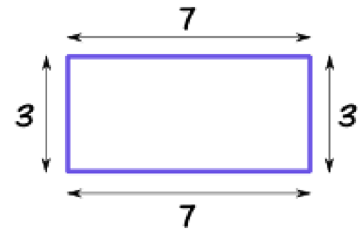
Unit 6 – Measurement and Geometry

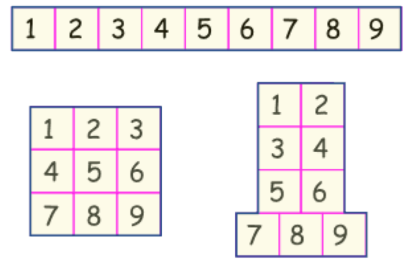
Understanding the Concepts of Perimeter, Area and Volume

Perimeter



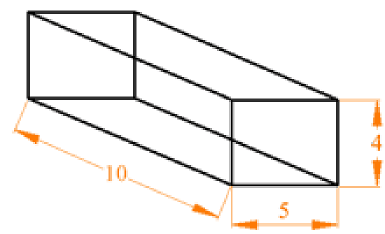
- The ***distance*** around a two-dimensional shape.
- **Example:** the perimeter of this rectangle is 3+7+3+7 = 20
- The perimeter of a circle is called the ***circumference***.
- Perimeter is measured in ***linear units*** such as mm, cm, m, km.

Area



- The "size" or "amount of space" inside the boundary of a two-dimensional surface, including curved surfaces. In the case of a curved surface, the area is usually called ***surface area***.
- **Example:** If each small square at the left has an area of 1 cm^2, the larger shapes all have an area of 9 cm^2.
- Area is measured in ***square units*** such as mm^2, cm^2, m^2, km^2.

Volume



- The "amount of space" contained within the interior of a three-dimensional object. (The ***capacity*** of a three-dimensional object.)
- **Example:** The volume of the "box" at the right is 4×5×10 = 200 m^3. This means, for instance, that 200 m^3 of water could be poured into the box.
- Volume is measured in ***cubic units*** such as mm^3, cm^3, m^3, km^3, mL, L.
  Note: 1 mL = 1 cm^3

Questions

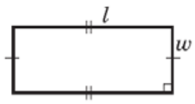
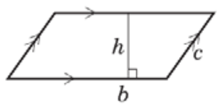
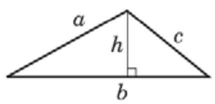
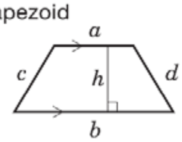
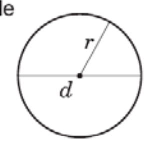
1. You have been hired to renovate an old house. For each of the following jobs, state whether you would measure perimeter, area or volume and explain why.

Job	***Perimeter, Area or Volume?***	***Why?***
Replace the baseboards in a room. 		
Paint the walls.		
Pour a concrete foundation. 		

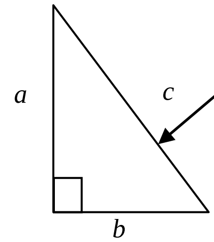
2. Convert 200 m^3 to litres. (**Hint:** Draw a picture of 1 m^3.)

Calculating Perimeter, Area and Volume

Perimeter and Area Equations

Geometric Figure	Perimeter	Area
Rectangle 	$P = l + l + w + w$ or $P = 2(l + w)$	$A = lw$
Parallelogram 	$P = b + b + c + c$ or $P = 2(b + c)$	$A = bh$
Triangle 	$P = a + b + c$	$A = \frac{bh}{2}$ or $A = \frac{1}{2}bh$
Trapezoid 	$P = a + b + c + d$	$A = \frac{(a+b)h}{2}$ or $A = \frac{1}{2}(a + b)h$
Circle 	$C = \pi d$ or $C = 2\pi r$	$A = \pi r^2$

Pythagorean Theorem



The ***hypotenuse*** is the ***longest side*** of a right triangle. It is always found ***opposite*** the right angle.

In ***any*** right triangle, **the square of the hypotenuse** is equal to **the sum of the squares of the other two sides**. That is,

$$c^2 = a^2 + b^2$$

By using your knowledge of rearranging equations, you can rewrite this equation as follows:

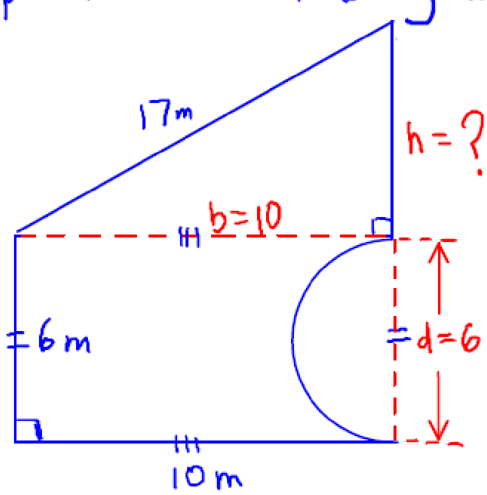
$$b^2 = c^2 - a^2 \qquad a^2 = c^2 - b^2$$

Note:
- The hypotenuse is always found OPPOSITE the right angle
- Any letters at all can be used. For example, we could write
  $x^2 = y^2 + z^2$,
  where x represents the length of the hypotenuse and y and z represent the lengths of the other two sides

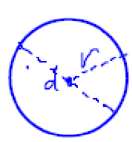
Example – Finding the Perimeter and Area of a Composite Shape

Find the area and perimeter of the following composite shape

Note:
To solve this problem, you must MAKE CONNECTIONS!
i.e.
(a) width of rectangle = diameter of semi-circle
(b) base of Δ = length of rectangle
(c) height of rectangle can be calculated using Pyth. Thm.



- Original diagram is in blue
- Red part added to diagram while "making connections"



r = radius
d = diameter
$C = \pi d = 2\pi r$
$A = \pi r^2$

Solution

(a) By the Pythagorean theorem,

$17^2 = 10^2 + h^2$

$\therefore 289 = 100 + h^2$

$\therefore 289 - 100 = 100 + h^2 - 100$

$\therefore 189 = h^2$

$\therefore h = \sqrt{189}$

$\therefore h \doteq 13.7$

(b) The circumference of the semi-circle

$C = \frac{\pi d}{2}$ ← semi-circle is half of a circle

$\therefore C \doteq \frac{3.14(6)}{2}$

$\therefore C \doteq 9.4$

$\therefore$ perimeter $\doteq 17 + 6 + 10 + 9.4 + 13.7 = 56.1$ m

Area of composite shape

= area of ▭ + area of ◺ − area of ◖

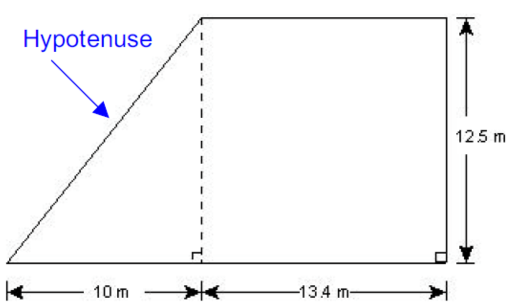
$= lw + \frac{bh}{2} - \frac{\pi r^2}{2}$

$\doteq 10(6) + \frac{10(13.7)}{2} - \frac{3.14(3)^2}{2} \doteq 114.4\ m^2$

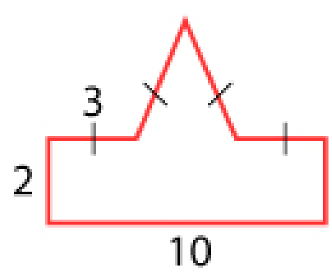
Perimeter and Area Questions

1. Calculate the ***perimeter*** and ***area*** of each of the following shapes.

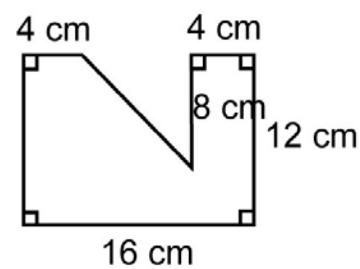
(a)



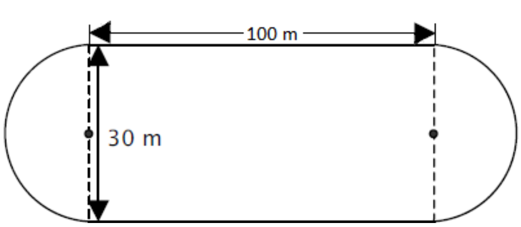
(b)



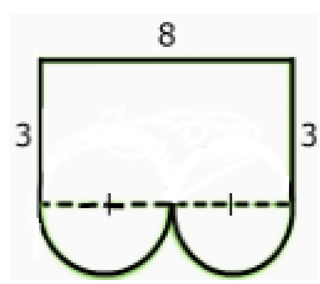
(c)



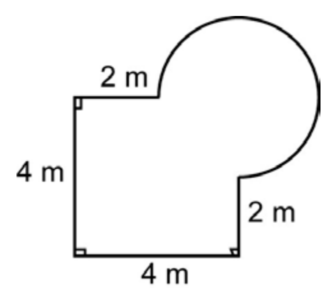
(d)



(e)

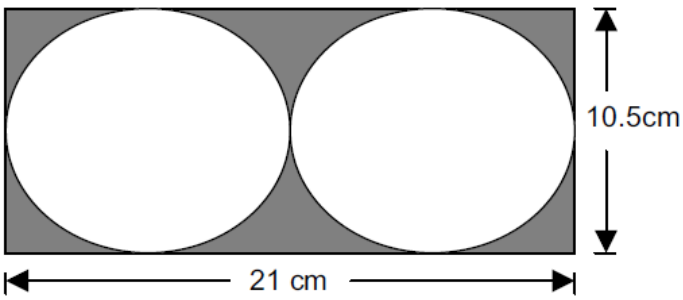


(f)

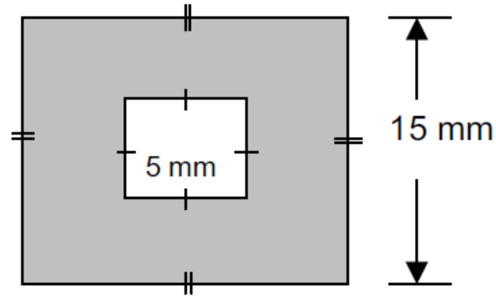


2. Calculate the area of the ***shaded region***.

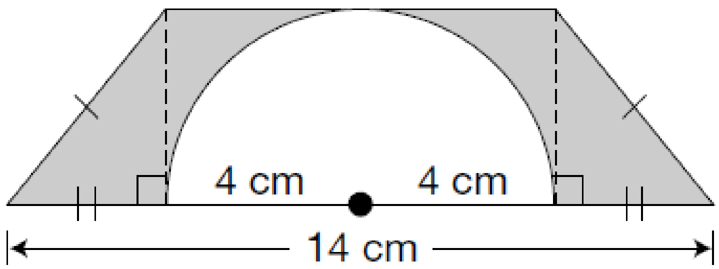
(a)



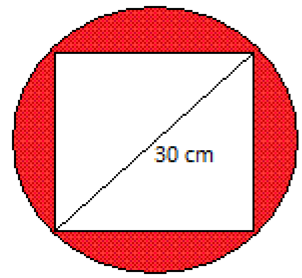
(b)



(c)



(d)

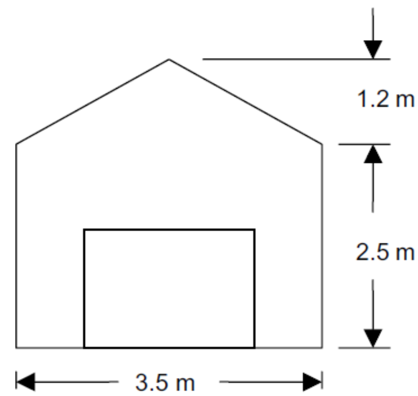


3. The front of a garage, excluding the door, needs to be painted.

(a) Calculate the area of the region that needs to be painted assuming that the door is 1.5m high and 2 m wide. (See the diagram at the right for all other dimensions.)

