# Unit 5 – Measurement and Geometry – Justifications



## *Textbook Homework for this Unit*

| | | Pages | Textbook Questions |
|---|---|---|---|
| Day 1 | 8.2 Perimeter and Area of Composite Shapes | | **p. 432** #1, 2, 4, 8, 10, 13, 17 |
| Day 2 | 8.3 Surface Area and Volume of Prisms and Pyramids<br>8.4 Surface Area of a Cone | | **p. 442** #7b, 11bc<br>**p. 448** #4, 5, 8, 10, 11 |
| Day 3 | 8.3 Surface Area and Volume of Prisms and Pyramids<br>8.5 Volume of a Cone | | **p. 441** #8, 10, 11, 12<br>**p. 454** #3, 7, 10, 13 |
| Day 4 | 8.6 Surface Area of a Sphere<br>8.7 Volume of a Sphere | | **p. 459** #4, 5, 7, 8<br>**p. 466** #6, 7, 8, 11, 15, 16 |
| Day 5 | 7.1 Angle Relationships in Triangles<br>7.2 Angle Relationships in Quadrilaterals<br>7.3 Angle Relationships in Polygons | | **p. 371** #1→5, 11, 14, 15<br>**p. 381** #1, 2, 5, 7, 12, 13<br>**p. 391** #4, 7, 20 |
| Day 6 | 9.2 Perimeter and Area Relationships of a Triangle<br>9.3 Minimize the Surface Area of a Square Prism | | **p. 488** #3, 5, 9 (technology)<br>**p. 495** 1, 4, 5, 6, 11 |
| Day 7 | 9.4 Maximize the Volume of a Square Pyramid<br>9.5 Maximize the Volume of a Cylinder<br>9.6 Minimize the Surface Area of a Cylinder | | **p. 502** #2, 4, 9<br>**p. 508** # C2, 1, 4, 5, 8<br>**p. 513** #C2, 1, 3, 6 |

| | | | |
|---|---|---|---|
| Day 8 | Review | | **p. 408** #1$\rightarrow$7 **p. 470** #4, 6, 9, 11, 15<br>**p. 516** #4, 8, 14 |

# UNDERSTANDING *WHY* THE EQUATIONS ARE CORRECT

## *Understanding the Meaning of π and how it Relates to the Circumference of a Circle*

The following is an example of a typical conversation between Mr. Nolfi and a student who blindly memorizes formulas:

***Student:*** Sir, I can't remember whether the area of a circle is $\pi r^2$ or $2\pi r$. Which one is it?

***Mr. Nolfi:*** If you remember the meaning of $\pi$, you should be able to figure it out.

***Student:*** How can 3.14 help me make this decision? It's only a number!

***Mr. Nolfi:*** How dare you say something so disrespectful about one of the most revered numbers in the mathematical lexicon! (Just kidding. I wouldn't really say that.) It's true that the number 3.14 is an approximate value of $\pi$. But I asked you for its ***meaning***, not its value.

***Student:*** I didn't know that $\pi$ has a meaning. I thought that it was just a "magic" number.

***Mr. Nolfi:*** Leave magic to the magicians. In mathematics, every term (except for primitive terms) has a very precise definition. Read the following carefully and you'll never need to ask your original question ever again!

In ***any*** circle, the ***ratio*** of the ***circumference*** to the ***diameter*** is equal to a ***constant*** value that we call $\pi$. That is,

$$C : d = \pi .$$

Alternatively, this may be written as

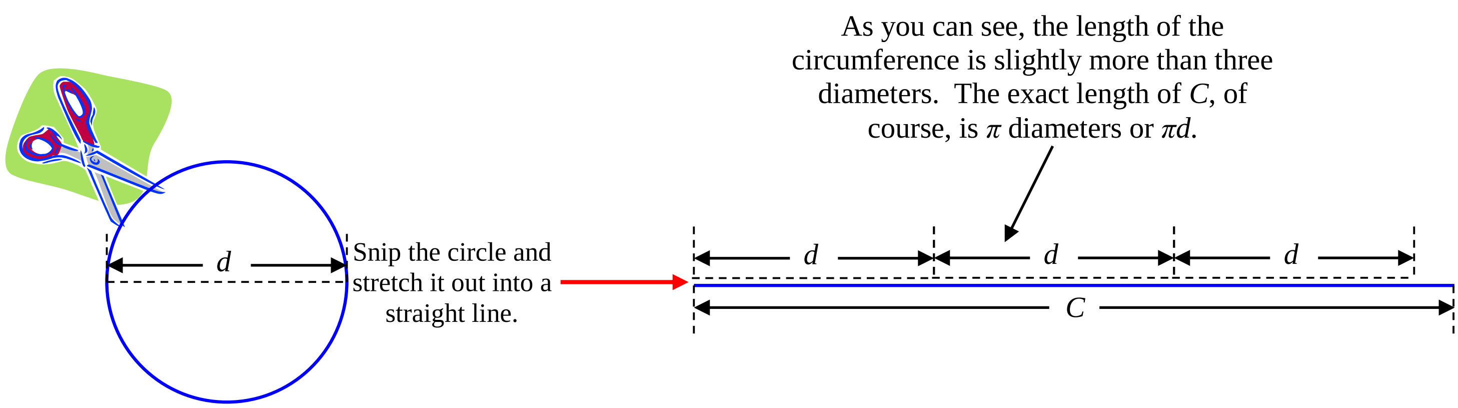
$$\frac{C}{d} = \pi$$

or, by multiplying both sides by $d$, in the more familiar form

$$C = \pi d .$$

If we recall that $d = 2r$, then we finally arrive at the most common form of this ***relationship***,

$$C = 2\pi r .$$




***Mr. Nolfi:*** So you see, by understanding the meaning of $\pi$, you can ***deduce*** that $C = 2\pi r$. Therefore, the formula for the area must be $A = \pi r^2$. Furthermore, it is not possible for the expression $2\pi r$ to yield units of area. The number $2\pi$ is dimensionless and $r$ is measured in units of distance such as metres. Therefore, the expression $2\pi r$ must result in a value measured in units of distance. On the other hand, the expression $\pi r^2$ must give a value measured in units of area because $r^2 = r(r)$, which involves multiplying a value measured in units of distance by itself. Therefore, by considering units alone, we are drawn to the inescapable conclusion that the area of a circle must be $\pi r^2$ and **not** $2\pi r$!

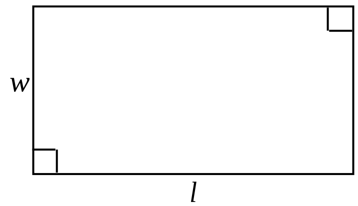
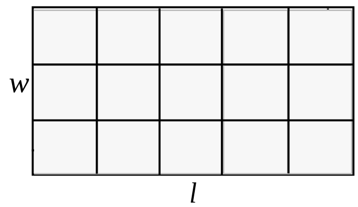
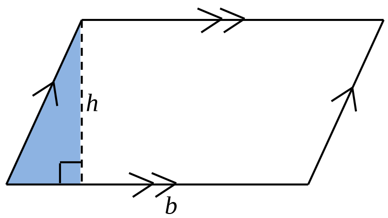
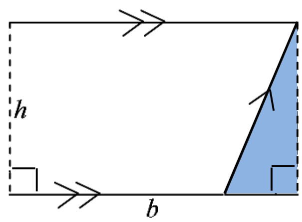
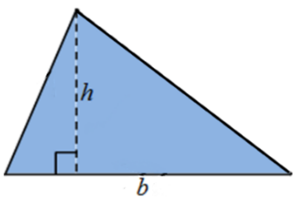
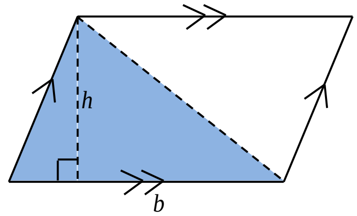
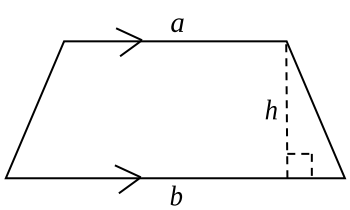
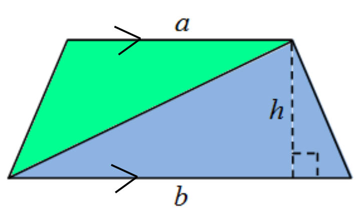
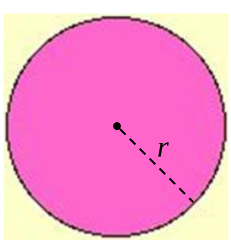
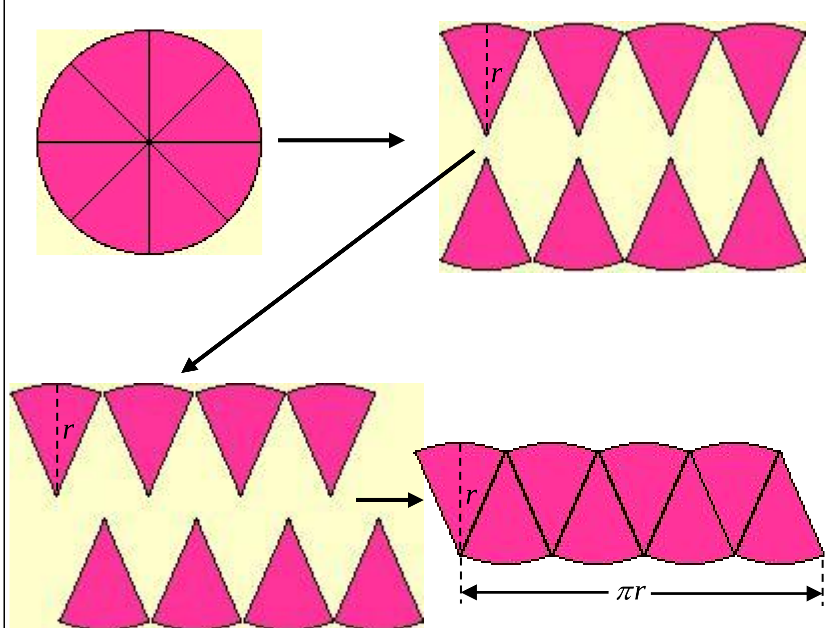
### *Examples*

$2\pi r \doteq 2(3.14)(3.6 \text{ cm}) = 22.608 \text{ cm} \rightarrow$ This answer cannot possibly measure area because cm is a unit of distance.

