Questions

1. What is an integer?

An integer is a whole number that can be positive, negative or zero. The set of all integers is denoted $\mathbb{Z}$.

2. What is a rational number? Why are such numbers called rational?

A rational number is a fraction that can be positive, negative or zero. Such numbers are called rational because they are expressed as the RATIO of two integers. The set of all rational numbers is denoted $\mathbb{Q}$ and it includes $\mathbb{Z}$ (e.g. $-2=\frac{-2}{1}$)

+(+) add a positive: GAIN
-(-) subtract a negative: GAIN

+(-) add a negative: LOSS
-(+) subtract a positive: LOSS

Gains more than losses: + answer
Losses more than gains: - answer

Rule for Determining the Sign of the Answer when Multiplying and Dividing

Two Numbers
Signs Same: + answer
Signs Different: - answer

More than Two Numbers
Even # of Negatives: + answer
Odd # of Negatives: - answer

Adding/Subtracting Fractions

- Express each fraction with a common denominator.
- Add/subtract the numerators.
- Keep the denominator!
- If possible, reduce to lowest terms.

$$\frac{3}{10}+\frac{8}{15}=\frac{9}{30}+\frac{16}{30}=\frac{25}{30}=\frac{5}{6}$$

Multiplying Fractions

- Multiply the numerators and multiply the denominators.
- If possible, reduce to lowest terms.

OR

- Reduce first (vertically and diagonally).
- Multiply the numerators and multiply the denominators.

$$\frac{3}{10}\left(\frac{8}{15}\right)=\frac{24}{150}=\frac{4}{25} \quad \textbf{OR} \quad \frac{\cancel{3}^{1}}{\cancel{10}_{5}}\left(\frac{\cancel{8}^{4}}{\cancel{15}_{5}}\right)=\frac{4}{25}$$

Dividing Fractions

- ***Do not*** change the 1st fraction.
- Change ÷ to ×.
- Find the reciprocal of the 2nd fraction (i.e. "***flip***").
- **Summary:** Multiply by the reciprocal (i.e. "flip 'n multiply")

$$\frac{3}{10}\div\frac{8}{15}=\frac{3}{10}\times\frac{15}{8}=\frac{45}{80}=\frac{9}{16}$$

3. ***Evaluate*** each of the following expressions ***without*** using a calculator.

(a) $-3+5$
$=2$

(b) $-3-5$
$=-8$

(c) $-3+(-5)$
$=-3-5$
$=-8$

(d) $-3-(-5)$
$=-3+5$
$=2$

(e) $-3+5-6+1$
$=2-6+1$
$=-4+1$
$=-3$

(f) $-3-5-6+1$
$=-13$

(g) $-3-(-5)+(-6)+1$
$=-3+5-6+1$
$=-3$

(h) $3+(-5)+(-6)-(-1)$
$=3-5-6+1$
$=-7$

4. ***Evaluate*** each of the following expressions ***without*** using a calculator.

(a) $-3(5)$
$=-15$

(b) $-3(+5)$
$=-15$

(c) $-3(-5)$
$=15$

(d) $-3(5)$
$=-15$

(e) $-3(5)(-6)(1)$
$=90$

(f) $-3(5)(-6)(-1)$
$=-90$

(g) $3(5)(6)(-1)$
$=-90$

(h) $-3(-5)(-6)(-1)$
$=90$

5. ***Evaluate*** each of the following expressions ***without*** using a calculator.

(a) $\frac{36}{-12}$ $= -3$

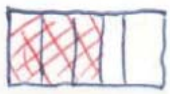
(b) $\frac{-36}{-12}$ $= 3$

(c) $\frac{+49}{+7}$ $= 7$

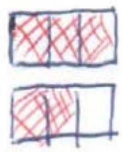
(d) $\frac{-64}{16}$ $= -4$

6. Draw diagrams to represent each of the following fractions.

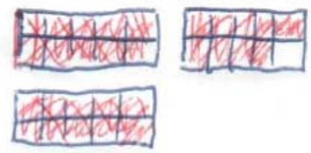
(a) $\frac{3}{5}$



(b) $\frac{5}{3}$



(c) $2\frac{9}{10}$



(d) $\frac{6}{2} = 3$

7. ***Evaluate*** each of the following ***expressions***.

(a) $\frac{3}{5}-\frac{2}{5}$

$= \frac{1}{5}$

(b) $\frac{-5}{3}+\frac{5}{6}$

$= \frac{-10}{6}+\frac{5}{6}$

$= \frac{-5}{6}$

(c) $-\frac{5}{14}+\left(-\frac{8}{21}\right)$

$= \frac{-15}{42}-\frac{16}{42}$

$= \frac{-31}{42}$

(d) $-\frac{5}{14}-\left(-\frac{8}{21}\right)$

$= \frac{-15}{42}+\frac{16}{42}$

$= \frac{1}{42}$

8. ***Evaluate*** each of the following ***expressions***.

(a) $\frac{3}{5}\left(-\frac{2}{5}\right)$

$= -\frac{6}{25}$

(b) $\frac{-5}{3}\left(+\frac{5}{6}\right)$

$= \frac{-25}{18}$

(c) $-\frac{5}{\cancel{14}_{7}}\left(-\frac{\cancel{8}^{4}}{21}\right)$

$= \frac{20}{147}$

(d) $\frac{5}{14}\left(-\frac{8}{21}\right)$

$= -\frac{20}{147}$

9. ***Evaluate*** each of the following ***expressions***.