(b) If one can of paint covers an area of 2.5 m^2, how many cans will need to be purchased?

(c) If one can of paint sells for $19.89, how much will it cost to buy the paint? (Include 13% HST.)



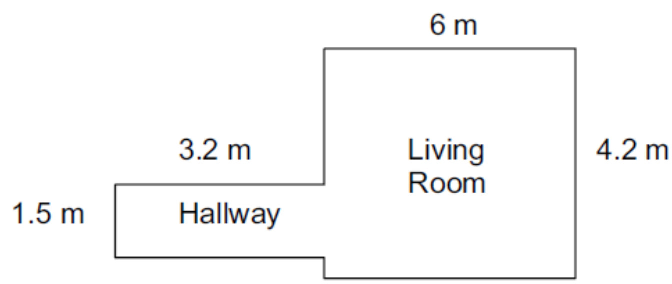
4. Bill wishes to replace the carpet in his living room and hallway with laminate flooring. A floor plan is shown at the right.

(a) Find the total area of floor to be covered.

(b) Laminate flooring comes in boxes that contain 2.15m^2 of material. How many boxes will Bill require?

(c) One box costs $43.25. How much will the flooring cost? (Include 13% HST.)

(d) When laying laminate flooring, it is estimated there will be 5% waste. How much waste can Bill expect on his project?

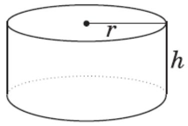
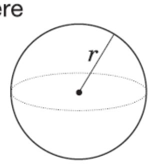
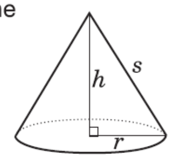
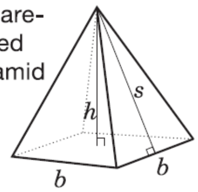
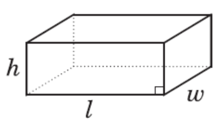
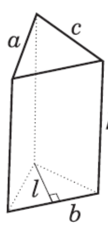


Answers

1. (a) P $\doteq$ 65.3 m, A=230 m^2 (b) P=26, A $\doteq$ 24.5 (c) P $\doteq$ 67.3 cm, A=160 cm^2 (d) P $\doteq$ 294.2 m, A $\doteq$ 3706.5 m^2 (e) P $\doteq$ 26.56, A $\doteq$ 36.56 (f) P $\doteq$ 21.42 m, A $\doteq$ 26.23 m^2	2. (a) A $\doteq$ 47.3 m^2 (b) A=200 mm^2 (c) A $\doteq$ 18.9 cm^2 (d) A $\doteq$ 257 cm^2
3. (a) A=7.85 m^2 (b) 4 cans (c) $89.90	4. (a) A=30 m^2 (b) At least 14 boxes, probably 15 (c) $684.22 (d) About 1.5 m^2 of waste, meaning that 15 boxes are needed.

Volume and Surface Area Equations

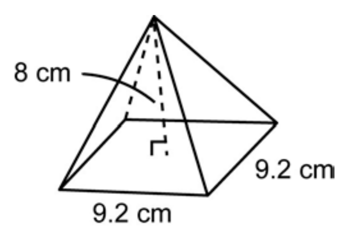
If you need additional help, Google "area volume solids."

Geometric Figure	Surface Area	Volume
Cylinder 	$A_{\text{base}} = \pi r^2$ $A_{\text{lateral surface}} = 2\pi r h$ $A_{\text{total}} = 2A_{\text{base}} + A_{\text{lateral surface}}$ $= 2\pi r^2 + 2\pi r h$	$V = (A_{\text{base}})(\text{height})$ $V = \pi r^2 h$
Sphere 	$A = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$ or $V = \frac{4\pi r^3}{3}$
Cone 	$A_{\text{lateral surface}} = \pi r s$ $A_{\text{base}} = \pi r^2$ $A_{\text{total}} = A_{\text{lateral surface}} + A_{\text{base}}$ $= \pi r s + \pi r^2$	$V = \frac{(A_{\text{base}})(\text{height})}{3}$ $V = \frac{1}{3}\pi r^2 h$ or $V = \frac{\pi r^2 h}{3}$
Square-based pyramid 	$A_{\text{triangle}} = \frac{1}{2}bs$ $A_{\text{base}} = b^2$ $A_{\text{total}} = 4A_{\text{triangle}} + A_{\text{base}}$ $= 2bs + b^2$	$V = \frac{(A_{\text{base}})(\text{height})}{3}$ $V = \frac{1}{3}b^2 h$ or $V = \frac{b^2 h}{3}$
Rectangular prism 	$A = 2(wh + lw + lh)$	$V = (\text{area of base})(\text{height})$ $V = lwh$
Triangular prism 	$A_{\text{base}} = \frac{1}{2}bl$ $A_{\text{rectangles}} = ah + bh + ch$ $A_{\text{total}} = A_{\text{rectangles}} + 2A_{\text{base}}$ $= ah + bh + ch + bl$	$V = (A_{\text{base}})(\text{height})$ $V = \frac{1}{2}blh$ or $V = \frac{blh}{2}$

Volume and Surface Area Questions

1. A cone has radius of 8 cm and slant height of 10 cm. What is the surface area of the cone to the nearest tenth of a square centimetre?

A 670.2 cm^2 **B** 804.2 cm^2 **C** 452.4 cm^2 **D** 640 cm^2



2. What is the volume of this pyramid, to the nearest tenth of a cubic centimetre?

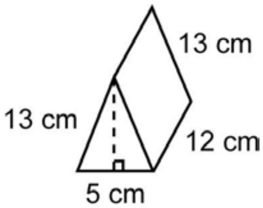
A 677.1 cm^3 **B** 231.8 cm^3 **C** 338.6 cm^3 **D** 225.7 cm^3

3. A sphere has radius 7 cm. What is the volume of the sphere to the nearest tenth of a cubic centimetre?

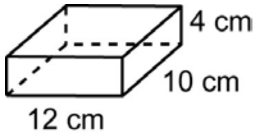
A 1436.8 cm^3 **B** 615.8 cm^3 **C** 4310.3 cm^3 **D** 205.3 cm^3

4. Find the surface area and volume of each object. Round your answers to one decimal place.

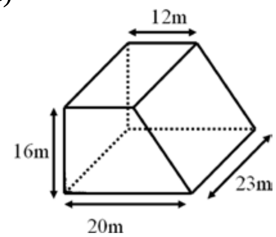
(a)



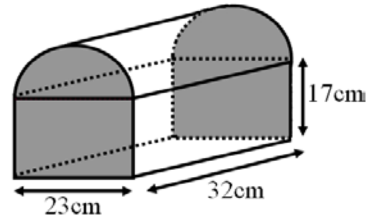
(b)



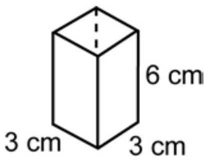
(c)



(d)



5. What is the maximum volume of a cone that would fit in this box?

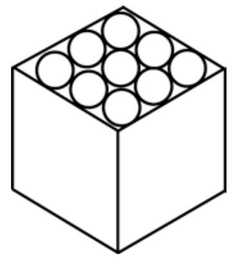


6. Carmine is packing 27 superballs in 3 square layers. Each ball has diameter 4 cm.

(a) What is the minimum volume of the box?

(b) What is the surface area of the box?

(c) How much empty space is in the box?



Answers

1. C **2.** D **3.** A

4. (a) A $\doteq$ 435.8 cm^2, V $\doteq$ 382.7 cm^3 (b) A=416 cm^2, V=480 cm^3 (c) A $\doteq$ 2027.7 m^2, V=5888 m^3 (d) A=61411 cm^2, V=19156 cm^3

5. 14.1 cm^3 **6.** (a) 1728 cm^3 (b) 864 cm^2 (c) 823.2 cm^3

Naming the Solids

UNDERSTANDING *WHY* THE EQUATIONS ARE CORRECT

Understanding the Meaning of π and how it Relates to the Circumference of a Circle

The following is an example of a typical conversation between Mr. Nolfi and a student who blindly memorizes formulas:

Student: Sir, I can't remember whether the area of a circle is πr^2 or $2\pi r$. Which one is it?

Mr. Nolfi: If you remember the meaning of π, you should be able to figure it out.

Student: How can 3.14 help me make this decision? It's only a number!

Mr. Nolfi: How dare you say something so disrespectful about one of the most revered numbers in the mathematical lexicon! (Just kidding. I wouldn't really say that.) It's true that the number 3.14 is an approximate value of π. But I asked you for its ***meaning***, not its value.

Student: I didn't know that π has a meaning. I thought that it was just a "magic" number.

Mr. Nolfi: Leave magic to the magicians. In mathematics, every term (except for primitive terms) has a very precise definition. Read the following carefully and you'll never need to ask your original question ever again!

In ***any*** circle, the ***ratio*** of the ***circumference*** to the ***diameter*** is equal to a ***constant*** value that we call π. That is,

$$C : d = \pi .$$

Alternatively, this may be written as

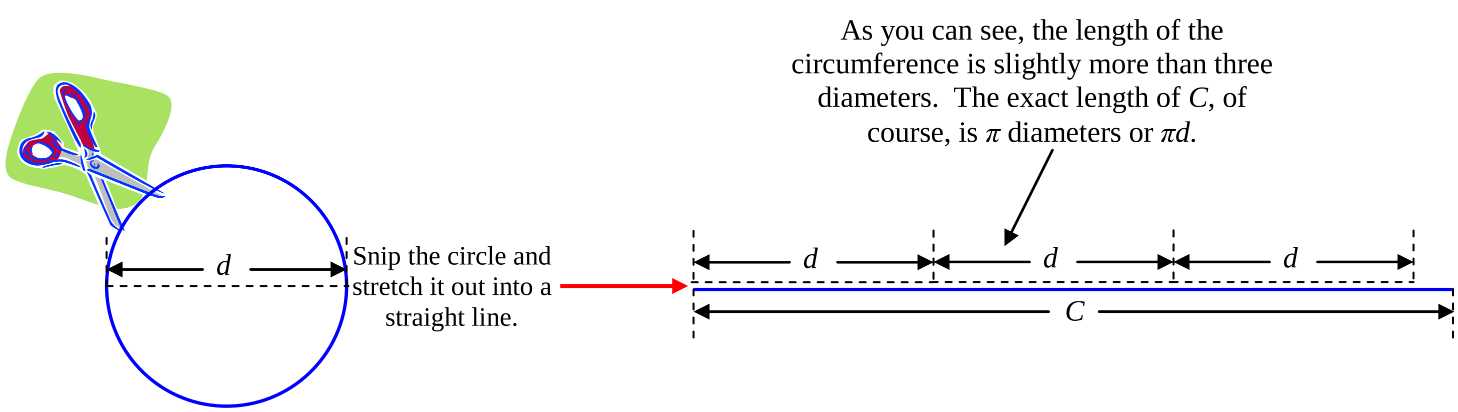
$$\frac{C}{d} = \pi$$

or, by multiplying both sides by *d*, in the more familiar form

$$C = \pi d .$$

If we recall that $d = 2r$, then we finally arrive at the most common form of this ***relationship***,

$$C = 2\pi r .$$



Mr. Nolfi: So you see, by understanding the meaning of π, you can ***deduce*** that $C = 2\pi r$. Therefore, the formula for the area must be $A = \pi r^2$. Furthermore, it is not possible for the expression $2\pi r$ to yield units of area. The number 2π is dimensionless and *r* is measured in units of distance such as metres. Therefore, the expression $2\pi r$ must result in a value measured in units of distance. On the other hand, the expression πr^2 must give a value measured in units of area because $r^2 = r(r)$, which involves multiplying a value measured in units of distance by itself. Therefore, by considering units alone, we are drawn to the inescapable conclusion that the area of a circle must be πr^2 and **not** $2\pi r$!