Therefore, $\pi r^2$ must be the correct expression for calculating the area of a circle.

$\pi r^2 \doteq 3.14(3.6 \text{ cm})^2 = 3.14(3.6 \text{ cm})(3.6 \text{ cm}) = 40.6944 \text{ cm}^2 \rightarrow$ Notice that the unit "cm$^2$" is appropriate for area.

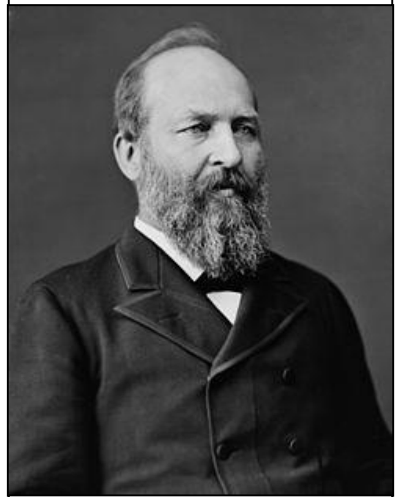
## *Understanding Area Equations*

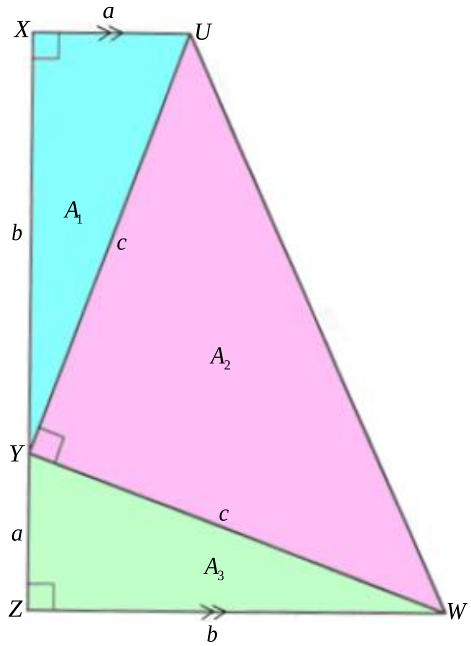
| **Shape** | **Explanation** | | **Equation for Area** |
|---|---|---|---|
|  |  | • Each square has an area of 1 unit$^2$<br>• Total number of squares<br>= (# of squares in each row)×(# rows)<br>= 5(3)<br>= $lw$ | $A = lw$ |
|  |  | • Cut off the shaded right triangle at the left end of the parallelogram.<br>• Attach the cut-off right triangle at the right side.<br>• A rectangle is formed. Its length is $b$ and its width is $h$, which means that its area must be $bh$. | $A = bh$ |
|  |  | • Begin with a parallelogram having the same base and height.<br>• Cut the parallelogram ***in half*** along the dashed diagonal.<br>• Since the parallelogram's area is $bh$, the triangle's area must be $\frac{bh}{2}$. | $A = \frac{bh}{2}$<br>or<br>$A = \frac{1}{2}bh$ |
|  |  | • Divide the trapezoid into two triangles as shown.<br>• Both triangles have a height of $h$. One triangle has a base of $a$ and the other has a base of $b$.<br>• Total area<br>$= \frac{ah}{2} + \frac{bh}{2} = \frac{ah+bh}{2} = \frac{h(a+b)}{2}$ | $A = \frac{h(a+b)}{2}$<br>or<br>$A = \frac{1}{2}h(a+b)$ |
| <br>The argument presented here for the area of a circle is lacking in rigour. It is based on the assumption that we can use the equation for the area of a parallelogram ( $A = bh$ ) to calculate the area of a shape that is similar to a parallelogram. A more rigorous argument can be constructed using calculus. |  | | • As shown in the diagrams, divide the circle into an even number of sectors, all of which have the same size.<br>• Rearrange the sectors as shown, then fit them together.<br>• The resulting shape is very close to a parallelogram, so its area should be about<br>$bh = (\pi r)r = \pi r^2$<br>$A = \pi r^2$ |

## *Understanding the Pythagorean Theorem*

## *Pythagorean Theorem Proof 1 – President James Garfield's Brilliant Proof*

James A. Garfield was the 20th president of the United States. In addition to being a highly successful statesman and soldier, President Garfield was also a noted scholar. Among his many scholarly accomplishments is his beautiful proof of the Pythagorean Theorem. It is outlined below.

| James A. Garfield |
| --- |
|  |
| 20th President of the U.S.A. |
| **In Office:** March 4, 1881- September 19, 1881 |
| Assassinated at the age of 49 |
| One of four assassinated presidents |




**(a)** Calculate the area of trapezoid *XZWU* by summing the areas of $\Delta UXY$, $\Delta YZW$ and $\Delta UYW$. Simplify fully!

$A_{Trap} = A_1 + A_2 + A_3$

$=$

**(b)** Calculate the area of the trapezoid by using the equation for the area of a trapezoid. Simplify fully!

$A_{Trap} = \frac{h(a+b)}{2}$

$=$

| Think! What is the height of the trapezoid? |
| --- |

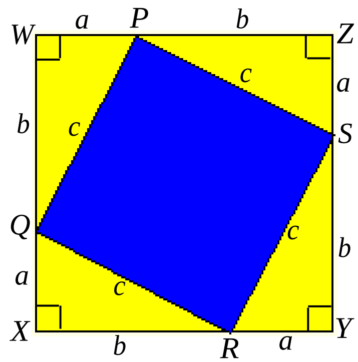
**(c)** In parts (a) and (b) you developed two different expressions for the area of trapezoid *XZWU*. Since both expressions give the area of the ***same shape***, they must be equal to each other! Set the expressions equal to each other and solve for $c^2$.

$A_{Trap} = A_{Trap}$

$\therefore$

### *Pythagorean Theorem Proof 2*

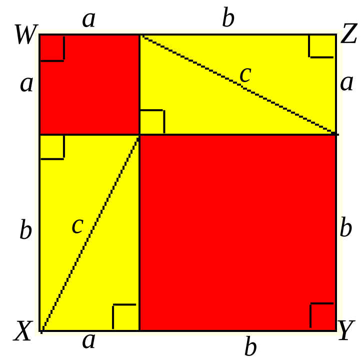
**(a)** Explain why quadrilateral *PQRS* *must* be a square.




**(b)** Use the above diagram to develop a proof of the Pythagorean Theorem. (**Hint:** The line of reasoning is similar to that of President Garfield's proof.)

### *Pythagorean Theorem Proof 3*

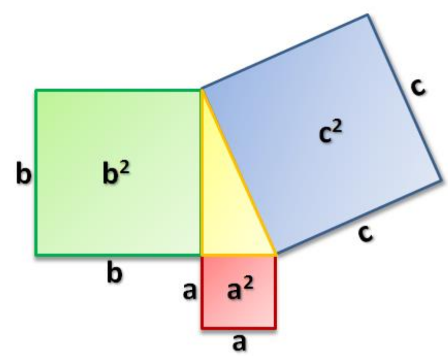
**(a)** Explain how the diagram at the right is simply a rearrangement of the pieces in the diagram at the left.




**(b)** Explain why the two diagrams together make it ***obvious*** that $a^2 + b^2 = c^2$. (This proof was devised in 1939 by Maurice Laisnez, a high school student in the Junior-Senior High School of South Bend, Indiana.)

### *Consequence of the Pythagorean Theorem*

Although the Pythagorean Theorem is an equation that relates the lengths of the sides of a right triangle, it can also be ***interpreted*** in terms of areas.

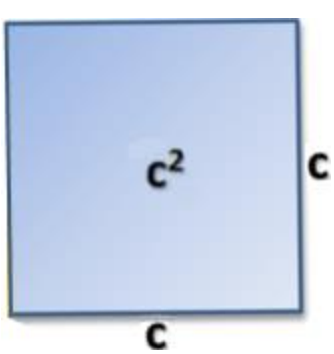


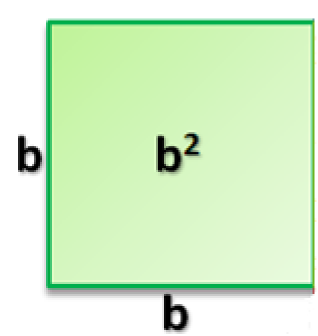

- We have ***proved*** that in a right triangle, the square of the hypotenuse must equal the sum of the squares of the other two sides.

