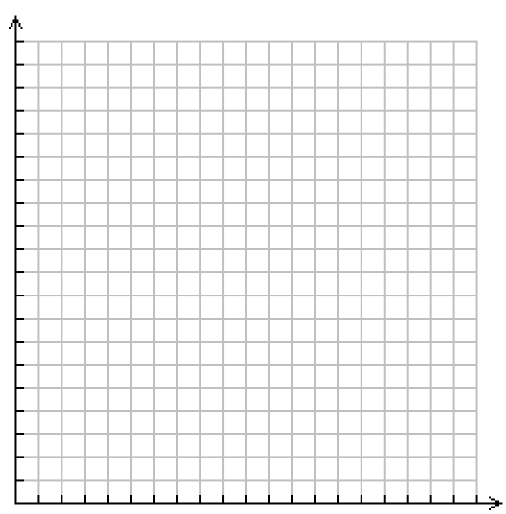
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Vertical Intercepts: ____________
Independent Variable: __________
Dependent Variables: ____________
Linear or Non-Linear? __________
Direct or Partial Variation? ____________
Constants of Variation: ________
Initial Values: _____
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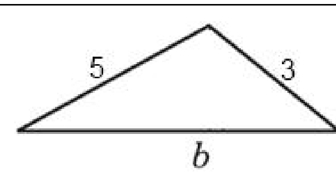
Question

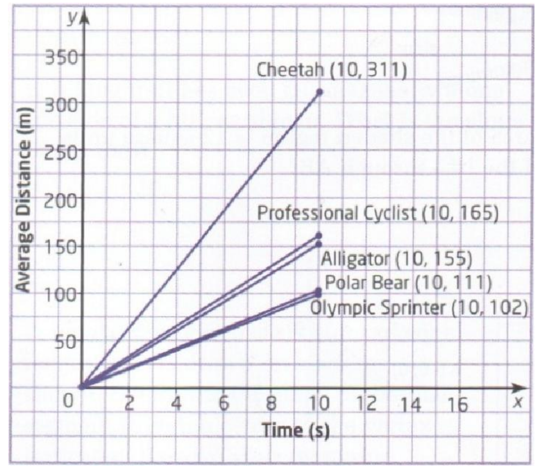
For the triangle shown at the right, explain why the value of b must be greater than 2 and less than 8.



ANALYTIC GEOMETRY: REVIEW PROBLEMS

1. Consider the graphs shown at the right. Each graph gives a typical example of how ***average distance*** varies over time for a ten-second sprint performed by various animals, an Olympic sprinter and a professional cyclist.

(a) ***Without using the given co-ordinates***, estimate the slope of each line segment (calculate rise/run for each line segment). Show how you arrived at your estimate. In addition, estimate the ***average speed*** in each case.



	Cheetah	Cyclist	Alligator	Polar Bear	Sprinter
Estimated Slope					
Estimated Average Speed					

(b) Now ***calculate the exact slope*** of each line segment as well as the ***exact average speed***. Show all calculations.

	Cheetah	Cyclist	Alligator	Polar Bear	Sprinter
Exact Slope					
Exact Average Speed					

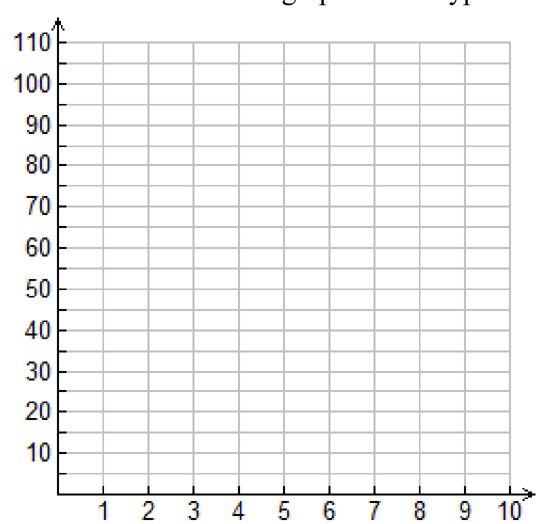
(c) Let d represent average distance in metres and t represent time in seconds. Write an equation of each line.