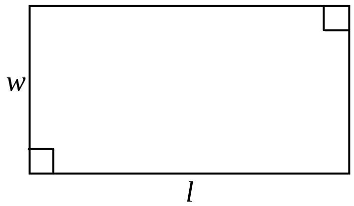
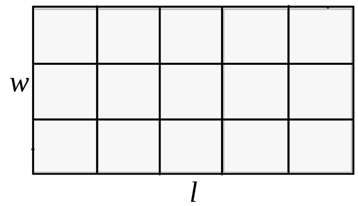
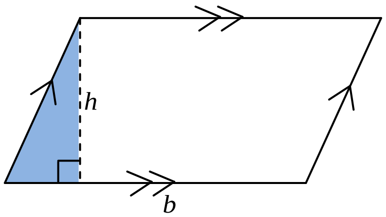
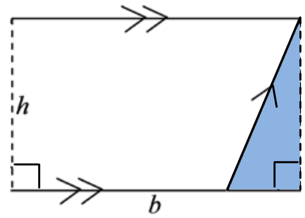
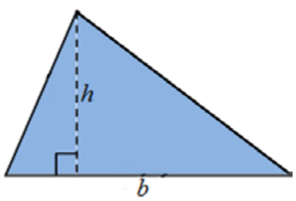
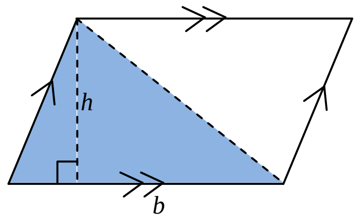
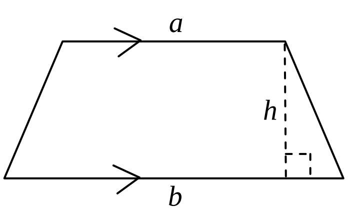
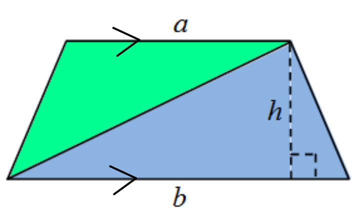
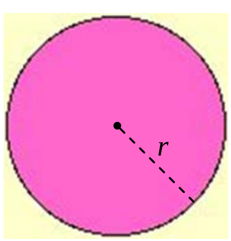
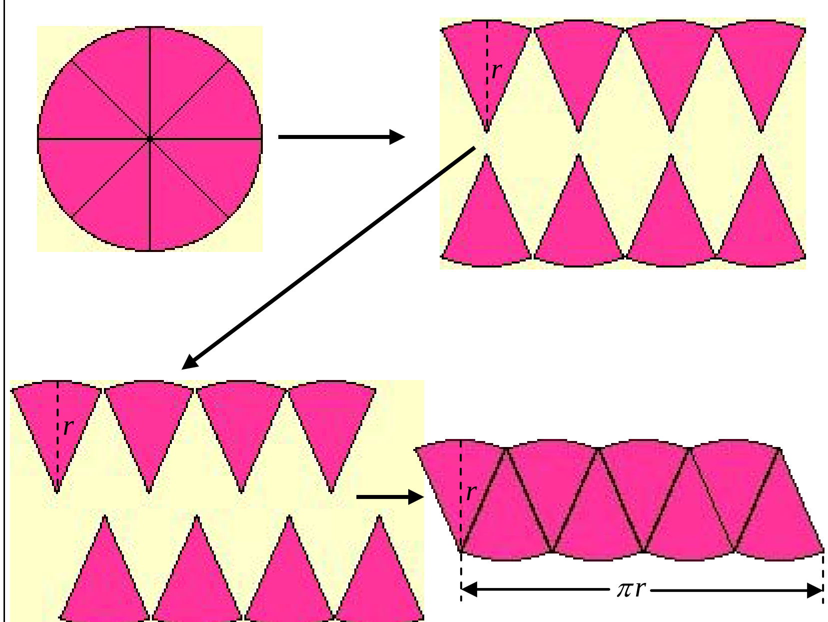
Examples

$2\pi r \doteq 2(3.14)(3.6 \text{ cm}) = 22.608 \text{ cm} \rightarrow$ This answer cannot possibly measure area because cm is a unit of distance.

Therefore, πr^2 must be the correct expression for calculating the area of a circle.

$\pi r^2 \doteq 3.14(3.6 \text{ cm})^2 = 3.14(3.6 \text{ cm})(3.6 \text{ cm}) = 40.6944 \text{ cm}^2 \rightarrow$ Notice that the unit "cm^2" is appropriate for area.

Understanding Area Equations

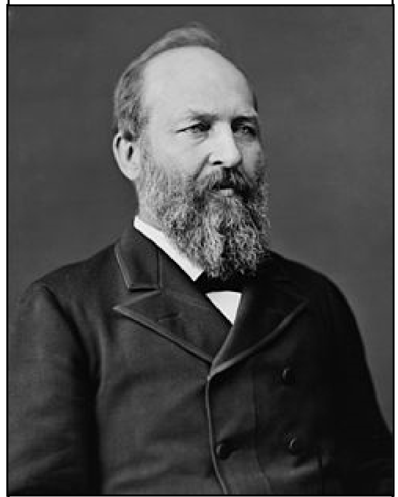
Shape	*Explanation*	*Equation for Area*
	 • Each square has an area of 1 unit^2 • Total number of squares = (# of squares in each row)×(# rows) = 5(3) = lw	$A = lw$
	 • Cut off the shaded right triangle at the left end of the parallelogram. • Attach the cut-off right triangle at the right side. • A rectangle is formed. Its length is b and its width is h, which means that its area must be bh.	$A = bh$
	 • Begin with a parallelogram having the same base and height. • Cut the parallelogram ***in half*** along the dashed diagonal. • Since the parallelogram's area is bh, the triangle's area must be $\frac{bh}{2}$.	$A = \frac{bh}{2}$ or $A = \frac{1}{2}bh$
	 • Divide the trapezoid into two triangles as shown. • Both triangles have a height of h. One triangle has a base of a and the other has a base of b. • Total area $= \frac{ah}{2} + \frac{bh}{2} = \frac{ah+bh}{2} = \frac{h(a+b)}{2}$	$A = \frac{h(a+b)}{2}$ or $A = \frac{1}{2}h(a+b)$
 The argument presented here for the area of a circle is lacking in rigour. It is based on the assumption that we can use the equation for the area of a parallelogram ($A = bh$) to calculate the area of a shape that is similar to a parallelogram. A more rigorous argument can be constructed using calculus.		• As shown in the diagrams, divide the circle into an even number of "wedges," all of which have the same size. • Rearrange the wedges as shown, then fit them together. • The resulting shape is very close to a parallelogram, so its area should be about $bh = (\pi r)r = \pi r^2$ $A = \pi r^2$

Understanding the Pythagorean Theorem

Pythagorean Theorem Proof 1 – President James Garfield's Brilliant Proof

James A. Garfield was the 20th president of the United States. In addition to being a highly successful statesman and soldier, President Garfield was also a noted scholar. Among his many scholarly accomplishments is his beautiful proof of the Pythagorean Theorem. It is outlined below.

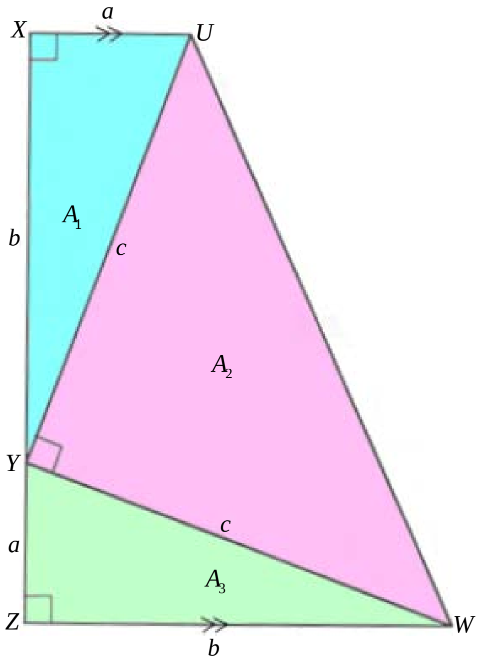
James A. Garfield



20th President of the U.S.A.

In Office:
March 4, 1881- September 19, 1881

Assassinated at the age of 49
One of four assassinated presidents



(a) Calculate the area of trapezoid *XZWU* by summing the areas of ΔUXY, ΔYZW and ΔUYW. Simplify fully!

$$A_{Trap} = A_1 + A_2 + A_3$$
$$=$$

(b) Calculate the area of the trapezoid by using the equation for the area of a trapezoid. Simplify fully!

$$A_{Trap} = \frac{h(a+b)}{2}$$
$$=$$

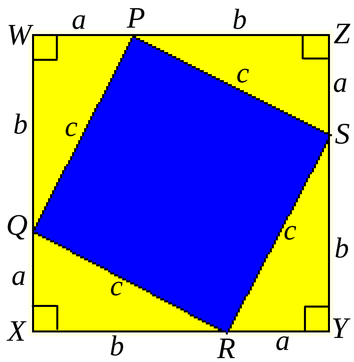
Think! What is the height of the trapezoid?

(c) In parts (a) and (a) you developed two different expressions for the area of trapezoid *XZWU*. Since both expressions give the area of the ***same shape***, they must be equal to each other! Set the expressions equal to each other and solve for c^2.

$$A_{Trap} = A_{Trap}$$
$$\therefore$$

Pythagorean Theorem Proof 2

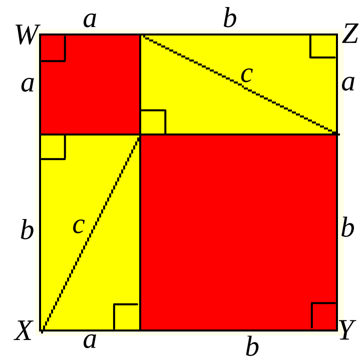
(a) Explain why quadrilateral *PQRS must* be a square.



(b) Use the above diagram to develop a proof of the Pythagorean Theorem. (**Hint:** The line of reasoning is similar to that of President Garfield's proof.)

Pythagorean Theorem Proof 3

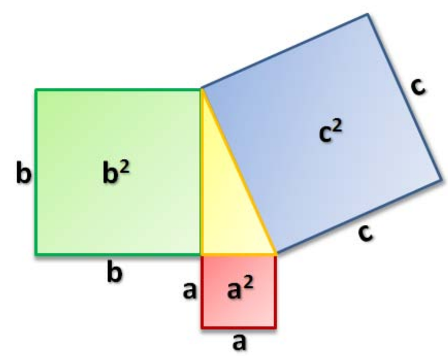
(a) Explain how the diagram at the right is simply a rearrangement of the pieces in the diagram at the left.



(b) Explain why the two diagrams together make it ***obvious*** that $a^2 + b^2 = c^2$. (This proof was devised in 1939 by Maurice Laisnez, a high school student in the Junior-Senior High School of South Bend, Indiana.)

Consequence of the Pythagorean Theorem

Although the Pythagorean Theorem is an equation that relates the lengths of the sides of a right triangle, it can also be ***interpreted*** in terms of areas.