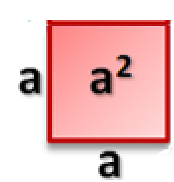
  That is, if *c* represents the length of the hypotenuse and *a* and *b* respectively represent the lengths of the other two sides, then $c^2 = a^2 + b^2$.

- By examining the diagram at the right, one can easily see that the expressions $a^2$, $b^2$ and $c^2$ are all areas of squares!

  Since $c^2 = a^2 + b^2$, ***it must also be true*** that

**Area of** 


**=** **Area of** 


**+** **Area of** 

## *Research*

**1.** There are literally hundreds of different proofs of the Pythagorean Theorem. Find a proof that you are able to understand and explain it in your own words. Include diagrams in your explanation.

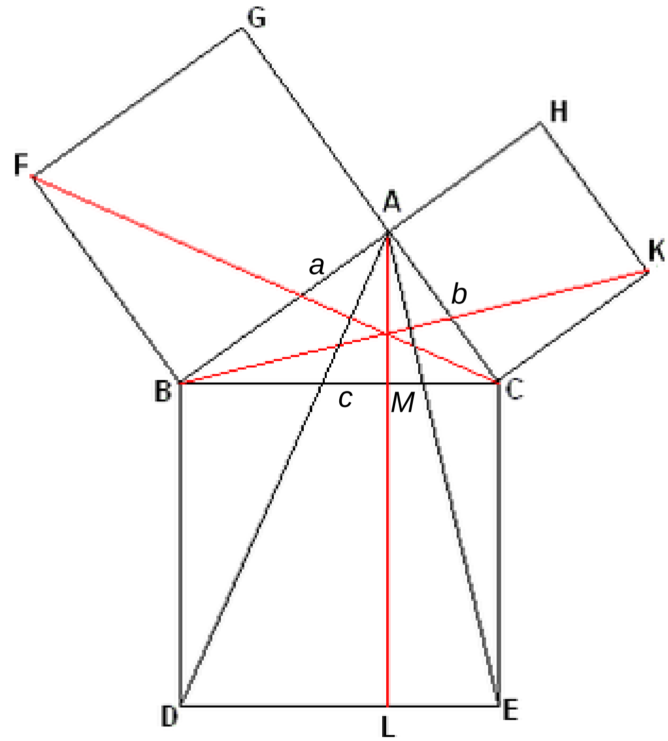
**2.** Do some research to answer the following questions:

**(a)** Who was Euclid?

**(b)** Euclid created a mathematical treatise, consisting of thirteen books, called the *Elements*. Why are Euclid's *Elements* considered so important?

**(c)** Find Euclid's proof of the Pythagorean Theorem and explain it in your own words. (See next page for a guided tour of Euclid's proof.)

## *A Guided Tour of Euclid's Beautiful Proof of the Pythagorean Theorem*




### *Given*

Mark all the following on the diagram (if not already done so):

- $\Delta ABC$ is a right triangle with $\angle BAC = 90°$
- Quadrilaterals *BCDE*, *AHKC* and *FGAB* are squares built upon sides *BC*, *AC* and *AB* respectively
- $AL \parallel BD$
- $a = \overline{AB}$, $b = \overline{AC}$, and $c = \overline{BC}$
- *M* is the point of intersection of *BC* and *AL*.

### *Guided Tour*

**1.** Why must *BAH* and *CAG* both be straight lines?

**2.** Explain why $\angle DBA = \angle FBC$.

**3.** Explain why $\overline{AD} = \overline{FC}$.

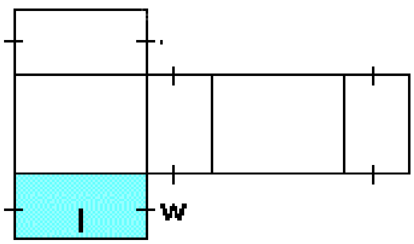
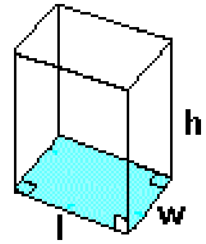
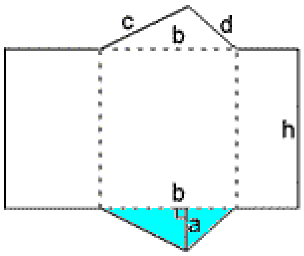
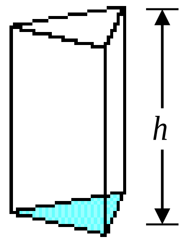
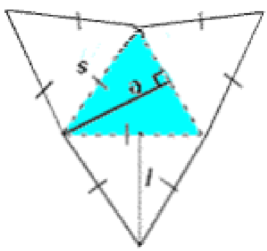
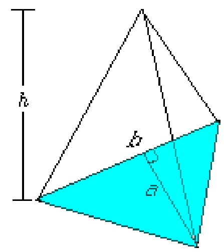
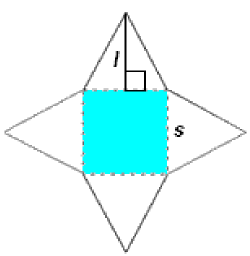
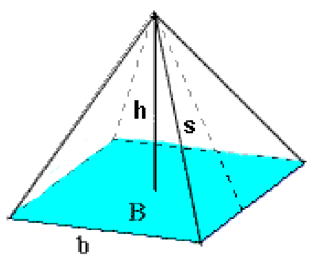
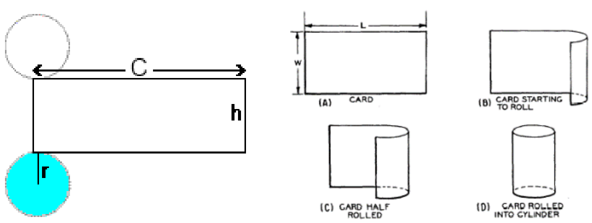
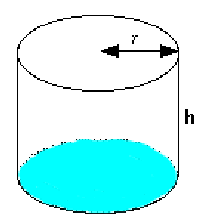
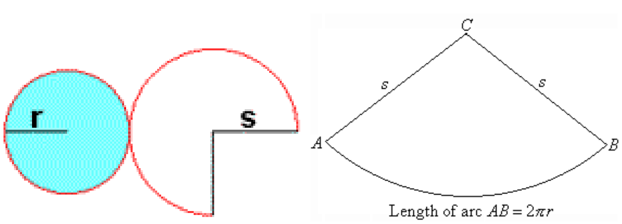
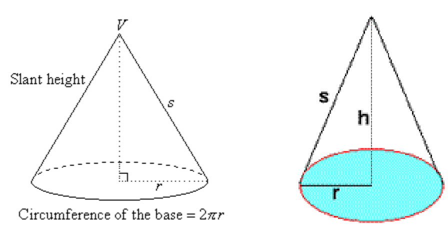
**4.** Explain why the area of $\Delta ABD$ equals the area of $\Delta FBC$.

**5.** Explain why rectangle *BMLD* has twice the area of $\Delta ABD$ and the same area as square *GFBA*.

**6.** Use a similar argument to show that rectangle *CMLE* must have the same area as square *AHKC*.

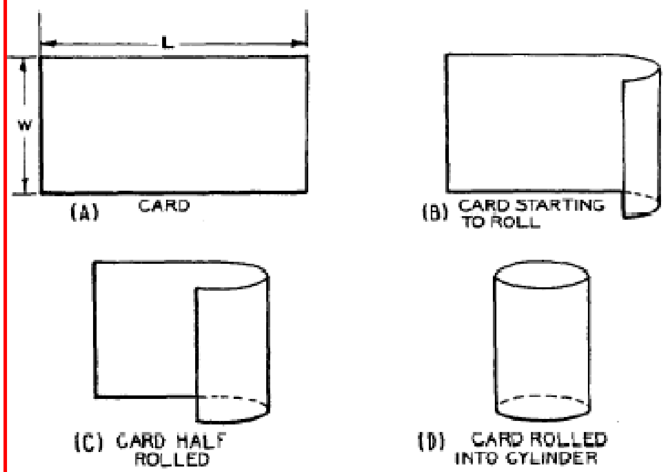
**7.** Use the results of points 5 and 6 to draw the conclusion.

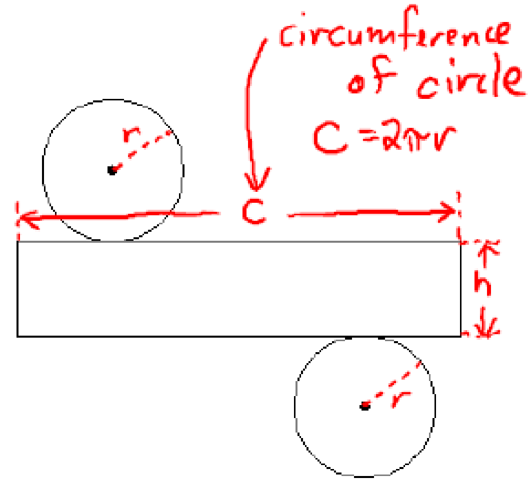
## *Understanding Surface Area Equations*

| *Net* | *Solid* | *Surface Area Equation* |
|---|---|---|
|  |  | |
|  |  | |
|  |  | |
|  |  | |
|  |  | |
|  |  | |

January 11-10
11:25 AM

# Surface Area of Cylinder



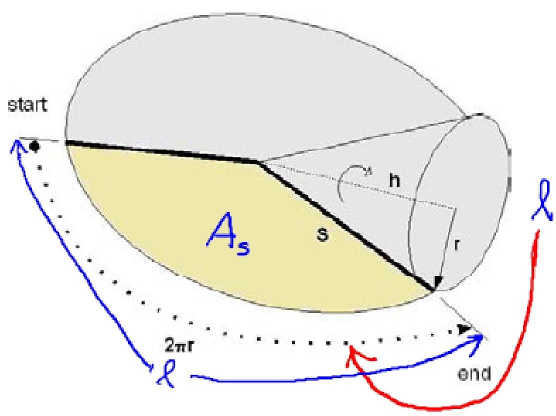




$SA = A_{Bases} + A_{Lateral Face}$

$\therefore$ SA = area of 2 circles + area of rectangle

$\therefore SA = 2\pi r^2 + 2\pi rh$

## Surface Area of a Cone




A cone can be made by rolling up a SECTOR of a circle. (A slice of pizza is an example of a sector.)

