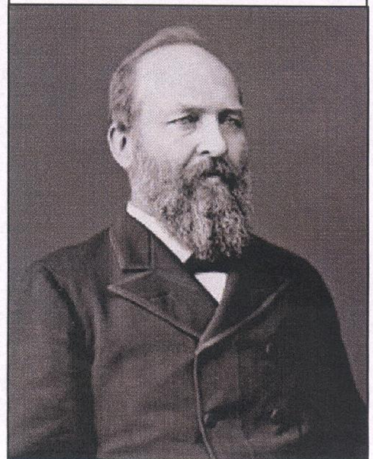
Understanding the Pythagorean Theorem

Pythagorean Theorem Proof 1 – President James Garfield's Brilliant Proof

James A. Garfield was the 20th president of the United States. In addition to being a highly successful statesman and soldier, President Garfield was also a noted scholar. Among his many scholarly accomplishments is his beautiful proof of the Pythagorean Theorem. It is outlined below.

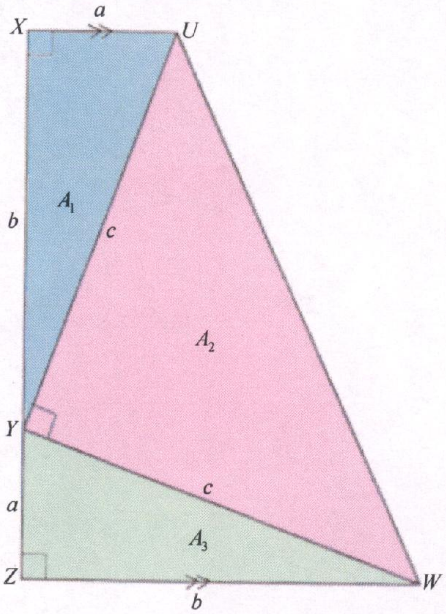
James A. Garfield



20th President of the U.S.A.

In Office:
March 4, 1881- September 19, 1881

Assassinated at the age of 49
One of four assassinated presidents



(a) Calculate the area of trapezoid $XZWU$ by summing the areas of ΔUXY, ΔYZW and ΔUYW. Simplify fully!

$$A_{Trap} = A_1 + A_2 + A_3$$
$$= \frac{ab}{2} + \frac{c(c)}{2} + \frac{ab}{2}$$
$$= \frac{ab + c^2 + ab}{2}$$
$$= \frac{c^2 + 2ab}{2}$$

(b) Calculate the area of the trapezoid by using the equation for the area of a trapezoid. Simplify fully!

Think! What is the height of the trapezoid?

$$A_{Trap} = \frac{h(a+b)}{2}$$
$$= \frac{(a+b)(a+b)}{2}$$
$$= \frac{a^2 + 2ab + b^2}{2}$$

$$(a+b)(a+b)$$
$$= a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

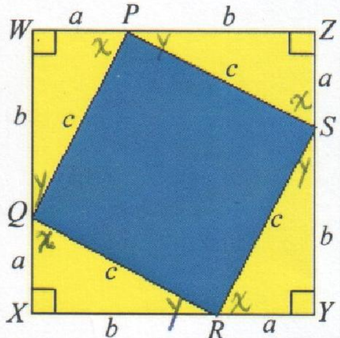
(c) In parts (a) and (a) you developed two different expressions for the area of trapezoid $XZWU$. Since both expressions give the area of the ***same shape***, they must be equal to each other! Set the expressions equal to each other and solve for c^2.

$$A_{Trap} = A_{Trap}$$
$$\therefore \frac{c^2 + 2ab}{2} = \frac{a^2 + 2ab + b^2}{2}$$
$$\therefore \frac{2}{1}\left(\frac{c^2 + 2ab}{2}\right) = \frac{2}{1}\left(\frac{a^2 + 2ab + b^2}{2}\right)$$
$$\therefore c^2 + 2ab = a^2 + 2ab + b^2$$
$$\therefore c^2 + 2ab - 2ab = a^2 + 2ab + b^2 - 2ab$$
$$\therefore c^2 = a^2 + b^2$$

This argument DEMONSTRATES that in a right triangle, it MUST be the case that $c^2 = a^2 + b^2$

Pythagorean Theorem Proof 2

(a) Explain why quadrilateral *PQRS must* be a square.



In any of the triangles, $x+y=90°$ because the sum of all the angles must be $180°$. Therefore, the interior angles of PQRS must all be right angles.

(b) Use the above diagram to develop a proof of the Pythagorean Theorem. (**Hint:** The line of reasoning is similar to that of President Garfield's proof.)

