2005 Canadian Open Mathematics Challenge – Part B, Question Four

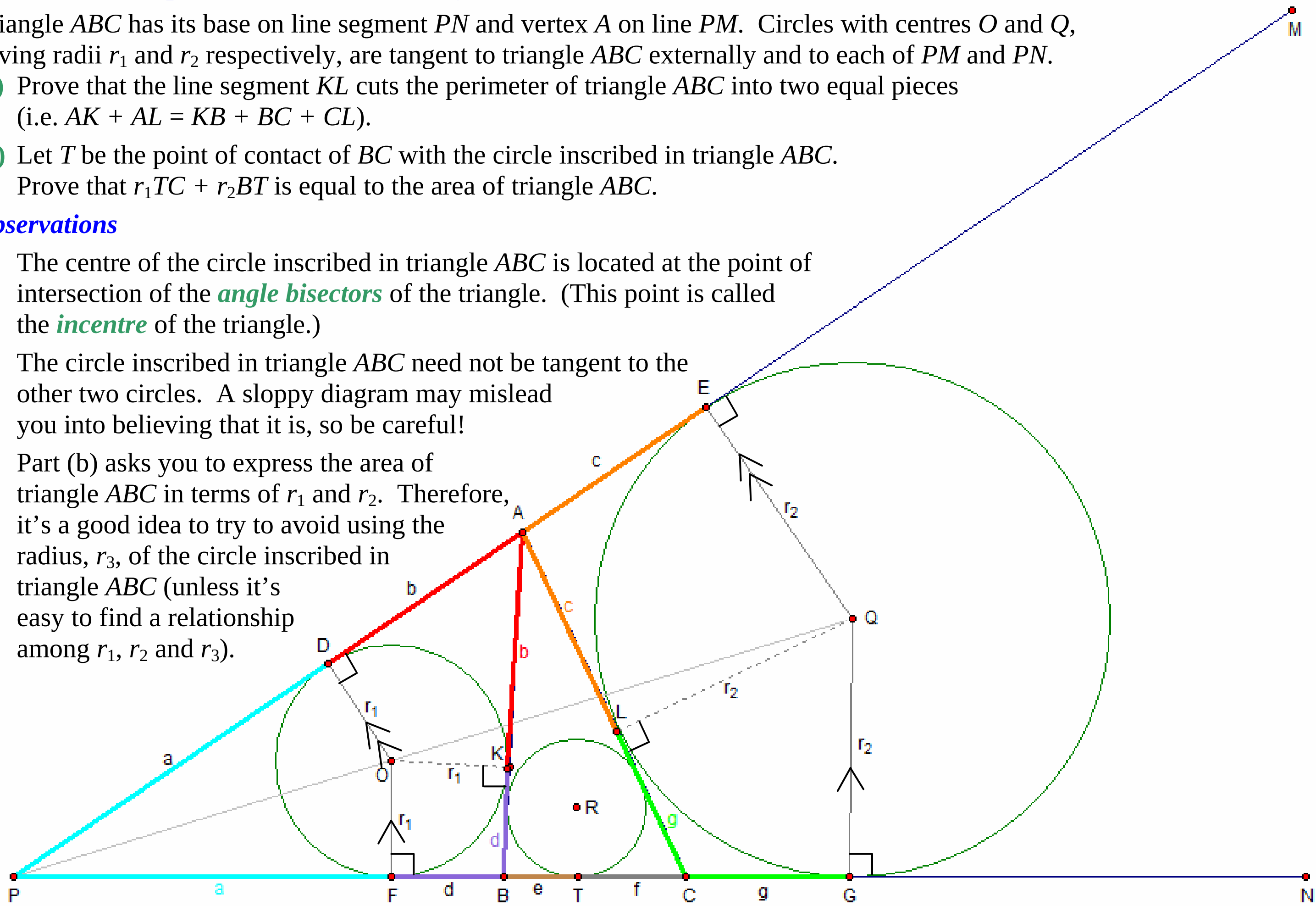
Triangle ABC has its base on line segment PN and vertex A on line PM. Circles with centres O and Q, having radii r_1 and r_2 respectively, are tangent to triangle ABC externally and to each of PM and PN.

(a) Prove that the line segment KL cuts the perimeter of triangle ABC into two equal pieces (i.e. $AK + AL = KB + BC + CL$).

(b) Let T be the point of contact of BC with the circle inscribed in triangle ABC. Prove that $r_1TC + r_2BT$ is equal to the area of triangle ABC.

Observations

1. The centre of the circle inscribed in triangle ABC is located at the point of intersection of the ***angle bisectors*** of the triangle. (This point is called the ***incentre*** of the triangle.)
2. The circle inscribed in triangle ABC need not be tangent to the other two circles. A sloppy diagram may mislead you into believing that it is, so be careful!
3. Part (b) asks you to express the area of triangle ABC in terms of r_1 and r_2. Therefore, it's a good idea to try to avoid using the radius, r_3, of the circle inscribed in triangle ABC (unless it's easy to find a relationship among r_1, r_2 and r_3).