$\ell$ = length of arc = circumference of circular base of cone $= 2\pi r$

$A_s$ = area of sector with arc length $2\pi r$

$A_c$ = area of entire circle (radius s)

C = circumference of circle of radius s

$\dfrac{\ell}{C} = \dfrac{A_s}{A_c}$ (ratio of arc length to circumference must equal ratio of sector area to area of the circle)

$\therefore \dfrac{2\pi r}{2\pi s} = \dfrac{A_s}{\pi s^2}$

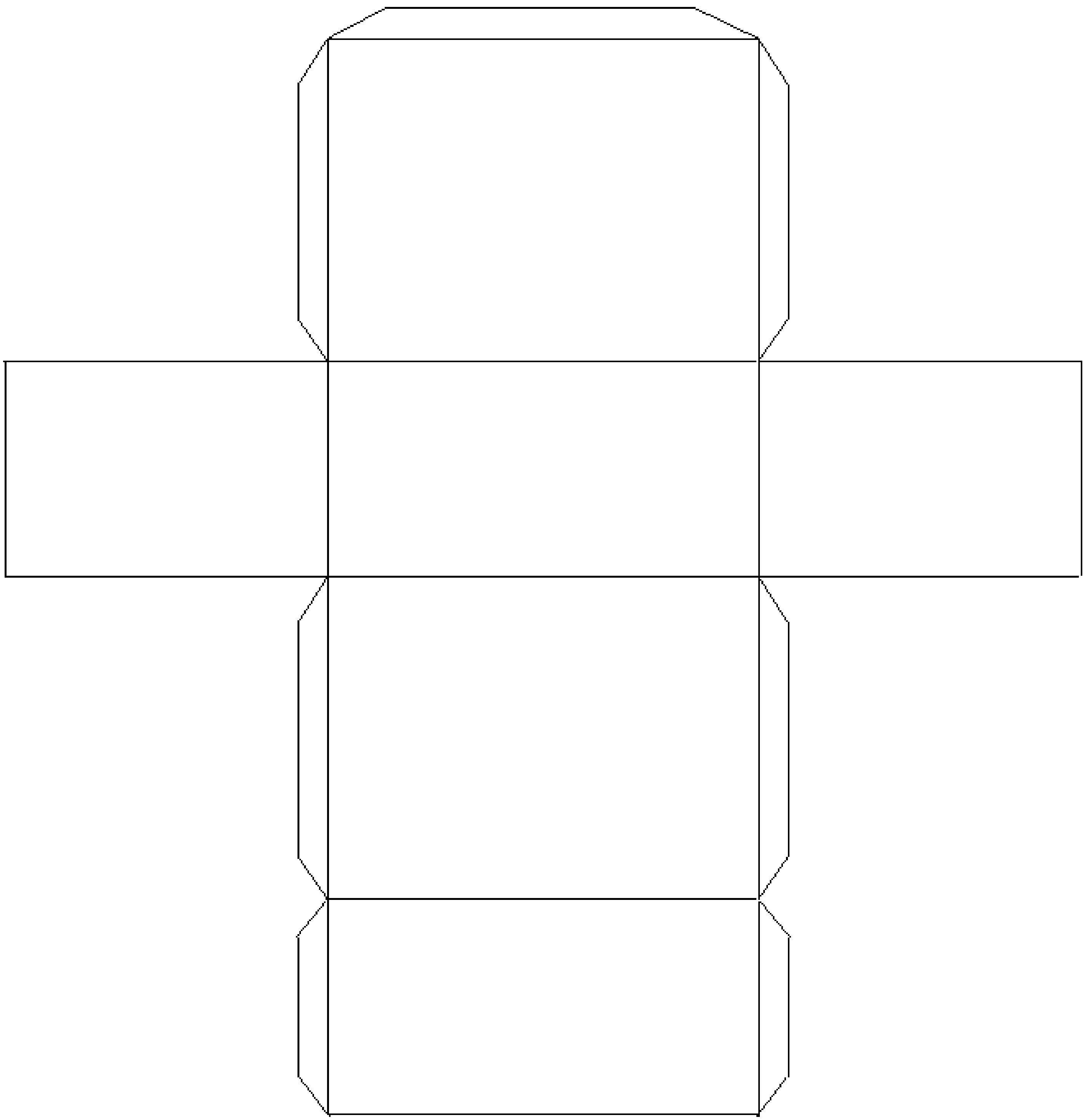
Multiplying both sides by $\pi s^2$, we obtain

$A_s = \dfrac{2\pi r(\pi s^2)}{2\pi s} = \dfrac{2\pi^2 r s^2}{2\pi s} = \pi rs$

$\therefore$ the area of the lateral surface of the cone must be $\pi rs$

***Rectangular Prism Net***

1. Mark the dimensions of a rectangular prism on the net.
2. Explain how the net shows that the surface area of a rectangular prism must be given by the equation $A = 2wh + 2lw + 2lh$. In addition, explain why this equation can also be written as $A = 2(wh + lw + lh)$.
3. If desired, cut out the net and assemble it into a rectangular prism.

