Unit 0 – Introduction to Mathematical Thinking

Sneak Preview: What you will Learn in this Unit

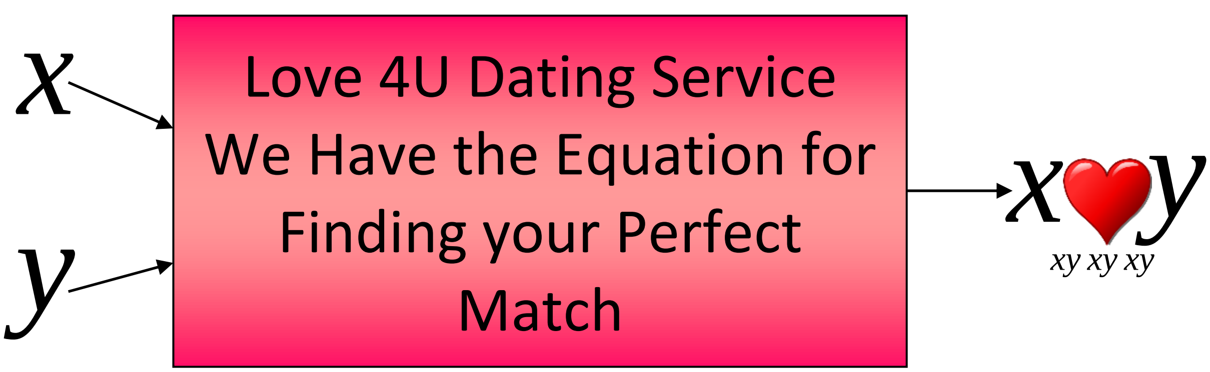
The main purpose of this unit is to introduce you to mathematical thinking. Among other things, you will learn that…

- ***Formulas*** are the ***finished products*** of mathematical thinking. They provide us with convenient ***algorithms*** for solving particular kinds of problems. However, formulas in and of themselves do not constitute mathematical thinking! To become a true mathematical thinker, it is necessary to move far beyond a purely formulaic approach!
- The mathematician's main goal is to ***discover*** how quantities are ***related*** to one another. The Pythagorean Theorem is an iconic illustration of what we mean by this. Every right triangle, no matter how large or small, must obey the equation $c^2 = a^2 + b^2$. Once again, however, it is not enough just to know the equation. A true mathematician also understands ***why*** this equation describes the relationship among the sides of a right triangle and can prove it in a highly rigorous fashion.
- The mathematics that you learn in high school can be reduced to three basic concepts:
  - Mathematical Objects (e.g. numbers, geometric shapes, etc)
  - Mathematical Operations (e.g. +, −, ×, ÷)
  - Mathematical Relationships (e.g. $c^2 = a^2 + b^2$)
- In keeping with the focus on mathematical relationships, several examples are given in this unit including…
  - The Pythagorean Theorem
  - Measurement relationships for several two-dimensional and three-dimensional shapes
  - Angle relationships in polygons

Math is like a Dating Service…

Math Is Like a Dating Service.

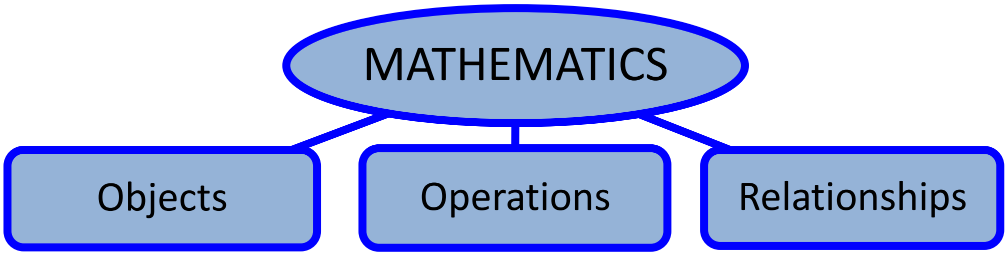
It's all about Relationships!



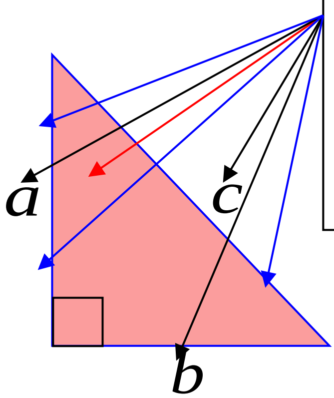
A Framework for Understanding Mathematics

As shown below, mathematics can be reduced to ***three basic concepts***:

1. Mathematical Objects (e.g. ***numbers*** are mathematical objects)
2. Mathematical Operations (e.g. $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, etc)
3. Mathematical Relationships (e.g. $c^2 = a^2 + b^2$)



The Pythagorean Theorem – Probably the Most Famous Mathematical Relationship



The Mathematical Objects

- **Right Triangle**
  Variables (a, b and c)
  Line Segments
  Numbers
- The ***variables*** represent the ***lengths*** of the sides of the right triangle. Therefore, the variables a, b and c represent ***numbers***.

The Mathematical Operations

- **Addition** (+)
- **Squaring** (raise to the exponent 2)

$$c^2 = a^2 + b^2$$

The Mathematical Relationship

- The ***equation*** $c^2 = a^2 + b^2$ expresses how the quantities represented by a, b and c are related to one another by means of mathematical operations.
- This relationship holds for all right triangles, no matter how large or small.

Exercise

Identify the mathematical objects, operations and relationships for the surface area of a cylinder.

Hint: $A = 2\pi r^2 + 2\pi rh$

What is Problem Solving?

Introduction: What is the Difference between Solving a Problem and Performing an Exercise?

- **Performing an Exercise:** This requires you to *follow a procedure* that you have learned.
  Performing mathematical exercises is analogous to executing drills like "suicides" when practicing for a sport or playing scales when practicing a musical instrument. Very little original thinking is required.
- **Solving a Problem:** This requires you to *think and be imaginative*.
  You can consider yourself a problem solver *only when you devise the strategy*. If you are following a strategy devised by someone else, then you are merely performing an exercise NOT solving a problem. This is analogous to playing a game in sports or improvising on a musical instrument. At any point in time, you can never be certain of exactly what will happen next. You must adapt to the circumstances as they change. The "game plan" evolves as the game is played!

Performing an Exercise

- Mechanical ("Auto Pilot")
- Very Predictable
- Follows a Set Procedure or Strategy Devised by Someone Else
- The Path to the Destination is Known and entirely Clear
- Little Thinking or Imagination Needed
- Like Doing Drills in Sports or Playing Scales in Music
- Also like a Police Officer Writing out a Ticket for a Traffic Violation

Solving a Problem

- Not Mechanical
- Somewhat Unpredictable
- Does not Follow a Strategy Devised by Someone Else
- The Problem Solver Devises the Strategy
- The Path to the Destination is not entirely Clear
- Thinking and Imagination are Required
- Like Improvising in Music or Playing a Game in Sports
- Also like a Police Detective trying to Solve a Crime

George Polya's Four Steps of Problem Solving

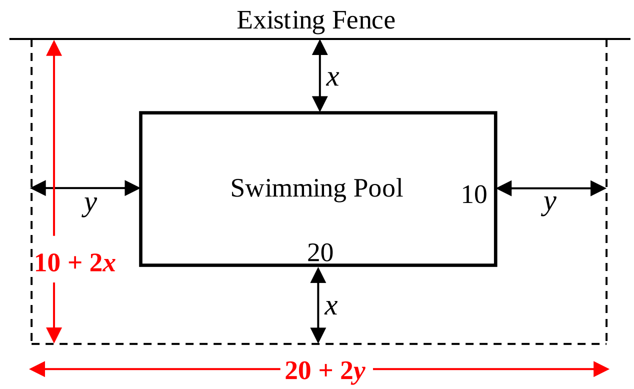
1. **Understand the Problem**
   Do you understand all the terminology used in the question? Do you understand what are you being asked to do? What information is given? Is all the given information required? Is there any missing information? What would a reasonable answer look like? Can you represent the problem in different ways? (**e.g.** diagram, graph, model, table of values, equation, etc.) …
2. **Devise a Strategy**
   What mathematical concepts are relevant and do you understand them? Do you know any strategies that could work? Do you need to invent a new strategy? Can you solve a simplified version of the problem? Can you solve a related problem? Can you work backwards and then reverse the steps? ...
3. **Carry out the Strategy**
   Carefully carry out the strategy that you devised in step 2.
4. **Check the Solution**
   Carefully check your solution. Does your answer make sense? Does it agree with the prediction you made in step 1? Have others arrived at the same answer?

Example

To help prevent drowning accidents, a protective fence is to be erected around a pool whose dimensions are 20 m by 10 m. Since there is an existing fence parallel to one of the 20-m sides of the pool, new fencing is only required around three sides of the pool. In addition, the gap between the fence and the edge of the pool must be uniform on opposite sides of the pool. If 100 m of fencing material is available, what is the maximum area that can be enclosed by the fencing?

Solution

1. Understand the Problem



Given

- pool is 20 m × 10 m
- existing fence ∥ to 20-m side of pool
- gap is uniform on opposite sides of pool
- 100 m of fencing available
  → $10 + 2x + 20 + 2y + 10 + 2x = 100$ [*]

Required to Find

- max area that can be enclosed by fencing
- i.e. max value of $A = (20 + 2y)(10 + 2x)$ [**]

Reasonable Prediction

Distribute the fencing according to the ratio of length to width of the swimming pool. Since the width is twice the length, we might expect the length and width of the rectangular area to be 50 m and 25 m respectively, yielding an area of 1250 m^2.
This turns out to be *flawed* reasoning but it still produces the correct answer in this specific case. (See question 1 on p. 7)

2. Devise a Strategy

- Find a relationship between x and y (see [*] above).
- Use the relationship to rewrite equation [**] in terms of a single unknown.
- Graphically or algebraically find the maximum value of A.

3. Carry Out the Strategy

Solve for One of the Unknowns in Terms of the Other

$10 + 2x + 20 + 2y + 10 + 2x = 100$

$\therefore 4x + 2y + 40 = 100$

$\therefore 4x + 2y = 60$

$\therefore 2y = 60 - 4x$

Write the Equation for *A* in terms of a Single Unknown

$A = (20 + 2y)(10 + 2x)$

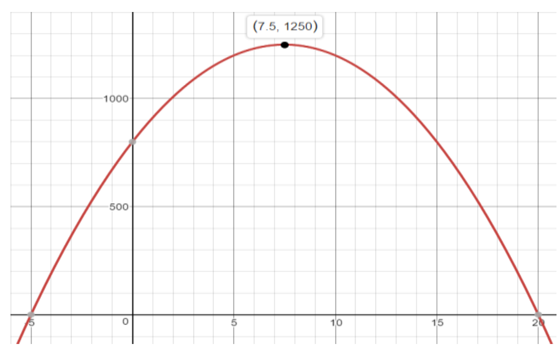
$\therefore A = (20 + 60 - 4x)(10 + 2x)$

$\therefore A = (80 - 4x)(10 + 2x)$

Conclusion

The maximum area that can be enclosed by the fence is 1250 m^2.

