Conceptual Retrospective: Number Sense and Algebra

1. Complete the following statements:

Two algebraic expressions are said to be ***equivalent*** if they can be simplified to EXACTLY the same expression. (Equivalent expressions must agree for ALL values of the unknowns.)

The following is an example of two equivalent expressions: $2x$, $5x-3x$

2. The expressions $2x$ and x^2 are ***not equivalent***. Show this in the following ways.

(a) $2x$ ***means*** 2 times a number while x^2 ***means*** a number times itself.

For example, if $x=-7$,

$2x =$ $2(-7)$ $=$ -14

and $x^2 =$ $(-7)(-7)$ $=$ 49.

(b) $2x$ ***means*** 2 groups of x while x^2 ***means*** x groups of x.

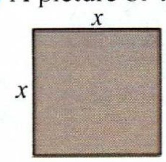
For example, if $x=4$, $2x=2(4)$, which means 2 groups of 4 and $x^2=4^2=(4)(4)$, which means 4 groups of 4.

(c) Complete the table. Then draw conclusions by completing the statement to the right of the table.

x	$2x$	x^2
−5	−10	25
−4	−8	16
−3	−6	9
−2	−4	4
−1	−2	1
0	0	0
1	2	1
2	4	4
3	6	9
4	8	16
5	10	25

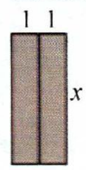
From the table, we can see that $2x$ and x^2 agree ***only when*** $x =$ 0 and when $x =$ 2. For all other values of x, $2x$ and x^2 DO NOT agree. Therefore, $2x$ and x^2 ***cannot be*** equivalent.

(d) A picture of x^2 could look like the following:



This picture can represent x^2 because its area is equal to x^2.

On the other hand, a picture of $2x$ could look like the following:



This picture can represent $2x$ because its area is equal to $2x$.

From these pictures we must conclude that $2x$ and x^2 are ***not equivalent*** because the areas are not equal unless $x=2$.

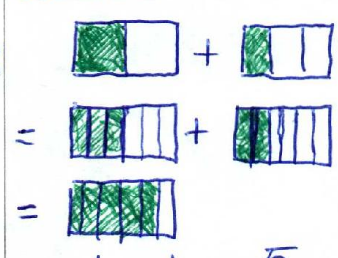
3. Using both a ***logical argument*** and ***pictures***, explain why $\frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$. Then complete the statement.

(a) ***Logical Argument***

$\frac{1}{2}+\frac{1}{3}$ must be GREATER than $\frac{1}{2}$ but $\frac{2}{5}$ is LESS THAN $\frac{1}{2}$.

$\therefore \frac{1}{2}+\frac{1}{3}\neq\frac{2}{5}$

(b) ***Pictures***



$\therefore \frac{1}{2}+\frac{1}{3}=\frac{5}{6}$

Whenever I ***add*** or ***subtract fractions***, I must always remember to express each fraction with a common denominator because each whole must be divided into the same # of equal parts

4. First complete the statements found below. Then ***evaluate*** the expression shown at the right. Show all steps!

$$\frac{-2\left[4^2-3(-7)^2\right]-\left(3^2-2^4\right)}{-6^2+(-6)^2+3(-7)(-8)-4(3-7)}$$

Whenever I ***evaluate expressions***, I must always remember:

1. ***Adding*** and ***subtracting*** involve gains and losses.
2. ***Multiplying*** and ***dividing*** involve groups of or how many groups of.
3. I ***should not*** use the distributive property because it usually makes the process more complicated.
4. I ***should*** use BEDMAS so that I'll know how to apply the operations in the correct order.
5. I ***should*** separate the expression into terms.

$$= \frac{-2[16-3(49)]-(9-16)}{-36+36+168-4(-4)}$$

$$= \frac{-2(16-147)-(-7)}{168-(-16)}$$

$$= \frac{-2(-131)+7}{168+16}$$

$$= \frac{262+7}{184}$$

$$= \frac{269}{184}$$

5. First complete the statements found below. Then ***substitute*** the given values into the expression shown at the right and ***evaluate***. Show all steps!

$$\frac{-a^2+3ab^3-6ab^2}{(a-b)(a+b)},\ a=4,\ b=-\frac{1}{2}$$

Whenever I ***substitute values into expressions***, I must:

1. Replace the ***variables*** with empty brackets, taking care to ensure that operations are not changed and exponents remain the same.
2. Then the given values should be inserted into the empty brackets, taking care to ensure that the correct values are used.
3. Finally, the resulting expression should be evaluated using BEDMAS and keeping in mind all the points made in question 4.