$A_{WXYZ}=(a+b)(a+b)=a^2+2ab+b^2$

A_{WXYZ} = Area of blue square + Area of 4 triangles

$=c^2+\frac{\overset{2}{\cancel{4}}}{1}\left(\frac{ab}{\underset{1}{\cancel{2}}}\right)$

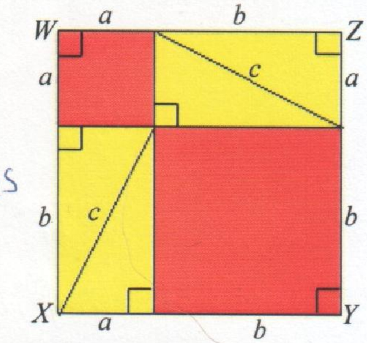
$=c^2+2ab$

$\therefore\ a^2+2ab+b^2=c^2+2ab$

$\therefore\ a^2+b^2=c^2$ (subtract 2ab from B.S.)

Pythagorean Theorem Proof 3

(a) Explain how the diagram at the right is simply a rearrangement of the pieces in the diagram at the left.



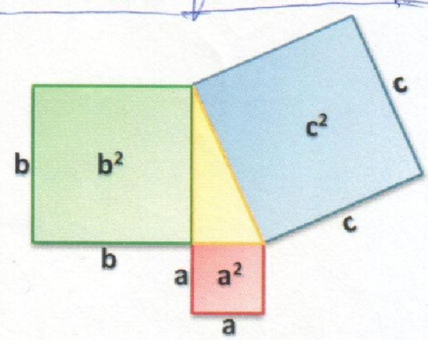
The right triangles are rearranged to form two rectangles.

(b) Explain why the two diagrams together make it ***obvious*** that $a^2+b^2=c^2$. (This proof was devised in 1939 by Maurice Laisnez, a high school student in the Junior-Senior High School of South Bend, Indiana.)

The combined area of the two red squares (a^2+b^2) must equal the area of the blue square (c^2) because the two red squares together occupy exactly the same amount of space as the blue square.

Consequence of the Pythagorean Theorem

Although the Pythagorean Theorem is an equation that relates the lengths of the sides of a right triangle, it can also be ***interpreted*** in terms of areas.



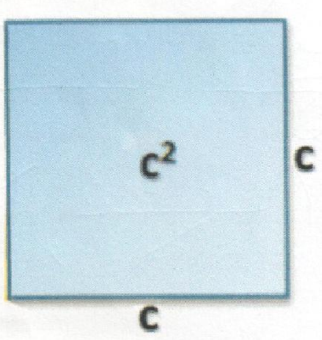
- We have ***proved*** that in a right triangle, the square of the hypotenuse must equal the sum of the squares of the other two sides.

  That is, if *c* represents the length of the hypotenuse and *a* and *b* respectively represent the lengths of the other two sides, then $c^2=a^2+b^2$.

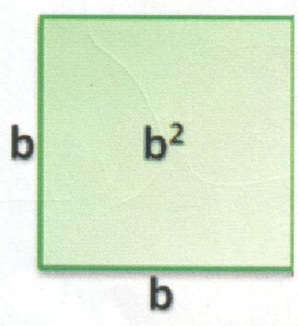
- By examining the diagram at the right, one can easily see that the expressions a^2, b^2 and c^2 are all areas of squares!