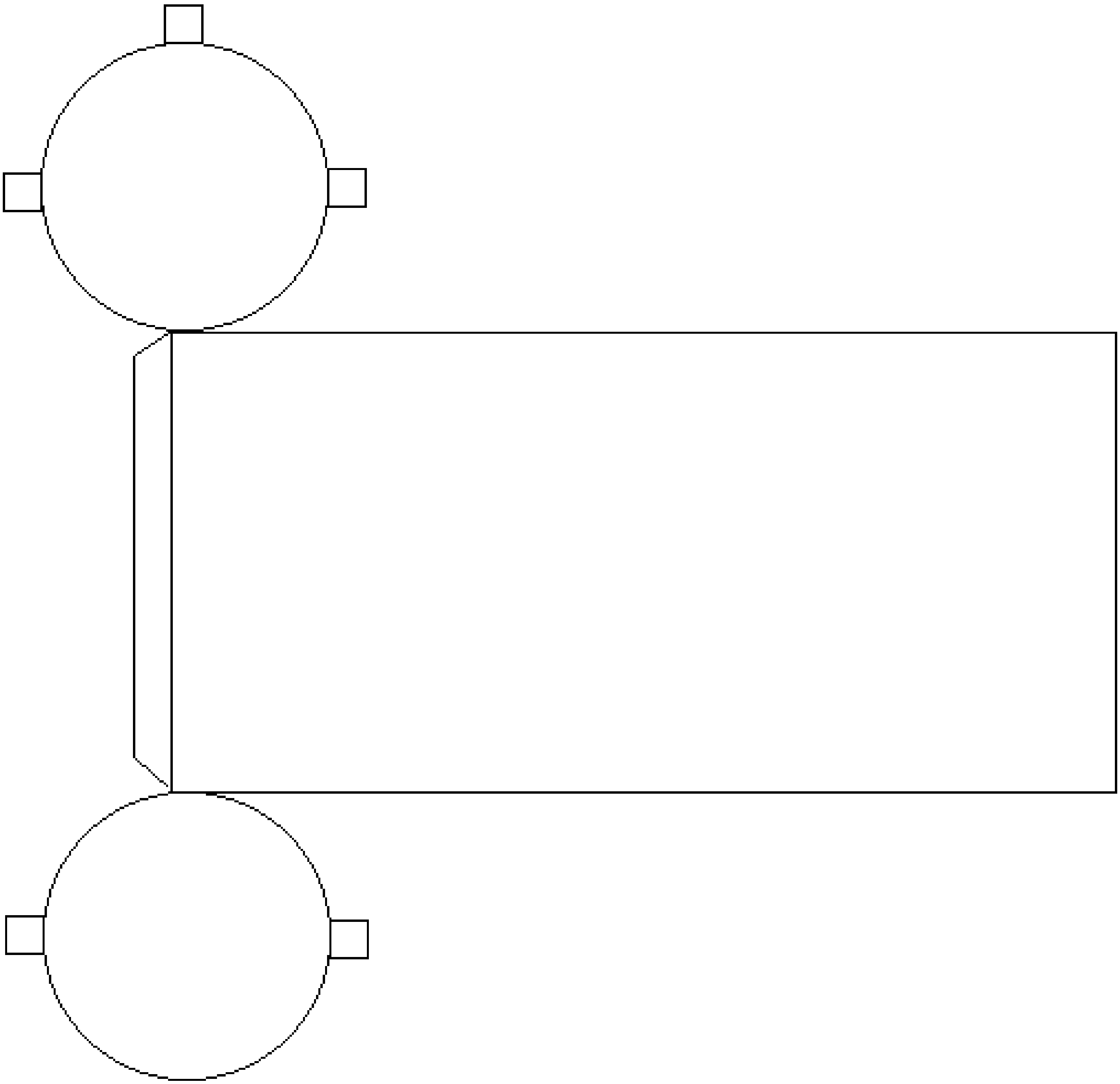


### *Right Circular Cylinder Net*

**1.** Mark the dimensions of a cylinder on the net.

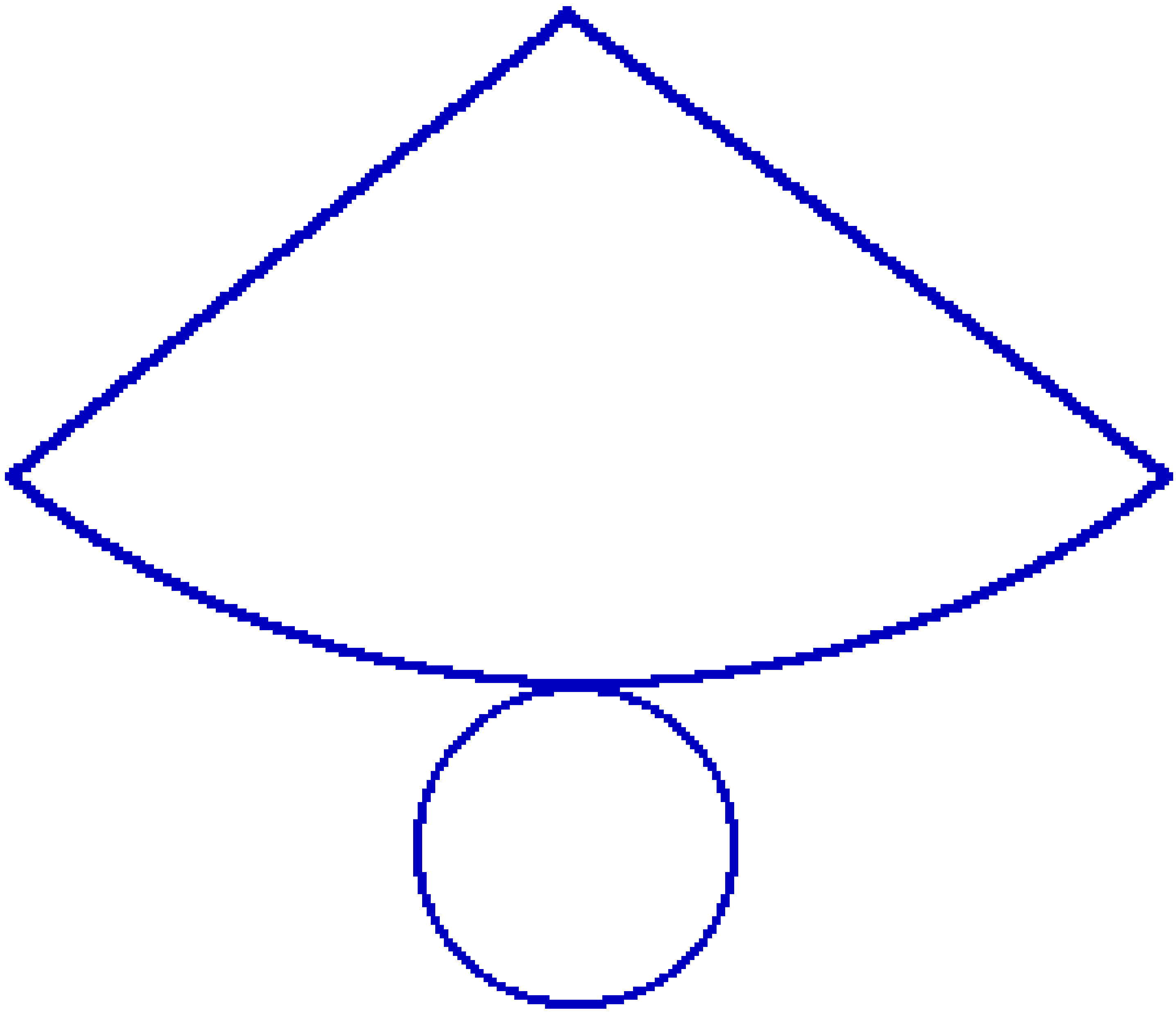
**2.** Explain how the net shows that the surface area of a cylinder must be given by the equation $A = 2\pi r^2 + 2\pi rh$ .

**3.** If desired, cut out the net and assemble it into a cylinder.



*Right Circular Cone Net*

1. Mark the dimensions of a cone on the net.
2. Explain how the net shows that the surface area of a cone must be given by the equation $A = \pi r^2 + \pi rs$ .
3. If desired, cut out the net and assemble it into a cone.



## *Understanding Volume Equations*

### Volume of Any Prism

$V = (\text{area of base})(\text{height})$

### Volume of any Pyramid

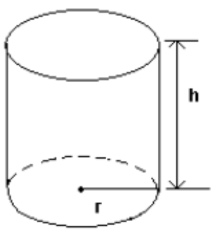
$V = \frac{1}{3}(\text{area of base})(\text{height})$

**OR**

$V = \frac{(\text{area of base})(\text{height})}{3}$

### Volume of Cylinder

Cylinder: has 2 parallel congruent circular faces called bases. The lateral surface is a rolled up rectangle

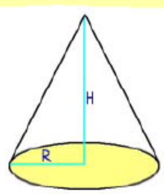



$\text{Volume} = (\text{area of base})(\text{height})$

$\therefore V = \pi r^2 h$

$\pi r^2$: area of circular base; $h$: height

### Volume of Cone



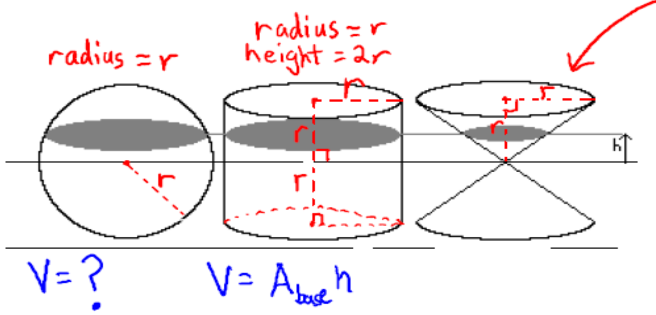

$\text{Volume} = \frac{1}{3}(\text{area of base})(\text{height})$

$\therefore V = \frac{1}{3}\pi r^2 h = \frac{\pi r^2 h}{3}$

**Note:** The following is a reasonably convincing argument that for a sphere of radius $r$,

$V = \frac{4}{3}\pi r^3$

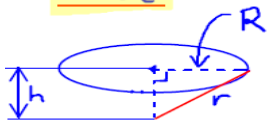
Height of one cone $= r$
Height of double cone $= 2r$




Let $h$ represent the distance from the "centre line" to the point at which the cross-section is taken

## Cross-Sections

### Circle




$h^2 + R^2 = r^2$

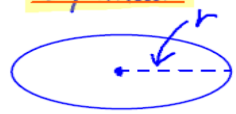
$\therefore R^2 = r^2 - h^2$

$A_{sp} = \pi R^2$

$\therefore A_{sp} = \pi(r^2 - h^2)$

$\therefore A_{sp} = \pi r^2 - \pi h^2$

### Cylinder




$A_{cyl} = \pi r^2$

**Notice that**

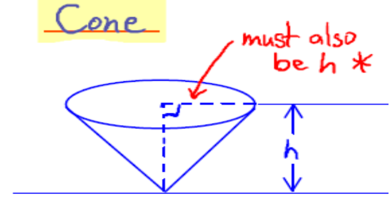
$A_{sp} + A_{co}$

$= \pi r^2 - \pi h^2 + \pi h^2$

$= \pi r^2$

$= A_{cyl}$

### Cone




* The cone is constructed in such a way that the height = radius $\therefore$ radius = h

$A_{co} = \pi h^2$

This **suggests** that

That is, our argument is strong but it does not prove definitively that $V_{sp} = \frac{4}{3}\pi r^3$

$$V_{cyl} = V_{sp} + V_{co}$$

$$\therefore V_{sp} = V_{cyl} - V_{co}$$

$$= 2\pi r^3 - \frac{2}{3}\pi r^3$$

$$= \frac{6}{3}\pi r^3 - \frac{2}{3}\pi r^3$$

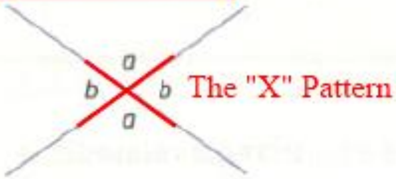
$$= \frac{4}{3}\pi r^3$$

# Understanding Geometric Relationships

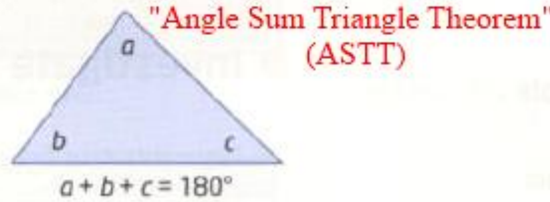
***Angle and Triangle Properties***

## Angle Properties

When two lines intersect, the **opposite angles** are equal.