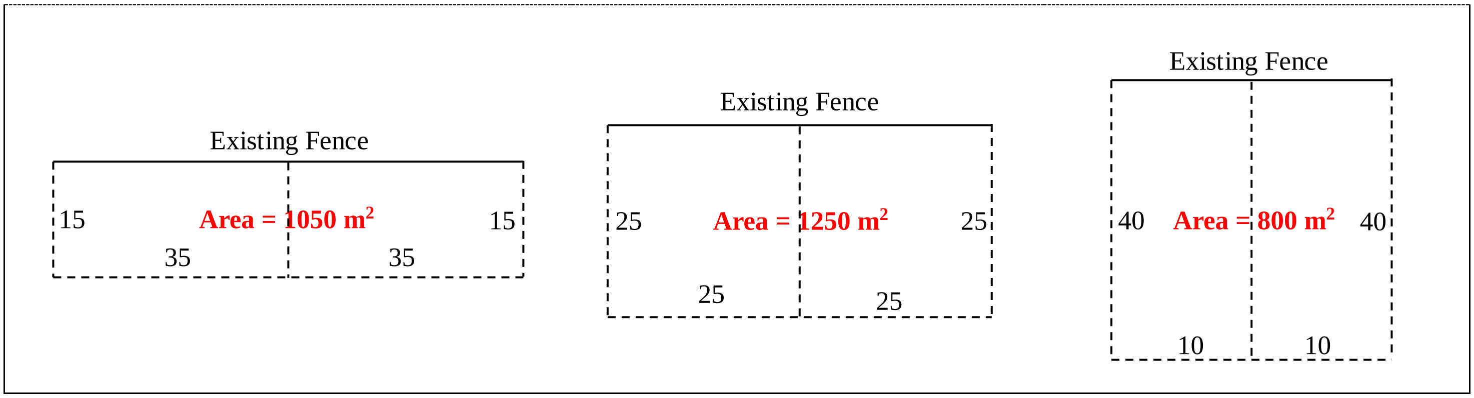
Use Graphing Software to Create a Graph



4. Check the Solution

It is well known that the area of a rectangle with a given perimeter is maximized when it's a square. However, we must be careful in this situation not to jump to conclusions. Since the pool is not a square and the fence is installed only along three sides of the rectangular area, we ***cannot assume*** that the area of the rectangular region is maximized if it's a square. If we do assume this, we arrive at an area of about 1100 m^2, which is clearly ***not*** the maximum area.

The diagrams on the next page show that the rectangular area can be divided into two smaller congruent rectangles, ***each of which has a perimeter of 100 m***. Since each of these rectangles has a fixed perimeter of 100 m, the maximum area is obtained when each rectangle is a square that is 25 m by 25 m. Therefore, the maximum area of each square is 625 m^2, yielding a maximum total area of 1250 m^2.



Questions

1. To help prevent drowning accidents, a protective fence is to be erected around a pool whose dimensions are 30 m by 10 m. Since there is an existing fence parallel to one of the 30-m sides of the pool, new fencing is only required around three sides of the pool. In addition, the gap between the fence and the edge of the pool must be uniform on opposite sides of the pool. If 100 m of fencing material is available, what is the maximum area that can be enclosed by the fencing? ***Solve this problem using both of the methods described above.***

2. Repeat question 1 but change the dimensions of the pool to 40 m by 10 m. Do you notice anything unexpected?

3. Repeat question 1 but this time, all four sides of the rectangular area need to be fenced off and the dimensions of the pool are once again 20 m by 10 m. Do you expect a different answer this time?

Answers

1. 1250 m^2 **2.** 1250 m^2 https://www.desmos.com/calculator/xl1ivv2t6l **3.** 625 m^2 https://www.desmos.com/calculator/an6iya7wdz

<table>
<tr><th>Question</th><th>Pattern?</th><th>Explanation</th></tr>
<tr><td>1. How many regions are there in the fourteenth diagram?

1 2 3</td><td>How is the number of regions (r) related to the diagram number (d)?
<table><tr><th>d</th><th>r</th></tr><tr><td></td><td></td></tr></table></td><td></td></tr>
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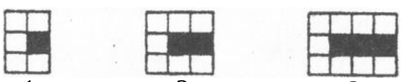
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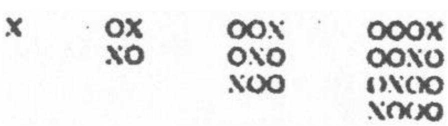
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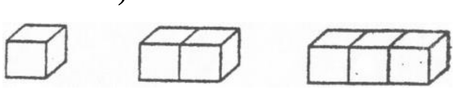
<tr><td>4. How many faces are visible in the twentieth diagram? (Include the back and sides.)

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<table>
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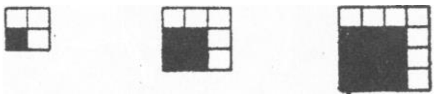
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(i) 16 days (ii) 49 days</td>
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<tr>
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180° 360° 540°

3 4 5</td>
<td>How is the sum of the interior angles (s) related to the number of sides (n)?
<table><tr><th>n</th><th>s</th></tr><tr><td></td><td></td></tr></table></td>
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<tr>
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1 2 3</td>
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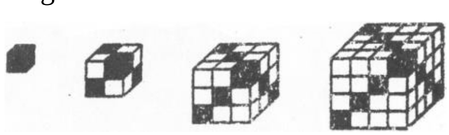
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</table>

Solutions to Pattern Finding Activity

<table>
<tr><th>Question</th><th colspan="2">Solutions (Including Equations that Describe the Relationships)</th></tr>
<tr><td>1. How many regions are there in the fourteenth diagram?

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<table><tr><th>d</th><th>$r=d+1$</th></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>3</td></tr><tr><td>3</td><td>4</td></tr><tr><td>.
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.</td></tr><tr><td>14</td><td>15</td></tr></table></td><td>From the table we can see that the number of regions is always one more than the diagram number. This relationship can be described by the following equation:
$r=d+1$</td></tr>
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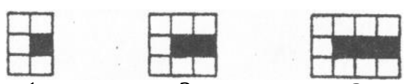
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<table><tr><th>d</th><th>$s=d$</th><th>$u=2d+3$</th></tr><tr><td>1</td><td>1</td><td>5</td></tr><tr><td>2</td><td>2</td><td>7</td></tr><tr><td>3</td><td>3</td><td>9</td></tr><tr><td>.
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.</td></tr><tr><td>8</td><td>8</td><td>19</td></tr></table></td><td>From the table we can see that the number of shaded squares and the number of unshaded squares are related to the diagram number according to the following equations:
$s=d$
(The # of shaded squares is equal to the diagram #.)
$u=2d+3$</td></tr>
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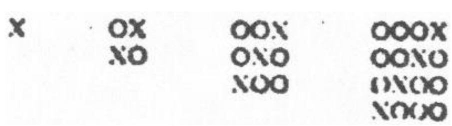
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$X=d$
$O=d^2-d$
The second equation can also be written as
$O=d(d-1)$.</td></tr>
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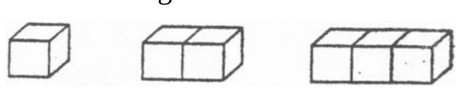
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.</td></tr><tr><td>20</td><td>62</td></tr></table></td><td>From the table we can see that the number of visible faces is always two more than triple the diagram number. This relationship can be described by the following equation:
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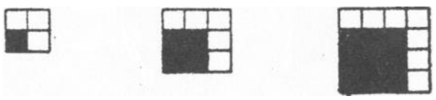
5. How many shaded squares are there in the twelfth diagram? How many unshaded squares are there in the twelfth diagram?



1 2 3

How is the number of shaded squares (s) ***related to*** the diagram number (d)? How is the number of unshaded squares (u) ***related to*** the diagram number (d)?

d	$s=d^2$	$u=2d+1$
1	1	3
2	4	5
3	9	7
⋮	⋮	⋮
12	144	25

From the table we can see that the number of shaded squares and the number of unshaded squares are ***related to*** the diagram number according to the following equations:

$$s=d^2$$

$$u=2d+1$$

6. A cow is milked twice a day. Each time she gives 11 kg of milk. Calculate the total milk production after
(i) 16 days (ii) 49 days

How is the total milk production (m) ***related to*** the time in days (t)?

t	$m=22t$
1	22
2	44
3	66
⋮	⋮
16	352
49	1078

From the table we can see that the total milk production is 22 times the number of days. This ***relationship*** can be described by the following equation:

$$m=22t$$

7. The sum of the interior angles of each polygon is shown. What is the sum of the interior angles of a decagon (a polygon with 10 sides).

180° 360° 540°

3 4 5

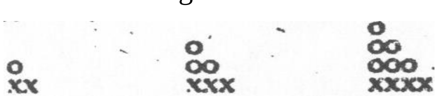
How is the sum of the interior angles (s) ***related to*** the number of sides (n)?

n	$s=180(n-2)$
3	180
4	360
5	540
⋮	⋮
10	1440

From the table we can see that the sum of the interior angles is the product of 180 and two less than the number of sides. This ***relationship*** can be described by the following equation:

$$s=180(n-2)$$

8. How many "X's" are in the tenth diagram? How many "O's" are there in the tenth diagram?



1 2 3

How is the number of "X's" (X) ***related to*** the diagram number (d)? How is the number of "O's" (O) ***related to*** the diagram number (d)?

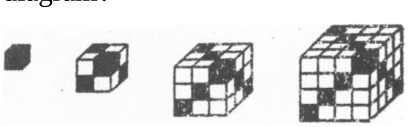
d	$X=d+1$	$O=\frac{d(d+1)}{2}$
1	2	1
2	3	3
3	4	6
⋮	⋮	⋮
10	11	55

From the table we can see that the number of "X's" and the number of "O's" are ***related to*** the diagram number according to the following equations:

$$X=d+1$$

$$O=\frac{d(d+1)}{2}=\frac{d^2+d}{2}$$

9. The cubes along one diagonal of each cube of a face are coloured (including the faces that can't be seen). How many cubes are coloured on the fifth diagram?



1 2 3 4

How is the number of coloured cubes (c) ***related to*** the diagram number (d)?

d	$c=6(d-2)+4, d\neq 1$
1	1
2	4
3	10
4	16
5	22

From the table we can see that the number of coloured cubes is six times, 2 less than the diagram number, all increased by four. This ***relationship*** can be described by the following equation:

$$c=6(d-2)+4, d\neq 1.$$

By simplifying, we obtain $c=6d-8, d\neq 1$.

Summary of Main Ideas

- Exercises versus Problems
  - Exercises Help Build Specific Skills
  - Problem Solving is the Ultimate Goal
  - Focus on Problem Solving not Exercises

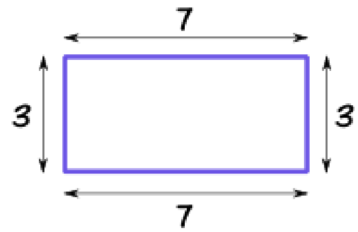
- Understanding versus Memorizing
  - Memorizing can be Helpful but only up to a Point
  - Understanding is the Ultimate Goal
  - Focus on a Small Number of Important Ideas
  - View Ideas from a Variety of Perspectives
  - Learn in Depth
  - Focus on Relationships not Procedures

- Proficiency Requires Frequent Practice
  - Our Brains Adapt to our Environment
  - Neural Pathways Strengthen with Frequent Practice
  - Learning is a Gradual Process
  - Sloppy Practice Leads to Sloppy Performance
  - Careful Practice Leads to Great Performance

- Keys to Success
  - Do Homework Every Day!
  - Always Write Solutions with Excellent Form!
  - Solve Problems without Assistance as Much as Possible!
  - Get Help when you are Struggling to Understand!
  - Always try to go Beyond Expectations!

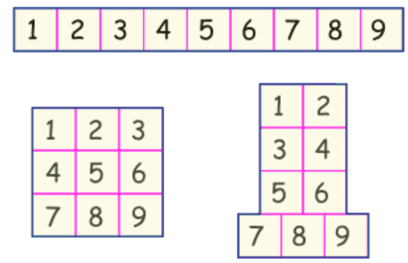
UNDERSTANDING THE CONCEPTS OF PERIMETER, AREA AND VOLUME

Perimeter



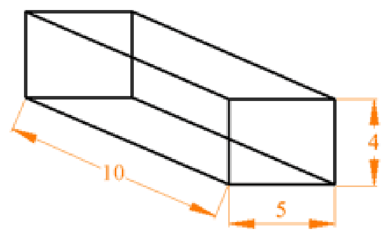
- The ***distance*** around a two-dimensional shape.
- **Example:** the perimeter of this rectangle is 3+7+3+7 = 20
- The perimeter of a circle is called the ***circumference***.
- Perimeter is measured in ***linear units*** such as mm, cm, m, km.