	Cheetah	Cyclist	Alligator	Polar Bear	Sprinter
Equation					

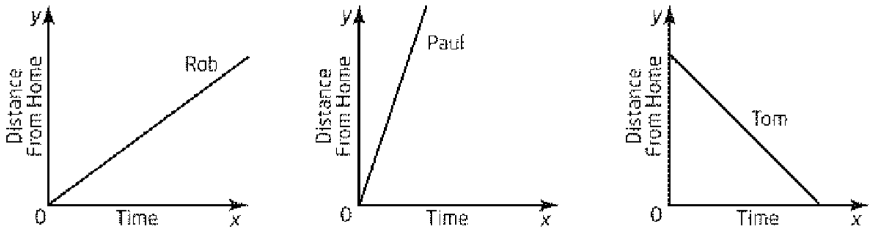
(d) Being lines, each of the given graphs represents a linear relation. This might suggest to some that the ***speed is constant*** (not average speed) in each case. (Of course, average speed over an interval of time ***must be constant***.)

(i) Explain why it is not realistic for the speed to be constant.

(ii) Sketch a more realistic graph for the typical Olympic sprinter.



2. Consider the graphs below. Whose graph slopes downward? What does this indicate?



(a) Tom's graph slopes downward. This indicates that his distance from home is decreasing.
(b) Paul's graph slopes downward. This indicates that his distance from home is decreasing.
(c) Rob's graph slopes downward. This indicates that his distance from home is increasing.
(d) Paul's graph slopes downward. This indicates that his distance from home is increasing.

3. Which statement is true?

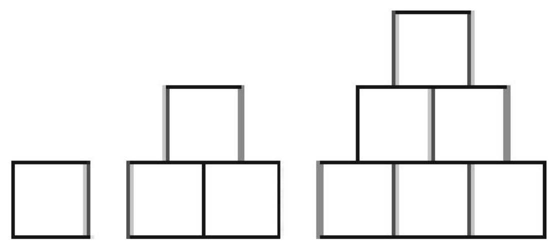
(a) The constant of variation of a linear relation equals its slope.

(b) $\dfrac{\Delta y}{\Delta x}$ is equal to the slope of a line as well as the rate of change of the dependent variable y with respect to x.

(c) A linear relation has an equation of the form $y = mx + b$.

(d) All of the above.

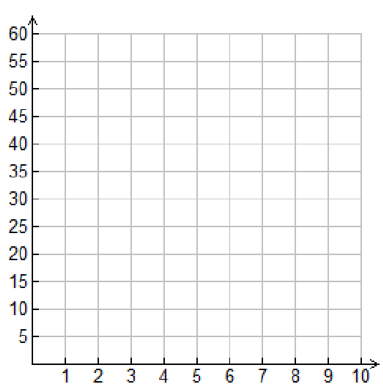
4. Consider the pattern shown at the right. Each square has a side length of 1 cm.

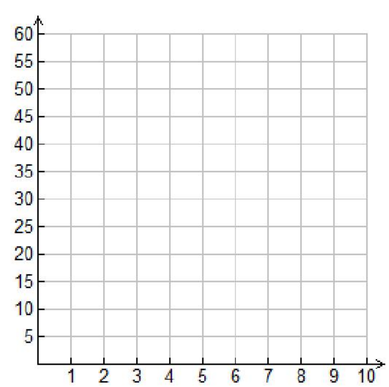


(a) Create a table comparing the diagram number with the area and perimeter for that diagram.

Diagram Number (d)	Area (A)	Perimeter (P)

(b) Sketch graphs of area versus diagram number and perimeter versus diagram number. Is either relation linear?



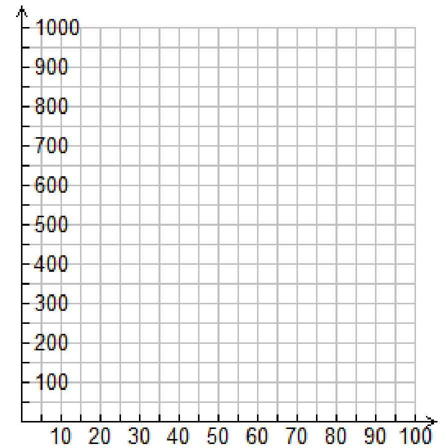


(c) Write equations for both A and P, the dependent variables, in terms of d, the independent variable.

