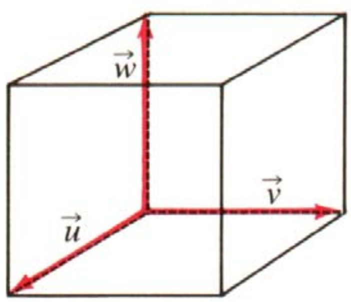
MGA4U0 – Final Culminating Activity and Review

1. Early in the course we used the methods of Cartesian geometry to prove that when the midpoints of any quadrilateral are joined, the resulting figure is a parallelogram (Varignon parallelogram). You should recall that the proof was quite long and messy. Give an alternative proof of the above statement that uses vector methods. (You should find that your proof is much shorter and far more elegant than the Cartesian geometry proof that we studied.)
2. At the beginning of the course, we studied a proof of the Pythagorean Theorem that involved similar triangles. Give an alternative proof of the Pythagorean Theorem that uses vector methods. (Use the dot product.)
3. You first learned about the Law of Cosines in grade 11. Your teacher may have given you a proof and also may have pointed out that the Law of Cosines is a generalization of the Pythagorean Theorem. Now that you have an arsenal of vector methods under your belt, use the dot product to prove the Law of Cosines.
4. Prove properties 1 to 9 of the dot product (see unit 2). (Most of these properties are easy to prove. You may encounter difficulties with the distributive law of the dot product over vector addition, however. Try projecting the vectors $\vec{v}$, $\vec{w}$ and $\vec{v}+\vec{w}$ onto $\vec{u}$.)
5. Prove properties 1 to 7 of the cross product (see unit 2).
6. A diagram of a ***unit cube*** is given. Evaluate each of the following. Use both geometric and algebraic arguments to support your answers.



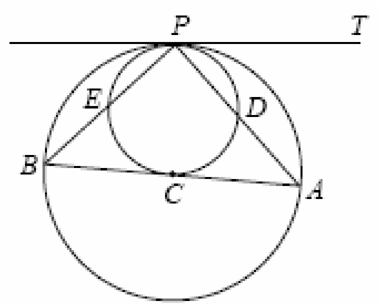
(a) $(\vec{u}+\vec{v})\cdot(\vec{v}+\vec{w})$

(b) $(\vec{u}+\vec{v}+\vec{w})\cdot(\vec{u}-\vec{v}+\vec{w})$

(c) $(\vec{u}+\vec{v})\times(\vec{v}+\vec{w})$

(d) $(\vec{u}+\vec{v}+\vec{w})\times(\vec{u}-\vec{v}+\vec{w})$

7. Two circles are tangent internally at *P*. A chord *AB* of the larger circle is tangent to the smaller circle at *C*. *PB* and *PA* cut the smaller circle at *E* and *D* respectively. Determine the length of *AC* in terms of the lengths of *AB*, *PE* and *PD*.



> ***Important Note***
> The given diagram is somewhat misleading because *AB* ***looks like*** it is a diameter of the circle and *C* ***looks like*** it is the midpoint of *AB*. Neither assumption is valid, however. You must use only the given information and be careful not to make any unjustifiable assumptions! (To avoid such problems, it is a good idea to sketch a better diagram!)

8. Explain why the triple scalar products $(\vec{v}\times\vec{w})\cdot\vec{v}=0$ and $(\vec{v}\times\vec{w})\cdot\vec{w}=0$. Use diagrams to support your statements.
9. Parallelogram *ABCD* has *H* and *K* as midpoints of *BC* and *CD* respectively. Prove that $3(\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD})=4(\overrightarrow{AH}-\overrightarrow{HK})$.
10. Pamela, whose weight is approximately 500 N, would like to sunbathe on a roof inclined at 15° to the horizontal. If there is a frictional force of 75 N exerted on Pamela by the roof, what force must be applied at an angle of 30° to keep her from slipping down the roof?

