Unit 6 – Measurement and Geometry – Justifications

UNIT 6 – MEASUREMENT AND GEOMETRY – JUSTIFICATIONS 1

UNDERSTANDING *WHY* THE EQUATIONS ARE CORRECT 2

Understanding the Meaning of π and how it Relates to the Circumference of a Circle 2
Examples 2
Understanding Area Equations 3
Understanding the Pythagorean Theorem 4
Pythagorean Theorem Proof 1 – President James Garfield's Brilliant Proof 4
Pythagorean Theorem Proof 2 5
Pythagorean Theorem Proof 3 5
Consequence of the Pythagorean Theorem 5
Research 6
Understanding Surface Area Equations 7
Understanding Volume Equations 9

UNDERSTANDING GEOMETRIC RELATIONSHIPS 11

Angle and Triangle Properties 11
Proof of ASTT 11
Angles in Isosceles and Equilateral Triangles 12
Exterior Angle Theorem (EAT) 12
Sum of the Exterior Angles of a Triangle 13
Sum of the Interior Angles of a Convex Polygon 13
Sum of the Exterior Angles of a Convex Polygon 14

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UNDERSTANDING *WHY* THE EQUATIONS ARE CORRECT

Understanding the Meaning of π and how it Relates to the Circumference of a Circle

The following is an example of a typical conversation between Mr. Nolfi and a student who blindly memorizes formulas:

Student: Sir, I can't remember whether the area of a circle is πr^2 or $2\pi r$. Which one is it?

Mr. Nolfi: If you remember the meaning of π, you should be able to figure it out.

Student: How can 3.14 help me make this decision? It's only a number!

Mr. Nolfi: How dare you say something so disrespectful about one of the most revered numbers in the mathematical lexicon! (Just kidding. I wouldn't really say that.) It's true that the number 3.14 is an approximate value of π. But I asked you for its ***meaning***, not its value.

Student: I didn't know that π has a meaning. I thought that it was just a "magic" number.

Mr. Nolfi: Leave magic to the magicians. In mathematics, every term (except for primitive terms) has a very precise definition. Read the following carefully and you'll never need to ask your original question ever again!

In ***any*** circle, the ***ratio*** of the ***circumference*** to the ***diameter*** is equal to a ***constant*** value that we call π. That is,

$$C : d = \pi \, .$$

Alternatively, this may be written as

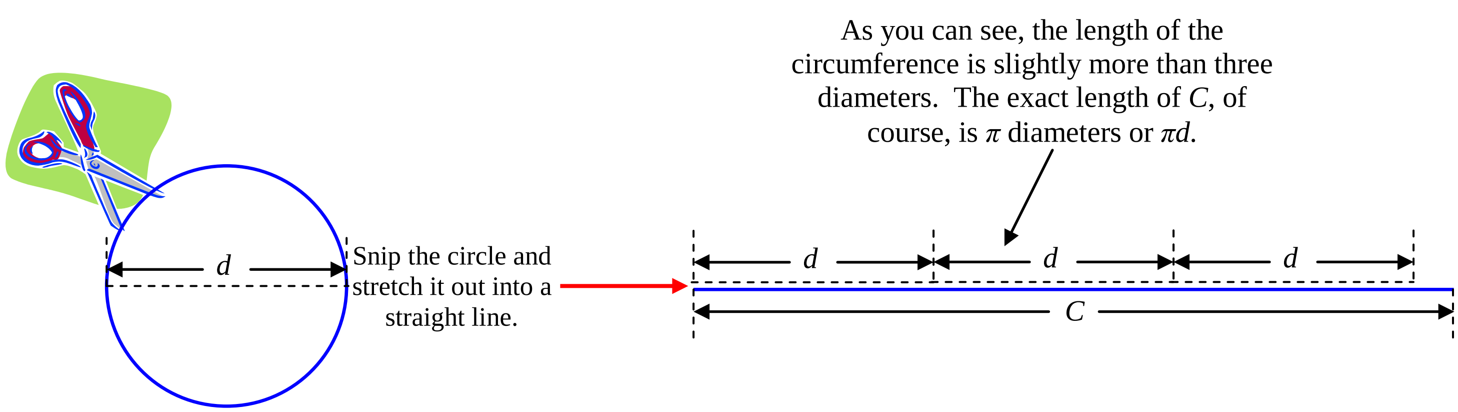
$$\frac{C}{d} = \pi$$

or, by multiplying both sides by d, in the more familiar form

$$C = \pi d \, .$$

If we recall that $d = 2r$, then we finally arrive at the most common form of this ***relationship***,

$$C = 2\pi r \, .$$



Mr. Nolfi: So you see, by understanding the meaning of π, you can ***deduce*** that $C = 2\pi r$. Therefore, the formula for the area must be $A = \pi r^2$. Furthermore, it is not possible for the expression $2\pi r$ to yield units of area. The number 2π is dimensionless and r is measured in units of distance such as metres. Therefore, the expression $2\pi r$ must result in a value measured in units of distance. On the other hand, the expression πr^2 must give a value measured in units of area because $r^2 = r(r)$, which involves multiplying a value measured in units of distance by itself. Therefore, by considering units alone, we are drawn to the inescapable conclusion that the area of a circle must be πr^2 and **not** $2\pi r$!

Examples

$2\pi r \doteq 2(3.14)(3.6 \text{ cm}) = 22.608 \text{ cm} \rightarrow$ This answer cannot possibly measure area because cm is a unit of distance.

Therefore, πr^2 must be the correct expression for calculating the area of a circle.

$\pi r^2 \doteq 3.14(3.6 \text{ cm})^2 = 3.14(3.6 \text{ cm})(3.6 \text{ cm}) = 40.6944 \text{ cm}^2 \rightarrow$ Notice that the unit "cm^2" is appropriate for area.