5. Alison and Lucy belong to different fitness clubs. Alison has a membership that cost her \$300 and she pays \$2 each time she visits the club. Lucy has a pay-as-you-go membership and she pays \$8 each time she visits her club.

(a) Let n represent the number of visits to the fitness club and let C represent the total cost in dollars. Write equations for C in terms of n for both Lucy and Alison. In addition, sketch the graph of each relation on a single grid.

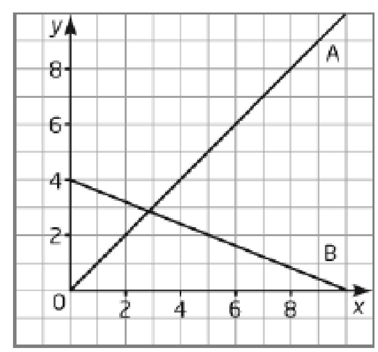
	Equation for C in terms of n
Lucy	
Alison	



(b) Use your graphs to ***estimate*** the values of n for which Lucy has a better deal and the values of n for which Alison has a better deal.

(c) Now solve an equation to determine the exact value of n at which Lucy and Alison pay exactly the same amount. Use your solution to determine the values of n for which Lucy has a better deal and the values of n for which Alison has a better deal.

6. The graph shows two relations, A and B, one direct and one partial variation.

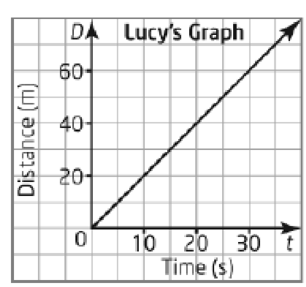


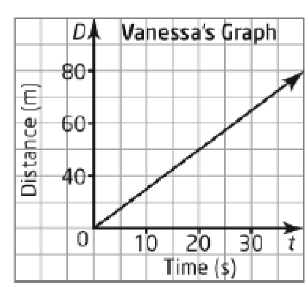
(a) Identify the partial variation.

(b) Give the fixed, or initial, value for the partial variation.

(c) Which relation has a greater constant of variation?

7. Lucy and Vanessa are walking home from school.





(a) How far did each person walk in 20 s?

(b) What is the slope of each graph?

(c) Who walked faster? Explain.

(d) Whose graph looks steeper?

(e) Why is it not true in this case that a steeper graph indicates a faster speed?

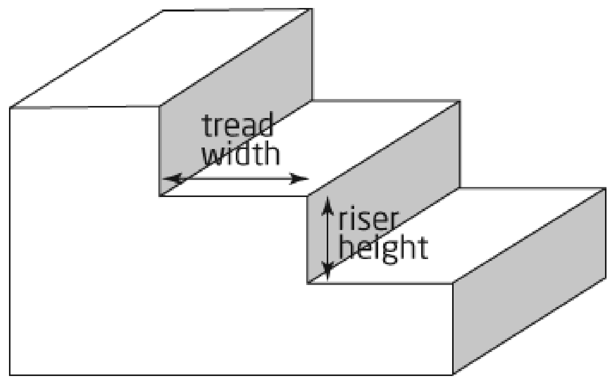
8. The length of a trip ***varies directly*** with the amount of gasoline used. Yael's car used 16 L for the first 145 km of his trip from Toronto to Montreal.

(a) How much gasoline, rounded to the nearest litre, should he expect to use in the remaining 400 km of his trip?

(b) If gasoline costs $1.13/L, can he complete the trip with a budget of $70?

9. Jorgen is designing a set of steps from his deck to the garden 2 m below. He knows that a comfortable slope for steps is about 0.6. In addition, he wants the tread width to be 30 cm.

(a) What should the height of each riser be?



(b) How many steps will the staircase have? Be sure to give an integer answer and explain the effects of your choice.

10. Modified True/False

Indicate whether each statement is true or false. If the statement is false, change the underlined part(s) to make the statement true.