Area



- The "size" or "amount of space" inside the boundary of a two-dimensional surface, including curved surfaces.
- In the case of the surface of a three-dimensional object, the area is usually called ***surface area***.
- **Example:** If each small square at the right has an area of 1 cm^2, the larger shapes all have an area of 9 cm^2.
- Area is measured in ***square units*** such as mm^2, cm^2, m^2, km^2.

Volume



- The "amount of space" contained within the interior of a three-dimensional object. (The ***capacity*** of a three-dimensional object.)
- **Example:** The volume of the "box" at the right is 4×5×10 = 200 m^3. This means, for instance, that 200 m^3 of water could be poured into the box.
- Volume is measured in ***cubic units*** such as mm^3, cm^3, m^3, km^3, mL, L.
  Note: 1 mL = 1 cm^3

Questions

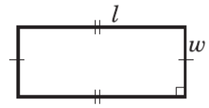
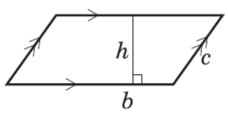
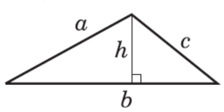
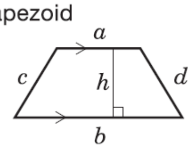
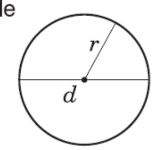
1. You have been hired to renovate an old house. For each of the following jobs, state whether you would measure perimeter, area or volume and explain why.

Job	*Perimeter, Area or Volume?*	*Why?*
Replace the baseboards in a room. 		
Paint the walls.		
Pour a concrete foundation. 		

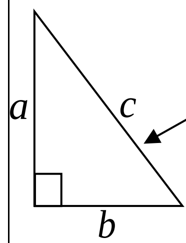
2. Convert 200 m^3 to litres. (**Hint:** Draw a picture of 1 m^3.)

MEASUREMENT RELATIONSHIPS

Perimeter and Area Equations

Geometric Figure	Perimeter	Area
Rectangle 	$P = l + l + w + w$ or $P = 2(l + w)$	$A = lw$
Parallelogram 	$P = b + b + c + c$ or $P = 2(b + c)$	$A = bh$
Triangle 	$P = a + b + c$	$A = \frac{bh}{2}$ or $A = \frac{1}{2}bh$
Trapezoid 	$P = a + b + c + d$	$A = \frac{(a + b)h}{2}$ or $A = \frac{1}{2}(a + b)h$
Circle 	$C = \pi d$ or $C = 2\pi r$	$A = \pi r^2$

Pythagorean Theorem



The ***hypotenuse*** is the ***longest side*** of a right triangle. It is always found ***opposite*** the right angle.

In ***any*** right triangle, **the square of the hypotenuse** is equal to **the sum of the squares of the other two sides**. That is,

$$c^2 = a^2 + b^2$$

By using your knowledge of rearranging equations, you can rewrite this equation as follows:

$$b^2 = c^2 - a^2$$

and

$$a^2 = c^2 - b^2$$

The Meaning of π

In ***any*** circle, the ***ratio*** of the ***circumference*** to the ***diameter*** is equal to a ***constant*** value that we call π. That is,

$$C : d = \pi .$$

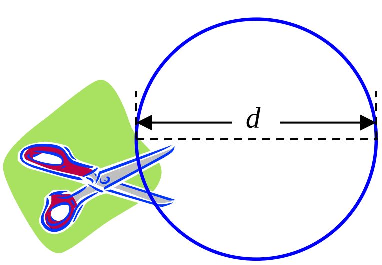
Alternatively, this may be written as

$$\frac{C}{d} = \pi$$

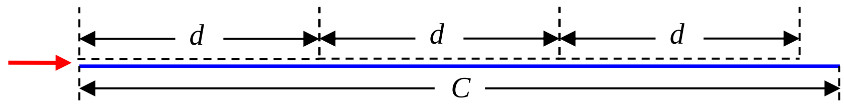
or, by multiplying both sides by d, in the more familiar form

$$C = \pi d .$$

If we recall that $d = 2r$, then we finally arrive at the most common form of this ***relationship***: $C = 2\pi r$.



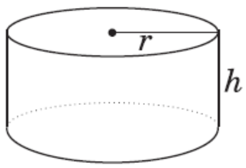
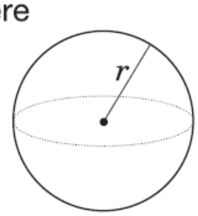
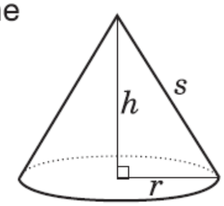
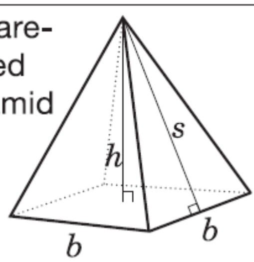
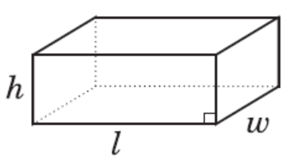
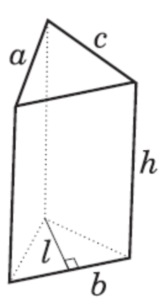
Snip the circle and stretch it out into a straight line.



As you can see, the length of the circumference is slightly more than three diameters. The exact length of C, of course, is π diameters or πd.

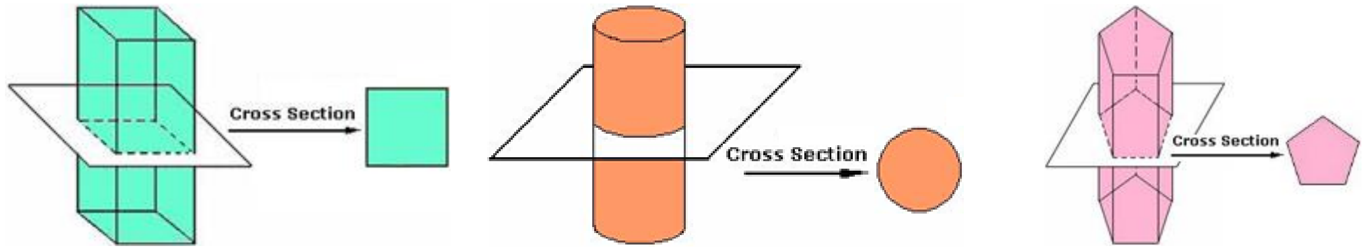
Volume and Surface Area Equations

If you need additional help, Google "area volume solids."

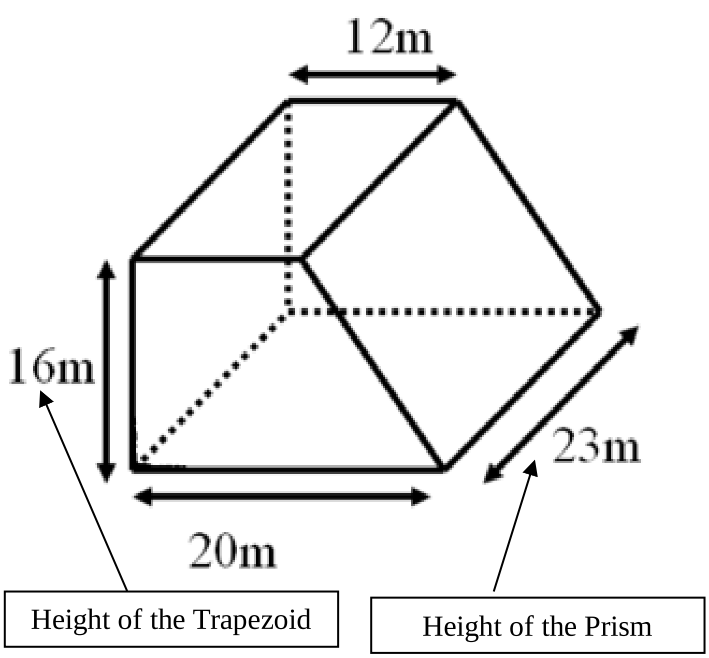
Geometric Figure	Surface Area	Volume
Cylinder 	$A_{\text{base}} = \pi r^2$ $A_{\text{lateral surface}} = 2\pi rh$ $A_{\text{total}} = 2A_{\text{base}} + A_{\text{lateral surface}}$ $= 2\pi r^2 + 2\pi rh$	$\boxed{V = (A_{\text{base}})(\text{height})}$ $V = \pi r^2 h$ This is true for all ***prisms*** and ***cylinders***.
Sphere 	$A = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$ or $V = \frac{4\pi r^3}{3}$ This is true for all ***pyramids*** and ***cones***.
Cone 	$A_{\text{lateral surface}} = \pi rs$ $A_{\text{base}} = \pi r^2$ $A_{\text{total}} = A_{\text{lateral surface}} + A_{\text{base}}$ $= \pi rs + \pi r^2$	$\boxed{V = \frac{(A_{\text{base}})(\text{height})}{3}}$ $V = \frac{1}{3}\pi r^2 h$ or $V = \frac{\pi r^2 h}{3}$
Square-based pyramid 	$A_{\text{triangle}} = \frac{1}{2}bs$ $A_{\text{base}} = b^2$ $A_{\text{total}} = 4A_{\text{triangle}} + A_{\text{base}}$ $= 2bs + b^2$	$\boxed{V = \frac{(A_{\text{base}})(\text{height})}{3}}$ $V = \frac{1}{3}b^2 h$ or $V = \frac{b^2 h}{3}$
Rectangular prism 	$A = 2(wh + lw + lh)$	$\boxed{V = (\text{area of base})(\text{height})}$ $V = lwh$ This is true for all ***prisms*** and ***cylinders***.
Triangular prism 	$A_{\text{base}} = \frac{1}{2}bl$ $A_{\text{rectangles}} = ah + bh + ch$ $A_{\text{total}} = A_{\text{rectangles}} + 2A_{\text{base}}$ $= ah + bh + ch + bl$	$\boxed{V = (A_{\text{base}})(\text{height})}$ $V = \frac{1}{2}blh$ or $V = \frac{blh}{2}$

Volumes of Solids with a Uniform Cross-Section

A solid has a ***uniform cross-section*** if any cross-section ***parallel to the base*** is ***congruent*** to the base (i.e. has exactly the same shape and size as the base). Prisms and cylinders have a uniform cross-section. Pyramids and cones do not.



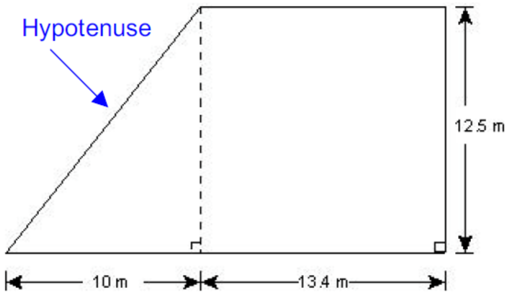
For all solids with a uniform cross-section, $V = (A_{\text{base}})(\text{height})$

Problem	***Solution***
Find the volume of the given trapezoidal prism. 	If we stand this solid on end, we can easily see that it is a solid with a uniform cross-section. Any cross-section parallel to the trapezoidal base is a trapezoid that is congruent to the base. Therefore, $V = (A_{\text{base}})(\text{height})$ $= \left(\frac{16(12+20)}{2}\right)(23)$ $= \left(\frac{16(32)}{2}\right)(23)$ $= 256(23)$ $= 5888$ The volume of the given trapezoidal prism is 5888 m^3.