(a) $\frac{3}{5}\div\left(-\frac{2}{5}\right)$

$= \frac{3}{\cancel{5}_{1}}\times\left(\frac{\cancel{5}^{1}}{-2}\right)$

$= -\frac{3}{2}$

(b) $\frac{-5}{3}\div\left(+\frac{5}{6}\right)$

$= \frac{-\cancel{5}^{1}}{\cancel{3}_{1}}\times\frac{\cancel{6}^{2}}{\cancel{5}_{1}}$

$= \frac{-2}{1} = -2$

(c) $-\frac{5}{14}\div\left(-\frac{8}{21}\right)$

$= \frac{-5}{\cancel{14}_{2}}\times\left(\frac{-\cancel{21}^{3}}{8}\right)$

$= \frac{15}{16}$

(d) $\frac{5}{14}\div\left(-\frac{8}{21}\right)$

$= \frac{5}{\cancel{14}_{2}}\times\left(\frac{-\cancel{21}^{3}}{8}\right)$

$= \frac{-15}{16}$

1 2 3 4

B E DM AS

Division/Multiplication - tied, left-to-right

Addition/Subtraction - tied, left-to-right

Terms are separated by + and - signs.

Separate each expression into terms. Then apply the operations in the correct order.

10. ***Evaluate*** each of the following ***expressions*** by applying the operations in the correct order.

(a) $-20\div\left(-4-(-8)\right)$

$= -20\div(-4+8)$

$= -20\div 4$

$= -5$

(b) $-20-4(-8)^2$

$= -20-4(64)$

$= -20-256$

$= -276$

(c) $-20+\left(-4-(-2)^3\right)$

$= -20+(-4-(-8))$

$= -20+(-4+8)$

$= -20+4 = -16$

(d) $2(-7)-\dfrac{10}{2^2-3^2}+2(-3)^4$

$= -14-\dfrac{10}{4-9}+2(81)$

$= -14-\left(\dfrac{10}{-5}\right)+162$

$= -14-(-2)+162$

$= -14+2+162$

$= 150$

(e) $-3\left[-2+2(6)-4(3)^3\right]^4$

$= -3[-2+12-4(27)]^4$

$= -3[10-108]^4$

$= -3(-98)^4$

$= -3(92236816)$

$= -276710448$

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(f) $\dfrac{\ldots(-3)}{[2-(-3)]^2}$

$= \dfrac{-10+(-15)}{[2+3]^2}$

$= \dfrac{-10-15}{5^2}$

$= \dfrac{-25}{25}$

$= -1$

(g) $-\dfrac{5}{14}+\left(-\dfrac{8}{21}\right)\left(\dfrac{7}{-4}\right)$

$= -\dfrac{5\times3}{14\times3}+\dfrac{2\times14}{3\times14}$

$= \dfrac{-15}{42}+\dfrac{28}{42}$

$= \dfrac{13}{42}$

(h) $-\dfrac{5}{3}\div\dfrac{10}{9}+\left(-\dfrac{8}{21}\right)\left(\dfrac{3}{-4}\right)^2$

$= -\dfrac{5}{3}\times\dfrac{9}{10}+\left(-\dfrac{8}{21}\right)\left(\dfrac{9}{16}\right)$

$= -\dfrac{3}{2}+\left(-\dfrac{3}{14}\right)$

$= \dfrac{-3\times7}{2\times7}-\dfrac{3}{14}$

$= \dfrac{-21}{14}-\dfrac{3}{14}$

$= \dfrac{-24\div2}{14\div2} = -\dfrac{12}{7}$

(i) $\dfrac{-10+5(-3)}{[2-(-3)]^2}-\left(\dfrac{-5}{3}\right)\left(+\dfrac{5}{6}\right)$

$= \dfrac{-10+(-15)}{(2+3)^2}-\left(\dfrac{-25}{18}\right)$

$= \dfrac{-25}{5^2}+\dfrac{25}{18}$

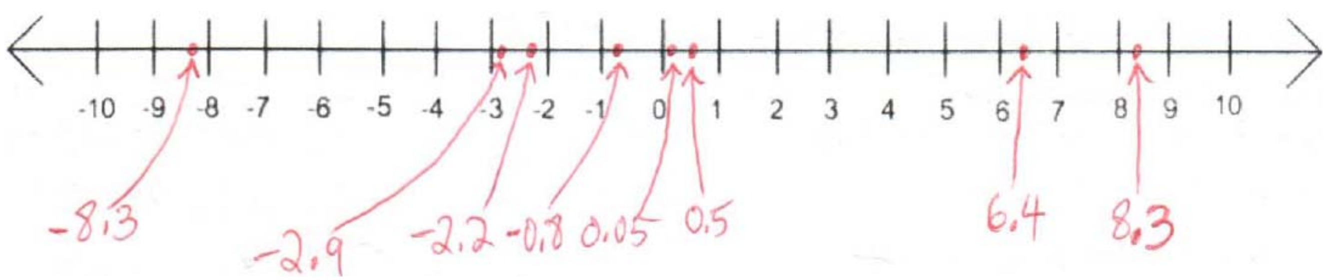
$= \dfrac{-25}{25}+\dfrac{25}{18}$

$= -1+\dfrac{25}{18}$

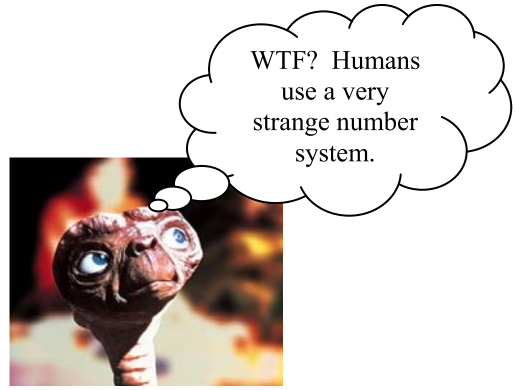
$= \dfrac{-18}{18}+\dfrac{25}{18} = \dfrac{7}{18}$

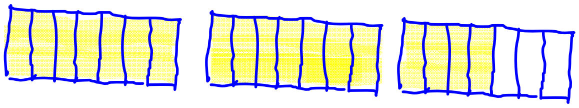
11. Use the given number line to arrange the following numbers in order from smallest to largest.