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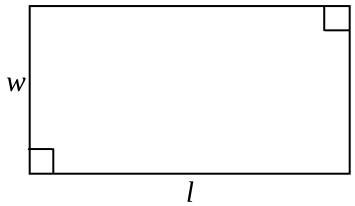
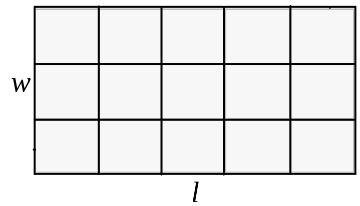
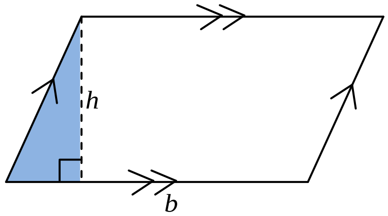
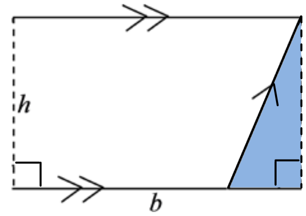
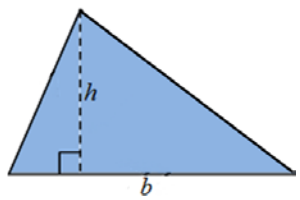
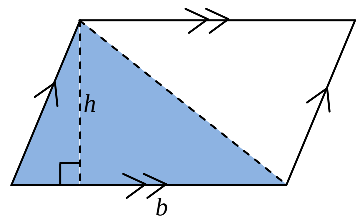
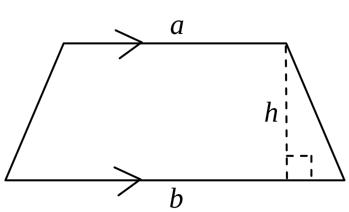
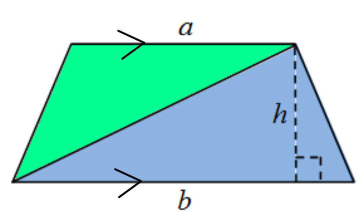
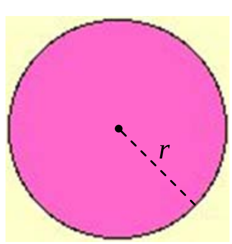
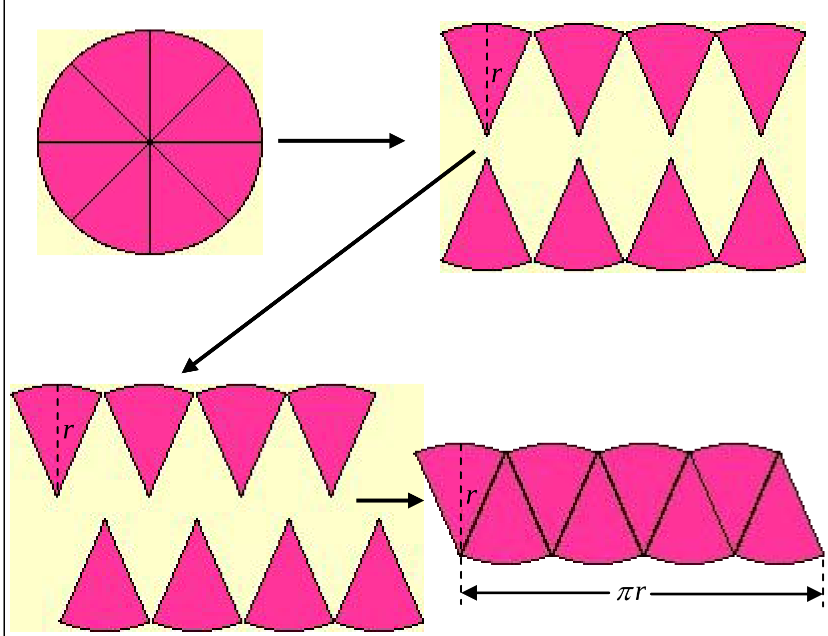
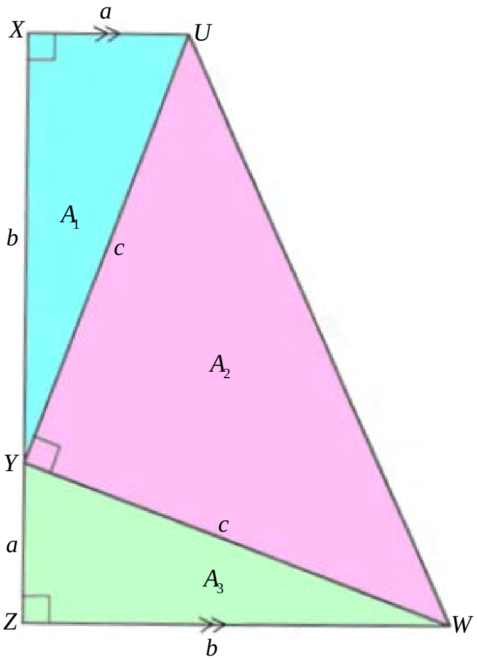
Understanding Area Equations

Shape	*Explanation*	*Equation for Area*
	 • Each square has an area of 1 unit^2 • Total number of squares = (# of squares in each row)×(# rows) = 5(3) = lw	$A = lw$
	 • Cut off the shaded right triangle at the left end of the parallelogram. • Attach the cut-off right triangle at the right side. • A rectangle is formed. Its length is b and its width is h, which means that its area must be bh.	$A = bh$
	 • Begin with a parallelogram having the same base and height. • Cut the parallelogram ***in half*** along the dashed diagonal. • Since the parallelogram's area is bh, the triangle's area must be $\frac{bh}{2}$.	$A = \frac{bh}{2}$ or $A = \frac{1}{2}bh$
	 • Divide the trapezoid into two triangles as shown. • Both triangles have a height of h. One triangle has a base of a and the other has a base of b. • Total area $= \frac{ah}{2} + \frac{bh}{2} = \frac{ah + bh}{2} = \frac{h(a+b)}{2}$	$A = \frac{h(a+b)}{2}$ or $A = \frac{1}{2}h(a+b)$
 The argument presented here for the area of a circle is lacking in rigour. It is based on the assumption that we can use the equation for the area of a parallelogram ($A = bh$) to calculate the area of a shape that is similar to a parallelogram. A more rigorous argument can be constructed using calculus.		• As shown in the diagrams, divide the circle into an even number of sectors, all of which have the same size. • Rearrange the sectors as shown, then fit them together. • The resulting shape is very close to a parallelogram, so its area should be about $bh = (\pi r)r = \pi r^2$ $A = \pi r^2$