  Since $c^2=a^2+b^2$, ***it must also be true*** that

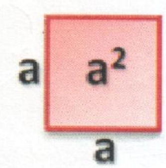
Area of



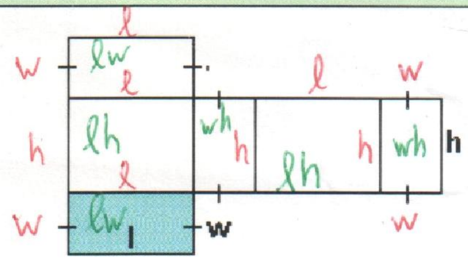
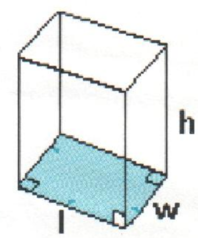
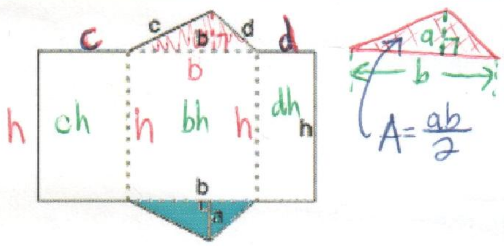
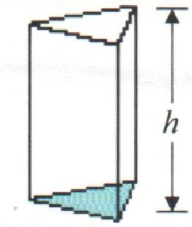
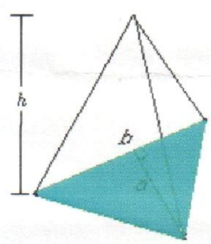
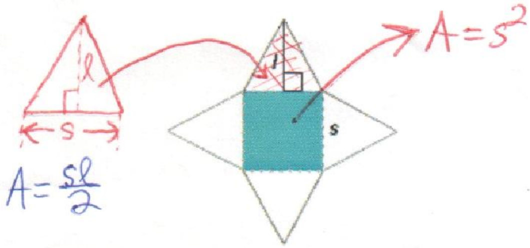
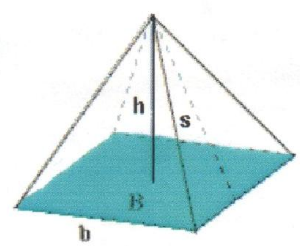
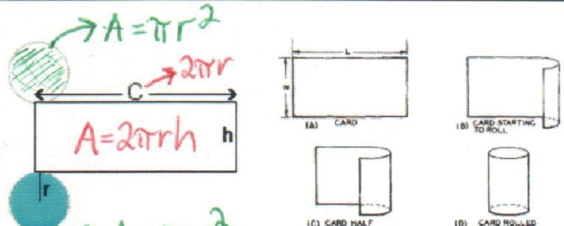
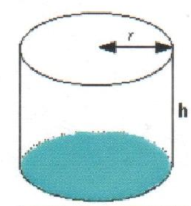
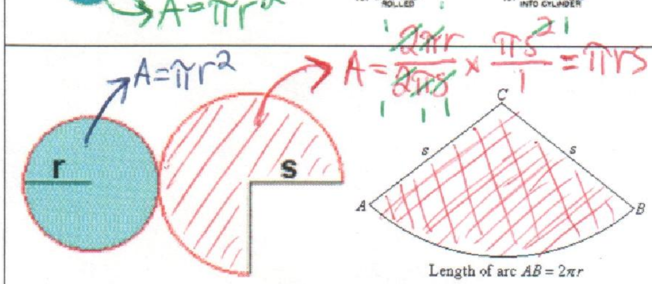
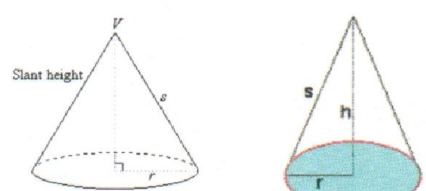
= **Area of**



+ **Area of**

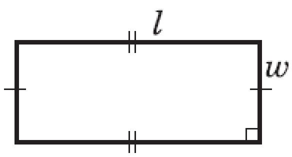
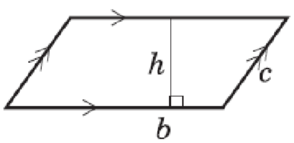
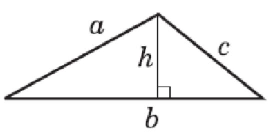
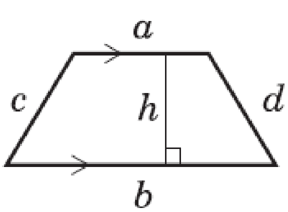
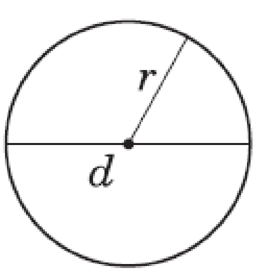


Understanding Surface Area Equations

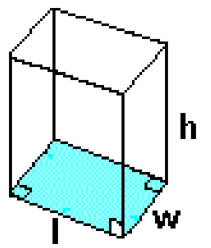
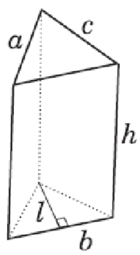
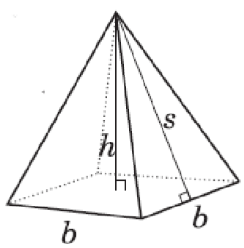
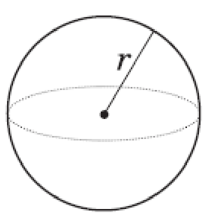
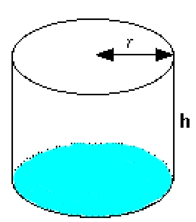
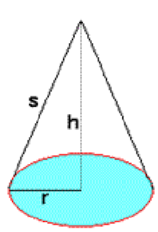
Net	*Solid*	*Surface Area Equation*
		$A = lw + lw + lh + lh + wh + wh$ $\therefore A = 2lw + 2lh + 2wh$ $\therefore A = 2(lw + lh + wh)$
		$A = bh + ch + dh + \frac{2}{1}\left(\frac{ab}{2}\right)$ $\therefore A = bh + ch + dh + ab$ $\therefore A = h(b + c + d) + ab$
		$A = \frac{4}{1}\left(\frac{sl}{2}\right)$ $\therefore A = 2sl$
		$A = \frac{4}{1}\left(\frac{sl}{2}\right) + s^2$ $\therefore A = 2sl + s^2$
		$A = \pi r^2 + \pi r^2 + 2\pi rh$ $\therefore A = 2\pi r^2 + 2\pi rh$
		$A = \pi r^2 + \pi rs$