8.3, –2.9, 0.05, 6.4, –0.8, –2.2 –8.3, 0.5

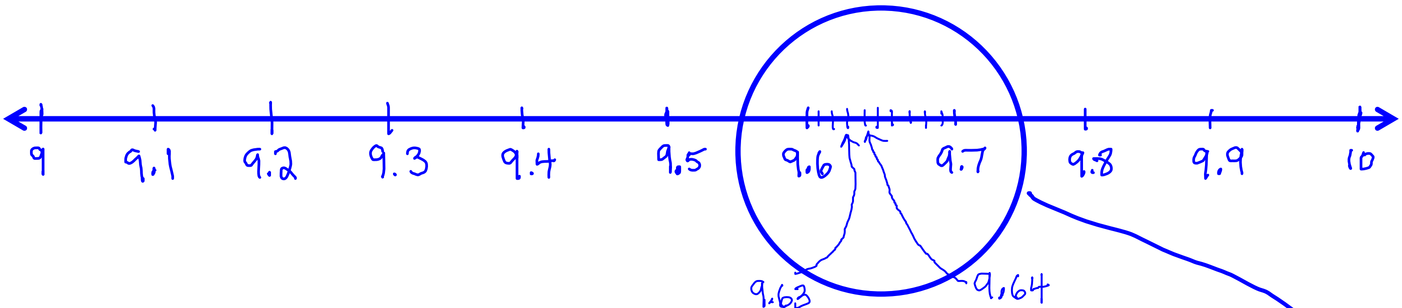


12. An extraterrestrial being is seeking your help in learning how to use the human number system. He/she/it asks you to explain the meaning of the numbers $2\frac{4}{7}$ and 9.637. Draw diagrams to help the extraterrestrial understand our number system.

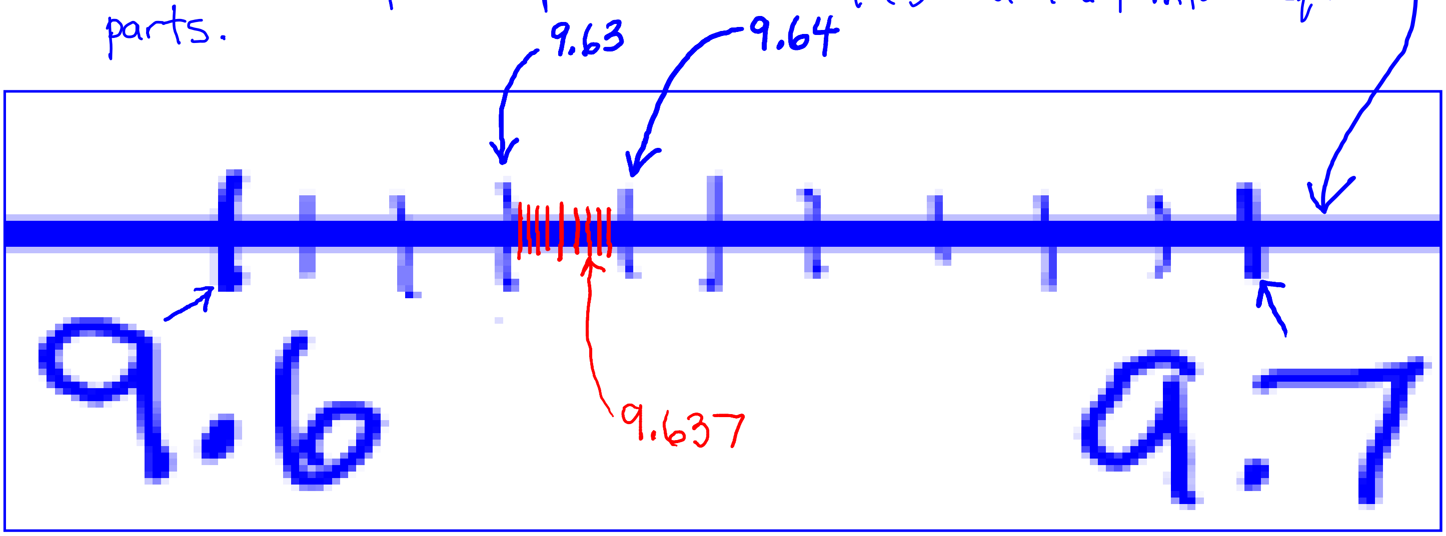




$2\frac{4}{7}$



- Divide up the space between 9 and 10 into 10 equal parts
- Then divide up the space between 9.6 and 9.7 into 10 equal parts
- Then divide up the space between 9.63 and 9.64 into 10 equal parts.



WHY, WHY, WHY? YOU MUST ASK WHY!

Important Reminder

If you would like the study of mathematics to contribute significantly to your intellectual growth, you ***MUST UNDERSTAND*** the ***MEANING*** of mathematical concepts such as the following:

1. **Numbers**
   e.g. small versus large numbers, negative versus positive numbers, whole numbers versus fractions
2. **Mathematical Operations**
   e.g. "+" means "gain," "−" means "lose," "×" means "groups of," "÷" means "how many groups of"
   "$100 \div \frac{1}{3}$ means "How many groups of $\frac{1}{3}$ can be formed from 100?"
3. **Variables and Expressions**
   e.g. "x" means "an *unknown* or *unspecified* number," "$2x$" means "2 times *any number*"
4. **Mathematical Relationships**
   e.g. "$c^2 = a^2 + b^2$" is an equation that expresses the relationship among the side lengths of a right triangle.