Understanding the Pythagorean Theorem

Pythagorean Theorem Proof 1 – President James Garfield's Brilliant Proof

James A. Garfield was the 20th president of the United States. In addition to being a highly successful statesman and soldier, President Garfield was also a noted scholar. Among his many scholarly accomplishments is his beautiful proof of the Pythagorean Theorem. It is outlined below.



(a) Calculate the area of trapezoid *XZWU* by summing the areas of ΔUXY, ΔYZW and ΔUYW. Simplify fully!

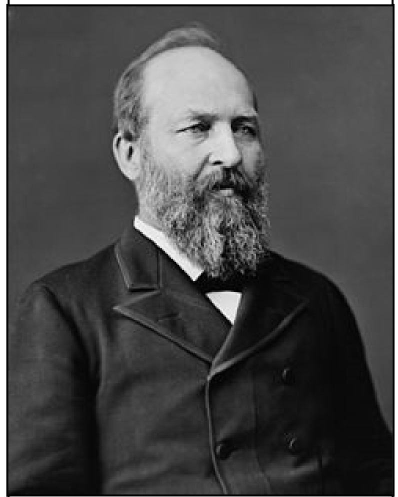
$$A_{Trap} = A_1 + A_2 + A_3$$
$$=$$

(b) Calculate the area of the trapezoid by using the equation for the area of a trapezoid. Simplify fully!

$$A_{Trap} = \frac{h\left(a+b\right)}{2}$$
$$=$$

Think! What is the height of the trapezoid?

James A. Garfield



20th President of the U.S.A.

In Office:
March 4, 1881- September 19, 1881

Assassinated at the age of 49
One of four assassinated presidents

(c) In parts (a) and (b) you developed two different expressions for the area of trapezoid *XZWU*. Since both expressions give the area of the ***same shape***, they must be equal to each other! Set the expressions equal to each other and solve for c^2.

$$A_{Trap} = A_{Trap}$$
$\therefore$

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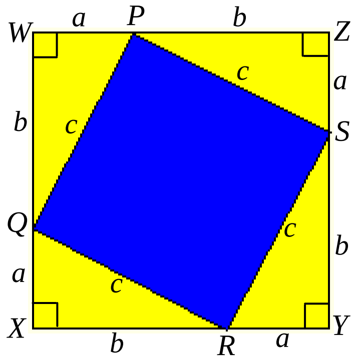
Pythagorean Theorem Proof 2

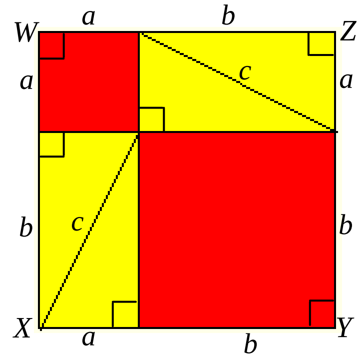
(a) Explain why quadrilateral *PQRS must* be a square.



(b) Use the above diagram to develop a proof of the Pythagorean Theorem. (**Hint:** The line of reasoning is similar to that of President Garfield's proof.)

Pythagorean Theorem Proof 3

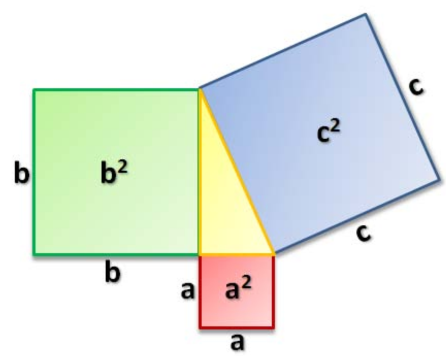
(a) Explain how the diagram at the right is simply a rearrangement of the pieces in the diagram at the left.



(b) Explain why the two diagrams together make it ***obvious*** that $a^2 + b^2 = c^2$. (This proof was devised in 1939 by Maurice Laisnez, a high school student in the Junior-Senior High School of South Bend, Indiana.)

Consequence of the Pythagorean Theorem

Although the Pythagorean Theorem is an equation that relates the lengths of the sides of a right triangle, it can also be ***interpreted*** in terms of areas.

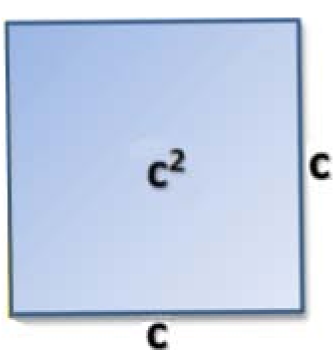


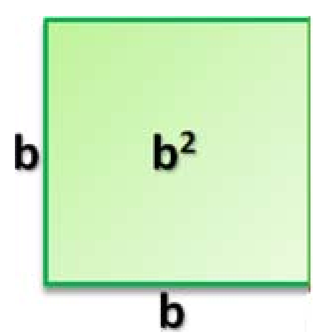
- We have ***proved*** that in a right triangle, the square of the hypotenuse must equal the sum of the squares of the other two sides.

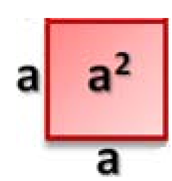
  That is, if *c* represents the length of the hypotenuse and *a* and *b* respectively represent the lengths of the other two sides, then $c^2 = a^2 + b^2$.