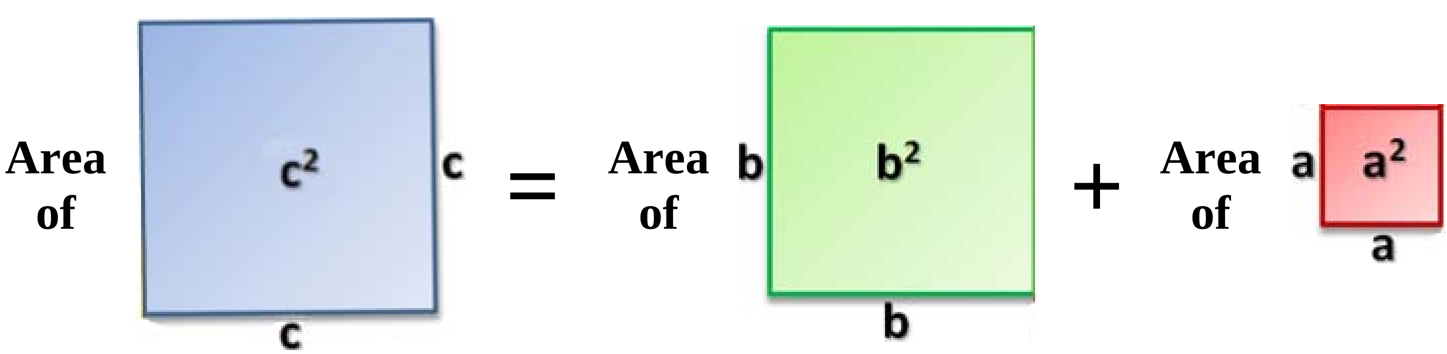


- We have ***proved*** that in a right triangle, the square of the hypotenuse must equal the sum of the squares of the other two sides.

  That is, if *c* represents the length of the hypotenuse and *a* and *b* respectively represent the lengths of the other two sides, then $c^2 = a^2 + b^2$.

- By examining the diagram at the right, one can easily see that the expressions a^2, b^2 and c^2 are all areas of squares!

  Since $c^2 = a^2 + b^2$, ***it must also be true*** that



Research

1. There are literally hundreds of different proofs of the Pythagorean Theorem. Find a proof that you are able to understand and explain it in your own words. Include diagrams in your explanation.

2. Do some research to answer the following questions:

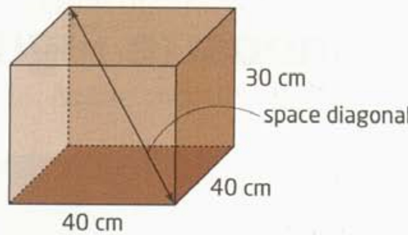
(a) Who was Euclid?

(b) Euclid created a mathematical treatise, consisting of thirteen books, called the *Elements*. Why are Euclid's *Elements* considered so important?

(c) Find Euclid's proof of the Pythagorean Theorem and explain it in your own words.

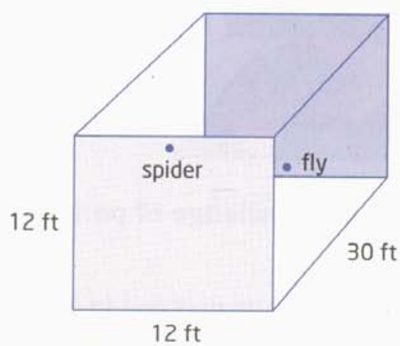
Extend

10. A cardboard box measures 40 cm by 40 cm by 30 cm. Calculate the length of the space diagonal, to the nearest centimetre.

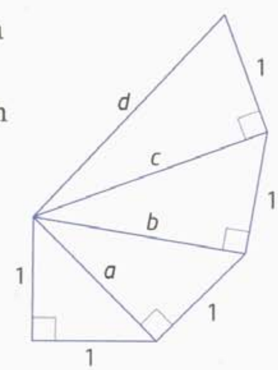


11. The Spider and the Fly Problem is a classic puzzle that originally appeared in an English newspaper in 1903. It was posed by H.E. Dudeney. In a rectangular room with dimensions 30 ft by 12 ft by 12 ft, a spider is located in the middle of one 12 ft by 12 ft wall, 1 ft away from the ceiling. A fly is in the middle of the *opposite* wall 1 ft away from the floor. If the fly does not move, what is the shortest distance that the spider can crawl along the walls, ceiling, and floor to capture the fly?

Hint: Using a net of the room will help you get the answer, which is less than 42 ft!



12. A spiral is formed with right triangles, as shown in the diagram.



a) Calculate the length of the hypotenuse of each triangle, leaving your answers in square root form. Describe the pattern that results.

b) Calculate the area of the spiral shown.

c) Describe how the expression for the area would change if the pattern continued.

13. Math Contest

a) The set of whole numbers (5, 12, 13) is called a *Pythagorean triple*. Explain why this name is appropriate.

b) The smallest Pythagorean triple is (3, 4, 5). Investigate whether multiples of a Pythagorean triple make Pythagorean triples.

c) Substitute values for m and n to investigate whether triples of the form $(m^2 - n^2, 2mn, m^2 + n^2)$ are Pythagorean triples.

d) What are the restrictions on the values of m and n in part c)?

Answers

10. 64 cm

11. 40 ft

12. a) $\sqrt{2}$; $\sqrt{3}$; $\sqrt{4}$; $\sqrt{5}$

b) $\frac{\sqrt{1}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{4}}{2}$

c) As you add right triangles to the spiral pattern, the area will increase by $\frac{\sqrt{\text{Number of Triangles}}}{2}$.

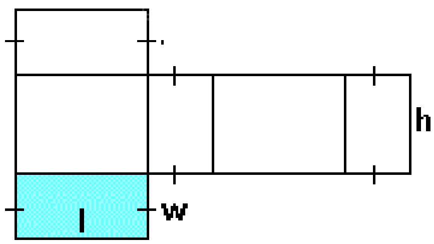
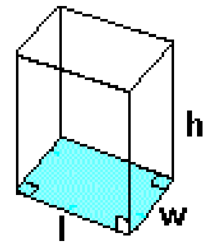
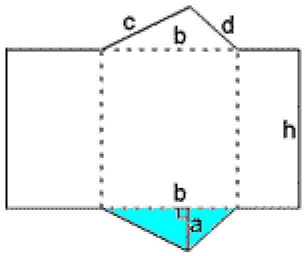
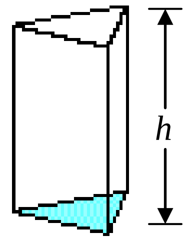
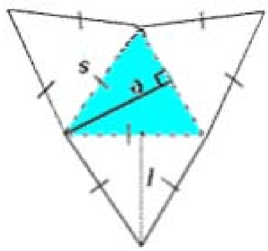
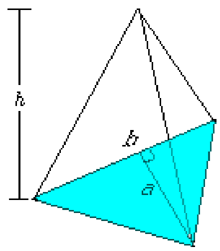
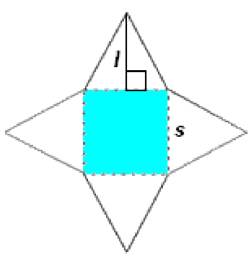
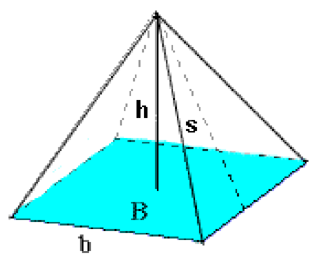
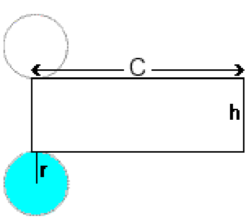
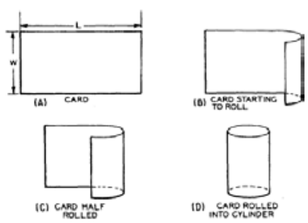
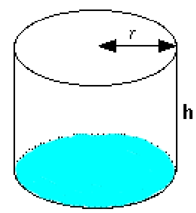
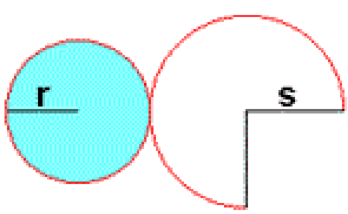
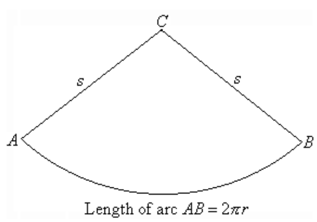
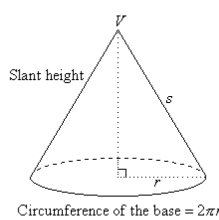
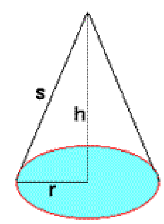
13. a) This name is appropriate because this set of three whole numbers satisfies the Pythagorean theorem.

b) Yes.

c) Yes, they are Pythagorean triples, with some restrictions on the values of m and n.

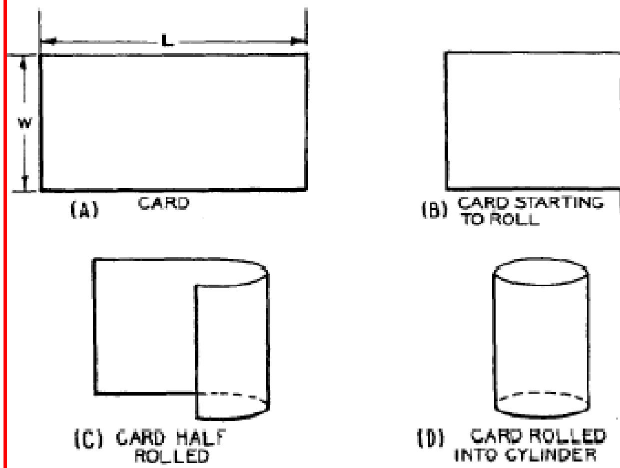
d) $m > n > 0$

Understanding Surface Area Equations

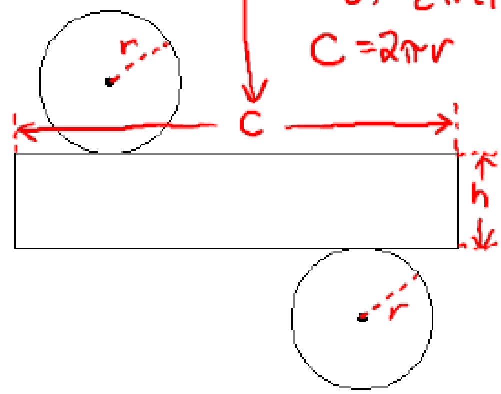
Net	*Solid*	*Surface Area Equation*
		
		
		
		
 		
 	 	

January 11-10
11:25 AM

Surface Area of Cylinder



circumference of circle
$C = 2\pi r$

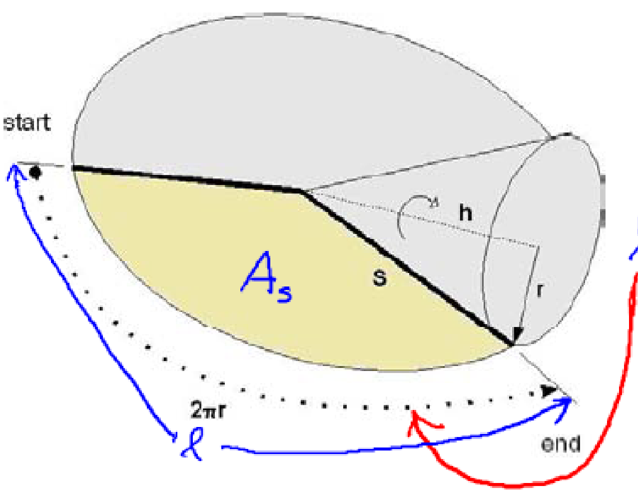


$SA = A_{Bases} + A_{Lateral Face}$

$\therefore SA =$ area of 2 circles + area of rectangle

$\therefore SA = 2\pi r^2 + 2\pi rh$

Surface Area of a Cone



A cone can be made by rolling up a SECTOR of a circle. (A slice of pizza is an example of a sector.)