____ <u>Partial variation</u> occurs when the ratio of the dependent variable to the independent variable is constant. **Change:** ________________

____ Any linear relation has an equation of the form $y = mx + b$, where m represents the <u>fixed, or initial value of y</u>, and b represents the <u>constant of variation</u>. **Change:** ________________

____ The <u>vertical intercept</u>, constant of variation and rate of change all represent the same concept for a linear relation. **Change:** ________________

____ The following are all units of <u>change</u>: kilometres per hour, dollars per kilogram, litres per 100 km, breaths per minute **Change:** ________________

ANALYTIC GEOMETRY – PREPARING FOR THE UNIT TEST

Practice Test 1

Multiple Choice

For each question, select the best answer.

1. Which relation is a direct variation?

A $y = 5x$ **B** $y = 2^x$

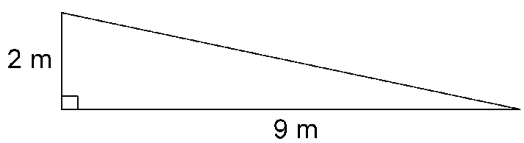
C $y = 5x^2$ **D** $y = 5x - 2$

2. The cost of tea varies directly with the mass. Liz bought 4.5 kg of tea for $10.35. What is the constant of variation?

A 0.43 **B** 14.85

C 5.85 **D** 2.30

3. What is the slope of this ramp?



A $\frac{2}{9}$ **B** $-\frac{2}{9}$

C $\frac{9}{2}$ **D** $-\frac{9}{2}$

4. Which equation represents this relation?

x	y
0	4
1	1
2	–2
3	–5
4	–8

A $y = -3x + 4$ **B** $y = 4x - 3$

C $y = 3x + 4$ **D** $y = 3x - 4$

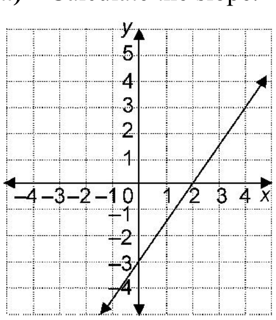
5. The cost of a newspaper advertisement is $750 plus $80 for each day it runs. Which equation represents this relation?

A $C = 80n - 750$ **B** $C = 80n + 750$

C $C = 750n + 80$ **D** $C = 750n - 80$

Short Response

6. **a)** Calculate the slope.



b) Find the vertical intercept.
c) Write an equation for the relation.

7. The cost to ship goods varies directly with the mass. Paul paid $20.40 to ship a package with mass 24 kg. Write an equation for this relationship.

8. Is this relation linear or non-linear? How can you tell without graphing?

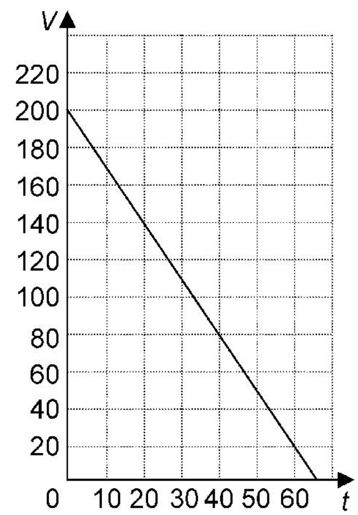
x	y
2	0.16
4	0.64
6	1.44
8	2.56

9. Sheila works in a bookstore. She earns $240 per week, plus $0.15 for every bestseller she sells.
a) Write an equation for this relationship.
b) Last week, Sheila sold 19 bestsellers. How much did she earn?

Extend

Show all your work.

10. This graph shows the volume of water in a child's pool over time as the pool is draining.



a) Calculate the rate of change of the volume of water. How does the rate of change relate to the graph?
b) Write an equation for the relationship.
c) Suppose the rate of change changes to –4 L/min. How long will it take the pool to empty?

Answers

1. A **2.** D **3.** B **4.** A **5.** B

6. **a)** $\frac{3}{2}$ **b)** –3 **c)** $y = \frac{3}{2}x - 3$ **7.** $C = 0.85m$

8. Non-linear. The first differences are not equal.

9. **a)** $E = 0.15n + 240$ **b)** $242.85

10. **a)** –3 L/min; the rate of change is the slope
b) $V = 200 - 3t$ (or $V = -3t + 200$) **c)** 50 min

Practice Test 2

Multiple Choice

For each question, select the best answer.