- By examining the diagram at the right, one can easily see that the expressions a^2, b^2 and c^2 are all areas of squares!

  Since $c^2 = a^2 + b^2$, ***it must also be true*** that

Area of 

= **Area of** 

+ **Area of** 

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Research

1. There are literally hundreds of different proofs of the Pythagorean Theorem. Find a proof that you are able to understand and explain it in your own words. Include diagrams in your explanation.

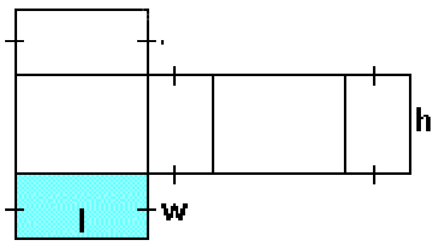
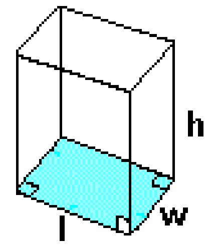
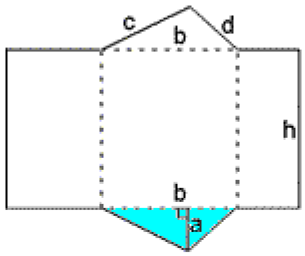
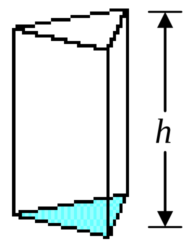
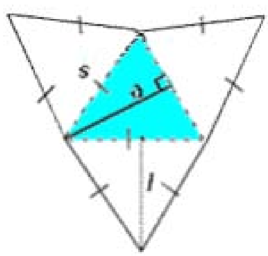
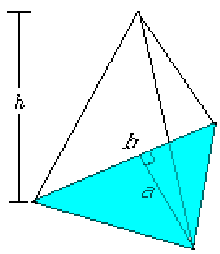
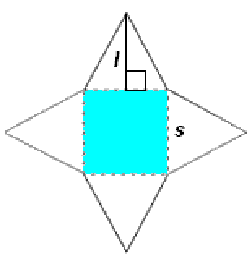
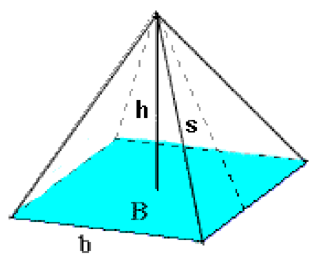
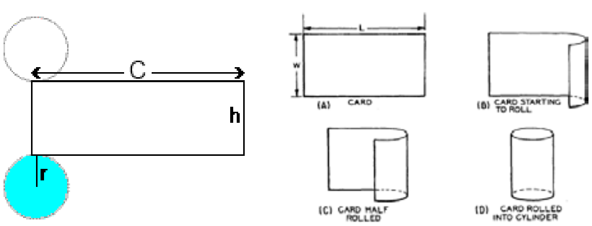
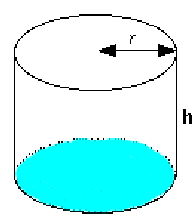
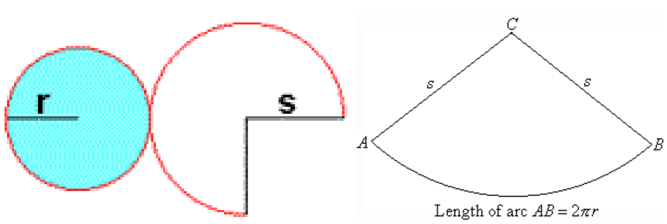
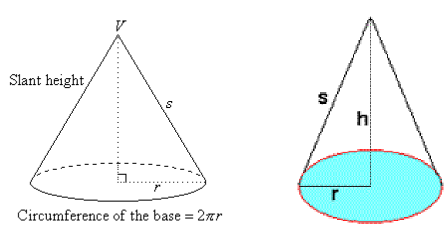
2. Do some research to answer the following questions:

(a) Who was Euclid?

(b) Euclid created a mathematical treatise, consisting of thirteen books, called the *Elements*. Why are Euclid's *Elements* considered so important?

(c) Find Euclid's proof of the Pythagorean Theorem and explain it in your own words.

Understanding Surface Area Equations

Net	*Solid*	*Surface Area Equation*
		
		
		
		
		
		

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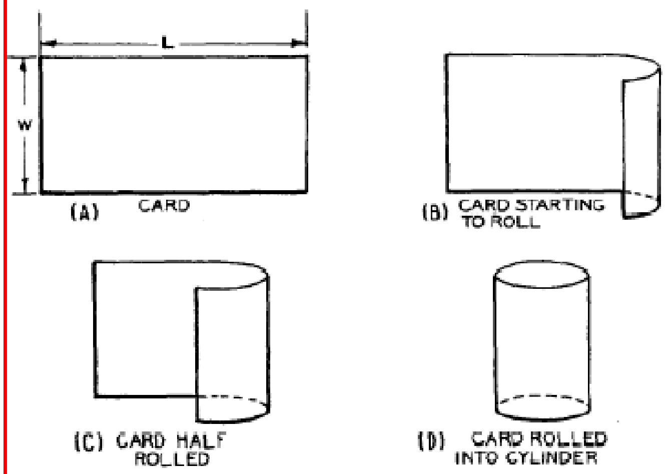
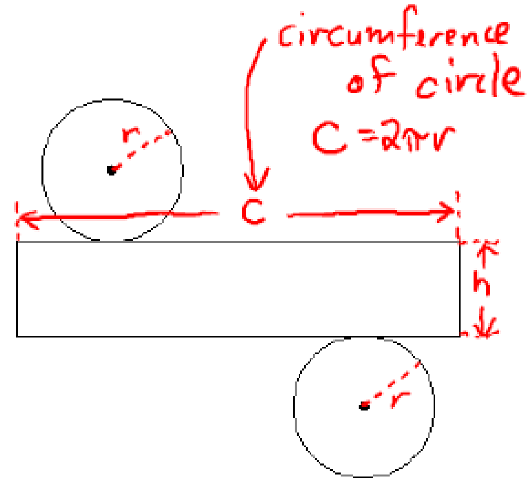
Surface Area of Cylinder