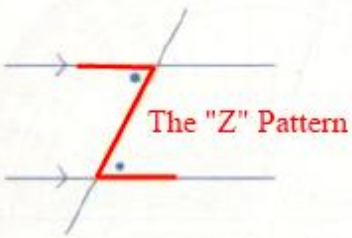



The sum of the interior angles of a triangle is 180°.

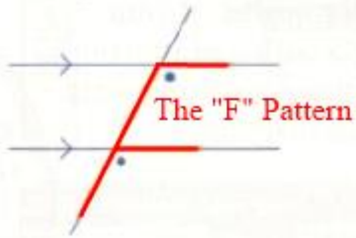



When a transversal crosses parallel lines, many pairs of angles are related.

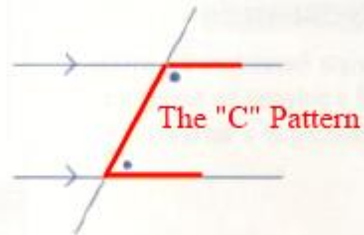
**alternate angles** are equal



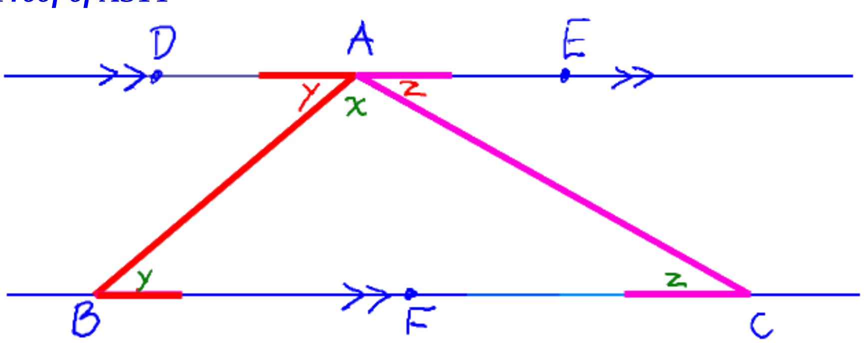

**corresponding angles** are equal




**co-interior angles** have a sum of 180°




***Proof of ASTT***




$\angle DAB = \angle ABF = y$ (alternate angles)
$\angle EAC = \angle ACF = z$ (alternate angles) } "Z"

But $\angle DAB + \angle BAC + \angle EAC = 180°$ (straight line)

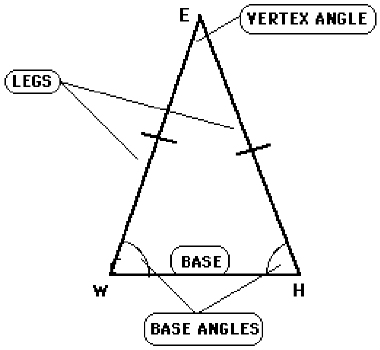
$\therefore y + x + z = 180°$

$\therefore$ the sum of the interior angles of a triangle MUST BE 180°

Angle Sum Triangle Theorem

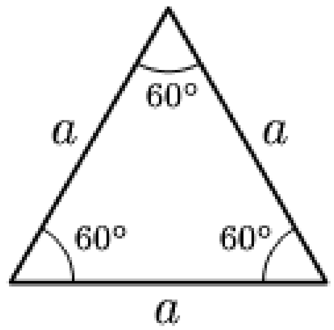
## *Angles in Isosceles and Equilateral Triangles*

- The ***Isosceles Triangle Theorem (ITT)*** asserts that a triangle is isosceles if and only if its ***base angles are equal***.
- This can be proved using ***triangle congruence theorems*** (not covered in this course).



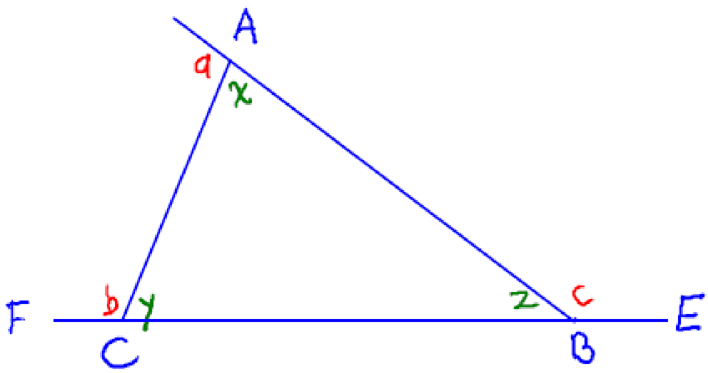

- Using ITT, it can be shown that an ***equilateral triangle*** is also ***equiangular*** (all three angles have the same measure).
- If $x$ represents the measure of each angle, then

$x+x+x=180°$ (ASTT)
$\therefore 3x=180°$
$\therefore x=60°$




## *Exterior Angle Theorem (EAT)*

### Exterior Angles of a Triangle




$x+y+z=180°$ (ASTT)
$c+z=180°$ (straight line)

$\therefore c=x+y$

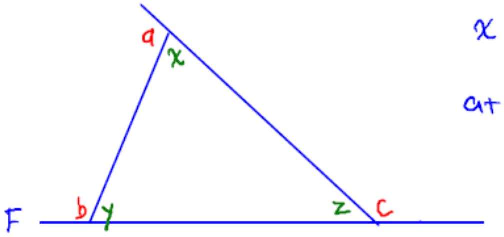
$\therefore$ exterior $\angle ABE$ is the sum of $\angle CAB$ and $\angle ACB$

Using a similar argument, we can show that $b=x+z$ and $a=y+z$

**Exterior Angle Theorem**

The exterior angle at each vertex of a triangle is equal to the sum of the interior angles at the other two vertices.

## *Sum of the Exterior Angles of a Triangle*




$x + y + z = 180^\circ$ (ASTT)

$a + x + b + y + c + z$
$= 180^\circ + 180^\circ + 180^\circ$ (straight lines)
$= 540^\circ$

But $a + x + b + y + c + z$
$= a + b + c + x + y + z$
$= a + b + c + 180^\circ$ (since $x + y + z = 180^\circ$)
$= 540^\circ$

$\therefore a + b + c + 180^\circ = 540^\circ$

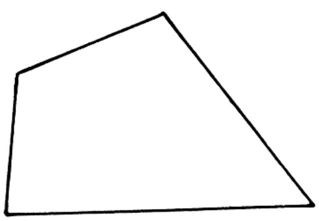
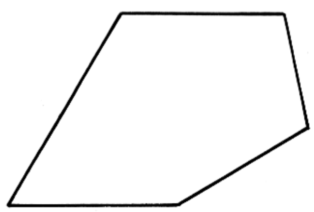
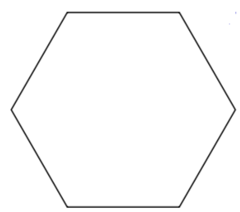
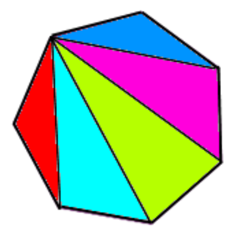
$\therefore a + b + c + 180^\circ - 180^\circ = 540^\circ - 180^\circ$

$\therefore a + b + c = 360^\circ$

In any triangle, the sum of the exterior angles MUST BE $360^\circ$

## *Sum of the Interior Angles of a Convex Polygon*

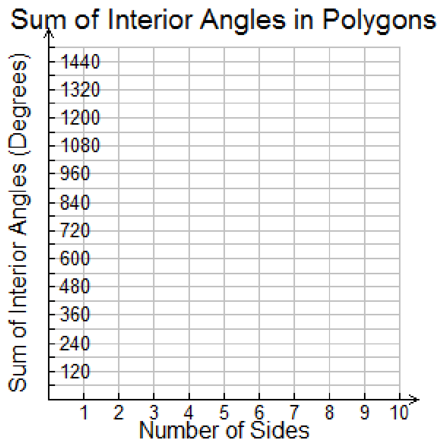
**1.** By dividing each polygon into triangles, calculate the sum of the interior angles of the following convex polygons. Note that one of the shapes has already been done for you.

| | | | |
|---|---|---|---|
|  |  |  |  |
| Name: ________________ | Name: ________________ | Name: ________________ | Name: Heptagon |
| Number of Sides: _______ | Number of Sides: _______ | Number of Sides: _______ | Number of Sides: 7 |
| Number of Triangles: ____ | Number of Triangles: ____ | Number of Triangles: ____ | Number of Triangles: 5 |
| Sum of Interior Angles: ____________________ | Sum of Interior Angles: ____________________ | Sum of Interior Angles: ____________________ | Sum of Interior Angles: $5(180^\circ) = 900^\circ$ |

**2.** Now summarize your results in the following table and sketch a graph relating the sum of the interior angles of a convex polygon to the number of sides. Then answer questions (a) to (f).

$n$ = number of sides in the polygon, $s$ = sum of the interior angles of the polygon

| $n$ | $s$ | $\Delta s$ (1st Differences) |
|---|---|---|
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | 900° | |




**(a)** Do you expect the pattern to continue indefinitely beyond $n = 7$? Explain.

**(b)** Write an equation relating $s$ to $n$. Explain why it is not surprising that the relation between $s$ and $n$ is linear.