You must always seek to understand *why* something is true. Every claim *requires* a justification!

"You do not really understand something unless you can explain it to your grandmother." (Albert Einstein)

Why on Earth do you do it like that?

1. <u>Why is subtracting a negative number a GAIN?</u> (**e.g.** $5-(-10)=5+10=15$)

   Suppose that you have just borrowed $100 in cash from a good friend and that you have decided to keep the borrowed money in your wallet. In addition, you have $500 of your own money, all of which you have deposited into a bank account. Now suppose further that your friend is so charitable that he/she decides that *you do not need to repay the debt*.

   Another way of putting this is that you have *$500 of your own* and your friend *takes away a debt of $100*. *Do you gain or lose money?* Write a mathematical expression to support your answer.

What you Have	What you Owe
$500	~~$100~~
$100	

∴ you GAIN $100

$500 − (debt of $100)
= $500 − (−$100) must be a gain (see table)
= $500 + $100
= $600

2. <u>Why is the product of a positive number and a negative number NEGATIVE?</u> (**e.g.** $5(-10)=-50$)
   <u>Why is the quotient of a positive number and a negative number NEGATIVE?</u> (**e.g.** $-50 \div 5 = -10$)
   Hints: "$5(-10)$" can be interpreted as "five groups of -10." Multiplication and division are *opposites* of each other.

$\left.\begin{array}{l} 5(-10) = 5 \text{ groups of } -10 = -50 \\ x(-y) = x \text{ groups of } -y = -xy \end{array}\right\}$ product must be NEGATIVE

$-50 \div 5$ must be -10 because $\times$ and $\div$ are opposites

$-50 \div 5 = -10$ MEANS that $5(-10) = -50$, which we already know to be true

3. <u>Why is the product of two negative numbers POSITIVE?</u> (**e.g.** $-5(-10)=50$)

<u>Why is the quotient of two negative numbers POSITIVE?</u> (**e.g.** $-50\div(-10)=5$)

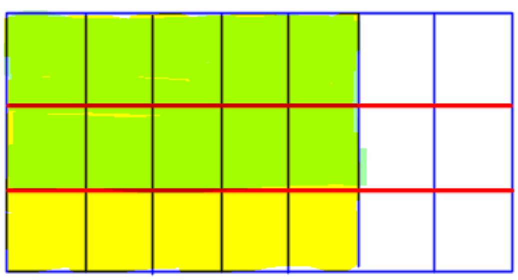
Hint: Multiplication and division are *opposites* of each other.

(a) $5(-10) = -50$ since 5 groups of -10 must be -50 (shown in 2)
Since $\div$ and $\times$ are opposites, it must be the case then that $-50\div(-10)=5$

(b) $50\div(-10)=-5$
Since $\div$ and $\times$ are opposites, it must follow that $(-5)(-10)=50$

4. <u>When multiplying fractions, why do we multiply the numerators and the denominators?</u>
(**e.g.** $\frac{2}{3}\times\frac{5}{7}=\frac{2\times 5}{3\times 7}=\frac{10}{21}$)

Hint: The operation "multiplication" is equivalent to the operation "of."



(yellow) $= \frac{5}{7}$

(green) $= \frac{2}{3}$ of $\frac{5}{7} = \frac{10}{21}$

$\frac{2}{3}\times\frac{5}{7} = \frac{2}{3}$ groups of $\frac{5}{7} = \frac{2}{3}$ of $\frac{5}{7}$

From the diagram we can clearly see that $\frac{2}{3}$ of $\frac{5}{7}$ must be $\frac{10}{21}$

The same result is obtained by evaluating $\frac{2\times 5}{3\times 7}$

5. <u>When dividing fractions, why do we multiply by the reciprocal?</u> (**e.g.** $\frac{2}{3}\div\frac{5}{7}=\frac{2}{3}\times\frac{7}{5}=\frac{14}{15}$)

Hint: Multiplication and division are *opposites* of each other. In addition, for the operations "$\times$" and "$\div$," the *opposite* of a number is its *reciprocal*. (Recall that for the operations "$+$" and "$-$," the *opposite* of a *number* is its negative.)

Every division can be rewritten as a multiplication because $\div$ and $\times$ are opposites of each other.

e.g. $10\div 2 = 10\times\frac{1}{2}$, $\frac{5}{3}\div 2 = \frac{5}{3}\times\frac{1}{2}$

$a\div b = \frac{a}{b} = \frac{a}{1}\left(\frac{1}{b}\right) = a\times\frac{1}{b}$

opposite operations

reciprocals (opposite of a # when $\times$ or $\div$)

Expressions and Equations – Mathematical Phrases and Sentences

- ***equation*** → L.H.S. = R.H.S. → a complete mathematical "***sentence***"
  e.g. "The sum of two consecutive numbers is 31." → $x + x + 1 = 31$
- ***expression*** → ***not*** a complete mathematical "sentence" → more like a ***phrase***
  e.g. "Ten more than a number" → $x + 10$

Solving the so-called "word problems" that you are given in school is usually just a matter of ***translating English sentences into mathematical equations***.