What Happens If…

1. Complete the following table. The first row has been done for you.

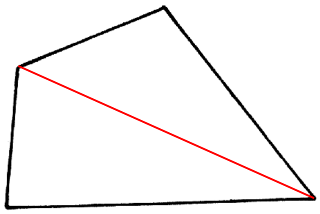
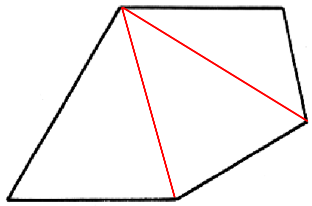
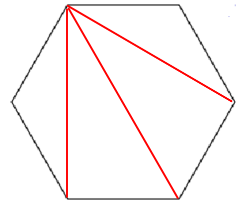
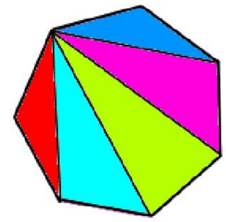
Shape	***Name of the Shape***	***What Happens to the Perimeter if …***	***What Happens to the Area if …***
	Rectangle	…the length is doubled ***Solution*** $P = 2l + 2w$ If the length is doubled, the new length is $2l$. Then, the perimeter becomes $P = 2(2l) + 2w = 4l + 2w = (2l + 2w) + 2l$ The perimeter increases by $2l$.	…the width is tripled ***Solution*** $A = lw$ If the width is tripled, the new width is $3w$. Then, the area becomes $A = l(3w) = 3lw = 3(lw)$ The area is also tripled.
	Paralel-logram	…the base is doubled ***Solution*** $P = 2b + 2c$ If the base is doubled, the new base is $2b$. Then, the perimeter becomes $P = 2(2b) + 2c = 4b + 2c = (2b + 2c) + 2b$ The perimeter increases by $2b$.	…the height is quadrupled ***Solution*** $A = bh$ If the height is quadrupled, the new height is $4h$. Then, the area becomes $A = b(4h) = 4bh = 4(bh)$ The area is also quadrupled.
	Triangle	…the base is tripled (if this can be done without changing the values of *a* and *c*) ***Solution*** $P = a + b + c$ If the base is doubled, the new base is $2b$. Then, the perimeter becomes $P = a + 2b + c = (a + b + c) + b$ The perimeter increases by b.	…the height is tripled ***Solution*** $A = \frac{bh}{2}$ If the height is tripled, the new height is $3h$. Then, the area becomes $A = \frac{b(3h)}{2} = \frac{3bh}{2} = \frac{3(bh)}{2}$ The area is also tripled.
	Trapezoid	…the base is tripled (if this can be done without changing the values of *c* and *d*) ***Solution*** $P = a + b + c + d$ If the base is doubled, the new base is $2b$. Then, the perimeter becomes $P = a + 2b + c + d = (a + b + c + d) + b$ The perimeter increases by b.	…the height is doubled ***Solution*** $A = \frac{h(a+b)}{2}$ If the height is tripled, the new height is $2h$. Then, the area becomes $A = \frac{2h(a+b)}{2} = 2\left(\frac{h(a+b)}{2}\right)$ The area is also doubled.
	Circle	…the radius is doubled ***Solution*** $C = 2\pi r$ If the radius is doubled, the new radius is $2r$. Then, the perimeter becomes $C = 2\pi r = 2\pi(2r) = 4\pi r = 2(2\pi r)$ The circumference is doubled.	…the radius is doubled ***Solution*** $A = \pi r^2$ If the radius is doubled, the new radius is $2r$. Then, the area becomes $A = \pi(2r)^2 = \pi(4r^2) = 4\pi r^2$ The area is ***quadrupled***.