    Now imagine that the roof is made of a transparent ("see-through") material that does not cause light rays passing through it to be "bent" (i.e. change direction). If the sun is directly overhead and Pamela is 1.7 m tall, what is the length of Pamela's shadow cast on the floor beneath the roof? Use vector projections to calculate the shadow length.
11. A canoeist wishes to cross a river 1 km in width. If the current flows at 10 km/h and the canoeist can paddle 12 km/h in still water, determine the direction in which she must steer to travel straight across the river. How long will it take her to cross the river?
12. Use the dot product to prove that the altitude of an isosceles triangle bisects the base.

13. The 2003 Lamborghini Murcielago has a ***power*** rating of 426 kW at 7500 rpm and a ***torque*** rating of 650 Nm at 5400 rpm. Explain how you can use these two specifications to evaluate the performance of the Murcielago. (Note that the Newton-metre is the same as the Joule.)

14. Explain the difference between a "definition" and an "undefined term." Give an example of each in Euclidean geometry.

15. Explain the terms "axiom," "postulate" and "theorem." What do all of these have in common? Give examples of each in Euclidean geometry.

16. Astronomers use simple geometry and trigonometry and a phenomenon known as ***parallax*** to calculate distances to nearby stars. Explain how this method works and show an example of how it is used. In addition, explain why this method will only work for stars that are relatively close to the Earth. (***This question involves research!***)

17. Explain what is meant by ***inductive reasoning***, ***deductive reasoning*** and ***indirect reasoning***. Give examples of each.

18. The following table summarizes the most common external forces that act on bridges.

Vertical Forces	*Longitudinal Forces*	*Transverse Forces*	*Erratic or Intermittent Forces*
□ Reaction to bridge weight □ Weight of traffic □ Lift or down-force caused by wind	□ Reaction to thrust of arch □ Reaction to tension of a suspension bridge □ Force generated by traffic, especially accelerating traffic	□ Wind pressure □ Water pressure (river flow, waves, tides, etc) □ Traffic forces (if road is curved)	□ Air turbulence □ Vibration from live loads □ Traffic variations

Using cardboard, wood or any other materials at your disposal, construct a three-dimensional model of the vertical, longitudinal and transverse forces acting on a bridge. In a good model, the sum of the force vectors should be $\vec{0}$ and their relative magnitudes should be as realistic as possible. (***This question involves a little research!***)

19. The following fallacious argument is used to "demonstrate" that every triangle is isosceles. Find the flaw(s).

Given ΔABC, construct the bisector of angle A and the perpendicular bisector of side BC opposite angle A. Now consider the various cases.

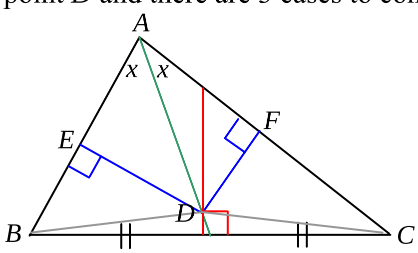
Case 1: The bisector of $\angle A$ and the perpendicular bisector of segment BC are either parallel or identical. In either case, the bisector of $\angle A$ is perpendicular to BC and hence, by definition, is an altitude. Therefore the triangle is isosceles. (The conclusion follows from the theorem in question 12.)

Suppose now that the bisector of $\angle A$ and the perpendicular bisector of the side opposite $\angle A$ are not parallel and do not coincide. Then they intersect in exactly one point D and there are 3 cases to consider.

Case 2: The point D is inside the triangle

Case 3: The point D is on the triangle

Case 4: The point D is outside the triangle



Note that the diagram to the left is for case 2. To ensure that you understand the given argument, you should construct diagrams for the other cases as well.