Mathematical Words

Symbol	***English Equivalent***
$+$	**sum**, plus, added to, more than, increased by, gain of, total of, combined with
$-$	**difference**, minus, subtracted from, less than, fewer than, decreased by, loss of
$\times$	**product**, times, multiplied by, of, factor of, double (×2), twice (×2), triple (×3)
$\div$	**quotient**, divided by, half of (÷2), one-third of (÷3), per, ratio of
$=$	**is**, are, was, were, will be, gives, yields

Translating from English into Algebraic Expressions and Equations

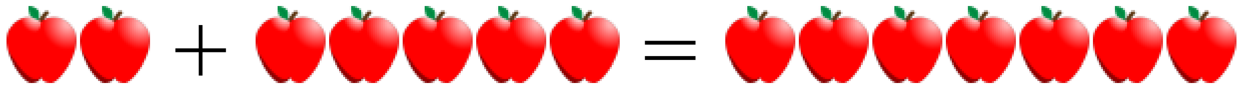
Complete the following table.

English	***Algebraic Expression***	***English***	***Algebraic Equation***
Six more than a number	$n+6$	Six more than a number is 5.	$n+6=5$
A number decreased by 7	$x-7$	A number decreased by 7 is –9.	$x-7=-9$
The product of a number and –3	$-3y$	The product of a number and –3 is 4.	$-3y=4$
Half of a number	$\frac{z}{2}$	Half of a number is 16.	$\frac{z}{2}=16$
Triple a number decreased by 5	$3x-5$	Triple a number decreased by 5 is 8.	$3x-5=8$
Double a number plus 5	$2n-5$	Double a number plus 5 gives 13.	$2n-5=13$
One-third of a number minus 2	$\frac{x}{3}-2$	One-third of a number minus 2 yields 16.	$\frac{1}{3}x-2=16$
Five less than one-fourth of a number.	$\frac{x}{4}-5$	Five less than one-fourth of a number is -1	$\frac{x}{4}-5=-1$
Sixty-five decreased by a number	$65-y$	Sixty-five decreased by a number gives 7.	$65-y=7$
A number divided by 7	$\frac{x}{7}$	A number divided by 7 is –10.	$\frac{x}{7}=-10$
Quadruple a number subtracted ***from*** 6	$6-4z$	Quadruple a number subtracted ***from*** 6 is 2.	$6-4z=2$
Quadruple a number subtracted from 2	$2-4t$	Quadruple a number subtracted from 2 is 3	$2-4t=3$
The quotient of 2, and a number decreased by 4	$\frac{2}{x-4}$	The quotient of 2, and a number decreased by 4 is -9	$\frac{2}{x-4}=-9$
The quotient of 6, and a number subtracted from 3	$\frac{6}{3-n}$	The quotient of 6 and a number subtracted from 3 is 2.	$\frac{6}{3-n}=2$
The product of 2, and a number increased by 7	$2(x+7)$	The product of 2, and a number increased by 7 is 13.	$2(x+7)=13$
The difference of triple a number, and a number increased by 3	$3y-(y+3)$	The difference of triple a number, and a number increased by 3 yields 21.	$3y-(y+3)=21$

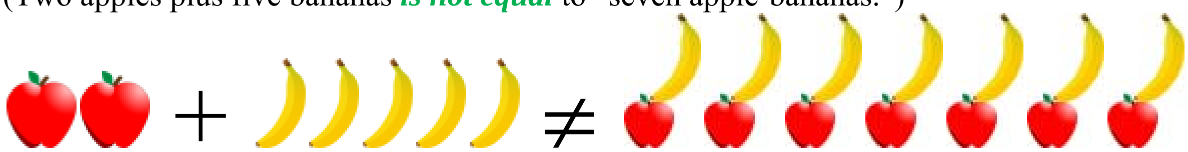
Simplifying Algebraic Expressions Involving Addition and Subtraction

Definition of "Simplify"
Use the ***rules of algebra and arithmetic*** to write an expression in the ***simplest possible form***.

e.g. The expression $2a+5a$ ***simplifies*** to $7a$. (Two apples plus five apples ***is equal to*** seven apples.)



On the other hand, $2a+5b$ ***cannot be simplified*** because $2a$ and $5b$ are ***unlike terms***.
(Two apples plus five bananas ***is not equal*** to "seven apple-bananas.")



Important Points to Remember when Simplifying Polynomials that don't Contain Brackets

- Keep in mind and apply correctly the rules for adding and subtracting integers. The main idea underlying addition and subtraction of integers is that these operations all boil down to either a ***loss*** (move down, move left, etc) or a ***gain*** (move up, move right, etc)

  $+(+)=+=$ ***add*** a ***positive*** value = ***gain*** $\qquad -(-)=+=$ ***subtract*** a ***negative*** value = ***gain***
  $+(-)=-=$ ***add*** a ***negative*** value = ***loss*** $\qquad -(+)=-=$ ***subtract*** a ***positive*** value = ***loss***

- Remember to look for ***like terms***. Do not fall into the trap of attempting to simplify the sum or difference of ***unlike terms***. (2 cows + 2 cows = 4 cows $\qquad$ 2 boys + 2 boys = 4 boys $\qquad$ 2 cows + 2 boys ≠ 4 cowboys

Simplifying Expressions involving Two or More Terms and NO Brackets

Examples

Simplify each of the following polynomials:

1.
$-5a+3a-6b+4b$
$=-2a-2b$

2.
$-2x^2y-7x^2y+3xy-8xy$
$=-9x^2y-5xy$

Note that x^2y and xy are ***NOT*** like terms. The term x^2y means xxy, which is different from xy.

3.
$-5a-6b+3a-4b$
$=-5a+3a-6b-4b$
$=-2a-10b$

Collect Like Terms
The operations *move* with the terms!

4.
$-5a+6b+3c-4d$

This polynomial cannot be simplified because ***there are no like terms***. (In your notes or on a test, you may write "CBS" as a short form for "cannot be simplified.")