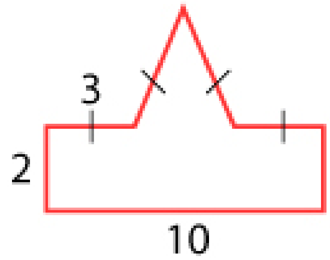
Perimeter and Area Problems

1. Calculate the ***perimeter*** and ***area*** of each of the following shapes.

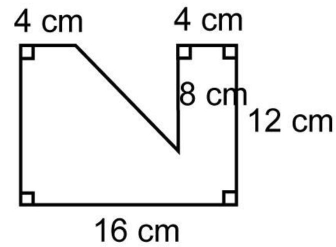
(a)



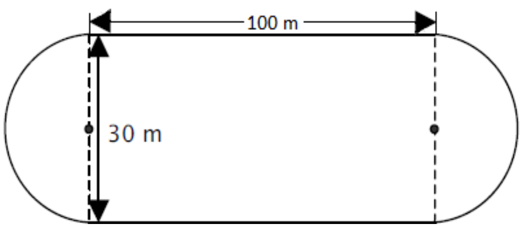
(b)



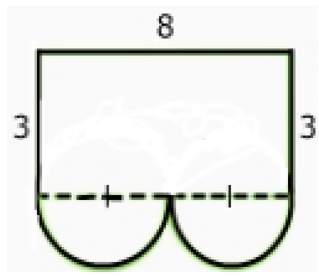
(c)



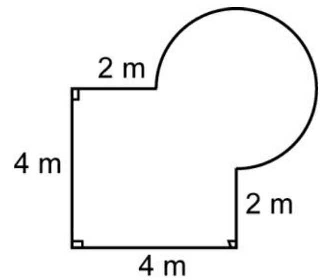
(d)



(e)

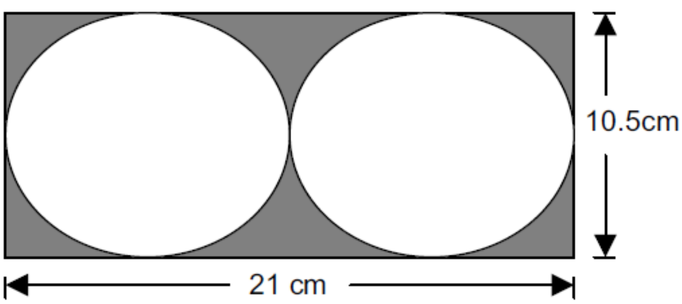


(f)

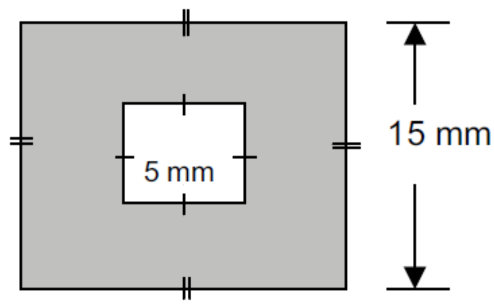


2. Calculate the area of the ***shaded region***.

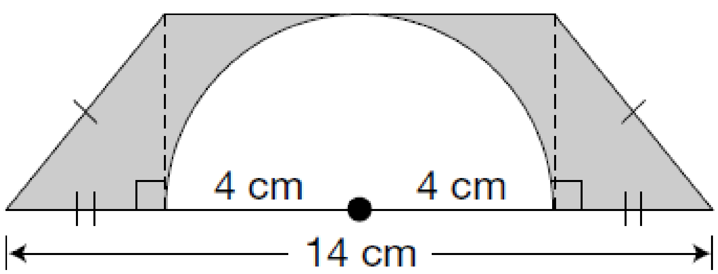
(a)



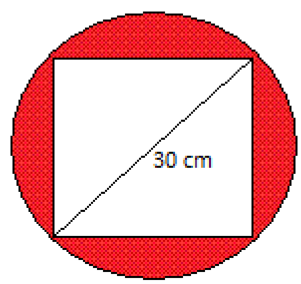
(b)



(c)



(d)

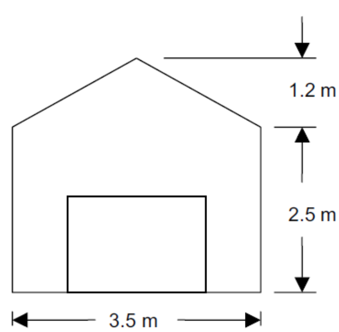


3. The front of a garage, excluding the door, needs to be painted.

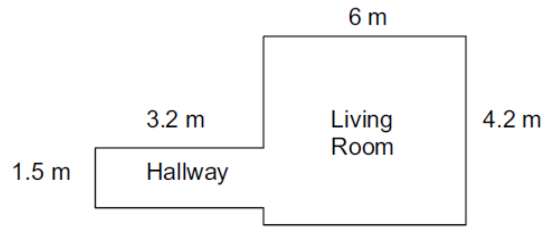
(a) Calculate the area of the region that needs to be painted assuming that the door is 1.5m high and 2 m wide. (See the diagram at the right for all other dimensions.)

(b) If one can of paint covers an area of 2.5 m^2, how many cans will need to be purchased?

(c) If one can of paint sells for $19.89, how much will it cost to buy the paint? (Include 13% HST.)



4. Bill wishes to replace the carpet in his living room and hallway with laminate flooring. A floor plan is shown at the right.



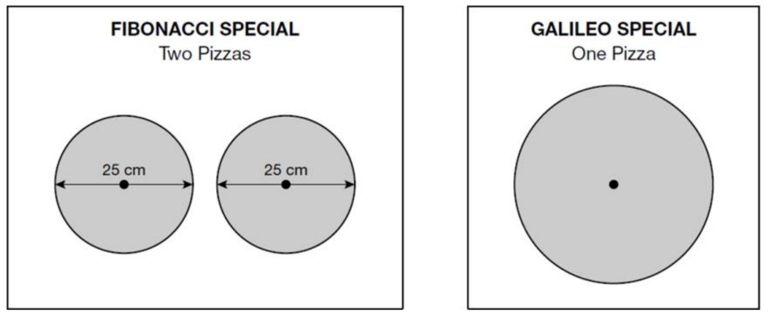
(a) Find the total area of floor to be covered.

(b) Laminate flooring comes in boxes that contain 2.15m^2 of material. How many boxes will Bill require?

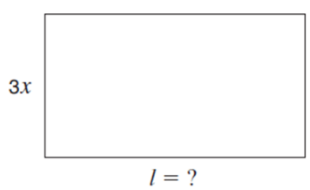
(c) One box costs $43.25. How much will the flooring cost? (Include 13% HST.)

(d) When laying laminate flooring, it is estimated there will be 5% waste. How much waste can Bill expect?

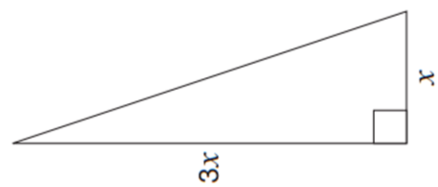
5. As shown in the diagram, a pizza shop has a weekend special. What should be the diameter of the Galileo Special pizza so that both specials contain the same amount of pizza? (You may assume that the pizzas all have the same thickness.)



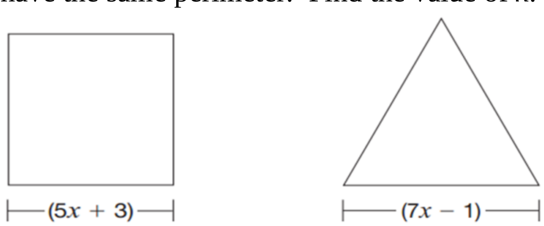
6. The area of the following rectangle is $6xy^2$ square units and its width is $3x$ units. What is its length?



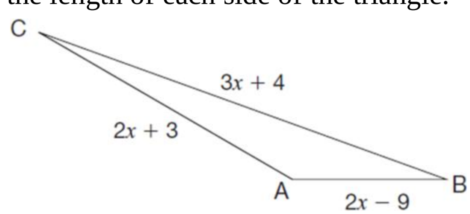
7. The area of the following triangular garden is $96\ \text{m}^2$. Determine the value of x.



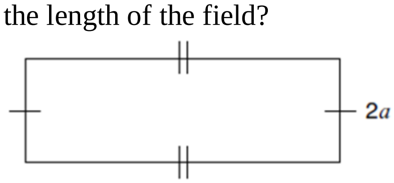
8. The following square and equilateral triangle both have the same perimeter. Find the value of x.



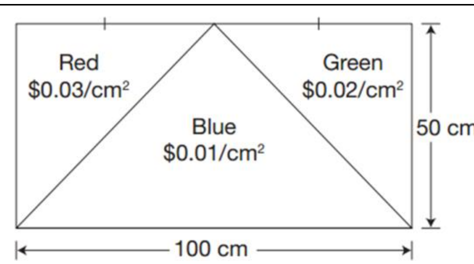
9. The perimeter of the following triangle is 75 m. Determine the length of each side of the triangle.



10. The following rectangular field has a width of $2a$ metres and a perimeter of $10a - 6$ metres. What is the length of the field?



11. A flag that is to be made of three different materials. What is the total cost of the flag?



Answers

1. **(a)** $P \doteq 65.3$ m, $A=230\ \text{m}^2$ **(b)** $P=26$, $A \doteq 24.5$ **(c)** $P \doteq 67.3$ cm, $A=160\ \text{cm}^2$ **(d)** $P \doteq 294.2$ m, $A \doteq 3706.5\ \text{m}^2$ **(e)** $P \doteq 26.56$, A□ 36.56 **(f)** $P \doteq 21.42$ m, $A \doteq 25.42\ \text{m}^2$	**2.** **(a)** $A \doteq 47.3\ \text{m}^2$ **(b)** $A=200\ \text{mm}^2$ **(c)** $A \doteq 18.9\ \text{cm}^2$ **(d)** $A \doteq 257\ \text{cm}^2$ **3.** **(a)** $A=7.85\ \text{m}^2$ **(b)** 4 cans **(c)** $89.90
4. **(a)** $A=30\ \text{m}^2$ **(b)** At least 14 boxes, probably 15 **(c)** $684.22 **(d)** About $1.5\ \text{m}^2$ of waste ($\therefore$ 15 boxes are needed) **5.** About 35.4 cm **6.** $l = 2y^2$ units **7.** $x = 8$	**8.** $x = 15$ **9.** $x = 11$ lengths of sides: 13, 25, 37 **10.** $l = 3a - 3$ units **11.** Red: $37.50 Green: $25.00 Blue: $25.00 **Total: $87.50**

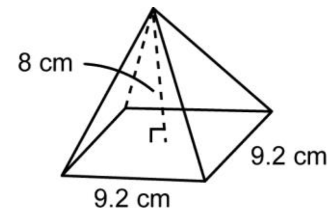
Volume and Surface Area Problems

1. A cone has radius of 8 cm and slant height of 10 cm. What is its surface area to the nearest tenth of a cm^2?

A 670.2 cm^2 **B** 804.2 cm^2 **C** 452.4 cm^2 **D** 640 cm^2

2. What is the volume of this pyramid, to the nearest tenth of a cubic centimetre?

A 677.1 cm^3 **B** 231.8 cm^3 **C** 338.6 cm^3 **D** 225.7 cm^3

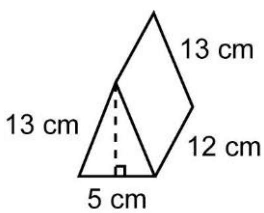


3. A sphere has radius 7 cm. What is its volume to the nearest tenth of a cubic centimetre?

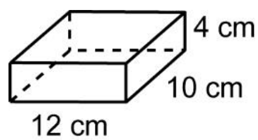
A 1436.8 cm^3 **B** 615.8 cm^3 **C** 4310.3 cm^3 **D** 205.3 cm^3

4. Find the ***surface area*** and ***volume*** of each object. Round your answers to one decimal place.

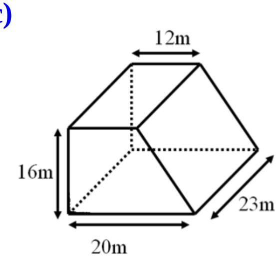
(a)



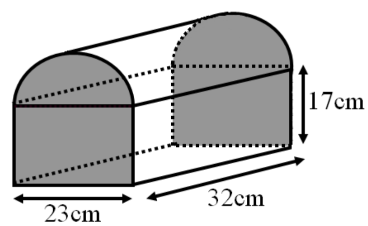
(b)



(c)



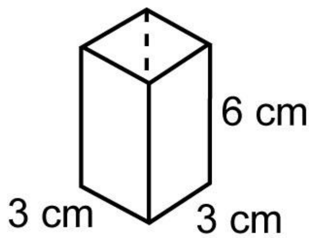
(d)



5. The shape at the right is a

________________________ .

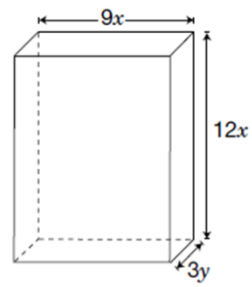
What is the maximum volume of a cone that would fit in this shape?