**(c)** State the ***meaning*** of the slope of the linear relation between $s$ and $n$.

**(d)** Does the vertical intercept of this linear relation have a meaning? Explain.

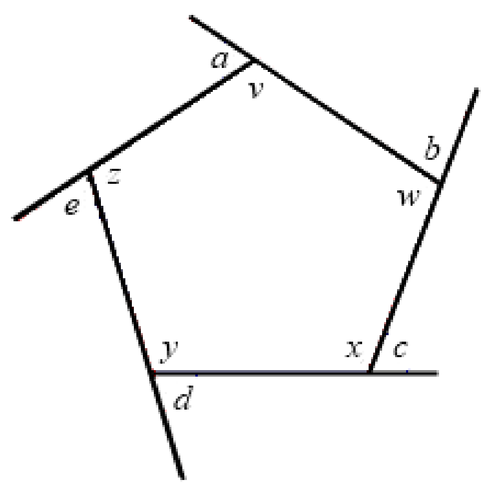
**(e)** Does it make sense to "connect the dots" in the above graph? Explain.

**(f)** State an easy way to remember how to calculate the sum of the interior angles of a polygon.

## *Sum of the Exterior Angles of a Convex Polygon*

The argument presented here for a pentagon can be used for a polygon with any number of sides.

In the pentagon at the right, ***there are five straight line angles***, that is, there are five 180° angles. Therefore,




$$(a+v)+(b+w)+(c+x)+(d+y)+(e+z)=180°+180°+180°+180°+180°$$

$$\therefore a+v+b+w+c+x+d+y+e+z=5(180°)$$

$$\therefore a+b+c+d+e+v+w+x+y+z=5(180°)$$

$\therefore a+b+c+d+e+3(180°)=5(180°)$ (Since $v+w+x+y+z$ is the sum of the interior angles of the pentagon.)

$$\therefore a+b+c+d+e+3(180°)-3(180°)=5(180°)-3(180°)$$

$$\therefore a+b+c+d+e=2(180°)$$

$$\therefore a+b+c+d+e=360°$$

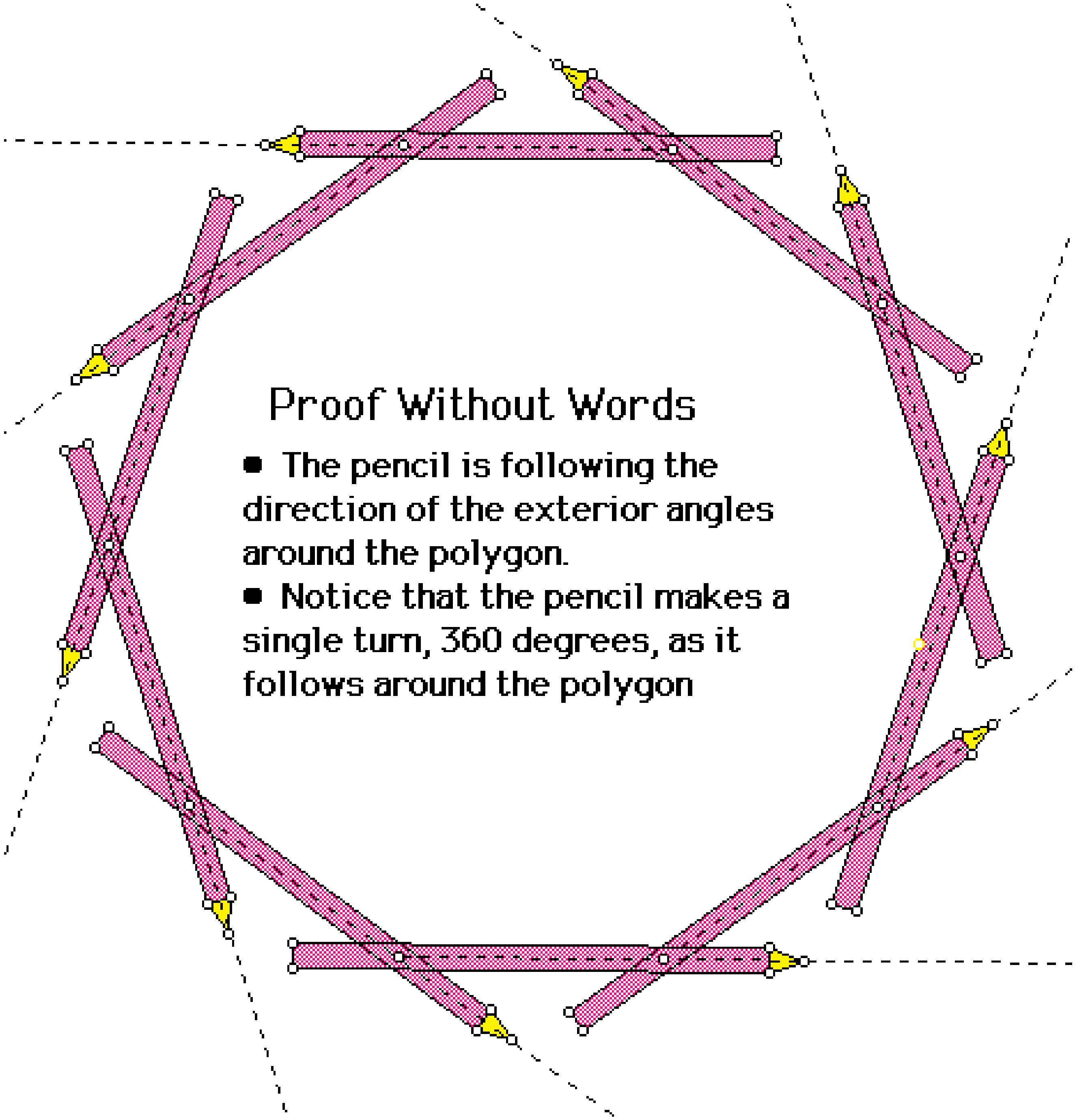
For a convex polygon with $n$ sides…

- There are $n$ straight line angles, for a total measure of $180°n$.
- The sum of the $n$ interior angles is equal to $180°(n-2)$.
- The sum of the exterior angles is equal to $180°n-180°(n-2)=180°n-180°n+360°=360°$

The ***sum of the exterior angles*** of ***any convex polygon*** is 360°.

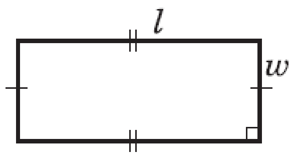
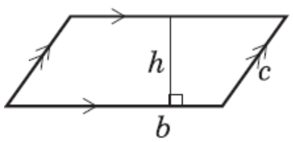
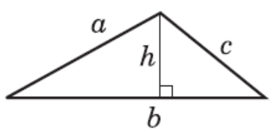
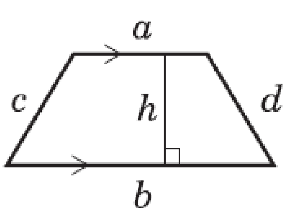
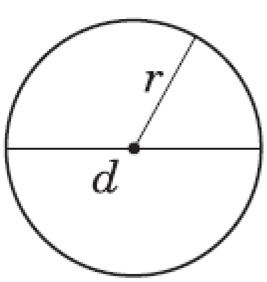
# Proof Without Words

- The pencil is following the direction of the exterior angles around the polygon.
- Notice that the pencil makes a single turn, 360 degrees, as it follows around the polygon

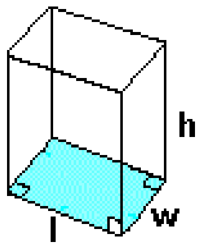
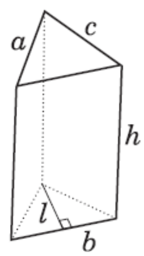
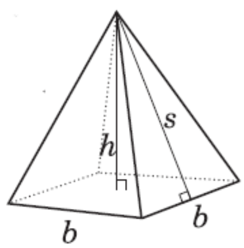
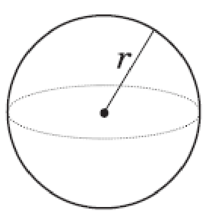
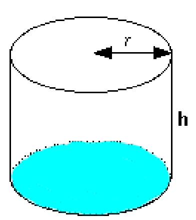
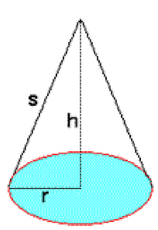


# WHAT HAPPENS IF…

**1.** Complete the following table. The first row has been done for you.

| ***Shape*** | ***Name of the Shape*** | ***What Happens to the Perimeter if …*** | ***What Happens to the Area if …*** |
|---|---|---|---|
|  | Rectangle | …the length is doubled?<br>***Solution***<br>$P = 2l + 2w$<br>If the length is doubled, the new length is $2l$. Then, the perimeter becomes<br>$P = 2(2l) + 2w = 4l + 2w = (2l + 2w) + 2l$<br>The perimeter increases by $2l$. | …the width is tripled?<br>***Solution***<br>$A = lw$<br>If the width is tripled, the new width is $3w$. Then, the area becomes<br>$A = l(3w) = 3lw = 3(lw)$<br>The area is also tripled. |
|  | | …the base is doubled? | …the height is quadrupled? |
|  | | …the base is tripled? (If this can be done without changing the values of *a* and *c*.) | …the height is tripled? |
|  | | …the base is tripled? (If this can be done without changing the values of *c* and *d*.) | …the height is doubled? |
|  | | …the radius is doubled? | …the radius is doubled? |