ℓ = length of arc = circumference of circular base of cone
$= 2\pi r$

A_s = area of sector with arc length $2\pi r$

A_c = area of entire circle (radius s)

C = circumference of circle of radius s

Understanding Volume Equations

Volume of Any Prism

$V = (\text{area of base})(\text{height})$

Volume of any Pyramid

$V = \frac{1}{3}(\text{area of base})(\text{height})$

OR

$V = \frac{(\text{area of base})(\text{height})}{3}$

In fact, the above relationships hold more generally:

Volume of Cylinder = (area of base)(height)

Volume of Cone = $\frac{1}{3}$(area of base)(height)

Volume of ANY Regular Solid

A regular solid has the following characteristics

- 2 parallel, congruent faces called bases
- any cross-section parallel to the base is congruent to the base

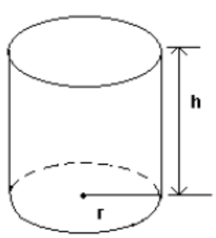
CONGRUENT ⟶ exactly the same in all respects

$V_{\text{regular solid}} = A_{\text{Base}}\, h$ $\qquad h = \text{height}$

Examples of Regular Solids: prisms, cylinders

Volume of Cylinder

Cylinder: has 2 parallel congruent circular faces called bases. The lateral surface is a rolled up rectangle

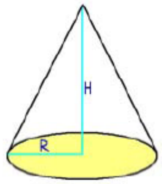


Volume = (area of base)(height)

$\therefore\ V = \pi r^2 h$

πr^2: area of circular base; h: height

Volume of Cone

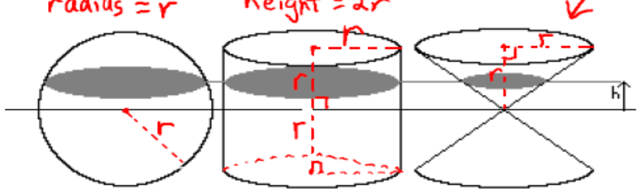


$$\text{Volume} = \frac{1}{3}(\text{area of base})(\text{height})$$

$$\therefore V = \frac{1}{3}\pi r^2 h = \frac{\pi r^2 h}{3}$$

Note: The following is a reasonably convincing argument that for a sphere of radius r,

$$V = \frac{4}{3}\pi r^3$$



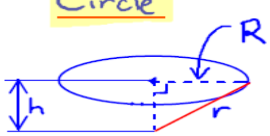
Height of one cone = r
Height of double cone = $2r$

Let h represent the distance from the "centre line" to the point at which the cross-section is taken

$V = ?$ $\qquad V = A_{base}h$

Cross-Sections

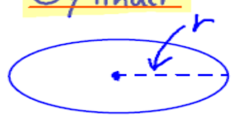
Circle



$h^2 + R^2 = r^2$
$\therefore R^2 = r^2 - h^2$

$A_{sp} = \pi R^2$
$\therefore A_{sp} = \pi(r^2 - h^2)$
$\therefore A_{sp} = \pi r^2 - \pi h^2$

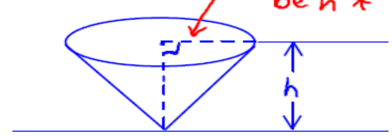
Cylinder



$A_{cyl} = \pi r^2$

Notice that
$A_{sp} + A_{co}$
$= \pi r^2 - \pi h^2 + \pi h^2$
$= \pi r^2$
$= A_{cyl}$

Cone



* The cone is constructed in such a way that the height = radius
$\therefore$ radius = h

$A_{co} = \pi h^2$

This suggests that

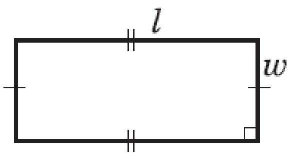
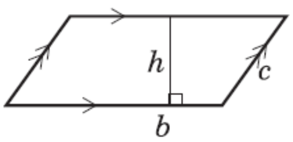
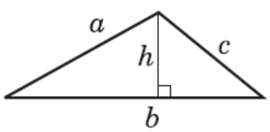
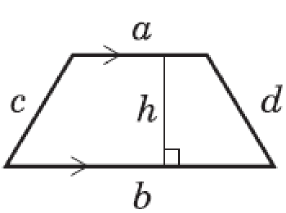
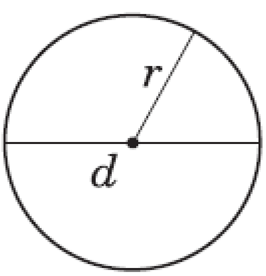
That is, our argument is strong but it does not prove definitively that $V_{sp} = \frac{4}{3}\pi r^3$

$$V_{cyl} = V_{sp} + V_{co}$$

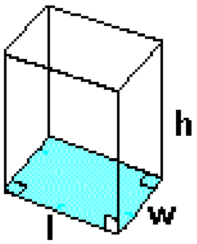
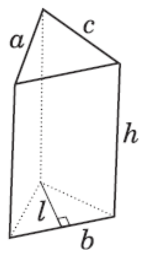
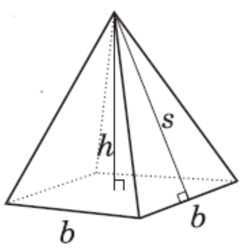
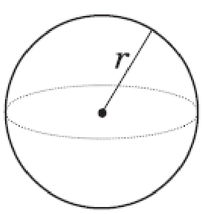
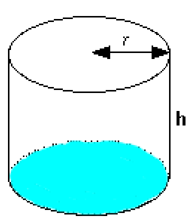
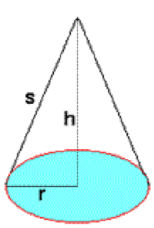
$$\therefore V_{sp} = V_{cyl} - V_{co}$$
$$= 2\pi r^3 - \frac{2}{3}\pi r^3$$
$$= \frac{6}{3}\pi r^3 - \frac{2}{3}\pi r^3$$
$$= \frac{4}{3}\pi r^3$$

WHAT HAPPENS IF…

1. Complete the following table. The first row has been done for you.

Shape	***Name of the Shape***	***What Happens to the Perimeter if …***	***What Happens to the Area if …***
	Rectangle	…the length is doubled ***Solution*** $P=2l+2w$ If the length is doubled, the new length is $2l$. Then, the perimeter becomes $P=2(2l)+2w=4l+2w=(2l+2w)+2l$ The perimeter increases by $2l$.	…the width is tripled ***Solution*** $A=lw$ If the width is tripled, the new width is $3w$. Then, the area becomes $A=l(3w)=3lw=3(lw)$ The area is also tripled.
		…the base is doubled	…the height is quadrupled
		…the base is tripled (if this can be done without changing the values of *a* and *c*)	…the height is tripled
		…the base is tripled (if this can be done without changing the values of *c* and *d*)	…the height is doubled
		…the radius is doubled	…the radius is doubled

2. Complete the following table. The first row has been done for you.

Shape	*Name of the Shape*	*What Happens to the Surface Area if …*	*What Happens to the Volume if …*
	Rectangular Prism	…the length is doubled *Solution* $A = 2lw + 2lh + 2wh$ If the length is doubled, the new length is $2l$. Then, the surface area becomes $A = 2(2l)w + 2(2l)h + 2wh$ $= 4lw + 4lh + 2wh$ $= (2lw + 2lh + 2wh) + 2lw + 2lh$ The surface area increases by $2lw + 2lh$.	…the width is tripled *Solution* $V = lwh$ If the width is tripled, the new width is $3w$. Then, the volume becomes $V = l(3w)h = 3lwh = 3(lwh)$ The volume is also tripled.
		…*b* is doubled (if this can be done without changing the values of *a* and *c*)	…the height is quadrupled
		…the slant height is tripled	…the height is tripled
		…the radius is doubled	…the radius is doubled
		…the radius is doubled	…the radius is doubled
		…the radius is doubled	…the radius is doubled

OPTIMIZATION

Definition: Optimize

- Make ***optimal*** (i.e. the best, most favourable or desirable, ***especially under some restriction***); get the most out of; use best
- In a mathematical context, to ***optimize*** means either to ***maximize*** (make as great as possible) or to ***minimize*** (make as small as possible), subject to a restriction called a ***constraint***.

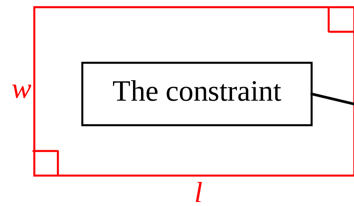
Optimization Problem 1

You have 400 m of fencing and you would like to enclose a rectangular region of ***greatest possible area***. What dimensions should the rectangle have?

(a) What is the ***constraint*** in this problem?

The constraint is the length of fencing available. Since only 400 m of fencing are available, the region enclosed by the fence will have a limited size.

(b) Draw a diagram describing this situation. Then write an equation that relates the unknowns to the constraint.



Length of fencing = 400 m

$\therefore$ perimeter = 400 m

$\therefore$ $\boxed{2l + 2w = 400}$

(c) What quantity needs to be optimized? State whether the quantity needs to be maximized or minimized. Then write an equation describing the quantity that needs to be optimized.

In this problem, the area needs to be ***maximized***. Therefore, the equation must describe the area of the rectangular region.

$$A = lw$$

(d) The equation in (c) cannot be used directly to maximize the area because there are too many variables. Use the constraint equation to solve for l in terms of w. Then rewrite the equation in (c) in such a way that A is expressed entirely in terms of w.

$\because 2l + 2w = 400$

$\therefore \frac{2l}{2} + \frac{2w}{2} = \frac{400}{2}$

$\therefore l + w = 200$

$\therefore l + w - w = 200 - w$

$\therefore l = 200 - w$

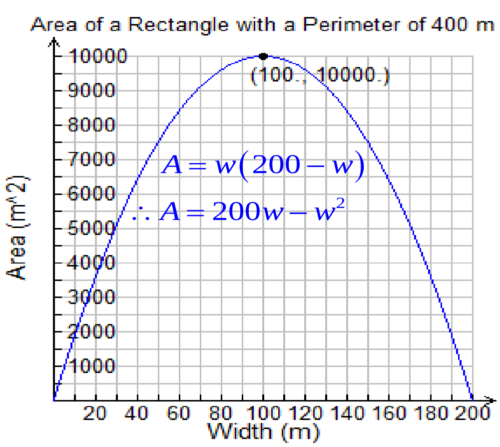
$\therefore A = lw$

$\therefore A = (200 - w)w$

$\therefore A = w(200 - w)$

Now the area has been expressed ***in terms of one variable only*** (i.e. the width).

(e) Sketch a graph of area of the rectangle versus width. Label the axes and include a title.



(f) Is the relationship between A and w linear or non-linear? Give ***three*** reasons to support your answer.

The relation is ***non-linear***. We know this because of the following reasons.

- The graph is curved.
- The equation has a squared term (w^2).
- The first differences are ***not constant***.

w	A	ΔA
20	3600	-
30	5100	1500
40	6400	1300
50	7500	1100
60	8400	900

(g) State the dimensions of the rectangular region having a perimeter of 400 m and a ***maximal*** area.

From the graph it can be seen that the maximum area is 10000 m^2, which is attained when the width is 100 m. Therefore, for maximum area,

$w = 100$ and $l = 200 - w = 200 - 100 = 100$.