2. Complete the following table. The first row has been done for you.

Shape	***Name of the Shape***	***What Happens to the Surface Area if …***	***What Happens to the Volume if …***
	Rectangular Prism	…the length is doubled ***Solution*** $A = 2lw + 2lh + 2wh$ If the length is doubled, the new length is $2l$. Then, the surface area becomes $A = 2(2l)w + 2(2l)h + 2wh$ $= 4lw + 4lh + 2wh$ $= (2lw + 2lh + 2wh) + 2lw + 2lh$ The surface area increases by $2lw + 2lh$.	…the width is tripled ***Solution*** $V = lwh$ If the width is tripled, the new width is $3w$. Then, the volume becomes $V = l(3w)h = 3lwh = 3(lwh)$ The volume is also tripled.
	Triangular Prism	…b is doubled (if this can be done without changing the values of a and c) ***Answer*** The surface area increases by bl (i.e. bl is **added** to the surface area).	…the height is quadrupled ***Answer*** The volume is also quadrupled (i.e. **multiplied** by 4).
	Square-based Pyramid	…the slant height is tripled ***Answer*** The surface area increases by $4bs$ (i.e. $4bs$ is **added** to the surface area).	…the height is tripled ***Answer*** The volume is also tripled (i.e. **multiplied** by 3).
	Sphere	…the radius is doubled ***Answer*** The surface area ***quadruples*** (i.e. **multiplied** by 4).	…the radius is doubled ***Answer*** The volume ***is multiplied by 8!***
	Cylinder	…the radius is doubled ***Answer*** The surface area increases by $6\pi r^2 + 2\pi rh$ (i.e. $6\pi r^2 + 2\pi rh$ is **added** to the surface area).	…the radius is doubled ***Answer*** The volume ***quadruples*** (i.e. **multiplied** by 4).
	Cone	…the radius is doubled ***Answer*** The surface area increases by $3\pi r^2 + \pi rs$ (i.e. $3\pi r^2 + \pi rs$ is **added** to the surface area).	…the radius is doubled ***Answer*** The volume ***quadruples*** (i.e. **multiplied** by 4).

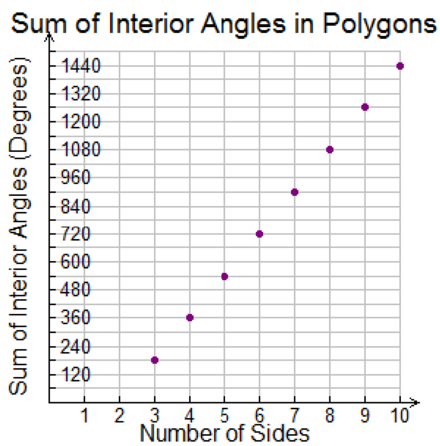
Sum of the Interior Angles of a Convex Polygon

1. By dividing each polygon into triangles, calculate the sum of the interior angles of the following convex polygons. Note that one of the shapes has already been done for you.

			
Name: Quadrilateral	Name: Pentagon	Name: Hexagon	Name: Heptagon
Number of Sides: 4	Number of Sides: 5	Number of Sides: 6	Number of Sides: 7
Number of Triangles: 2	Number of Triangles: 3	Number of Triangles: 4	Number of Triangles: 5
Sum of Interior Angles: $2(180°)=360°$	Sum of Interior Angles: $3(180°)=540°$	Sum of Interior Angles: $4(180°)=720°$	Sum of Interior Angles: $5(180°)=900°$

2. Now summarize your results in the following table and sketch a graph relating the sum of the interior angles of a convex polygon to the number of sides. Then answer questions (a) to (f).
n = number of sides in the polygon, s = sum of the interior angles of the polygon

n	s	Δs (1st Differences)
3	180°	-
4	360°	180°
5	540°	180°
6	720°	180°
7	900°	180°



(a) Do you expect the pattern to continue indefinitely beyond $n = 7$? Explain.
The pattern should continue indefinitely because increasing the number of sides by 1 also increases the number of triangles by 1. This means that the sum of the interior angles will grow by 180° for each increase by 1 in the number of sides.

(b) Write an equation relating s to n. Explain why it is not surprising that the relation between s and n is linear.
$s = 180°(n-2) = 180°n - 360°$

(c) State the ***meaning*** of the slope of the linear relation between s and n.
m = slope = 180 (from the equation)
The sum of the interior angles of an n-sided polygon increases at a rate of 180° per additional side.

(d) Does the vertical intercept of this linear relation have a meaning? Explain.
The vertical intercept has no meaning because there is no such thing as a zero-sided polygon.

(e) Does it make sense to "connect the dots" in the above graph? Explain.
It does not make sense to connect the dots because the number of sides in a polygon must be a whole number greater than or equal to 3.