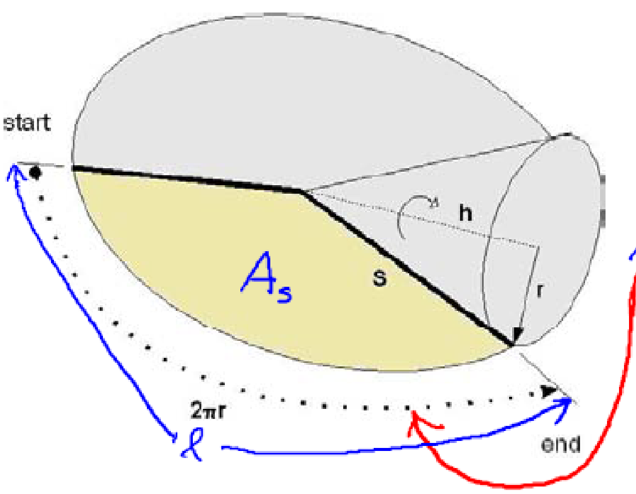


$SA = A_{Bases} + A_{Lateral Face}$

$\therefore SA =$ area of 2 circles + area of rectangle

$\therefore SA = 2\pi r^2 + 2\pi rh$

Surface Area of a Cone



A cone can be made by rolling up a SECTOR of a circle. (A slice of pizza is an example of a sector.)

ℓ = length of arc = circumference of circular base of cone

$= 2\pi r$

A_s = area of sector with arc length $2\pi r$

A_c = area of entire circle (radius s)

C = circumference of circle of radius s

Understanding Volume Equations

Volume of Any Prism

$V = (\text{area of base})(\text{height})$

Volume of any Pyramid

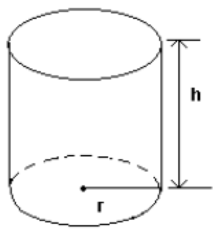
$V = \frac{1}{3}(\text{area of base})(\text{height})$

OR

$V = \frac{(\text{area of base})(\text{height})}{3}$

Volume of Cylinder

Cylinder: has 2 parallel congruent circular faces called bases. The lateral surface is a rolled up rectangle

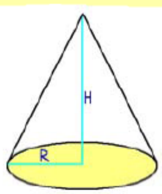


Volume = (area of base)(height)

$\therefore V = \pi r^2 h$

(πr^2 ← area of circular base; h ← height)

Volume of Cone



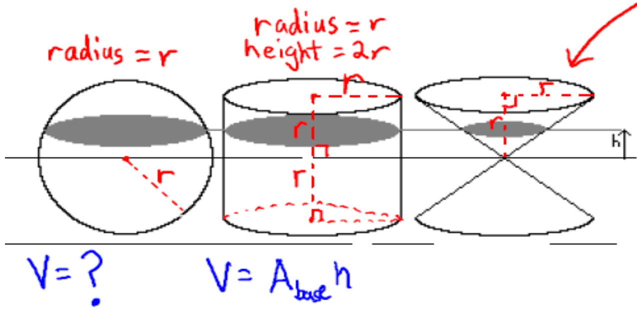
Volume $= \frac{1}{3}(\text{area of base})(\text{height})$

$\therefore V = \frac{1}{3}\pi r^2 h = \frac{\pi r^2 h}{3}$

Note: The following is a reasonably convincing argument that for a sphere of radius r,

$V = \frac{4}{3}\pi r^3$

Height of one cone $= r$
Height of double cone $= 2r$



Let h represent the distance from the "centre line" to the point at which the cross-section is taken

Cross-Sections

Circle

$h^2 + R^2 = r^2$

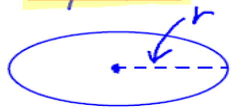
$\therefore R^2 = r^2 - h^2$

$A_{sp} = \pi R^2$

$\therefore A_{sp} = \pi(r^2 - h^2)$

$\therefore A_{sp} = \pi r^2 - \pi h^2$

Cylinder



$A_{cyl} = \pi r^2$

Notice that

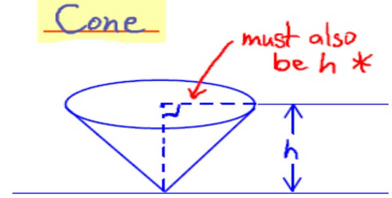
$A_{sp} + A_{co}$

$= \pi r^2 - \pi h^2 + \pi h^2$

$= \pi r^2$

$= A_{cyl}$

Cone



* The cone is constructed in such a way that the height = radius
$\therefore$ radius = h

$A_{co} = \pi h^2$

This suggests that

That is, our argument is strong but it does not prove definitively that $V_{sp} = \frac{4}{3}\pi r^3$