For maximal area, both the length and the width should be 100 m. In other words, the region should be a square with side length of 100 m.

Optimization Problem 2

Design a cylindrical pop can that has the ***greatest possible capacity*** but can be manufactured using at most 375 cm^2 of aluminum.

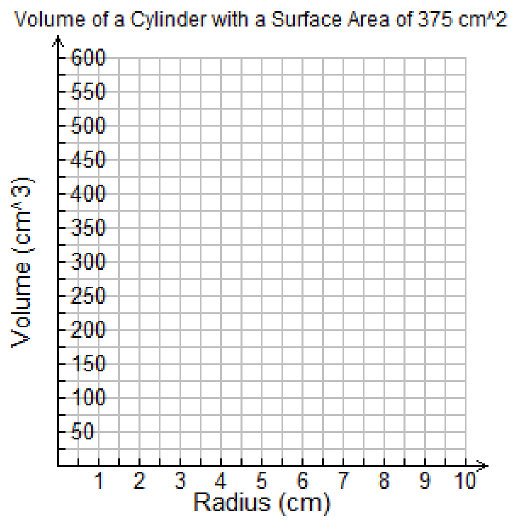
(a) What is the ***constraint*** in this problem?

(b) Draw a diagram describing this situation. Then write an equation that relates the unknowns to the constraint. (Let r represent the radius of the cylinder and let h represent its height.)

(c) What quantity needs to be optimized? State whether the quantity needs to be maximized or minimized. Then write an equation describing the quantity that needs to be optimized.

(d) The equation in (c) cannot be used directly to maximize the volume because there are too many variables. Use the constraint equation to solve for h in terms of r. Then rewrite the equation in (c) in such a way that V is expressed entirely in terms of r.

(e) Sketch a graph of volume of the cylindrical can versus radius. Label the axes and include a title.



(f) Is the relationship between V and r linear or non-linear? Give ***three*** reasons to support your answer.

r	V	ΔV

(g) State the dimensions of the cylindrical can having a surface area of 375 cm^2 and a ***maximal*** volume.

Optimization Problem 3

A container for chocolates must have the shape of a ***square prism*** and it must also have a volume of 8000 cm^3. Design the box in such a way that it can be manufactured using the ***least amount of material***.

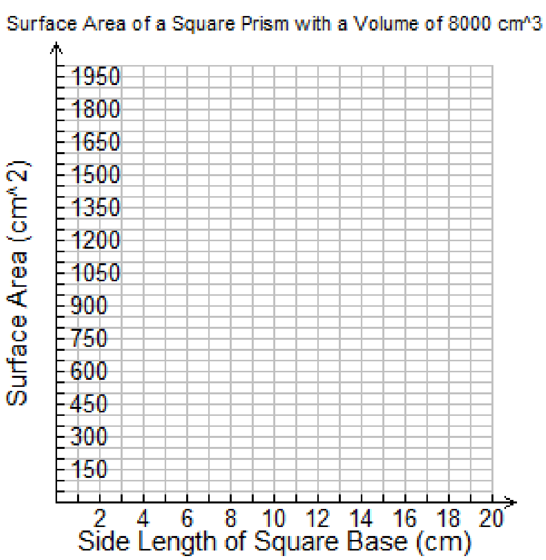
(a) What is the ***constraint*** in this problem?

(b) Draw a diagram describing this situation. Then write an equation that relates the unknowns to the constraint. (Let x represent the side length of the square base and let h represent the height.)

(c) What quantity needs to be optimized? State whether the quantity needs to be maximized or minimized. Then write an equation describing the quantity that needs to be optimized.

(d) The equation in (c) cannot be used directly to ***minimize*** the surface area because there are too many variables. Use the constraint equation to solve for h in terms of x. Then rewrite the equation in (c) in such a way that A is expressed entirely in terms of x.

(e) Sketch a graph of volume of the cylindrical can versus width. Label the axes and include a title.



(f) Is the relationship between A and x linear or non-linear? Give ***three*** reasons to support your answer.

x	A	ΔA

(g) State the dimensions of the square prism with a volume of 8000 cm^3 and a ***minimal*** surface area.

GEOMETRY

Classify Triangles

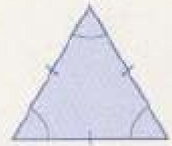
Classify Triangles

Triangles can be classified using their side lengths or their angle measures.

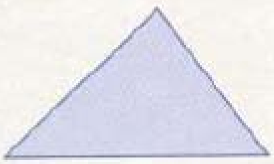
scalene triangle
- no equal sides
- no equal angles

isosceles triangle
- two equal sides
- two equal angles

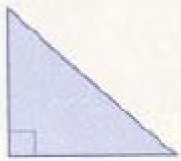
equilateral triangle
- three equal sides
- three equal angles



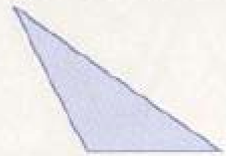
acute triangle
- three acute angles (less than 90°)



right triangle
- one right angle (90°)



obtuse triangle
- one obtuse angle (between 90° and 180°)



Classify Polygons

Classify Polygons

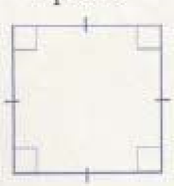
A **polygon** is a closed figure formed by three or more line segments.

A **regular polygon** has all sides equal and all angles equal.

Some quadrilaterals have special names. A regular quadrilateral is a **square**. An irregular quadrilateral may be a **rectangle**, a **rhombus**, a **parallelogram**, or a **trapezoid**.

Number of Sides	Name
3	triangle
4	quadrilateral
5	pentagon
6	hexagon

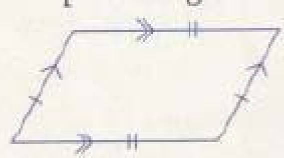
square



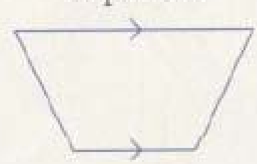
rhombus

rectangle

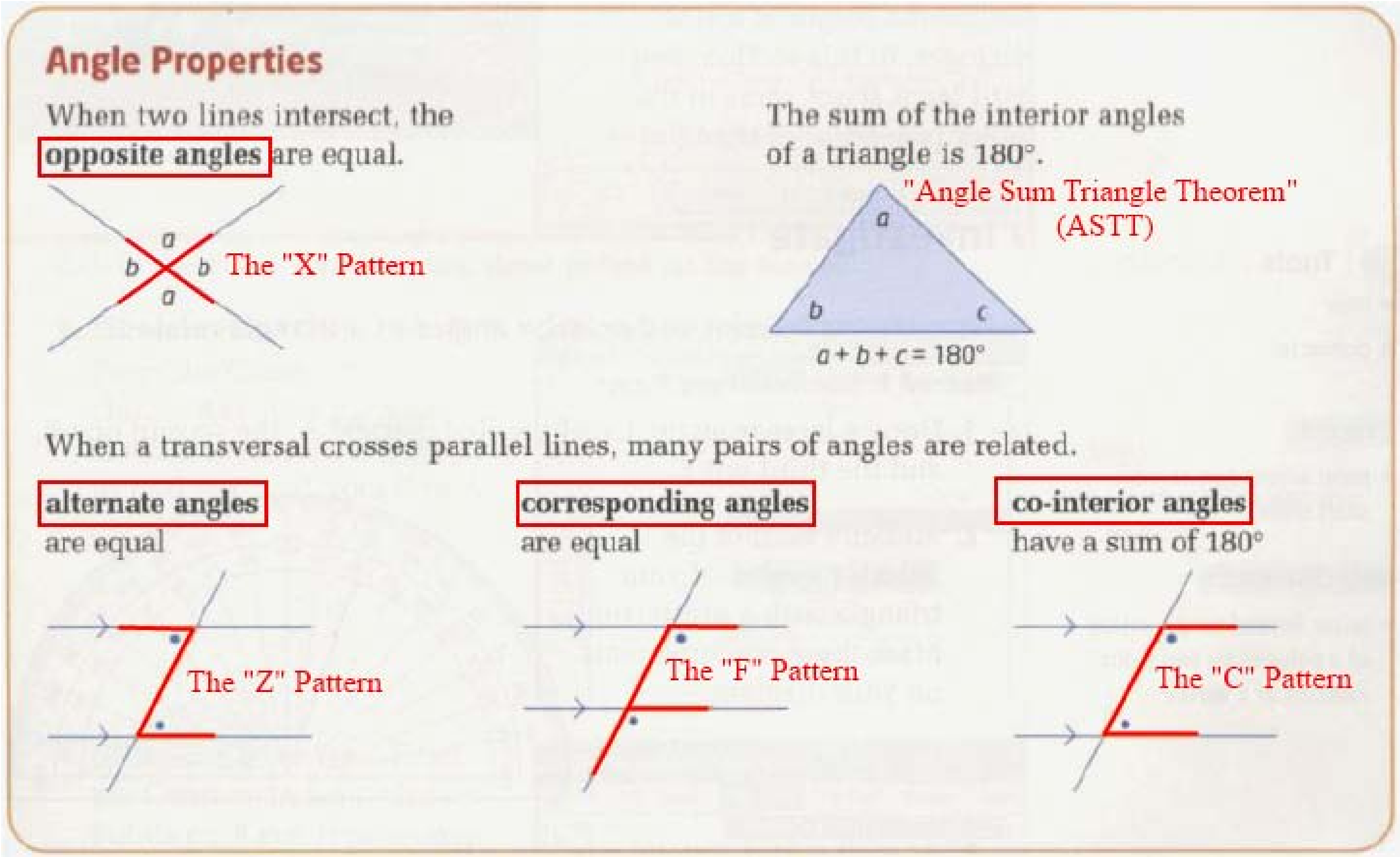
parallelogram



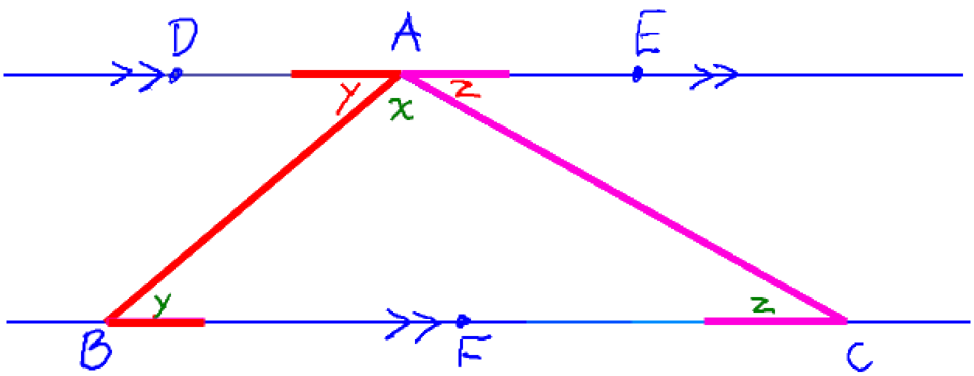
trapezoid



Angle and Triangle Properties



Proof of ASTT



$\angle DAB = \angle ABF = y$ (alternate angles)
$\angle EAC = \angle ACF = z$ (alternate angles)
} "Z"

But $\angle DAB + \angle BAC + \angle EAC = 180^\circ$ (straight line)

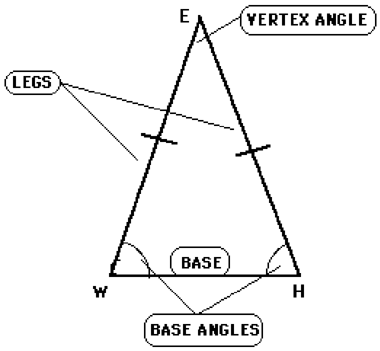
$\therefore y + x + z = 180^\circ$

$\therefore$ the sum of the interior angles of a triangle MUST BE 180°

Angle Sum Triangle Theorem

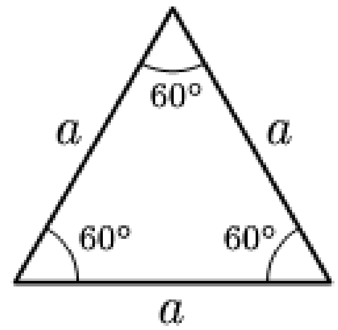
Angles in Isosceles and Equilateral Triangles

- The ***Isosceles Triangle Theorem (ITT)*** asserts that a triangle is isosceles if and only if its ***base angles are equal***.
- This can be proved using ***triangle congruence theorems*** (not covered in this course).