5.
$-15ab^2-6a^2b-13ab^2-4a^2b-10ab$
$=-15ab^2-13ab^2-6a^2b-4a^2b-10ab$
$=-28ab^2-10a^2b-10ab$

Summary of Main Ideas

Algebra as a Language

Complete the following statements:

(a) Languages like English are best suited to descriptions of a qualitative nature.

(b) The language of algebra is best suited to descriptions of a quantitative nature.

(c) Math is like a dating service because it's all about relationships.

(d) The language of algebra has many advantages when it comes to describing mathematical relationships. Some of the advantages include that it is a universal language for expressing mathematical ideas, it allows mathematical relationships to be expressed very concisely and it also allows relationships to be manipulated easily.

(e) Expressions and equations can be compared to phrases and sentences respectively.

Give an example of an expression: $2x-7$ Give an example of an equation: $3x-8=7$

An expression is like a phrase because it is NOT a complete mathematical "sentence".

An equation is like a sentence because it IS a complete mathematical "sentence".

(f) The Pythagorean Theorem is an example of an equation that describes the mathematical relationship among the sides of a right triangle.

(g) Math is much easier to understand when we keep in mind the meaning of the symbols, operations, expressions and equations. Also, it helps to have good control over one's mental auto-pilot.

Vocabulary of Algebra

Complete the following table:

Name	*Example*	*Name*	*Example*
Constant	3	Unlike Terms	$3x$, $3y$
Variable	x	Like Terms	$3ab^2$, $-10ab^2$
Expression	$-14c^2-5b$	Simplify an Expression	$3ab^2-10ab^2=-7ab^2$
Term	$-14b^2$	Evaluate an Expression	$3(-3)(-4)^2-10(3)(4)^2=-304$
(Numeric) Coefficient	$-14b^2$	Polynomial	$-4x^5+2x^3-7x^2+x-1$
Literal Coefficient (Variable Part)	$-14b^2$	Trinomial	$-4x^5+2x^3-7x^2$
Polynomial	$4x^2y-6y^5+3b$	Binomial	$3ab^2-10ab^2$
Monomial	$-6y$	Monomial	$-7x^2$
Binomial	$-6y+3b$	Expression	$-3x^2y+5abc-\frac{2xy^3z}{ab^2}-5\sqrt{z}$
Trinomial	x^2+3x+2	Term	$-5\sqrt{z}$
Evaluate an Expression	$3(2)-4=6-4=2$	Coefficient	$-5\sqrt{z}$
Simplify an Expression	$3a+2a=5a$	Variable Part	$-5\sqrt{z}$
Like Terms	$3a, 2a$	Variable	a
Unlike Terms	$3a, 2b$	Constant	-34553476348.467674737

Simplifying Algebraic Expressions

1. Complete the following statements:

(a) "Evaluate an expression" means to perform the operations in an expression.

(b) "Simplify an expression" means to write an expression in a simpler form.

(c) When $-3(-3)-10(-3)(5)^2$ is evaluated, the result is 759.

(d) The expression $3ab^2-10ab^2$ ***can*** be simplified because the terms are like.

(e) The expression $3ab-10ab^2$ ***cannot*** be simplified because the terms are unlike.

(f) One way to interpret the expression $2p+5p$ is two pizzas plus five pizzas. Using this interpretation, it ***makes sense*** that the simplified form is $7p$ because the total # of pizzas is 7.

(g) One way to interpret the expression $2h+5d$ is two hotties plus five doggies. Using this interpretation, it ***does not make sense*** that the simplified form is $7\cancel{hd}$ because 2 hotties plus 5 doggies certainly CANNOT equal 7 hottie-doggies!!

(h) When simplifying expressions, first the like terms should be collected. When this is being done, it is very important that the operations move with the terms.

(i) When simplifying expressions containing brackets, the brackets can be removed without making any other changes only if the bracket is preceded by a plus sign. This is so because addition can be performed in any order whatsoever without changing the sum. If a bracket is preceded by a minus sign, brackets cannot be removed without making other changes. This is so because the result of subtraction (i.e. the difference) ***is*** affected by the order in which it is performed. In this case, it is best to remove brackets by adding the opposite.

2. Simplify each of the following expressions.

a) $(7x-9)+(x-4)$

b) $(3y+8)+(-y-5)$

c) $(8c-6)-(c+7)$

d) $(k+2)-(3k-2)$

e) $(3p^2-8p+1)+(9p^2+4p-1)$

f) $(5xy^2+6x-7y)-(3xy^2-6x+7y)$

g) $(4x-3)+(x+8)-(2x-5)$

h) $(2uv^2-3v)-(v+3u)+(4uv^2-9u)$

Answers to Question 2

(a) $8x-13$
(b) $2y+3$
(c) $7c-13$
(d) $-2k+4$
(e) $12p^2-4p$
(f) $2xy^2+12x-14y$
(g) $3x+10$
(h) $6uv^2-12u-4v$

How to Read Powers

e.g. Consider the power 2^3. It can be read in a variety of different ways as shown below.

2^3
- "Two to the exponent three"
- "Two cubed"
- "The third power of two"
- "Two to the third"
- "Two raised to the exponent three"

Note that many people will also say, "Two to the ***power*** three." Technically, this is incorrect because the power is 2^3 ***not*** three. However, it is such a common practice to use the word "power" as if it were synonymous with "exponent" that we have no choice but to accept it.

A Common Mistake that you Should Never Make

NEVER confuse powers with multiplication

e.g. 2^3 ***means*** $2\times2\times2=8$ ***NOT*** $2\times3=6$

If you confuse powers with multiplication, then the mass of the sun would be only 600 kg, which is clearly nonsensical!