$V_{cyl} = V_{sp} + V_{co}$

$\therefore V_{sp} = V_{cyl} - V_{co}$

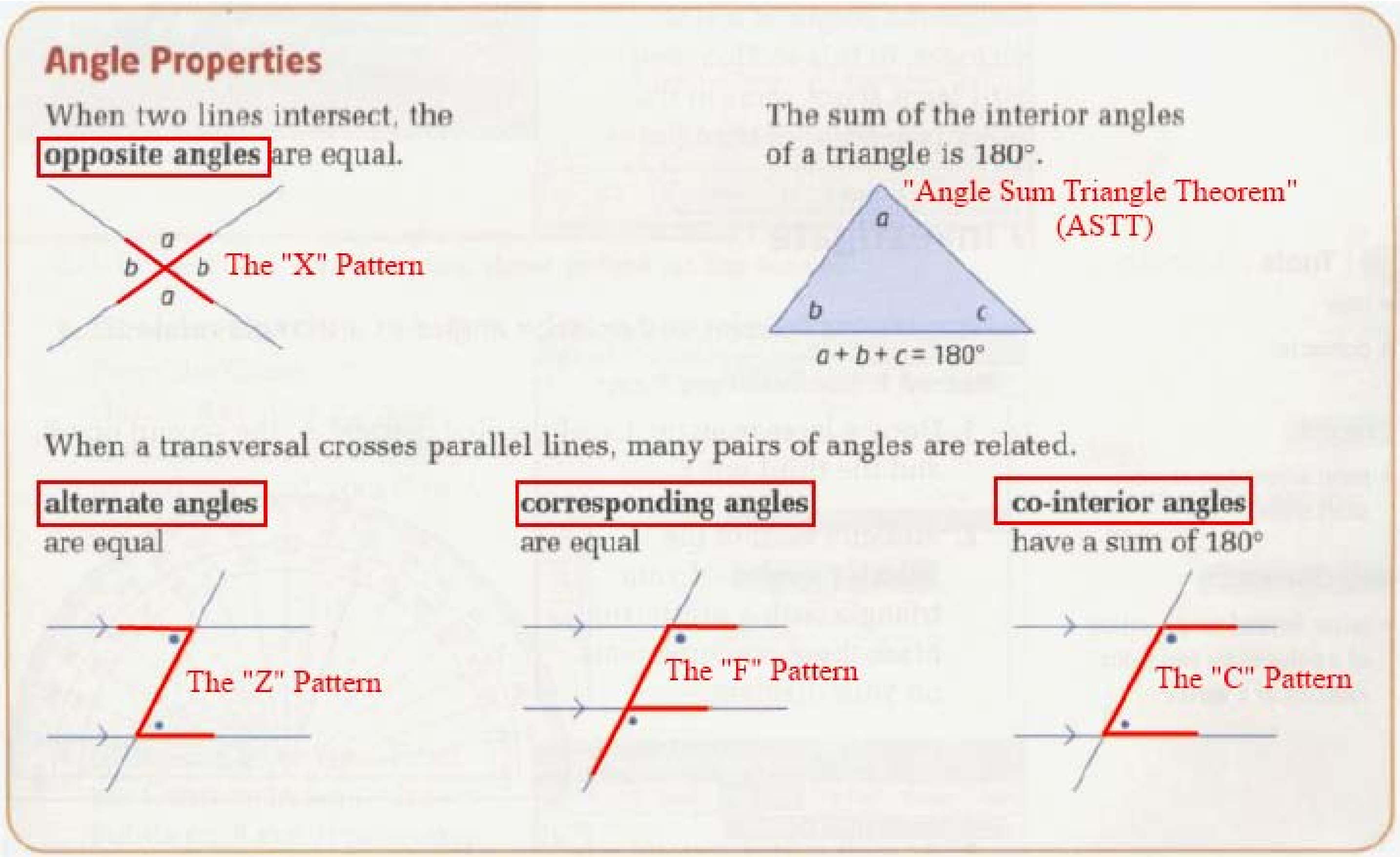
$= 2\pi r^3 - \frac{2}{3}\pi r^3$

$= \frac{6}{3}\pi r^3 - \frac{2}{3}\pi r^3$

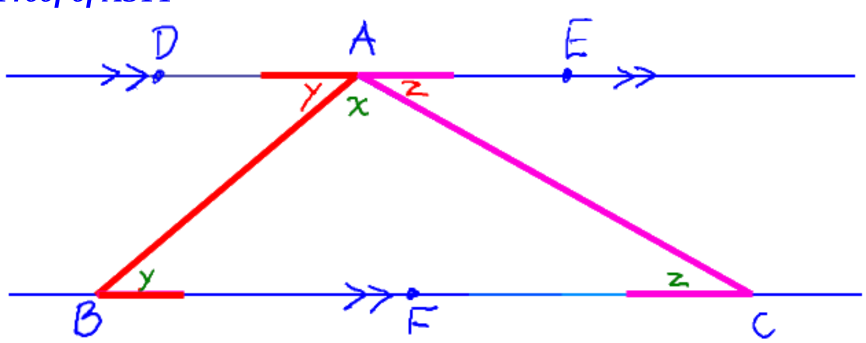
$= \frac{4}{3}\pi r^3$

UNDERSTANDING GEOMETRIC RELATIONSHIPS

Angle and Triangle Properties



Proof of ASTT



$\angle DAB = \angle ABF = y$ (alternate angles) } "Z"

$\angle EAC = \angle ACF = z$ (alternate angles) }

But $\angle DAB + \angle BAC + \angle EAC = 180^\circ$ (straight line)

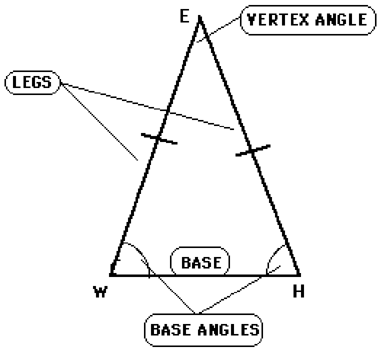
$\therefore y + x + z = 180^\circ$

$\therefore$ the sum of the interior angles of a triangle MUST BE 180°

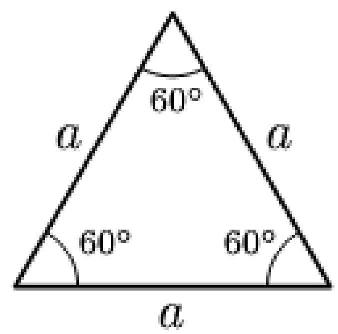
Angle Sum Triangle Theorem

Angles in Isosceles and Equilateral Triangles

- The ***Isosceles Triangle Theorem (ITT)*** asserts that a triangle is isosceles if and only if its ***base angles are equal***.
- This can be proved using ***triangle congruence theorems*** (not covered in this course).