6. The shape at the right is a

________________________ .

What is the volume of this shape?

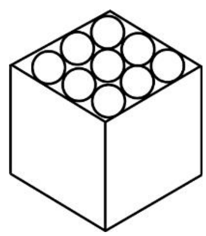


7. Carmine is packing 27 superballs in 3 square layers. Each ball has diameter 4 cm.

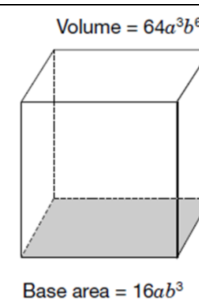
(a) What is the minimum volume of the box?

(b) What is the surface area of the box?

(c) How much empty space is in the box?

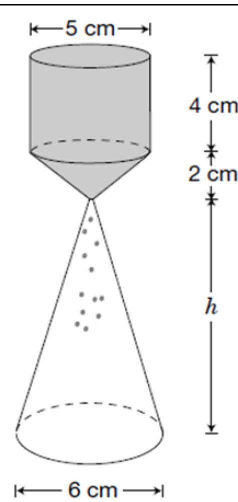


8. Expressions are given for the volume and base area of a rectangular prism. What is the height of the prism?



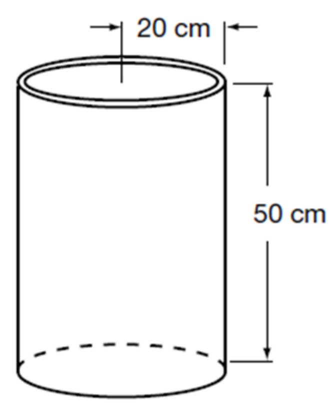
9. As shown below, sand is being poured from one container to another. The sand flows from the shaded part to the unshaded cone. The shaded part starts full of sand. By the time the shaded part is empty, the unshaded cone is filled to the top.

What is the height of the unshaded cone?

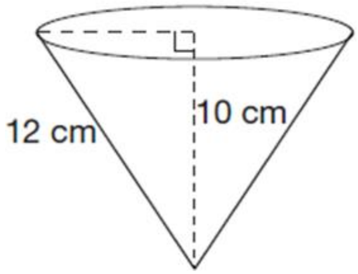


10. Brad has a cylindrical container that is open at the top. He wants to paint the outer surfaces of the container, including the bottom.

Calculate the area of that surface that needs to be painted.

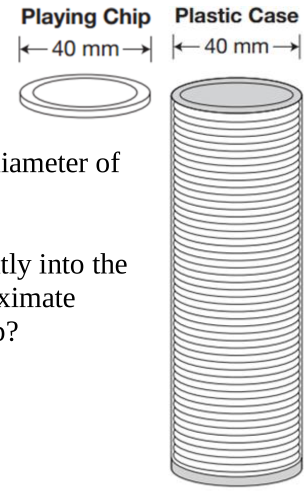


11. Calculate the surface area of the cone shown below.

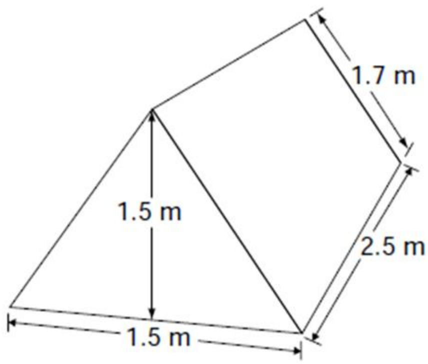


12. The playing chips of a board game are stored in cylindrical plastic cases. The plastic cases have a volume of 25120 mm^3 and a diameter of 40 mm.

If 50 playing chips can fit tightly into the plastic case, what is the approximate height (thickness) of each chip?

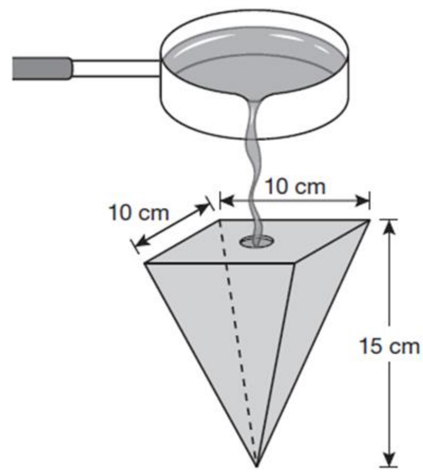


13. Including the ends and the floor, calculate the surface area of the tent shown below.



14. The mould shown at the right is used to make a candle in the shape of a square-based pyramid.

What is the volume of the mould?



Answers

1. C **2.** D **3.** A

4. **(a)** $A \doteq 435.8$ cm^2, $V \doteq 382.7$ cm^3 **(b)** A=416 cm^2, V=480 cm^3 **(c)** $A \doteq 2027.7$ m^2, V=5888 m^3 **(d)** $A \doteq 4178$ cm^2, $V \doteq 19160$ cm^3

5. square prism, 14.1 cm^3

6. rectangular prism, $324x^2y$ cubic units

7. **(a)** 1728 cm^3 **(b)** 864 cm^2 **(c)** 823.2 cm^3

8. $h = 4a^2b^3$ units

9. Vol. of shaded part = vol. of cylinder + vol. of small cone $= \pi(2.5)^2(4) + \dfrac{\pi(2.5)^2(2)}{3} \doteq 91.6 \doteq$ volume of unshaded cone

$\therefore$ height of unshaded cone = $h = \dfrac{3V}{\pi r^2} \doteq \dfrac{3(91.6)}{\pi(3)^2} \doteq 9.7$ cm

10. Area to be painted = area of bottom + area of lateral surface $= \pi(20)^2 + 2\pi(20)(50) \doteq 7540$ cm^2

11. First use the Pythagorean Theorem to calculate the radius of the cone ($r \doteq 6.6$). Then $A \doteq \pi(6.6)(12) + \pi(6.6)^2 \doteq 386$ cm^2

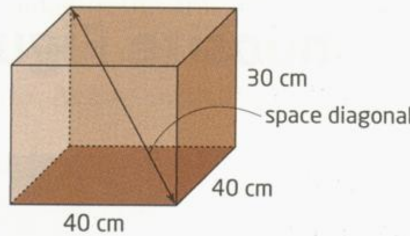
12. Each chip is about 0.4 mm thick.

13. The tent has a surface area of 14.5m^2

14. The mould has a volume of 500 cm^3

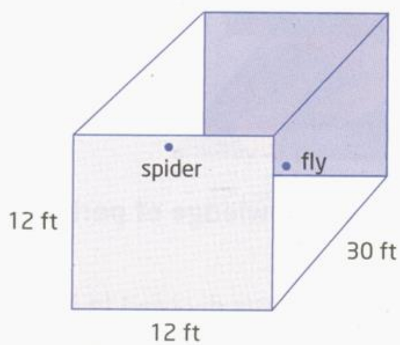
Extend

10. A cardboard box measures 40 cm by 40 cm by 30 cm. Calculate the length of the space diagonal, to the nearest centimetre.

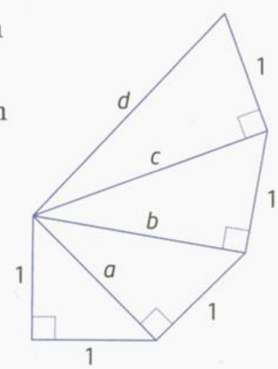


11. The Spider and the Fly Problem is a classic puzzle that originally appeared in an English newspaper in 1903. It was posed by H.E. Dudeney. In a rectangular room with dimensions 30 ft by 12 ft by 12 ft, a spider is located in the middle of one 12 ft by 12 ft wall, 1 ft away from the ceiling. A fly is in the middle of the *opposite* wall 1 ft away from the floor. If the fly does not move, what is the shortest distance that the spider can crawl along the walls, ceiling, and floor to capture the fly?

Hint: Using a net of the room will help you get the answer, which is less than 42 ft!



12. A spiral is formed with right triangles, as shown in the diagram.



a) Calculate the length of the hypotenuse of each triangle, leaving your answers in square root form. Describe the pattern that results.

b) Calculate the area of the spiral shown.

c) Describe how the expression for the area would change if the pattern continued.

13. Math Contest

a) The set of whole numbers (5, 12, 13) is called a *Pythagorean triple*. Explain why this name is appropriate.

b) The smallest Pythagorean triple is (3, 4, 5). Investigate whether multiples of a Pythagorean triple make Pythagorean triples.

c) Substitute values for m and n to investigate whether triples of the form $(m^2 - n^2, 2mn, m^2 + n^2)$ are Pythagorean triples.

d) What are the restrictions on the values of m and n in part c)?

Answers

10. 64 cm

11. 40 ft

12. a) $\sqrt{2}$; $\sqrt{3}$; $\sqrt{4}$; $\sqrt{5}$

b) $\frac{\sqrt{1}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{4}}{2}$

c) As you add right triangles to the spiral pattern, the area will increase by $\frac{\sqrt{\text{Number of Triangles}}}{2}$.

13. a) This name is appropriate because this set of three whole numbers satisfies the Pythagorean theorem.

b) Yes.