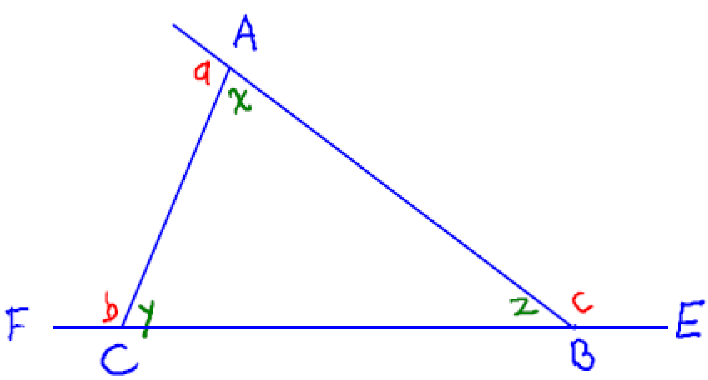
- Using ITT, it can be shown that an ***equilateral triangle*** is also ***equiangular*** (all three angles have the same measure).
- If x represents the measure of each angle, then

$x+x+x=180°$ (ASTT)
$\therefore 3x=180°$
$\therefore x=60°$



Exterior Angle Theorem (EAT)

Exterior Angles of a Triangle



$x+y+z=180°$ (ASTT)
$c+z=180°$ (straight line)

$\therefore c=x+y$

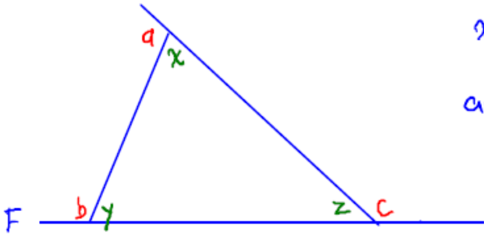
$\therefore$ exterior $\angle ABE$ is the sum of $\angle CAB$ and $\angle ACB$

Using a similar argument, we can show that $b=x+z$ and $a=y+z$

Exterior Angle Theorem

The exterior angle at each vertex of a triangle is equal to the sum of the interior angles at the other two vertices.

Sum of the Exterior Angles of a Triangle



$x + y + z = 180°$ (ASTT)

$a + x + b + y + c + z$
$= 180° + 180° + 180°$ (straight lines)
$= 540°$

But $a + x + b + y + c + z$
$= a + b + c + x + y + z$
$= a + b + c + 180°$ (since $x + y + z = 180°$)
$= 540°$

$\therefore a + b + c + 180° = 540°$

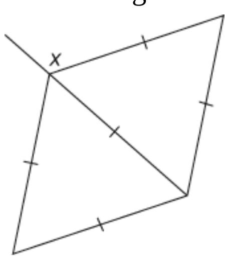
$\therefore a + b + c + 180° - 180° = 540° - 180°$

$\therefore a + b + c = 360°$

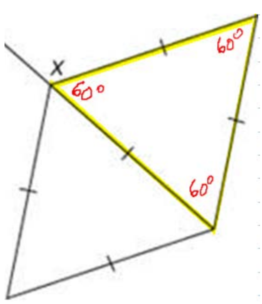
In any triangle, the sum of the exterior angles MUST BE 360°

Examples

1. Find the measure of the exterior angle labelled "x."

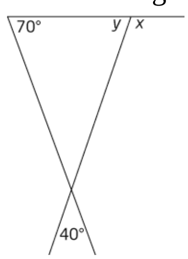


Solution



The interior angles of an equilateral triangle must have a measure of 60°. Therefore, by supplementary angles, $x = 180° - 60° = 120°$.

2. Find the measures of the unknown angles.

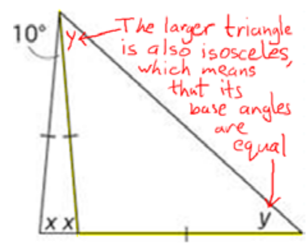


Solution

By the opposite angle theorem, the unlabelled missing angle in the triangle must have a measure of 40°. Then, by ASTT, $y = 180° - 70° - 40° = 70°$.

By supplementary angles, $x = 180° - 70° = 110°$

3. Find the measures of the unknown angles.



Solution

The "skinny" triangle is an isosceles triangle, which means that its base angles are equal.

Therefore,

$x + x + 10° = 180°$ (ASTT)

$\therefore 2x + 10° = 180°$

$\therefore 2x = 180° - 10°$

$\therefore 2x = 170°$

$\therefore x = 85°$

The larger triangle is also isosceles, which means that its base angles are equal. Therefore,

$x = y + y$ (EAT)

$\therefore x = 2y$

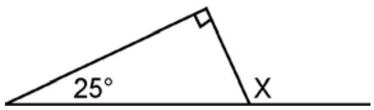
$\therefore \frac{x}{2} = y$

$\therefore y = \frac{85°}{2} = 42.5°$

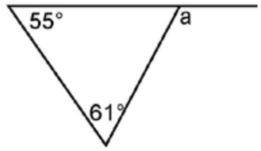
Practice: Angle Relationships in Triangles

1. Find the measure of each indicated exterior angle.

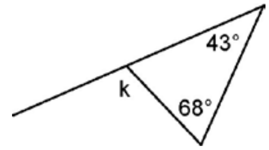
(a)



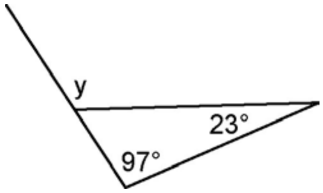
(b)



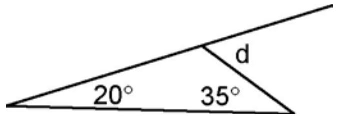
(c)



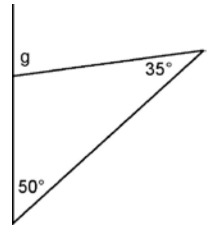
(d)



(e)

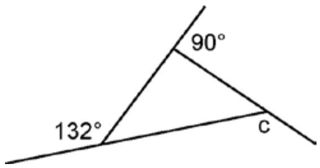


(f)

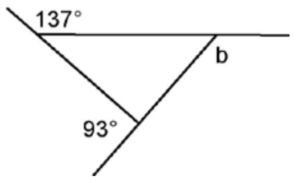


2. Find the measure of each indicated exterior angle.

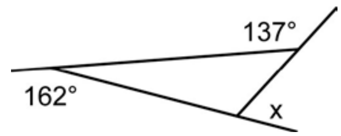
(a)



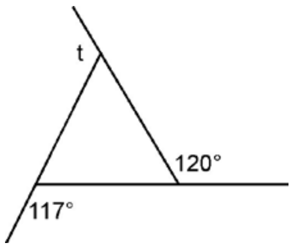
(b)



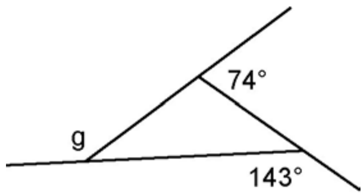
(c)



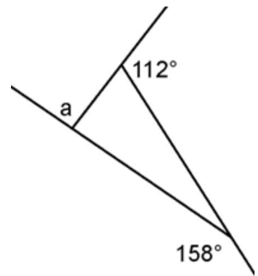
(d)



(e)

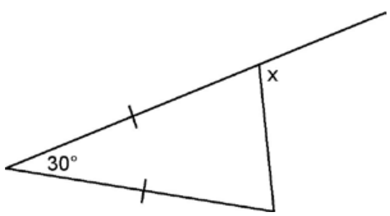


(f)

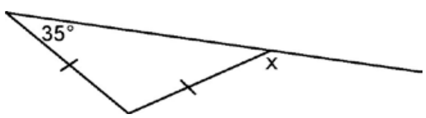


3. Find the measure of each indicated exterior angle.

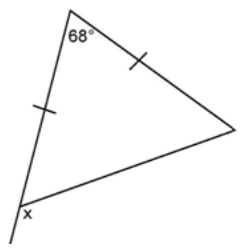
(a)



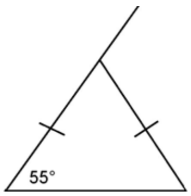
(b)



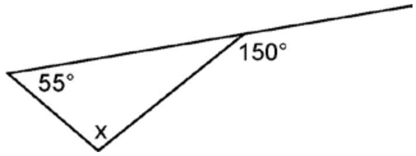
(c)



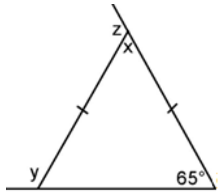
(d)



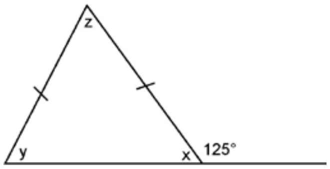
(e)



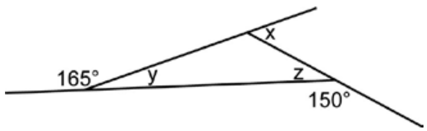
(f)



(g)



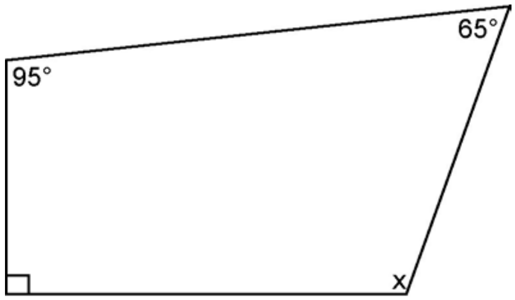
(h)



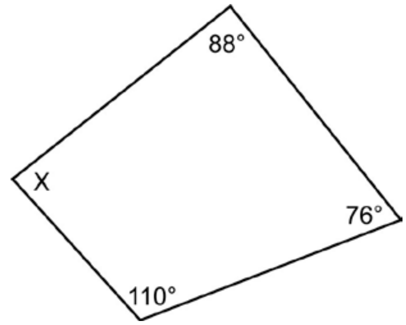
4. One interior angle in an isosceles triangle measures 42°. Find the possible measures for the exterior angles.