$$= \frac{-(4)^2+3(4)(-\frac{1}{2})^3-6(4)(-\frac{1}{2})^2}{[(4)-(-\frac{1}{2})][(4)+(-\frac{1}{2})]}$$

$$= \frac{-16+\frac{12}{1}(-\frac{1}{8})-\frac{24}{1}(\frac{1}{4})}{(\frac{4}{1}+\frac{1}{2})(\frac{4}{1}-\frac{1}{2})}$$

$$= \frac{-16+(-\frac{12}{8})-\frac{24}{4}}{(\frac{8}{2}+\frac{1}{2})(\frac{8}{2}-\frac{1}{2})}$$

$$= \frac{\frac{-16}{1}-\frac{3}{2}-\frac{6}{1}}{(\frac{9}{2})(\frac{7}{2})}$$

$$= \frac{\frac{-32}{2}-\frac{3}{2}-\frac{12}{2}}{(\frac{63}{4})}$$

$$= -\frac{47}{2}\div\frac{63}{4}$$

$$= -\frac{47}{\cancel{2}_1}\times\frac{\cancel{4}^2}{63} = \frac{-94}{63}$$

6. First complete the statements found below. Then ***simplify*** the expression shown at the right. Show all steps!

Whenever I ***simplify expressions***, I must remember:

1. When ***adding*** and ***subtracting expressions***, I must collect like terms.

   I should add the opposite ***only when*** an expression in brackets is preceded by a subtraction symbol.

2. When ***multiplying*** and ***dividing*** expressions, I must put like factors together and use the laws of exponents. In addition, if I multiply a monomial by a polynomial with two or more (unlike) terms, I should use the distributive property.

3. I ***must never confuse*** addition and subtraction with multiplication and division. For example, $x^2 + x^2 =$ $2x^2$ but $x^2(x^2) =$ x^4

$$\frac{(-ab^2)^3(-2a^2b)^4}{(2a)^{-3}} - 2a^5b^3(3a^9b^7 - 6a^{-3}b^{-3}) - (a-b)(2a+7b)$$

$$= \frac{(-a)^3(b^2)^3(-2)^4(a^2)^4(b^4)}{2^{-3}a^{-3}} - 6a^{14}b^{10} + 12a^2b^0 - (2a^2+7ab-2ab-7b^2)$$

$$= \frac{(-a^3)(b^6)(16)(a^8)(b^4)}{\frac{1}{8}a^{-3}} - 6a^{14}b^{10} + 12a^2 - (2a^2+5ab-7b^2)$$

$$= \frac{-16a^3a^8b^6b^4}{\frac{1}{8}a^{-3}} - 6a^{14}b^{10} + 12a^2 - 2a^2 - 5ab + 7b^2$$

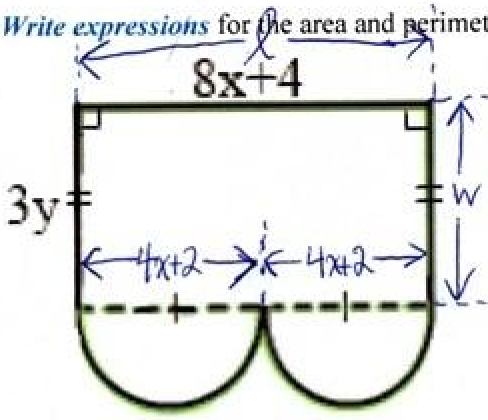
$$= \left(\frac{-16}{1} \div \frac{1}{8}\right)\left(\frac{a^{11}b^{10}}{a^{-3}}\right) - 6a^{14}b^{10} + 10a^2 - 5ab + 7b^2$$

$$= \left(\frac{-16}{1}\right) \times \left(\frac{8}{1}\right)\left(\frac{a^{11}}{a^{-3}}\right)\left(\frac{b^{10}}{1}\right) - 6a^{14}b^{10} + 10a^2 - 5ab + 7b^2$$

$$= -128a^{14}b^{10} - 6a^{14}b^{10} + 10a^2 - 5ab + 7b^2$$

$$= -134a^{14}b^{10} + 10a^2 - 5ab + 7b^2$$

7. ***Write expressions*** for the area and perimeter of the following shape.