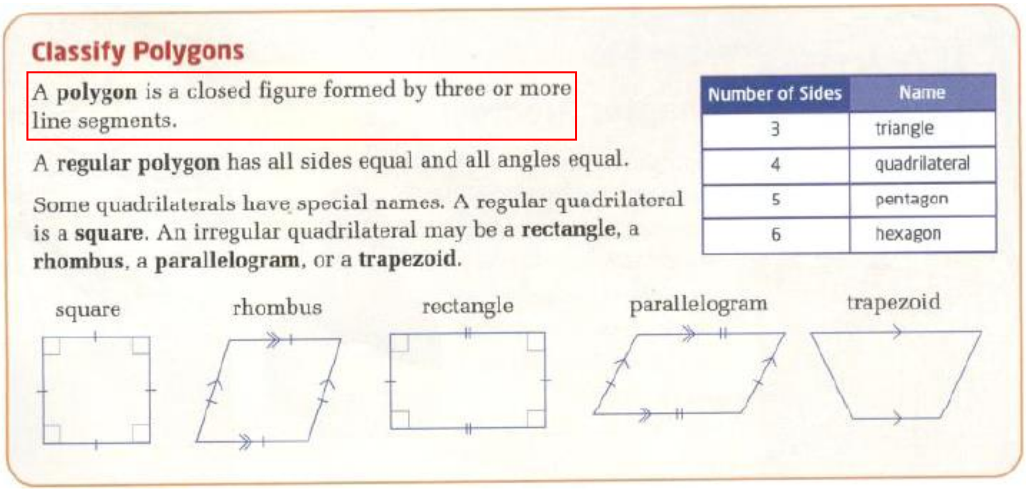
c) Yes, they are Pythagorean triples, with some restrictions on the values of m and n.

d) $m > n > 0$

ANGLE RELATIONSHIPS IN POLYGONS

Concepts

Classify Polygons



Classify Polygons

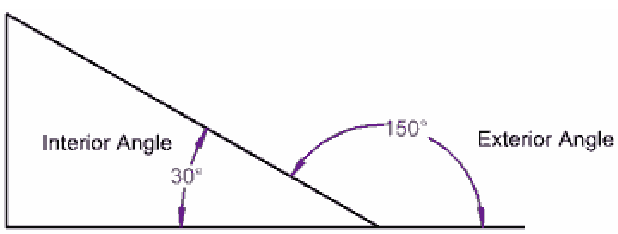
A **polygon** is a closed figure formed by three or more line segments.

A **regular polygon** has all sides equal and all angles equal.

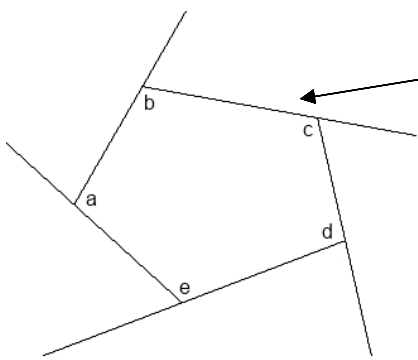
Some quadrilaterals have special names. A regular quadrilateral is a **square**. An irregular quadrilateral may be a **rectangle**, a **rhombus**, a **parallelogram**, or a **trapezoid**.

Number of Sides	Name
3	triangle
4	quadrilateral
5	pentagon
6	hexagon

Interior and Exterior Angles

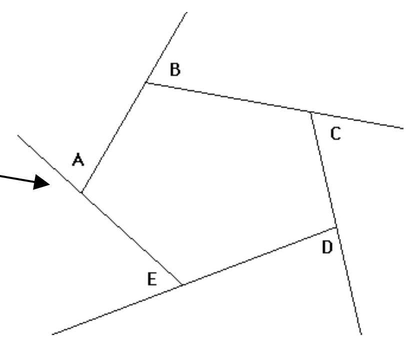


- The diagram at the left shows an ***interior angle*** of a triangle and its corresponding ***exterior angle***.
- Notice that corresponding interior and exterior angles must be ***supplementary*** (i.e. their sum must be 180°).



- The interior angles of a ***pentagon***
- The exterior angles of the same pentagon
- Notice that
  $a+A=180°$
  $b+B=180°$
  $c+C=180°$
  $d+D=180°$
  $e+E=180°$

The corresponding interior and exterior angles are ***supplementary***.



Polygon Definition

A ***polygon*** is a closed plane figure bounded by ***three or more*** line segments.

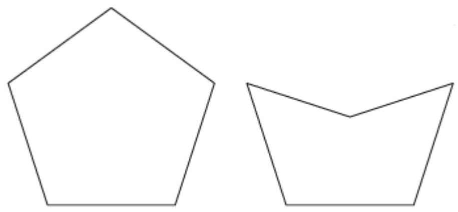
Regular Polygon Definition

A ***regular polygon*** is a polygon in which all sides have the same length (***equilateral***) and all angles have the same measure (***equiangular***).

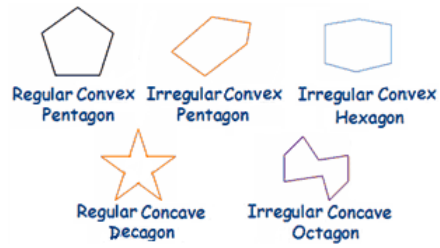
Irregular Polygon Definition

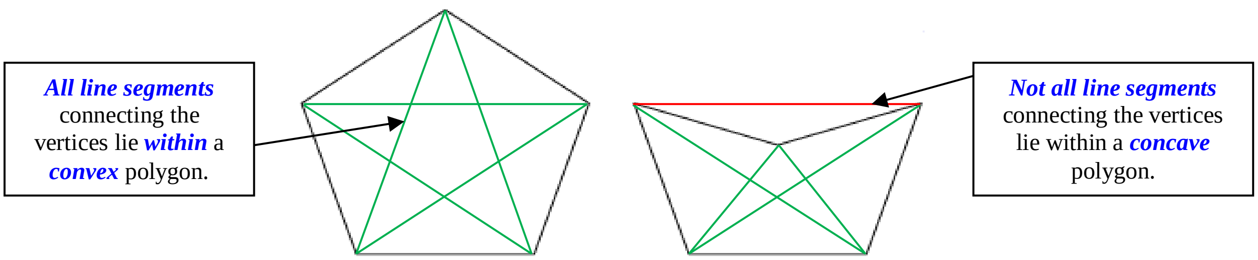
An ***irregular polygon*** is a polygon in which ***not*** all sides have the same length and ***not*** all angles have the same measure.

Convex Polygon Definition	Concave Polygon Definition
A ***convex polygon*** is a polygon that contains all line segments connecting any two of its vertices. In a convex polygon, the measure of ***each interior angle*** must be less than 180°.	A ***concave polygon*** is a polygon that ***does not*** contain all line segments connecting any two of its vertices. In a concave polygon, the measure of ***at least one interior angle*** is more than 180°. That is, a concave polygon must contain at least one ***reflex angle***.



convex polygon *concave polygon*

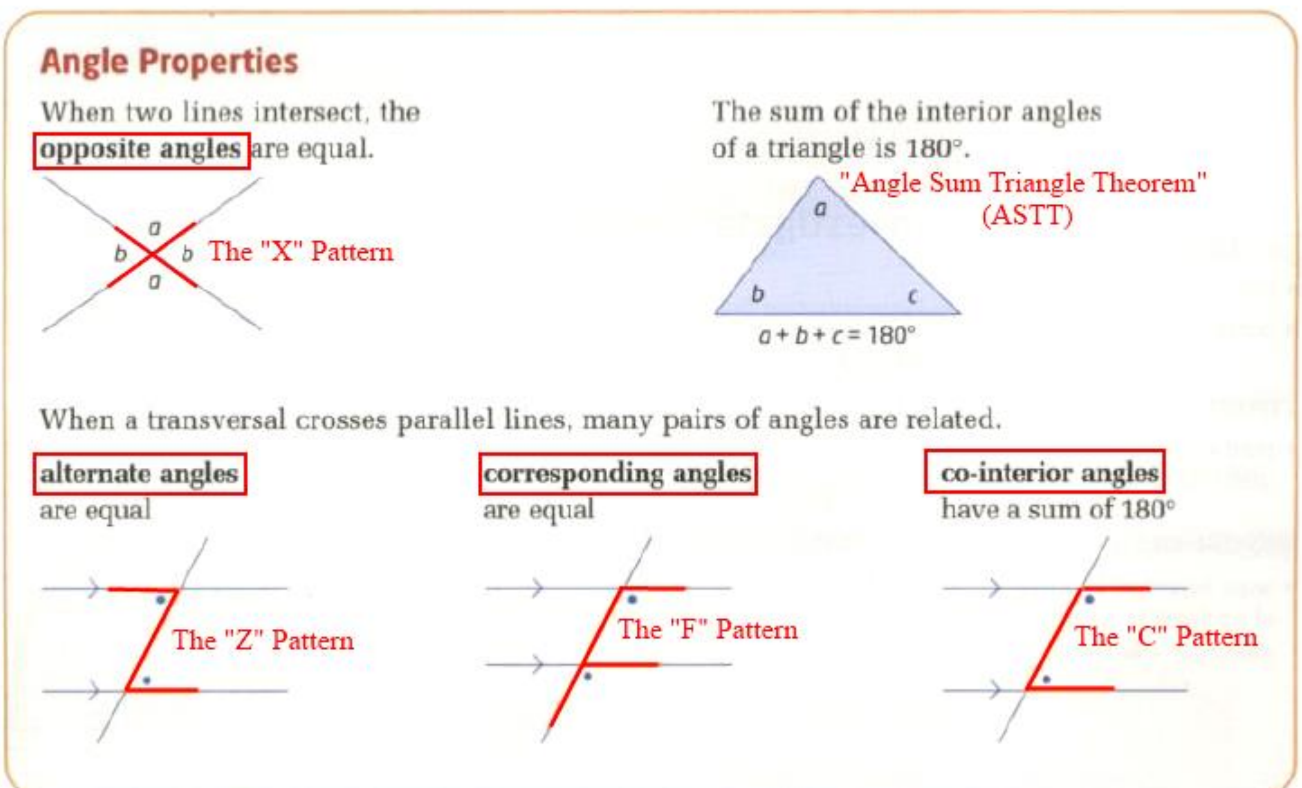




convex polygon *concave polygon*

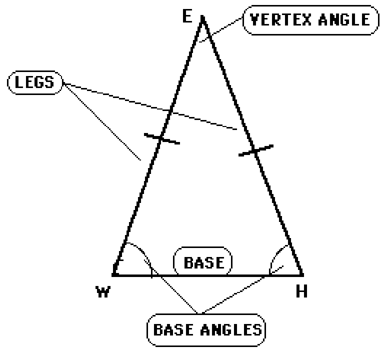
Relationships

Angle Properties – Intersecting Lines, Transversal Passing through a Pair of Parallel Lines, Triangles

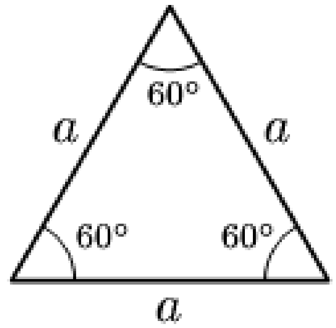


Angles in Isosceles and Equilateral Triangles

- The ***Isosceles Triangle Theorem (ITT)*** asserts that a triangle is isosceles if and only if its ***base angles are equal***.
- This can be proved using ***triangle congruence theorems*** (not covered in this course).