For each case, construct DE perpendicular to AB and DF perpendicular to AC, and for cases 2 and 4 join D to B and D to C. In each case, the following proof now holds:

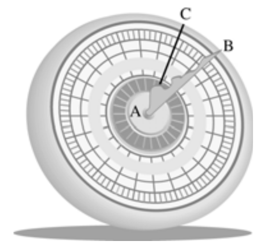
In ΔADE and ΔADF	In ΔDEB and ΔDFC
$DE = DF$ (ABT)	$DB = DC$ (RBT)
$AD = AD$ (common)	$DE = DF$ (ABT)
$\angle DEA = \angle DFA = 90°$	$\angle DEB = \angle DFC = 90°$
$\therefore \Delta ADE \cong \Delta ADF$ (HS)	$\therefore \Delta DEB \cong \Delta DFC$ (HS)
$\therefore AE = AF$	$\therefore EB = FC$

It follows that $AB = AC$ (in cases 2 and 3 by addition, and in case 4 by subtraction). Therefore, ΔABC is isosceles. //

20. What is the height of a regular pentagon with side length of a? Your answer ***may not*** be expressed in terms of trigonometric functions.

21. A city is surrounded by a circular wall with two gates, one on the North end and one on the South end. There is a yellow house 3 km north of the north gate and a blue house 9 km east of the south gate. If the yellow house is barely visible from the blue house (and vice versa), what is the radius of the circle formed by the wall?

22. The sundial shown at the right works on the principle that the sun casts a shadow from a central projecting pointer onto the surface of a calibrated circle. The edge of the shadow is used to indicate the time of day.



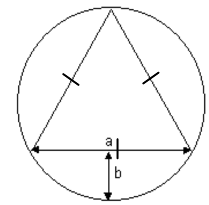
A coordinate system can be placed on this object so that A is at the origin and the coordinates of B and K are (0, 8, 3) and (−4, 4, 0) respectively.

(a) Determine the magnitude of the vector projection of the edge of the pointer AB onto the calibration line AC.

(b) Determine the size of $\angle BAC$.

(c) How is the accuracy of a sundial affected by its location (i.e. how far it is from the equator)?

23. In the diagram at the right, an equilateral triangle is inscribed in a circle. Given that the radius of the circle is R, what are the lengths of a and b?



24. ***Photonics*** is an area of study that involves the use of ***radiant energy*** (such as light), whose fundamental element is the photon. Photonic applications use the photon in the same way that electronic applications use the electron. Devices that run on light, however, have a number of advantages over those that use electricity. Light travels at about 10 times the speed of electricity, which means that data transmitted photonically can travel long distances in a fraction of the time. Furthermore, visible-light and infrared (IR) beams, unlike electric currents, pass through each other without interacting, eliminating the problem of interference. For instance, a single optical fibre has the capacity to carry three million telephone calls simultaneously. Photonic devices are also used to make biological measurements for medical or scientific purposes. This is done by measuring the ***decrease in intensity*** (called attenuation) of a laser beam as it passes through biological tissue. This question deals with a photonics problem from biomedical engineering ***whose solution involves a system of three linear equations in three unknowns***.

The data given in the table below were collected in a biomedical procedure involving three different wavelengths of laser light in the ***near infrared*** (NIR) region of the electromagnetic spectrum. NIR laser beams of wavelengths 758 nm, 798 nm and 898 nm were each propagated through a 2 cm sample of biological tissue. Using the data, some photonic theory and a few reasonable assumptions, we obtain the system of equations below. The system can be used to ***calculate the concentrations of oxygenated and deoxygenated haemoglobin*** (denoted C_{ox} and C_{dox} respectively).

$$0.444C_{ox} + 1.533C_{dox} + \mu_{other} = 0.293 \qquad (1)$$

$$0.702C_{ox} + 0.702C_{dox} + \mu_{other} = 0.1704 \qquad (2)$$

$$1.123C_{ox} + 0.721C_{dox} + \mu_{other} = 0.1903 \qquad (3)$$

(To simplify this problem, it is assumed that for the background chemicals, the decrease in intensity of the laser beam is due entirely to absorption. In addition, it is assumed that the molar extinction coefficient for the background chemicals is the same at all three wavelengths. This allows us to deal with only three unknowns instead of six.)