(f) State an easy way to remember how to calculate the sum of the interior angles of a polygon.
sum of interior angles = 180°×(# of triangles)

Optimization

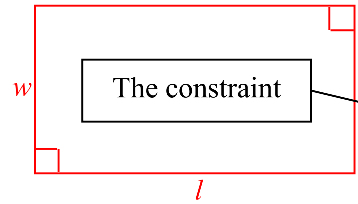
Optimization Problem 1 – Solved using an Algebraic and Graphical Model

You have 400 m of fencing and you would like to enclose a rectangular region of ***greatest possible area***. What dimensions should the rectangle have?

(a) What is the ***constraint*** in this problem?

The constraint is the length of fencing available. Since only 400 m of fencing are available, the region enclosed by the fence will have a limited size.

(b) Draw a diagram describing this situation. Then write an equation that relates the unknowns to the constraint.



Length of fencing = 400 m

$\therefore$ perimeter = 400 m

$\therefore \boxed{2l + 2w = 400}$

(c) What quantity needs to be optimized? State whether the quantity needs to be maximized or minimized. Then write an equation describing the quantity that needs to be optimized.

In this problem, the area needs to be ***maximized***. Therefore, the equation must describe the area of the rectangular region.

$$A = lw$$

(d) The equation in (c) cannot be used directly to maximize the area because there are too many variables. Use the constraint equation to solve for l in terms of w. Then rewrite the equation in (c) in such a way that A is expressed entirely in terms of w.

$\because 2l + 2w = 400$

$\therefore \frac{2l}{2} + \frac{2w}{2} = \frac{400}{2}$

$\therefore l + w = 200$

$\therefore l + w - w = 200 - w$

$\therefore l = 200 - w$

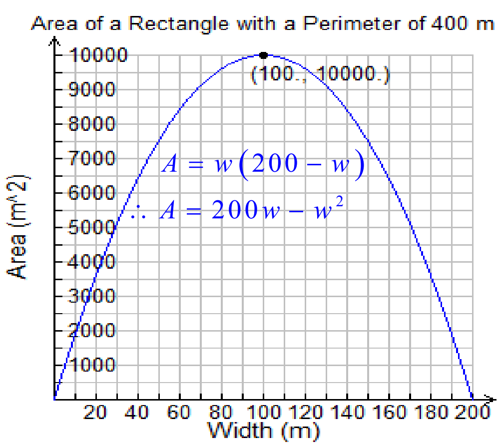
$\therefore A = lw$

$\therefore A = (200 - w)w$

$\therefore A = w(200 - w)$

Now the area has been expressed ***in terms of one variable only*** (i.e. the width).

(e) Sketch a graph of area of the rectangle versus width. Label the axes and include a title.



(f) Is the relation between A and w linear or non-linear? Give ***three*** reasons to support your answer.

w	A	ΔA
20	3600	-
30	5100	1500
40	6400	1300
50	7500	1100
60	8400	900

The relation is ***non-linear***. We know this because of the following reasons.

1. The graph is curved.
2. The equation has a squared term (w^2).
3. The first differences are ***not constant***.

(g) State the dimensions of the rectangular region having a perimeter of 400 m and a ***maximal*** area.

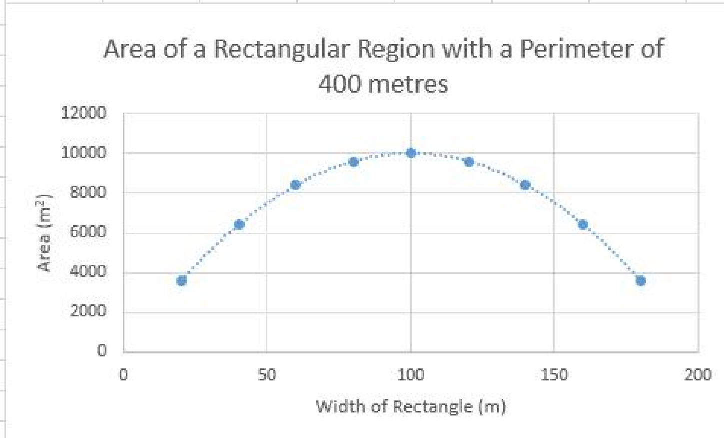
From the graph it can be seen that the maximum area is 10000 m^2, which is attained when the width is 100 m. Therefore, for maximum area,

$w = 100$ and $l = 200 - w = 200 - 100 = 100$.

For maximal area, both the length and the width should be 100 m. In other words, the region should be a square with side length of 100 m.