5. Find the measure of each indicated angle. **Hint:** Divide the quadrilaterals into triangles.

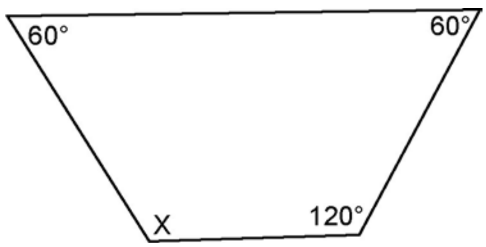
(a)



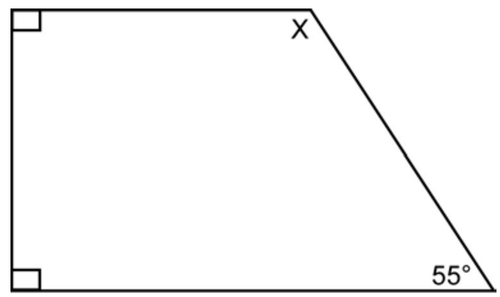
(b)



(c)



(d)



Answers

1. **a)** 115° **b)** 116° **c)** 111°
d) 120° **e)** 55° **f)** 85°

2. **a)** 138° **b)** 130° **c)** 61°
d) 123° **e)** 143° **f)** 90°

3. **a)** 105° **b)** 145° **c)** 124° **d)** 110°
e) $x = 95°$ **f)** $x = 50°$; $y = 115°$; $z = 130°$
g) $x = y = 55°$; $z = 70°$ **h)** $x = 45°$; $y = 15°$; $z = 30°$

4. 138°, 138°, 84° or 138°, 111°, 111°

5. **a)** 110° **b)** 86°
c) 120° **d)** 125°

Properties of Polygons

Polygon Definition

A ***polygon*** is a closed plane figure bounded by ***three or more*** line segments.

Regular Polygon Definition

A ***regular polygon*** is a polygon in which all sides have the same length (***equilateral***) and all angles have the same measure (***equiangular***).

Irregular Polygon Definition

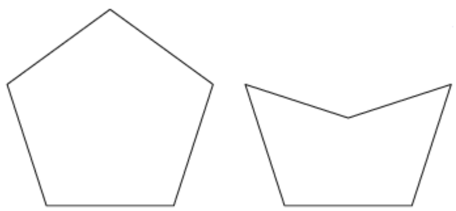
An ***irregular polygon*** is a polygon in which ***not*** all sides have the same length and ***not*** all angles have the same measure.

Convex Polygon Definition

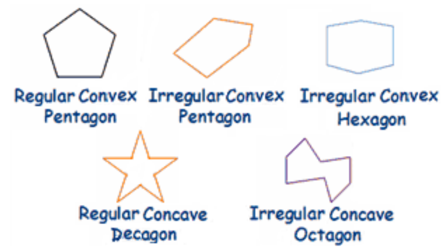
A ***convex polygon*** is a polygon that contains all line segments connecting any two of its vertices. In a convex polygon, the measure of ***each interior angle*** must be less than 180°.

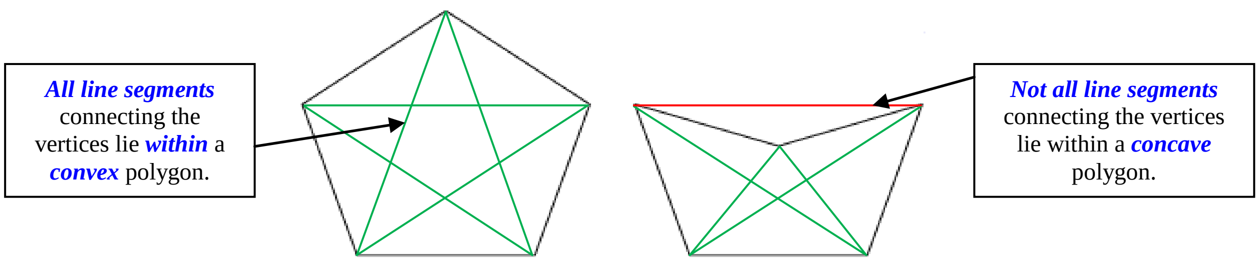
Concave Polygon Definition

A ***concave polygon*** is a polygon that ***does not*** contain all line segments connecting any two of its vertices. In a concave polygon, the measure of ***at least one interior angle*** is more than 180°. That is, a concave polygon must contain at least one ***reflex angle***.



convex polygon *concave polygon*

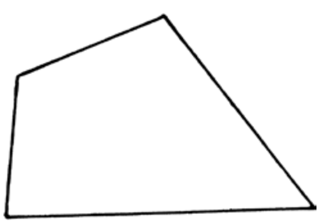
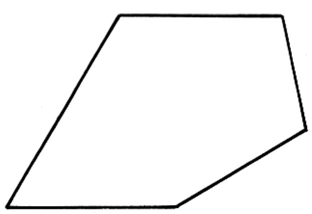
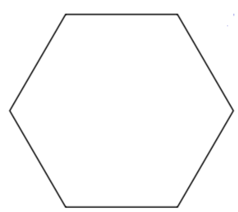
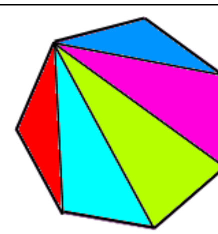




convex polygon *concave polygon*

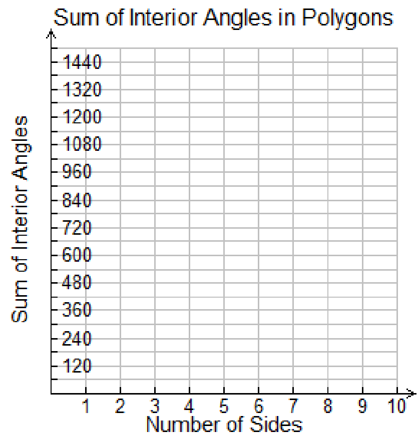
Sum of the Interior Angles of a Convex Polygon

1. By dividing each polygon into triangles, calculate the sum of the interior angles of the following convex polygons. Note that one of the shapes has already been done for you.

			
Name: ________________	Name: ________________	Name: ________________	Name: Heptagon
Number of Sides: _______	Number of Sides: _______	Number of Sides: _______	Number of Sides: 7
Number of Triangles: ____	Number of Triangles: ____	Number of Triangles: ____	Number of Triangles: 5
Sum of Interior Angles: ______________________	Sum of Interior Angles: ______________________	Sum of Interior Angles: ______________________	Sum of Interior Angles: $5(180°) = 900°$

2. Now summarize your results in the following table and sketch a graph relating the sum of the interior angles of a convex polygon to the number of sides. Then answer questions (a) to (f).
n = number of sides in the polygon, s = sum of the interior angles of the polygon

n	s	Δs (1st Differences)
3		
4		
5		
6		
7	900°	



(a) Do you expect the pattern to continue indefinitely beyond $n = 7$? Explain.

(b) Write an equation relating s to n. Explain why it is not surprising that the relation between s and n is linear.

(c) State the ***meaning*** of the slope of the linear relation between s and n.

(d) Does the vertical intercept of this linear relation have a meaning? Explain.

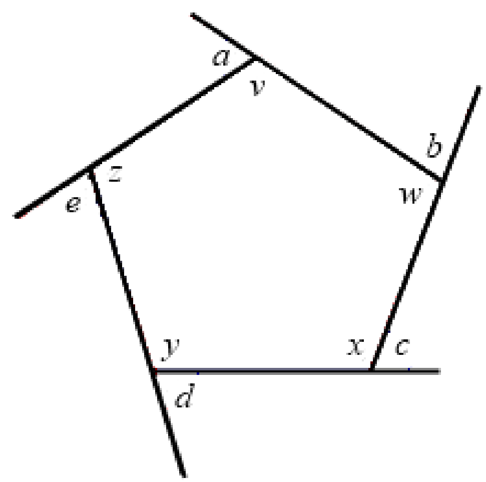
(e) Does it make sense to "connect the dots" in the above graph? Explain.

(f) State an easy way to remember how to calculate the sum of the interior angles of a polygon.

Sum of the Exterior Angles of a Convex Polygon

The argument presented here for a pentagon can be used for a polygon with any number of sides.

In the pentagon at the right, ***there are five straight line angles***, that is, there are five 180° angles. Therefore,



$$(a+v)+(b+w)+(c+x)+(d+y)+(e+z)=180°+180°+180°+180°+180°$$

$$\therefore a+v+b+w+c+x+d+y+e+z=5(180°)$$

$$\therefore a+b+c+d+e+v+w+x+y+z=5(180°)$$

$\therefore a+b+c+d+e+3(180°)=5(180°)$ (Since $v+w+x+y+z$ is the sum of the interior angles of the pentagon.)

$$\therefore a+b+c+d+e+3(180°)-3(180°)=5(180°)-3(180°)$$

$$\therefore a+b+c+d+e=2(180°)$$

$$\therefore a+b+c+d+e=360°$$

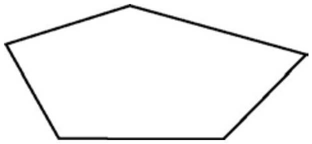
For a convex polygon with n sides…

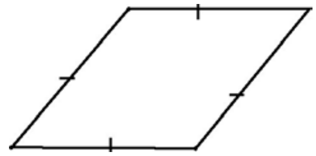
- There are n straight line angles, for a total measure of $180°n$.
- The sum of the n interior angles $180°(n-2)$.
- The sum of the exterior angles is equal to $180°n-180°(n-2)=180°n-180°n+360°=360°$

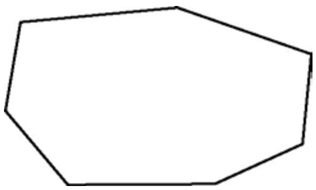
The ***sum of the exterior angles*** of ***any convex polygon*** is 360°.

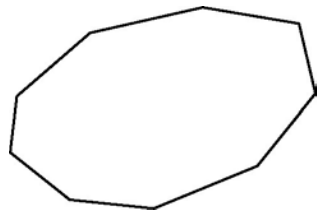
Practice: Angle Relationships in Polygons

1. Find the sum of the interior angles of each polygon.

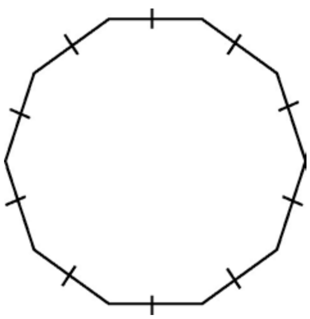
a) 

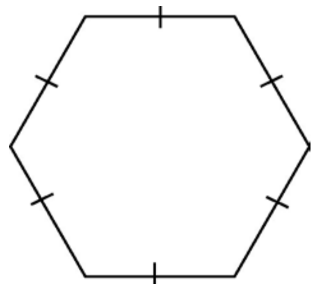
b) 

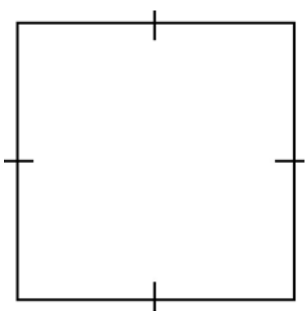
c) 

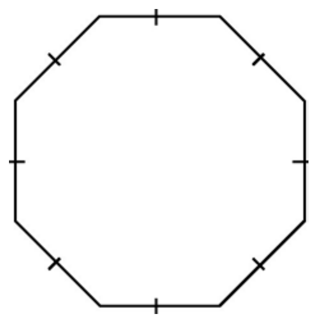
d) 

2. Find the sum of the interior angles of each polygon.

a) 

b) 

c) 

d) 

3. Find the sum of the interior angles of a polygon with each number of sides.

a) 11 sides **b)** 14 sides **c)** 18 sides **d)** 24 sides

4. **4.** Find the measure of each interior angle of a regular polygon with each number of sides.

a) 3 sides **b)** 20 sides **c)** 9 sides **d)** 16 sides

5. Find the number of sides each polygon has given the sum of its interior angles.

a) 720° **b)** 1980° **c)** 2340° **d)** 4140°

Answers

1. **a)** 540° **b)** 360° **c)** 900° **d)** 1260°
2. **a)** 1440° **b)** 720° **c)** 360° **d)** 1080°
3. **a)** 1620° **b)** 2160° **c)** 2880° **d)** 3960°

4. **a)** 60° **b)** 162° **c)** 140° **d)** 157.5°
5. **a)** 6 sides **b)** 13 sides
c) 15 sides **d)** 25 sides