Mass of sun = 2×10^{30} kg = 2 times10 multiplied by itself 30 times ***NOT*** $2\times10\times30=600$

Simplifying Expressions involving Powers by writing in Expanded Form

In the following examples, the expressions are simplified (written as a single power) by writing powers in expanded form. This is done to help you remember to ***think about the meaning of powers before you write your answers***.

(a) $3^2(3^4)=\underbrace{3(3)(3)(3)(3)(3)}_{\text{There are six factors of 3 altogether.}}=3^6$

(b) $\dfrac{4^5}{4^3}=\dfrac{4(4)(4)(4)(4)}{4(4)(4)}=\left[\dfrac{4(4)(4)}{4(4)(4)}\right]\left[\dfrac{4(4)}{1}\right]=[1][4(4)]=4^2$

(c) $(5^2)^3=(5^2)(5^2)(5^2)=5(5)(5)(5)(5)(5)=5^6$

(d) $x^2(x^4)=x(x)(x)(x)(x)(x)=x^6$

(e) $\dfrac{a^5}{a^3}=\dfrac{a(a)(a)(a)(a)}{a(a)(a)}=\left[\dfrac{a(a)(a)}{a(a)(a)}\right]\left[\dfrac{a(a)}{1}\right]=[1][a(a)]=a^2$

(f) $(s^2)^3=(s^2)(s^2)(s^2)=s(s)(s)(s)(s)(s)=s^6$

(g) $3(y^3)^2$

$=3(y^3)(y^3)$

$=3(y)(y)(y)(y)(y)(y)$

$=3y^6$

(h) $(3y^3)^2$

$=(3y^3)(3y^3)$

$=3(3)(y^3)(y^3)$

$=9y^6$

Understanding the Laws of Exponents

Name of Law	*Law Expressed in Algebraic Form*	*Law Expressed in Verbal Form*	*Example Showing why Law Works*
Product Rule	$a^x a^y = a^{x+y}$	To ***multiply*** two powers with the ***same base***, ***keep the base*** and ***add the exponents***.	$a^2a^4 = (a)(a)(a)(a)(a)(a) = a^6$ ***Two*** factors of a multiplied by ***four*** factors of a gives ***six*** factors of a.
Quotient Rule	$\dfrac{a^x}{a^y} = a^{x-y}$	To ***divide*** two powers with the ***same base***, ***keep the base*** and ***subtract the exponents***.	$\dfrac{a^5}{a^2} = \dfrac{(a)(a)(a)(a)(a)}{(a)(a)} = a^3$ ***Five*** factors of a divided by ***two*** factors of a leaves ***three*** factors of a. (Two factors of a in the numerator divide out with two factors of a in the denominator.)
Power of a Power Rule	$(a^x)^y = a^{xy}$	To ***raise*** a power to an exponent, ***keep the base*** and ***multiply*** the exponents.	$(a^3)^4 = (a^3)(a^3)(a^3)(a^3) = a^{12}$

Examples

Use the laws of exponents to simplify each of the following expressions.

(a) $x^2(x^4) = x^{2+4} = x^6$

(b) $\dfrac{y^5}{y^3} = y^{5-3} = y^2$

(c) $x^2(y^4)$ cannot be simplified because the bases are different

(d) $(a^3)^6 = a^{3\times 6} = a^{18}$

(e) $\dfrac{x^5}{y^3}$ cannot be simplified because the bases are different

(f) $5(x^2)^3 = 5(x^{2\times 3}) = 5x^6$

(g)
$$\begin{aligned}(5x^2)^3 &= (5x^2)(5x^2)(5x^2)\\ &= 5(5)(5)(x^2)(x^2)(x^2)\\ &= 125x^{2+2+2}\\ &= 125x^6\end{aligned}$$

Another Law of Exponents

Example (g) above suggests the following shortcut: $(5x^2)^3 = 5^3(x^2)^3 = 125x^{2\times 3} = 125x^6$. In general, this can be expressed as follows:

$$(ab)^x = a^x b^x$$

To ***raise a product to an exponent***, ***raise each factor*** in the product ***to the exponent***.

e.g. $(m^3n^4)^6 = (m^3)^6(n^4)^6 = m^{18}n^{24}$

A Big Example

Use the laws of exponents to simplify $\dfrac{2ab^2(3a^3b^3)}{(4ab^2)^4}$

$$\begin{aligned}\frac{2ab^2(3a^3b^7)}{(4ab^2)^4} &= \frac{2(3)a^1a^3b^2b^7}{4^4a^4(b^2)^4}\\ &= \frac{6a^{1+3}b^{2+7}}{256a^4b^{2\times 4}}\\ &= \frac{6a^4b^9}{256a^4b^8}\\ &= \left(\frac{6}{256}\right)\left(\frac{a^4}{a^4}\right)\left(\frac{b^9}{b^8}\right)\\ &= \left(\frac{3}{128}\right)(1)b^{9-8}\\ &= \frac{3}{128}b\end{aligned}$$

Since multiplication can be done in any order, the ***like factors*** can be put together.