1. Which relation is a partial variation?

A $y = 25x$ **B** $y = 2^x$

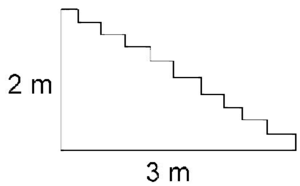
C $y = 5x^2$ **D** $y = 2x - 5$

2. Sophie's earnings vary directly with the number of hours she works. She earned \$25 in 4 h. What is the constant of variation?

A 0.16 **B** 6.25

C 100 **D** 21

3. What is the slope of this staircase?



A -6 **B** $-\frac{3}{2}$ **C** -2 **D** $-\frac{2}{3}$

4. Which equation represents this relation?

x	y
0	–1
1	–3
2	–5
3	–7
4	–9

A $y = -x - 2$ **B** $y = 2x - 1$

C $y = -2x - 1$ **D** $y = 2x + 1$

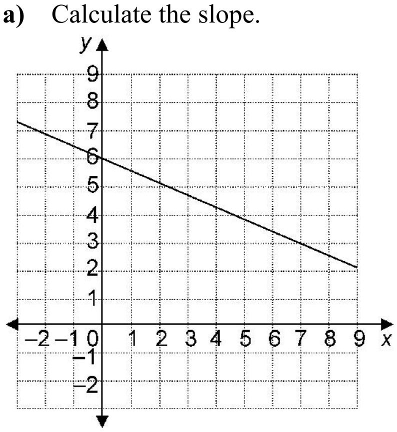
5. The cost to cater a party is \$200 plus \$15 for each guest. Which equation represents this relation?

A $C = 15n + 200$ **B** $C = 15n - 200$

C $C = 200n + 15$ **D** $C = 200n - 15$

Short Response

6. a) Calculate the slope.



b) Find the vertical intercept.
c) Write an equation for the relation.

7. The distance travelled varies directly with time. Anthony ran 49.6 m in 8 s.
a) Write an equation for this relationship.
b) Graph the relation.

8. Is this relation linear or non-linear? How can you tell without graphing?

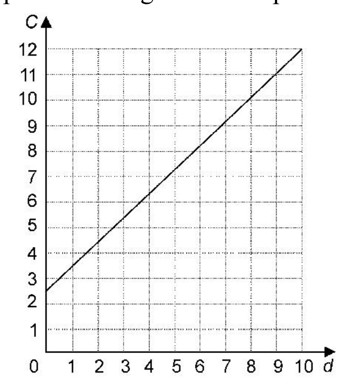
x	y
4	8.4
8	16.8
12	25.2
16	33.6

9. The cost to install wood trim is \$50, plus \$6/m of trim installed.
a) Write an equation for this relationship.
b) 18 m of trim were installed. What was the total cost?

Extend

Show all your work.

10. This graph shows the relationship between the cost of a taxi trip and the length of the trip.



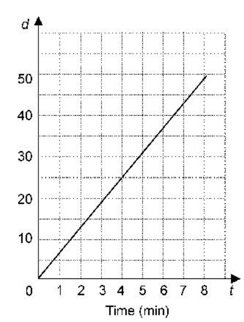
a) Calculate the rate of change of cost with respect to distance travelled. How does the rate of change relate to the graph?
b) Write an equation for the relationship.
c) Suppose the flat fee changed to \$3.00. How would the equation change? How would the graph change?

Answers

1. D **2.** B **3.** D **4.** C **5.** A

6. a) $-\frac{3}{7}$ **b)** 6 **c)** $y = -\frac{3}{7}x + 6$

7. a) $d = 6.2t$
b) See graph at the right



8. Linear. The first differences are constant. (The "x" values change by a constant amount and the "y" values also change by a constant amount.)

9. a) $C = 6l + 50$ (l = length of trim) **b)** \$158.00

10. a) 0.95 \$/km (equals the slope) **b)** $C = 0.95d + 2.5$
c) $C = 0.95d + 3$ The graph would be parallel to the original graph but have a vertical intercept of 3.