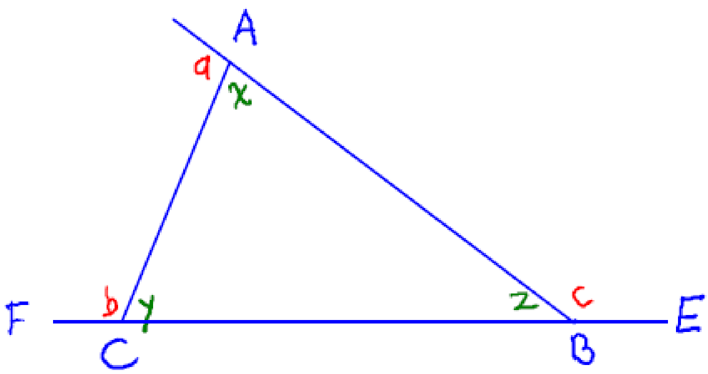


- Using ITT, it can be shown that an ***equilateral triangle*** is also ***equiangular*** (all three angles have the same measure).
- If x represents the measure of each angle, then
  $x+x+x=180°$ (ASTT)
  $\therefore 3x=180°$
  $\therefore x=60°$



Exterior Angle Theorem (EAT)

Exterior Angles of a Triangle



$x+y+z=180°$ (ASTT)

$c+z=180°$ (straight line)

$\therefore c=x+y$

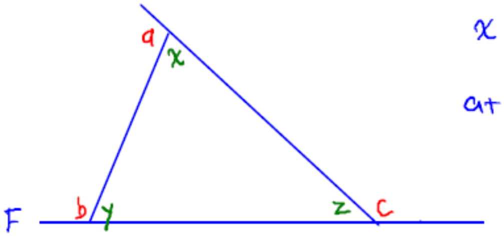
$\therefore$ exterior $\angle ABE$ is the sum of $\angle CAB$ and $\angle ACB$

Using a similar argument, we can show that $b=x+z$ and $a=y+z$

Exterior Angle Theorem

The exterior angle at each vertex of a triangle is equal to the sum of the interior angles at the other two vertices.

Sum of the Exterior Angles of a Triangle



$x + y + z = 180°$ (ASTT)

$a+x + b+y + c+z$
$= 180° + 180° + 180°$ (straight lines)
$= 540°$

But $a + x + b + y + c + z$
$= a + b + c + x + y + z$
$= a + b + c + 180°$ (since $x+y+z = 180°$)
$= 540°$

$\therefore a + b + c + 180° = 540°$

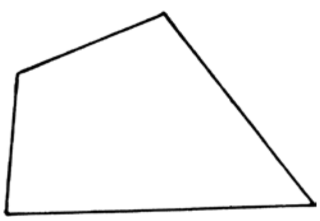
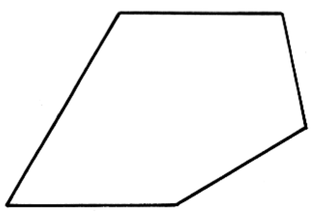
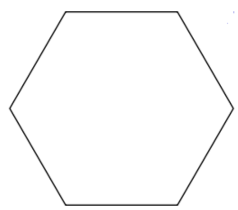
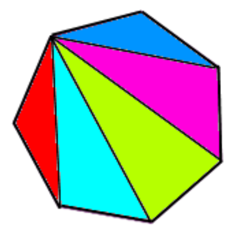
$\therefore a + b + c + 180° - 180° = 540° - 180°$

$\therefore a + b + c = 360°$

In any triangle, the sum of the exterior angles MUST BE 360°

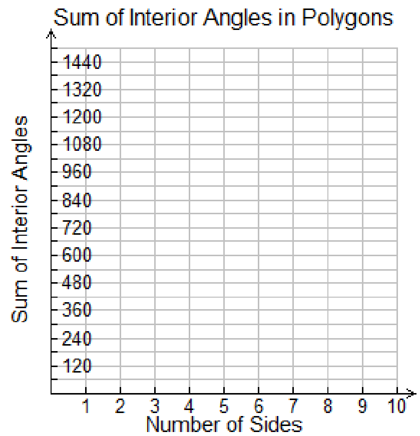
Sum of the Interior Angles of a Convex Polygon

1. By dividing each polygon into triangles, calculate the sum of the interior angles of the following convex polygons. Note that one of the shapes has already been done for you.

			
Name: ________________	Name: ________________	Name: ________________	Name: Heptagon
Number of Sides: _______	Number of Sides: _______	Number of Sides: _______	Number of Sides: 7
Number of Triangles: ____	Number of Triangles: ____	Number of Triangles: ____	Number of Triangles: 5
Sum of Interior Angles: ____________________	Sum of Interior Angles: ____________________	Sum of Interior Angles: ____________________	Sum of Interior Angles: $5(180°) = 900°$

2. Now summarize your results in the following table and sketch a graph relating the sum of the interior angles of a convex polygon to the number of sides. Then answer questions (a) to (f).
n = number of sides in the polygon, s = sum of the interior angles of the polygon

n	s	Δs (1st Differences)
3		
4		
5		
6		
7	900°	



(a) Do you expect the pattern to continue indefinitely beyond $n = 7$? Explain.

(b) Write an equation relating s to n. Explain why it is not surprising that the relation between s and n is linear.

(c) State the ***meaning*** of the slope of the linear relation between s and n.

(d) Does the vertical intercept of this linear relation have a meaning? Explain.