$d = 4x+2$
$r = 2x+1$

$$P = 2w + l + \pi d$$
$$= 2(3y) + 8x + 4 + \pi(4x+2)$$
$$= 6y + 8x + 4 + 4\pi x + 2\pi$$
$$= 8x + 4\pi x + 6y + 4 + 2\pi$$
$$= (8+4\pi)x + 6y + 4 + 2\pi$$

$$A = lw + \pi r^2$$
$$= 3y(8x+4) + \pi(2x+1)^2$$
$$= 24xy + 12y + \pi(4x^2+4x+1)$$
$$= 4\pi x^2 + 24xy + 4\pi x + 12y + \pi$$

Rough Work

$$(2x+1)^2$$
$$= (2x+1)(2x+1)$$
$$= 4x^2 + 2x + 2x + 1$$
$$= 4x^2 + 4x + 1$$

8. Complete the following table. By doing so ***correctly***, you will demonstrate your understanding of how basic mathematical principles can be used to ***justify*** or ***refute assertions***.

Assertion	***True or False?***	***If the assertion is true, justify it. If it is false, refute it. Then provide a correction.***
$\left(-3xy^3\right)^2$ $=(-3)^2\left(x^2\right)\left(y^3\right)^2$ $=9x^2y^6$	True	$\left(-3xy^3\right)^2$ $=\left(-3xy^3\right)\left(-3xy^3\right)$ $=(-3)(-3)xxy^3y^3$ $=9xxyyyyyy$ $=9x^2y^6$ ***Expanded form*** can always be used to verify a simplification performed by using laws of exponents. Although expanded form makes the simplification more time-consuming, it has the advantage of relying only on very basic principles such as the ***meaning*** of mathematical operations.
$2a+3a=5a^2$	False	When like terms are added/subtracted, the coefficients must be added/subtracted but the variable part ***must not change***. Thus, $2a+3a$ simplifies to $5a$, which is analogous to stating that 2 apples plus 3 apples is 5 apples. To strengthen the argument, we can use a ***counterexample*** as shown below. Suppose that $a=2$. Then $2a+3a=2(2)+3(2)=4+6=10$ but $5a^2=5(2)^2=5(4)=20$. Since we can confidently assert that $10\neq 20$, this example ***proves*** that $2a+3a$ ***is not*** equivalent to $5a^2$.
$\left(3x+y^3\right)^2$ $=(3x)^2+\left(y^3\right)^2$ $=9x^2+y^6$	False	Counterexample: Let $x=y=1$ $(3x+y^3)^2=[3(1)+1^3]^2$ $=4^2$ $=16$ ← Don't agree! → $9x^2+y^6$ $=9(1)^2+1^6$ $=9+1$ $=10$ Correction: $(3x+y^3)(3x+y^3)$ $=9x^2+3xy^3+3xy^3+y^6$ $=9x^2+6xy^3+y^6$
$2(x+3)(x+5)$ $=(2x+6)(2x+10)$ $=4x^2+32x+60$	False	Counterexample: Let $x=1$ $2(1+3)(1+5)$ $=2(4)(6)$ $=48$; $4(1)^2+32(1)+60$ $=4+32+60$ $=96$ ← Don't agree! Correction: See next row OR $2(x+3)(x+5)$ $=2(x^2+5x+3x+15)$ $=2(x^2+8x+15)=2x^2+16x+30$
$2(x+3)(x+5)$ $=(2x+6)(x+5)$ $=2x^2+16x+30$	True	The expression contains 3 factors, 2, x+3 and x+5. The factors can be multiplied in any order, left-to-right probably being easiest. $2(x+3)(x+5)$ $=(2x+6)(x+5)$ $=2x^2+10x+6x+30$ $=2x^2+16x+30$
$-2\left(-3c^7d^{-4}\right)+2\left(3c^7d^{-4}\right)$ $=6c^7d^{-4}+6c^7d^{-4}$ $=(6)(6)c^7c^7d^{-4}d^{-4}$ $=\dfrac{36c^{14}}{d^8}$	FALSE!	Counterexample: Let $c=d=1$ $-2(-3c^7d^{-4})+2(3c^7d^{-4})$ $=-2(-3)(1)^7(1)^{-4}+2(3)(1)^7(1)^{-4}$ $=6+6$ $=12$ ← Don't agree! → $\dfrac{36c^{14}}{d^8}=\dfrac{36(1)}{1^8}=36$ Correction: $-2(-3c^7d^{-4})+2(3c^7d^{-4})$ $=6c^7d^{-4}+6c^7d^{-4}$ $=12c^7d^{-4}=\dfrac{12c^7}{d^4}$
$3[5-4(7)]^2$ $=3(5)^2-3[4(7)]^2$ $=75-2352$ $=-2277$	False	Operations not performed in correct order! $3[5-4(7)]^2$ $=3(5-28)^2$ $=3(-23)^2$ $=3(529)$ $=1587$ ① ② ③ ④ BEDMAS L→R L→R