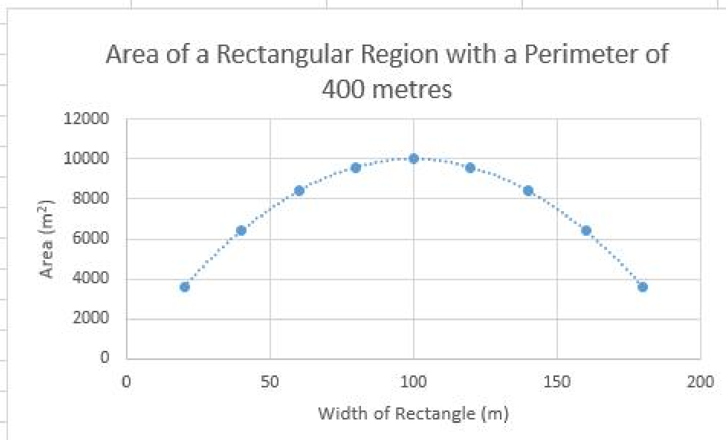
Optimization Problem 1 – Solved using a Numeric and Graphical Model

width	length	area
20	180	3600
40	160	6400
60	140	8400
80	120	9600
100	100	10000
120	80	9600
140	60	8400
160	40	6400
180	20	3600



Formulas used to Calculate the Length and the Area

width	length	area
20	=(400-2*B4)/2	=B4*C4
40	=(400-2*B5)/2	=B5*C5
60	=(400-2*B6)/2	=B6*C6
80	=(400-2*B7)/2	=B7*C7
100	=(400-2*B8)/2	=B8*C8
120	=(400-2*B9)/2	=B9*C9
140	=(400-2*B10)/2	=B10*C10
160	=(400-2*B11)/2	=B11*C11
180	=(400-2*B12)/2	=B12*C12



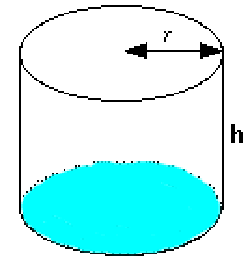
Optimization Problem 2 – Solved using an Algebraic and Graphical Model

Design a cylindrical pop can that has the ***greatest possible capacity*** but can be manufactured using at most 375 cm^2 of aluminum.

(a) What is the ***constraint*** in this problem?

The surface area of the pop can must be at most 375 cm^2.

(b) Draw a diagram describing this situation. Then write an equation that relates the unknowns to the constraint. (Let r represent the radius of the cylinder and let h represent its height.)



Constraint

Surface area = 375

$\therefore 2\pi r^2 + 2\pi rh = 375$

(c) What quantity needs to be optimized? State whether the quantity needs to be maximized or minimized. Then write an equation describing the quantity that needs to be optimized.

The volume needs to be maximized.

$V = \pi r^2 h$

(d) The equation in (c) cannot be used directly to maximize the volume because there are too many variables. Use the constraint equation to solve for h in terms of r. Then rewrite the equation in (c) in such a way that V is expressed entirely in terms of r.

$2\pi r^2 + 2\pi rh = 375$

$\therefore 2\pi rh = 375 - 2\pi r^2$

$\therefore h = \dfrac{375 - 2\pi r^2}{2\pi r}$

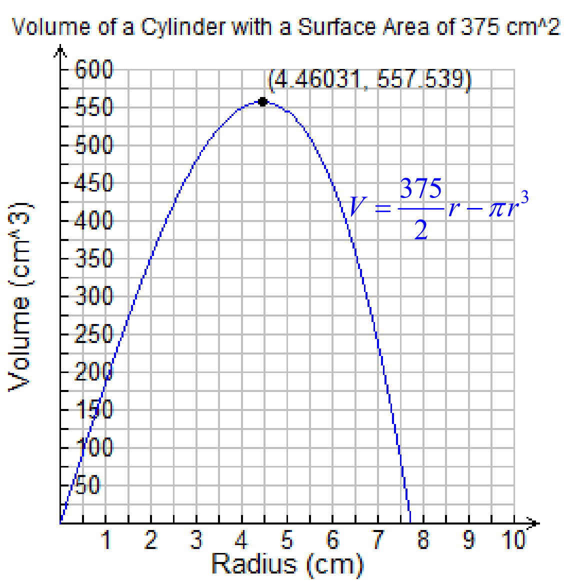
$V = \pi r^2 h$

$\therefore V = \dfrac{\pi r^2}{1}\left(\dfrac{375 - 2\pi r^2}{2\pi r}\right)$

$\therefore V = \dfrac{r\left(375 - 2\pi r^2\right)}{2} = \dfrac{375}{2}r - \pi r^3$

Now the volume has been expressed ***in terms of one variable only*** (i.e. the radius).

(e) Sketch a graph of volume of the cylindrical can versus radius. Label the axes and include a title.