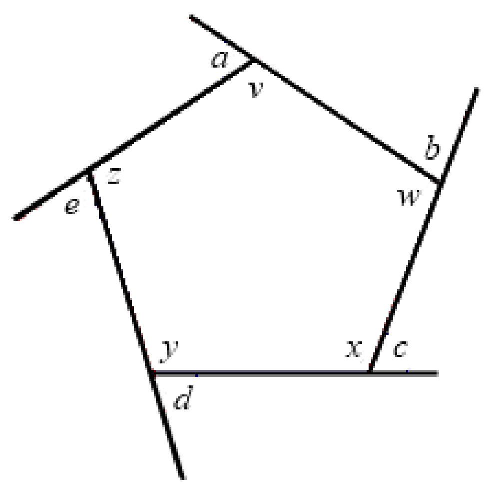
(e) Does it make sense to "connect the dots" in the above graph? Explain.

(f) State an easy way to remember how to calculate the sum of the interior angles of a polygon.

Sum of the Exterior Angles of a Convex Polygon

The argument presented here for a pentagon can be used for a polygon with any number of sides.

In the pentagon at the right, ***there are five straight line angles***, that is, there are five 180° angles. Therefore,



$$(a+v)+(b+w)+(c+x)+(d+y)+(e+z)=180°+180°+180°+180°+180°$$

$$\therefore a+v+b+w+c+x+d+y+e+z=5(180°)$$

$$\therefore a+b+c+d+e+v+w+x+y+z=5(180°)$$

$$\therefore a+b+c+d+e+3(180°)=5(180°)$$ (Since $v+w+x+y+z$ is the sum of the interior angles of the pentagon.)

$$\therefore a+b+c+d+e+3(180°)-3(180°)=5(180°)-3(180°)$$

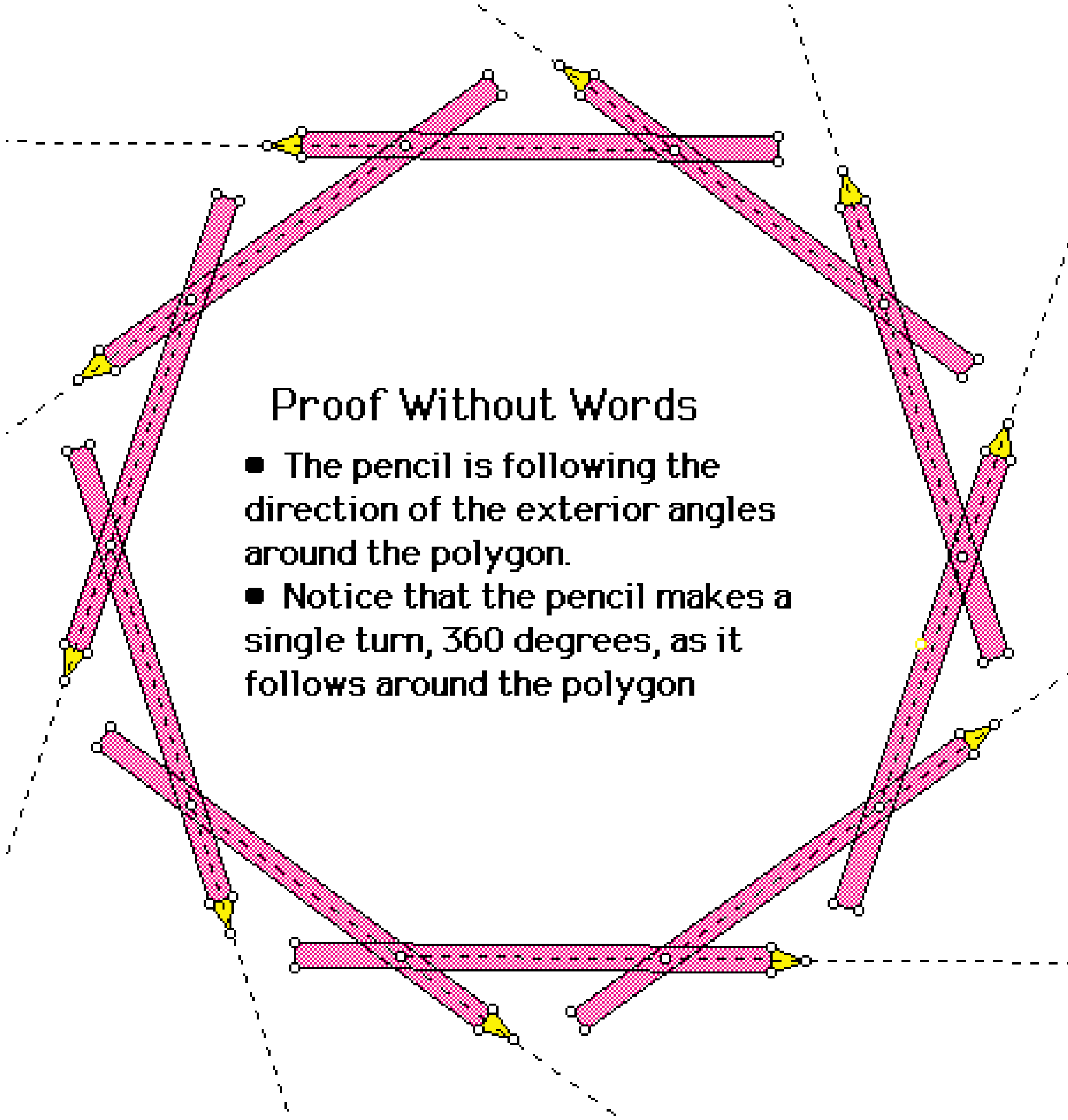
$$\therefore a+b+c+d+e=2(180°)$$

$$\therefore a+b+c+d+e=360°$$

For a convex polygon with n sides…

- There are n straight line angles, for a total measure of $180°n$.
- The sum of the n interior angles $180°(n-2)$.
- The sum of the exterior angles is equal to $180°n-180°(n-2)=180°n-180°n+360°=360°$

The ***sum of the exterior angles*** of ***any convex polygon*** is 360°.


Proof Without Words
The pencil is following the direction of the exterior angles around the polygon.
Notice that the pencil makes a single turn, 360 degrees, as it follows around the polygon