Proof

(a) Using the "Tangent from a Point Property" (TPP), we can justify the following assertions:

$PD = PF = a$, $AD = AK = b$, $AE = AL = c$, $BF = BK = d$, $CG = CL = g$ and $PE = PG$.

$\therefore a + b + c = a + d + e + f + g$ (since $PE = PG$)
$\therefore b + c = d + e + f + g$ (*)
$\therefore AK + AL = KB + BC + CL$ (since $AK = b$, $AL = c$, $BK = d$, $BC = e + f$ and $CL = g$)

Therefore, the line segment KL cuts the perimeter of triangle ABC into two equal pieces.

(b) To simplify matters, we shall use the notation "ΔABC" to mean "the area of triangle ABC, "*quad ABCD*" to mean "the area of quadrilateral $ABCD$" and "*trap ABCD*" to mean "the area of trapezoid $ABCD$."

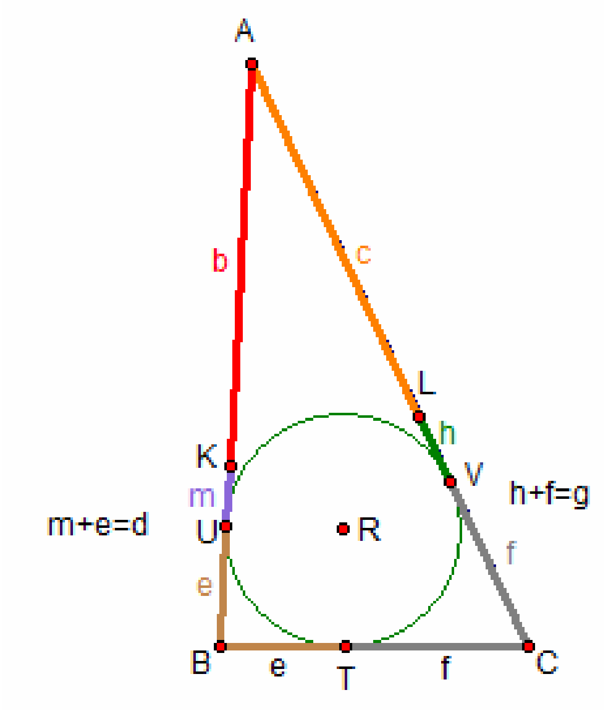
Using the "Tangent Radius Property" (TRP), we can justify the following assertions:

$OD \perp PE$, $QE \perp PE$, $OF \perp PG$, $QG \perp PG$, $OK \perp AB$ and $QL \perp AC$.

Therefore, $OD \parallel QE$ and $OF \parallel QG$ (PLT "F" pattern). Finally, from the diagram we see that ΔABC can be calculated as follows:

$$\begin{aligned}
\Delta ABC &= \textit{trap } QEDO + \textit{trap } GFOQ - \textit{quad } AEQL - \textit{quad } GCLQ - \textit{quad } ADOK - \textit{quad } OFBK \\
&= \tfrac{1}{2}(r_1+r_2)(b + c) + \tfrac{1}{2}(r_1+r_2)(d + e + f + g) - r_2c - r_2g - r_1b - r_1d \\
&= \tfrac{1}{2}(r_1+r_2)(b + c) + \tfrac{1}{2}(r_1+r_2)(b + c) - r_2c - r_2g - r_1b - r_1d \text{ (since } b + c = d + e + f + g \text{ by (*) above)} \\
&= (r_1+r_2)(b + c) - r_2c - r_2g - r_1b - r_1d \\
&= r_1b + r_1c + r_2b + r_2c - r_2c - r_2g - r_1b - r_1d \\
&= r_1c + r_2b - r_2g - r_1d \\
&= r_1(c - d) + r_2(b - g)
\end{aligned}$$

By examining the final expression above, we can see that a solution is close at hand! If what we are attempting to prove is true, then $c - d = TC = f$ and $b - g = BT = e$. This can be shown easily if we focus on ΔABC in the diagram given above.



$BU = BT = e$ (TPP)

$CV = CT = f$ (TPP)

$AU = AV = b + m = c + h$ (TPP) (**)

In addition, $d = KB = m + e$ and $g = LC = h + f$ (see diagram on page 1).

Now we can make use of the equations (*) and (**) proven above.

$b + m = c + h$ **(Equation 1)** (**)

$b + c = d + e + f + g$ (*)

$\therefore\ b + c = m + e + e + f + f + h$

$\therefore\ b + c = m + 2e + 2f + h$ **(Equation 2)**

Subtracting equation 1 from equation 2, we obtain

$c - m = m + 2e + 2f + h - c - h$

$\therefore 2c = 2m + 2e + 2f$

$\therefore c = m + e + f$

$\therefore c = d + f$

$\therefore f = c - d$

Adding equation 1 to equation 2, we obtain

$2b + m + c = c + h + m + 2e + 2f + h$

$\therefore 2b = 2e + 2f + 2h$

$\therefore b = e + f + h$

$\therefore b = e + g$

$\therefore e = b - g$

Continuing the derivation on the previous page, we have

$\Delta ABC = r_1(c - d) + r_2(b - g)$

$\quad = r_1 f + r_2 e$

$\quad = r_1 TC + r_2 BT$