**2.** Complete the following table. The first row has been done for you.

| *Shape* | *Name of the Shape* | *What Happens to the Surface Area if …* | *What Happens to the Volume if …* |
|---|---|---|---|
|  | Rectangular Prism | …the length is doubled?<br>***Solution***<br>$A = 2lw + 2lh + 2wh$<br>If the length is doubled, the new length is $2l$. Then, the surface area becomes<br>$A = 2(2l)w + 2(2l)h + 2wh$<br>$= 4lw + 4lh + 2wh$<br>$= (2lw + 2lh + 2wh) + 2lw + 2lh$<br>The surface area increases by $2lw + 2lh$. | …the width is tripled?<br>***Solution***<br>$V = lwh$<br>If the width is tripled, the new width is $3w$. Then, the volume becomes<br>$V = l(3w)h = 3lwh = 3(lwh)$<br>The volume is also tripled. |
|  | | …*b* is doubled? (If this can be done without changing the values of *a* and *c*.) | …the height is quadrupled? |
|  | | …the slant height is tripled? | …the height is tripled? |
|  | | …the radius is doubled? | …the radius is doubled? |
|  | | …the radius is doubled? | …the radius is doubled? |
|  | | …the radius is doubled? | …the radius is doubled? |

# OPTIMIZATION PROBLEMS

***Definition: Optimize***

- Make ***optimal*** (i.e. the best, most favourable or desirable, ***especially under some restriction***); get the most out of; use best
- In a mathematical context, to ***optimize*** means either to ***maximize*** (make as great as possible) or to ***minimize*** (make as small as possible), subject to a restriction called a ***constraint***.

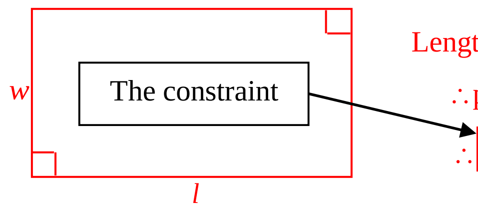
## *Optimization Problem 1*

You have 400 m of fencing and you would like to enclose a rectangular region of ***greatest possible area***. What dimensions should the rectangle have?

**(a)** What is the ***constraint*** in this problem?

The constraint is the length of fencing available. Since only 400 m of fencing are available, the region enclosed by the fence will have a limited size.

**(b)** Draw a diagram describing this situation. Then write an equation that relates the unknowns to the constraint.




Length of fencing = 400 m

$\therefore$ perimeter = 400 m

$\therefore$ $\boxed{2l + 2w = 400}$

**(c)** What quantity needs to be optimized? State whether the quantity needs to be maximized or minimized. Then write an equation describing the quantity that needs to be optimized.

In this problem, the area needs to be ***maximized***. Therefore, the equation must describe the area of the rectangular region.

$$A = lw$$

**(d)** The equation in (c) cannot be used directly to maximize the area because there are too many variables. Use the constraint equation to solve for $l$ in terms of $w$. Then rewrite the equation in (c) in such a way that $A$ is expressed entirely in terms of $w$.

$\because 2l + 2w = 400$

$\therefore \frac{2l}{2} + \frac{2w}{2} = \frac{400}{2}$

$\therefore l + w = 200$

$\therefore l + w - w = 200 - w$

$\therefore l = 200 - w$

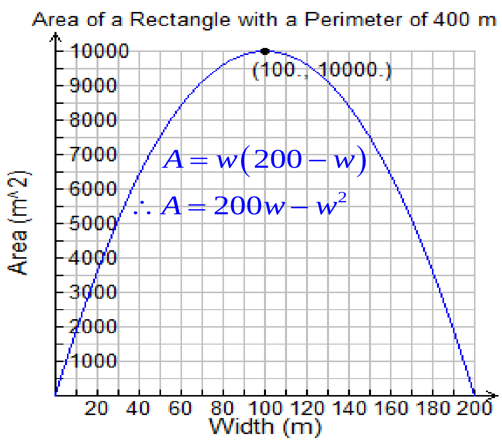
$\therefore A = lw$

$\therefore A = (200 - w)w$

$\therefore A = w(200 - w)$

Now the area has been expressed ***in terms of one variable only*** (i.e. the width).

**(e)** Sketch a graph of area of the rectangle versus width. Label the axes and include a title.




**(f)** Is the relationship between $A$ and $w$ linear or non-linear? Give ***three*** reasons to support your answer.

| $w$ | $A$ | $\Delta A$ |
|---|---|---|
| 20 | 3600 | - |
| 30 | 5100 | 1500 |
| 40 | 6400 | 1300 |
| 50 | 7500 | 1100 |
| 60 | 8400 | 900 |

The relation is ***non-linear***. We know this because of the following reasons.

- The graph is curved.
- The equation has a squared term ( $w^2$ ).
- The first differences are ***not constant***.

**(g)** State the dimensions of the rectangular region having a perimeter of 400 m and a ***maximal*** area.

From the graph it can be seen that the maximum area is 10000 m$^2$, which is attained when the width is 100 m. Therefore, for maximum area,

$w = 100$ and $l = 200 - w = 200 - 100 = 100$ .

For maximal area, both the length and the width should be 100 m. In other words, the region should be a square with side length of 100 m.

## *Optimization Problem 2*

Design a cylindrical pop can that has the ***greatest possible capacity*** but can be manufactured using at most 375 $cm^2$ of aluminum.

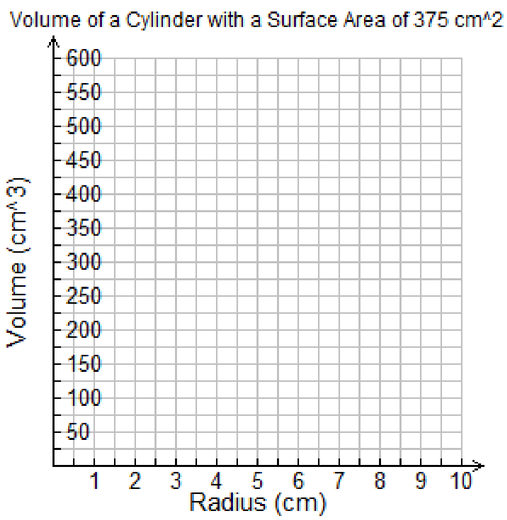
**(a)** What is the ***constraint*** in this problem?

**(b)** Draw a diagram describing this situation. Then write an equation that relates the unknowns to the constraint. (Let $r$ represent the radius of the cylinder and let $h$ represent its height.)

**(c)** What quantity needs to be optimized? State whether the quantity needs to be maximized or minimized. Then write an equation describing the quantity that needs to be optimized.

**(d)** The equation in (c) cannot be used directly to maximize the volume because there are too many variables. Use the constraint equation to solve for $h$ in terms of $r$. Then rewrite the equation in (c) in such a way that $V$ is expressed entirely in terms of $r$.

**(e)** Sketch a graph of volume of the cylindrical can versus radius. Label the axes and include a title.




**(f)** Is the relationship between $V$ and $r$ linear or non-linear? Give ***three*** reasons to support your answer.

| $r$ | $V$ | $\Delta V$ |
|---|---|---|
| | | |

**(g)** State the dimensions of the cylindrical can having a surface area of 375 $cm^2$ and a ***maximal*** volume.

## *Optimization Problem 3*

A container for chocolates must have the shape of a ***square prism*** and it must also have a volume of 8000 cm$^3$. Design the box in such a way that it can be manufactured using the ***least amount of material***.

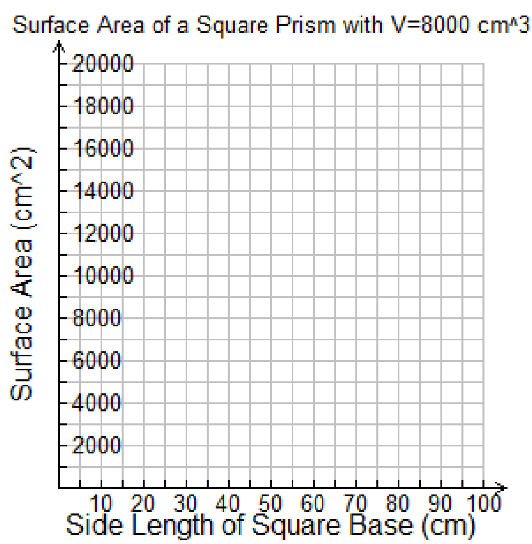
**(a)** What is the ***constraint*** in this problem?

**(b)** Draw a diagram describing this situation. Then write an equation that relates the unknowns to the constraint. (Let $x$ represent the side length of the square base and let $h$ represent the height.)

**(c)** What quantity needs to be optimized? State whether the quantity needs to be maximized or minimized. Then write an equation describing the quantity that needs to be optimized.

**(d)** The equation in (c) cannot be used directly to ***minimize*** the surface area because there are too many variables. Use the constraint equation to solve for $h$ in terms of $x$. Then rewrite the equation in (c) in such a way that $A$ is expressed entirely in terms of $x$.

**(e)** Sketch a graph of volume of the square prism versus width. Label the axes and include a title.




**(f)** Is the relationship between $A$ and $x$ linear or non-linear? Give ***three*** reasons to support your answer.

| $x$ | $A$ | $\Delta A$ |
|---|---|---|
| | | |

**(g)** State the dimensions of the square prism with a volume of 8000 cm$^3$ and a ***minimal*** surface area.