- Using ITT, it can be shown that an ***equilateral triangle*** is also ***equiangular*** (all three angles have the same measure).
- If x represents the measure of each angle, then
  $x+x+x=180°$ (ASTT)
  $\therefore 3x=180°$
  $\therefore x=60°$

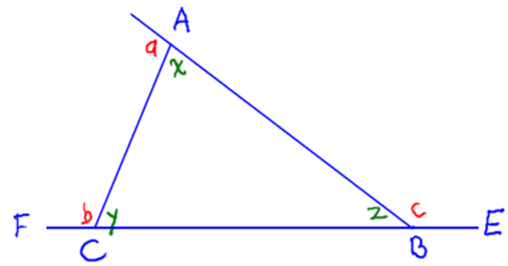


Exterior Angles of a Triangle

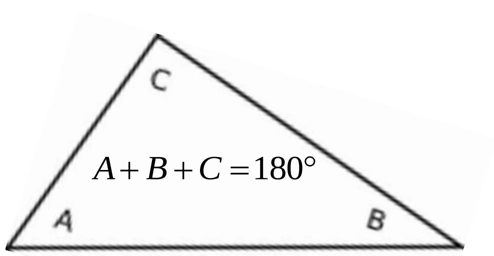
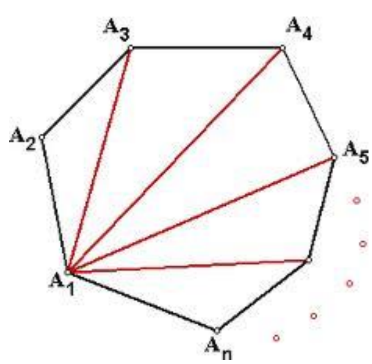
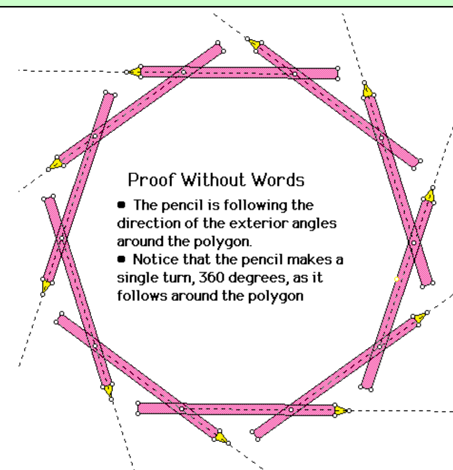
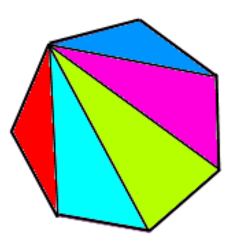
The exterior angle at each vertex of a triangle is equal to the sum of the interior angles at the other two vertices.

Using the diagram at the right, we can rewrite the above statement as follows:

$$a=y+z$$
$$b=x+z$$
$$c=x+y$$



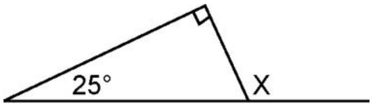
Sum of Interior and Exterior Angles of Polygons

The Relationships	***Understanding Why…***
 The ***sum*** of the ***interior angles*** of any ***triangle*** is 180°.  The ***sum*** of the ***exterior angles*** of ***any convex polygon*** is 360°  The ***sum*** of the ***interior angles*** of ***an n-sided convex polygon*** $=(n-2)\times 180°$ = (number of triangles) × 180°	  Name: Heptagon Number of Sides: 7 Number of Triangles: 5 Sum of Interior Angles: $5(180°)=900°$

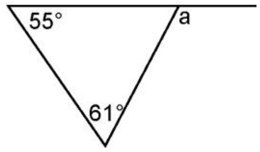
Problems on Angle Relationships in Triangles

1. Find the measure of each indicated exterior angle.

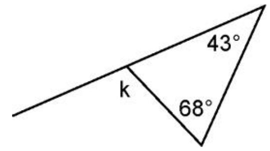
(a)



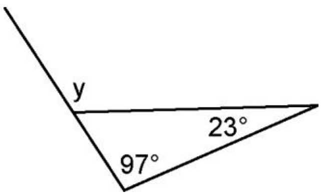
(b)



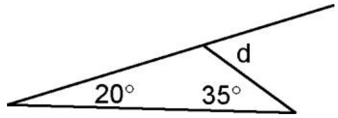
(c)



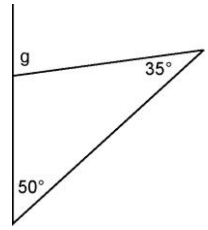
(d)



(e)

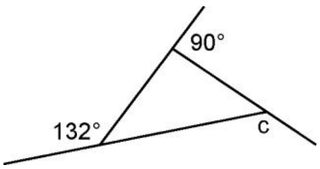


(f)

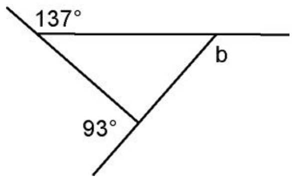


2. Find the measure of each indicated exterior angle.

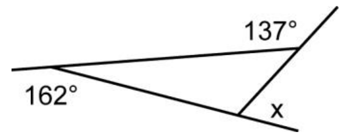
(a)



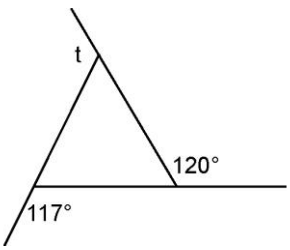
(b)



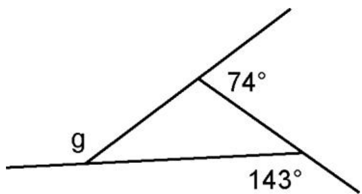
(c)



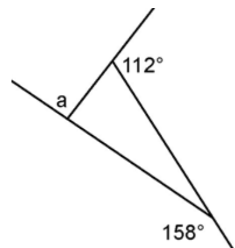
(d)



(e)

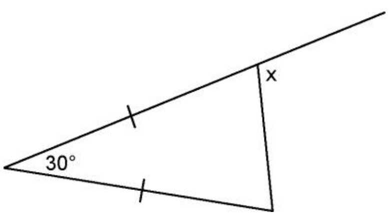


(f)

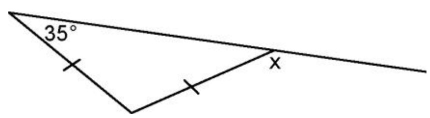


3. Find the measure of each indicated angle.

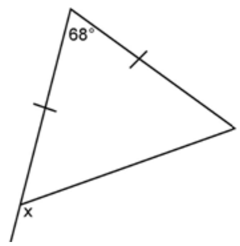
(a)



(b)



(c)



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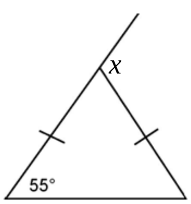

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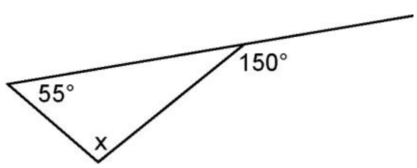

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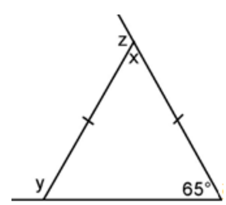
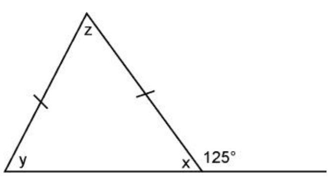
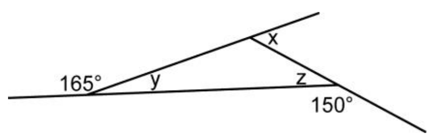


(g)



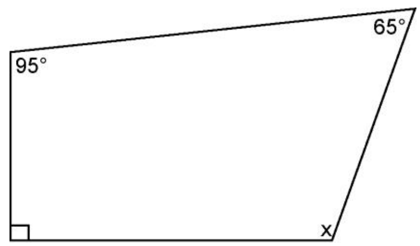
(h)



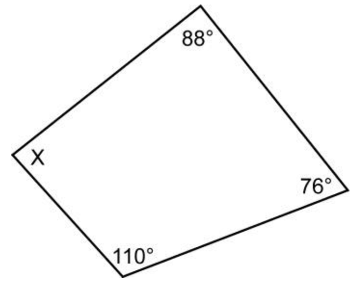
4. One interior angle in an isosceles triangle measures 42°. Find the possible measures for the exterior angles.

5. Find the measure of each indicated angle. **Hint:** Divide the quadrilaterals into triangles.

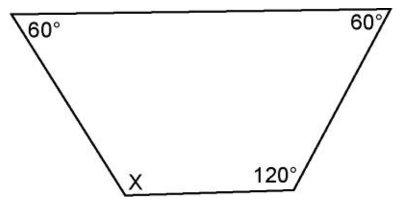
(a)



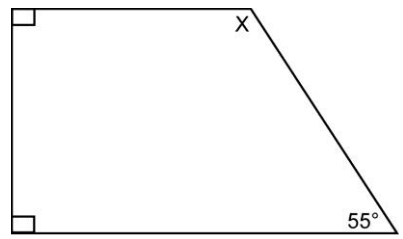
(b)



(c)

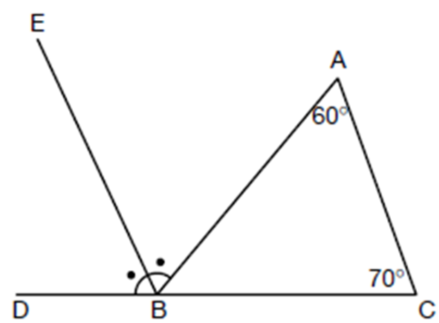


(d)

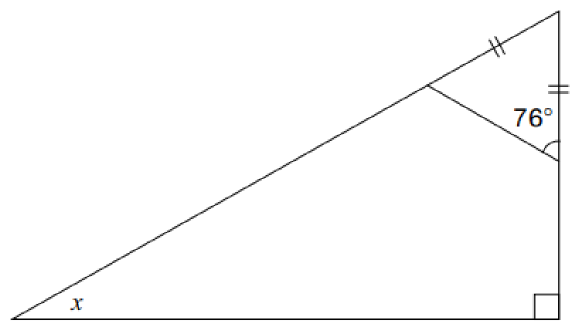


6. Find the measure of each indicated angle.

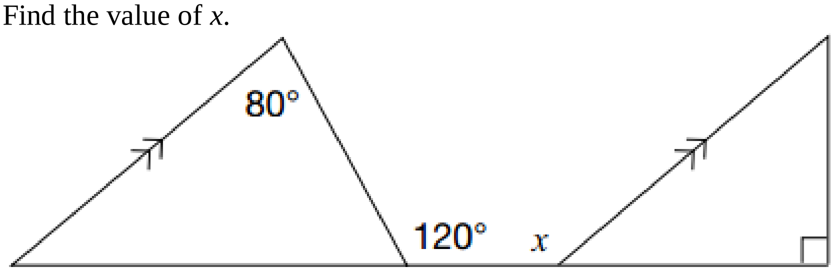
(a) In the following diagram, line segment *EB* ***bisects*** (divides into two equal angles) $\angle ABD$. What is the measure of $\angle ABE$.



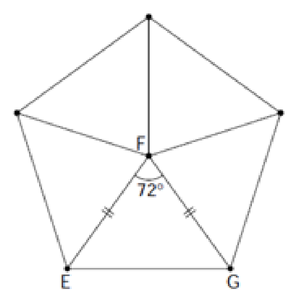
(b) Find the value of *x*.



(c) Find the value of *x*.



(d) What is the measure of $\angle FEG$?