(f) Is the relation between V and r linear or non-linear? Give ***three*** reasons to support your answer.

r	V	ΔV
1	184.4	-
2	349.9	165.5
3	477.7	127.8
4	548.9	71.2
5	544.8	−4.1

The relation between V and r is ***non-linear***

Reasons

1. The graph is curved.
2. The equation has a polynomial term of degree three ($-\pi r^3$).
3. The first differences are ***not constant***.

(g) State the dimensions of the cylindrical can having a surface area of 375 cm^2 and a ***maximal*** volume.

The dimensions for maximal volume are as follows:

$r \doteq 4.5$ cm (estimated from graph)

$h = \dfrac{375 - 2\pi r^2}{2\pi r} \doteq \dfrac{375 - 2(3.14)(4.5)^2}{2(3.14)(4.5)} \doteq 8.8$ cm

For maximal volume, the diameter is equal to the height!

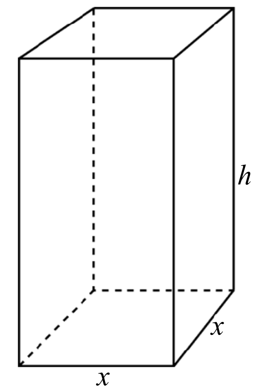
Optimization Problem 3 – Solved using an Algebraic and Graphical Model

A container for chocolates must have the shape of a ***square prism*** and it must also have a volume of 8000 cm^3. Design the box in such a way that it can be manufactured using the ***least amount of material***.

(a) What is the ***constraint*** in this problem?

The volume of the container must be 8000 cm^3.

(b) Draw a diagram describing this situation. Then write an equation that relates the unknowns to the constraint. (Let x represent the side length of the square base and let h represent the height.)



Constraint

Volume = 8000

$\therefore x^2 h = 8000$

(c) What quantity needs to be optimized? State whether the quantity needs to be maximized or minimized. Then write an equation describing the quantity that needs to be optimized.

The surface area needs to be minimized.

$A = 2x^2 + 4xh$

(d) The equation in (c) cannot be used directly to ***minimize*** the surface area because there are too many variables. Use the constraint equation to solve for h in terms of x. Then rewrite the equation in (c) in such a way that A is expressed entirely in terms of x.

$\therefore x^2 h = 8000$

$\therefore h = \dfrac{8000}{x^2}$

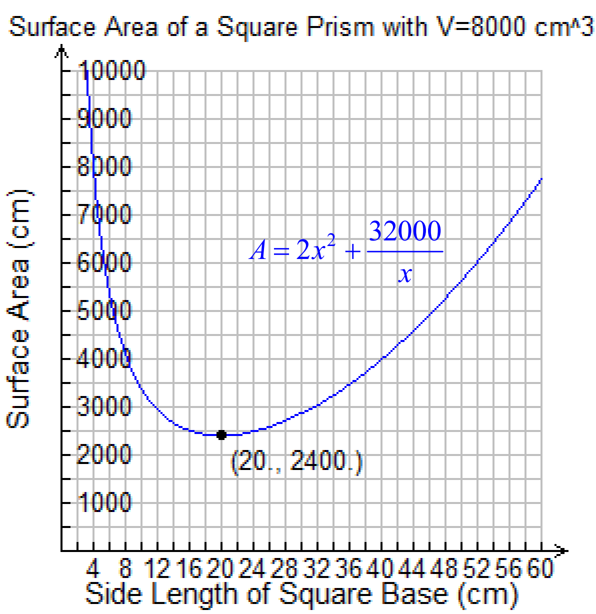
$A = 2x^2 + 4xh$

$\therefore A = 2x^2 + \dfrac{4x}{1}\left(\dfrac{8000}{x^2}\right)$

$\therefore A = 2x^2 + \dfrac{32000}{x}$

Now the surface area has been expressed ***in terms of one variable only*** (i.e. x).

(e) Sketch a graph of volume of the square prism versus width. Label the axes and include a title.



(f) Is the relation between A and x linear or non-linear? Give ***three*** reasons to support your answer.

x	A	ΔA
1	32002	-
2	16008	−15994
3	10684.7	−5323.3
4	8032	−2652.7
5	6450	−1582

The relation between A and x is ***non-linear***

Reasons

1. The graph is curved.
2. The equation has a squared term ($2x^2$).
3. The first differences are ***not constant***.

(g) State the dimensions of the square prism with a volume of 8000 cm^3 and a ***minimal*** surface area.

The dimensions for minimal surface area are as follows:

$x = 20$ cm (estimated from graph)

$\therefore h = \dfrac{8000}{x^2} = \dfrac{8000}{20^2} = \dfrac{8000}{400} = 20$ cm

For minimal surface area, the square prism must be a cube!