Summary of Simplifying Algebraic Expressions

Review of Simplifying Algebraic Expressions

Adding and Subtracting Polynomials with no Brackets

1. Collect like terms. ***Remember that the operation must "travel" with the term!***
2. Use the rules for adding and subtracting integers. (GAINS/LOSSES)

Examples

$$\begin{aligned} &-5a-6b+3a-4b \\ &=-5a+3a-6b-4b \\ &=-2a-10b \end{aligned}$$

$$\begin{aligned} &-2x^2y+3xy-7x^2y-8xy \\ &=-2x^2y-7x^2y+3xy-8xy \\ &=-9x^2y-5xy \end{aligned}$$

Adding and Subtracting Polynomials with Brackets

1. If a bracket is preceded by a "+" sign or no sign, the brackets can simply be removed because addition is ***insensitive*** to order.
2. If a bracket is preceded by a "–" sign, brackets cannot be removed because subtraction is sensitive to order. After ***adding the opposite***, the brackets can be removed. This works because − = + (−).
3. Collect like terms.
4. Use the rules for adding and subtracting integers. (GAINS/LOSSES)

Example

$$\begin{aligned} &(-5a+6b)-(3a-4b) \\ &=-5a+6b+(-3a+4b) \\ &=-5a+6b+(-3a)+4b \\ &=-5a+6b-3a+4b \\ &=-5a-3a+6b+4b \\ &=-8a+10b \end{aligned}$$

Add the opposite of $3a-4b$

Brackets can be removed now because the operation is "+."

The Distributive Property

1. Look for an expression in brackets containing two or more terms (usually the terms are unlike) and a factor outside the brackets.
2. Multiply each term in the brackets by the ***factor*** outside the brackets.

Examples

$$\begin{aligned} x(-3x^2-y^2) &= x(-3x^2)-x(y^2) \\ &=-3x^3-xy^2 \end{aligned}$$

$$\begin{aligned} &(-5a+6b)-(3a-4b) \\ &=-5a+6b-1(3a-4b) \\ &=-5a+6b-3a+4b \\ &=-5a-3a+6b+4b \\ &=-8a+10b \end{aligned}$$

The distributive property can be used as an ***alternative*** to adding the opposite. This is based on the property $1x=x$ (i.e. multiplying by 1 has no effect).

Multiplying and Dividing Monomials

1. Make sure there is ***only one term***. If there are two or more terms, make sure that you work on each term separately.
2. Put ***like factors*** together. You are allowed to do this because multiplication can be performed in any order.
3. Use the ***laws of exponents***.

Example

$$\frac{2ab^2(3a^3b^7)}{(4ab^2)^4}=\frac{2(3)a^1a^3b^2b^7}{4^4a^4(b^2)^4}=\frac{6a^{1+3}b^{2+7}}{256a^4b^{2\times4}}=\frac{6a^4b^9}{256a^4b^8}$$

$$=\left(\frac{6}{256}\right)\left(\frac{a^4}{a^4}\right)\left(\frac{b^9}{b^8}\right)=\left(\frac{3}{128}\right)(1)b^{9-8}=\frac{3}{128}b$$

A more Complicated Example Involving the Distributive Property

$$\begin{aligned} &-3a^2b(-6abc+9a^3b^4-7bc) \\ &=3a^2b(6abc)-3a^2b(9a^3b^4)+3a^2b(7bc) \\ &=18a^3b^2c-27a^5b^5+21a^2b^2c \end{aligned}$$

An Important Reflection on Order of Operations: Does Order always Matter?

Overview of the Standard Order of Operations (Operator Precedence)

- The operations in ***ALL*** mathematical expressions must be performed in a specific ***standard*** order.
- This ensures that every mathematical expression evaluates to ***exactly one value***.
- We use the mnemonic "**BEDMAS**" to help us remember the standard order. This mnemonic has limitations, however, because it only includes the basic operations +, −, ×, ÷ and powers. As you expand your repertoire of mathematical operations ("functions"), it will be necessary to go beyond "BEDMAS" to understand the order in which the operations and functions must be applied.
- Parentheses ("brackets") are used to ***override*** the standard order.

  e.g. $2+3\times5=2+15=17$ ("×" before "+") $(2+3)\times5=5(5)=25$ ("+" before "×")
- Most of the time, ***ORDER MATTERS***! However, there are ***exceptions***, some of which are listed below.

Exceptions: When Order doesn't Matter

1. **Commutative Property of "+" and "×":** $a+b=b+a$, $ab=ba$
2. **Associative Property of "+" and "×":** $(a+b)+c=a+(b+c)$, $(ab)c=a(bc)$
3. **Distributive Property:** $a(b+c)=ab+ac$ Adding before multiplying gives the same result as multiplying before adding.
4. **Power of a Product:** $(ab)^c=a^cb^c$ Multiplying before raising to the exponent gives the same result as raising to the exponent before multiplying.
5. **Quotient of a Product:** $\left(\frac{a}{b}\right)^c=\frac{a^c}{b^c}$ Dividing before raising to the exponent gives the same result as raising to the exponent before dividing.

Exercises

Determine whether the following pairs of expressions are equivalent. If so, provide proof. Otherwise, give a counterexample.

Pairs of Expressions		***Equivalent?***	***Proof or Counterexample***
x^2+y^2	$(x+y)^2$	No	Let $x=1$ and $y=1$. Then $x^2+y^2=1^2+1^2=1+1=2$. However, $(x+y)^2=(1+1)^2=2^2=4$. Therefore, $x^2+y^2\neq(x+y)^2$.
x^2y^2	$(xy)^2$	Yes	Power of a product rule! $(ab)^x=a^xb^x$
x^3+y^3	$(x+y)^3$	No	Let $x=1, y=1$. Then $x^3+y^3=1^3+1^3=1+1=2$ But $(x+y)^3=(1+1)^3=2^3=8$
$\sqrt{x+y}$	$\sqrt{x}+\sqrt{y}$	No	Let $x=16, y=9$ $\sqrt{x+y}=\sqrt{16+9}=\sqrt{25}=5$ But $\sqrt{x}+\sqrt{y}=\sqrt{16}+\sqrt{9}=4+3=7$