Answers

1. **a)** 115° **b)** 116° **c)** 111°
d) 120° **e)** 55° **f)** 85°

2. **a)** 138° **b)** 130° **c)** 61°
d) 123° **e)** 143° **f)** 90°

3. **a)** 105° **b)** 145° **c)** 124° **d)** 110°
e) $x = 95°$ **f)** $x = 50°$; $y = 115°$; $z = 130°$
g) $x = y = 55°$; $z = 70°$ **h)** $x = 45°$; $y = 15°$; $z = 30°$

4. 138°, 138°, 84° or 138°, 111°, 111°

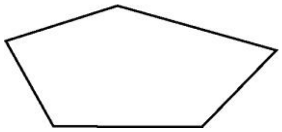
5. **a)** 110° **b)** 86°
c) 120° **d)** 125°

6. **a)** 65° **b)** 62°
c) 140° **d)** 54°

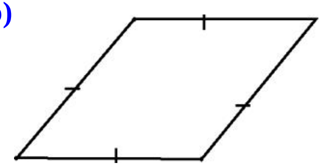
Problems on Angle Relationships in Polygons

1. Find the sum of the interior angles of each polygon.

a)



b)



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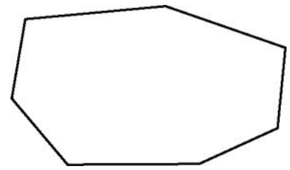
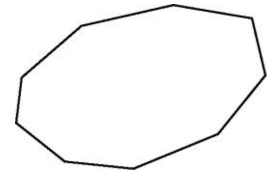
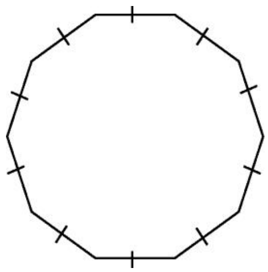


d)



2. Find the sum of the interior angles of each polygon.

a)



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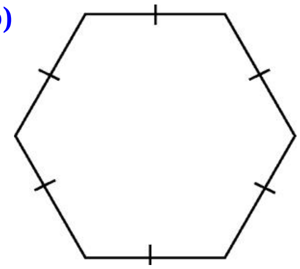

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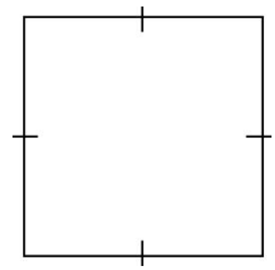
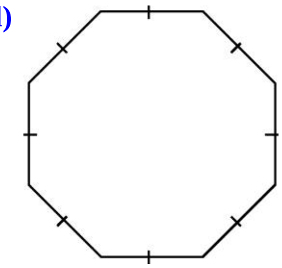


d)



3. Find the sum of the interior angles of a polygon with each number of sides.

a) 11 sides **b)** 14 sides
c) 18 sides **d)** 24 sides

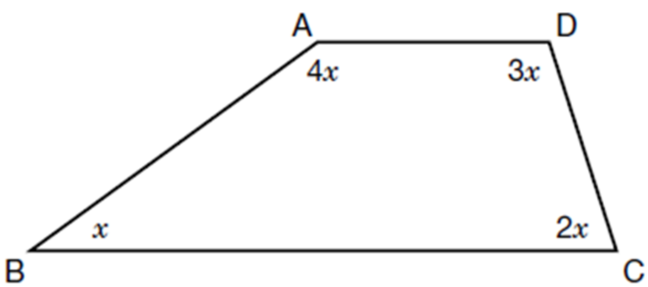
4. Find the measure of each interior angle of a ***regular polygon*** with each number of sides.

a) 3 sides **b)** 20 sides
c) 9 sides **d)** 16 sides

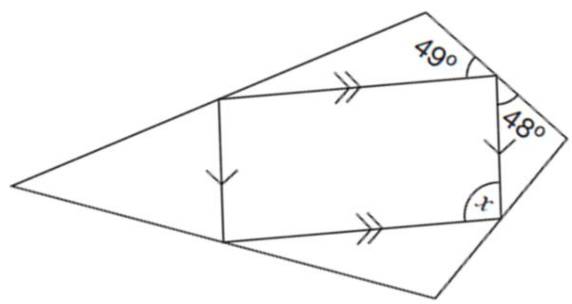
5. Find the number of sides in each polygon given the sum of its interior angles.

a) 720° **b)** 1980°
c) 2340° **d)** 4140°

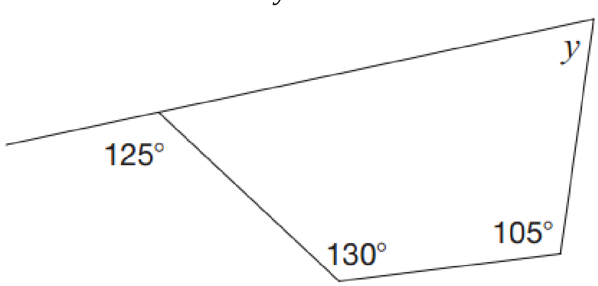
6. Determine the value of x.



7. Determine the value of x.

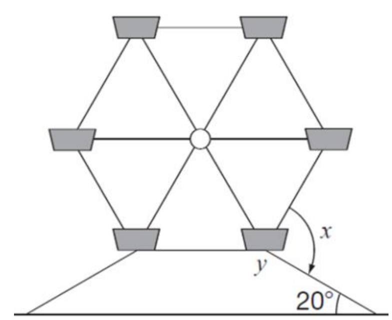


8. Determine the value of y.



9. A Ferris wheel has six sides of equal length. The exit ramp of the Ferris wheel is in the shape of a trapezoid and has an angle of incline of 20°.

Determine the values of x and y.



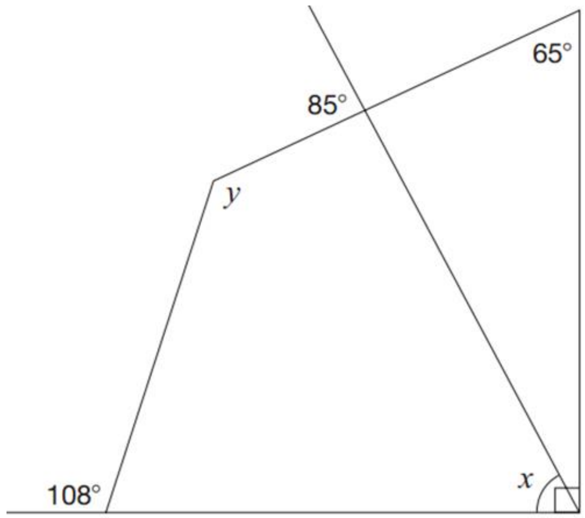
Answers

1. **a)** 540° **b)** 360° **c)** 900° **d)** 1260°
2. **a)** 1440° **b)** 720° **c)** 360° **d)** 1080°
3. **a)** 1620° **b)** 2160° **c)** 2880° **d)** 3960°
4. **a)** 60° **b)** 162° **c)** 140° **d)** 157.5°
5. **a)** 6 sides **b)** 13 sides
c) 15 sides **d)** 25 sides

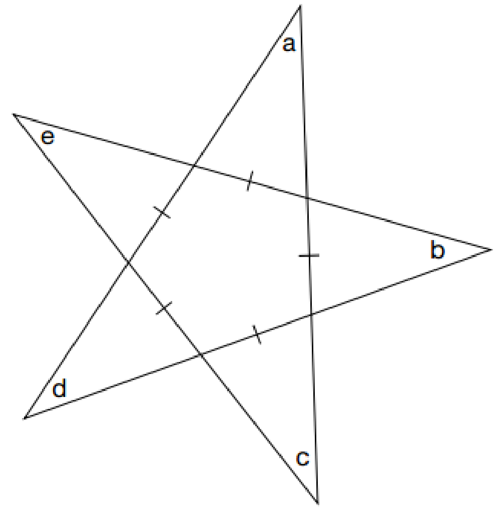
6. $x = 36°$
7. $x = 97°$
8. $y = 70°$
9. $x = 80°$, $y = 160°$

More Challenging Problems on Angle Relationships in Polygons

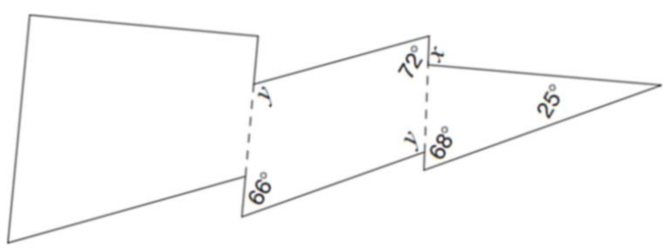
1. Determine the values of x and y.



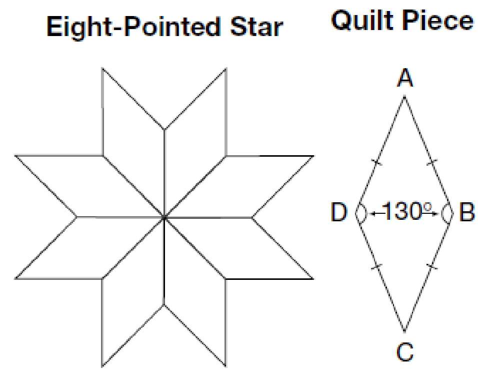
2. Determine the value of $a+b+c+d+e$.



3. Pravin designs a lightning bolt using two quadrilaterals and one triangle as shown below. Determine the values of x and y.



4. An eight-pointed star quilt is to be made using quilt pieces exactly like the one shown at the far right.



Is it possible to make the quilt using pieces like the given piece? Explain.

Answers

1. $x=60°$, $y=133°$

2. $a+b+c+d+e=180°$

3. $x=93°$, $y=111°$

4. Since eight pieces are needed to make the quilt, $\angle DAB$ and $\angle DCB$ should both have a measure of $\dfrac{360°}{8}=45°$. Using the fact that the sum of the interior angles of a quadrilateral is 360°, it follows that $\angle DAB$ and $\angle DCB$ actually both have a measure of $\dfrac{360°-2(130°)}{2}=50°$. Therefore, it is ***not possible*** to make the quilt using the given pieces.

Unit 0 Reflection

The main purpose of this unit was to introduce you to mathematical thinking. You have learned that…

- ***Formulas*** are the ***finished products*** of mathematical thinking. They provide us with convenient ***algorithms*** for solving particular kinds of problems. However, formulas, of themselves, do not constitute mathematical thinking! To become a true mathematical thinker, it is necessary to move far beyond a purely formulaic approach!
- The mathematician's main goal is to ***discover*** how quantities are ***related*** to one another. The Pythagorean Theorem is an iconic illustration of what we mean by this. Every right triangle, no matter how large or small, must obey the equation $c^2 = a^2 + b^2$. Once again, however, it is not enough just to know the equation. A true mathematician also understands ***why*** this equation describes the relationship among the sides of a right triangle and can prove it in a highly rigorous fashion.
- The mathematics that you learn in high school can be reduced to three basic concepts:
  - Mathematical Objects (e.g. numbers, geometric shapes, etc)
  - Mathematical Operations (e.g. +, −, ×, ÷)
  - Mathematical Relationships (e.g. $c^2 = a^2 + b^2$)
- In keeping with the focus on mathematical relationships, several examples were given in this unit including…
  - The Pythagorean Theorem
  - Measurement relationships for several two-dimensional and three-dimensional shapes
  - Angle relationships in polygons

Reflection Questions

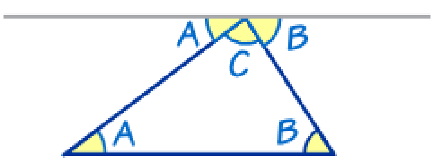
1. Use the diagrams shown below to explain why…

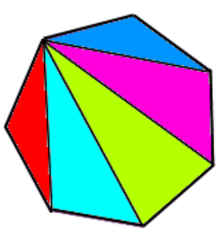
(a) …the sum of the interior angles of a triangle must be 180°.

(b) …the sum of the interior angles of an n-sided convex polygon must be $(n-2)\times 180°$.

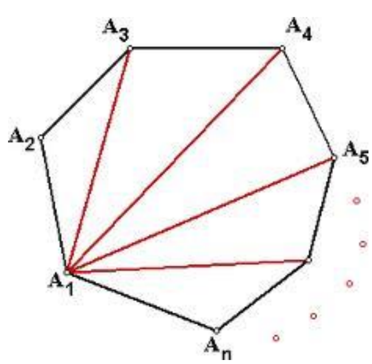
(c) …the sum of the exterior angles of any convex polygon must be 360°.

2. Reflect upon your answers to question 1. Then explain how these answers help you to keep in mind the main ideas of this unit.





Name: Heptagon
Number of Sides: 7
Number of Triangles: 5
Sum of Interior Angles: $5(180°) = 900°$



The ***sum*** of the ***interior angles*** of ***an n-sided convex polygon***

$= (n-2)\times 180°$

= (number of triangles) × 180°