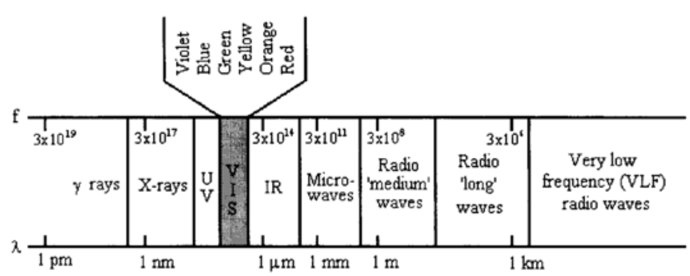
Absorption Coefficient (μ)

A measure of the decrease in intensity of a wave caused by ***absorption of energy*** (the conversion of energy to a different form such as thermal energy) that results from its passage through a medium.

Molar Extinction Coefficient (ε)

Absorbance of light per unit path length (usually the centimetre) and per unit of concentration (usually moles per litre). In other words, the molar extinction coefficient is the absorbance of a 1 mole/L solution with a 1 cm light path.

Chemical	**Molar Extinction Coefficient (ε) at given Wavelength of NIR Light** (mmoles L^{-1} cm^{-1})		
	758nm	**798nm**	**898nm**
Oxyhaemoglobin	0.444	0.702	1.123
Deoxyhaemoglobin	1.533	0.702	0.721
Other (background)	?	?	?
Total Absorption Coefficient (μ)	0.293 cm^{-1}	0.1704 cm^{-1}	0.1903 cm^{-1}



Note: The NIR region (ranging from about 0.7 μm to 3 μm) consists of the shortest wavelengths of the infrared (IR) region (ranging from about 0.7 μm to 100 μm). That is, the NIR region is the portion of the IR region closest to that of visible light (ranging from about 0.4 μm to 0.7 μm).

(a) Since we are interested in calculating only the values of C_{ox} and C_{dox}, describe a strategy for solving the above system that would allow you to find answers as quickly as possible.

(b) Apply your strategy from (a) and the method of elimination to solve the system.

(c) Now solve the system using Gaussian elimination (augmented matrix method).

(d) Use the augmented matrix and Gauss-Jordan elimination to solve the system. Use a graphing calculator to verify your answers.

(e) Interpret the system of linear equations given above as a series of three planes in $\mathbb{R}^3$. Determine a vector equation of the line of intersection of the planes described by equations (1) and (2).

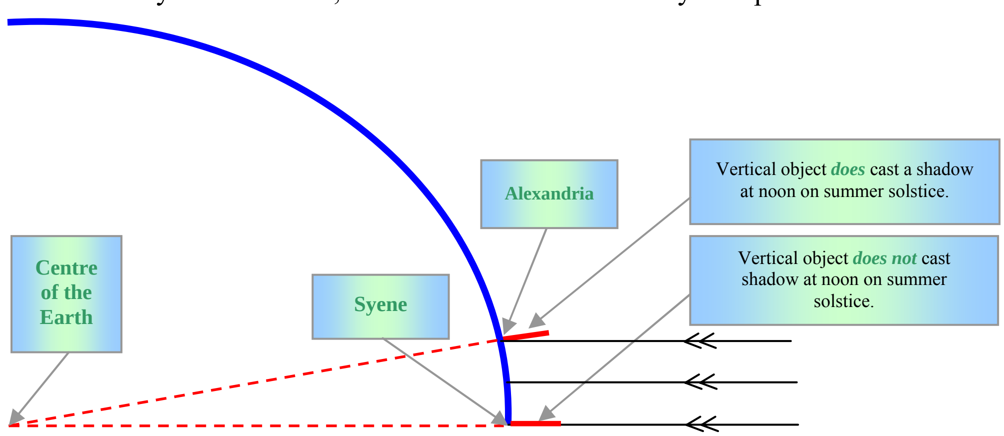
(f) Determine a vector equation of the plane described by equation (3).

(g) The first successful ***laser*** (light amplification by the stimulated emission of radiation) was demonstrated by its co-inventor Ted Maiman in 1960. Although initially the laser was believed to be of little practical value, there are now a multitude of different applications. List as many of these applications as you can.

25. It is a popular misconception that until the time of Christopher Columbus, people believed that the Earth was a flat plate. In reality, the spherical shape of the Earth was known to the ancient Greeks and quite likely, even to earlier civilizations. In the fourth century B.C., Aristotle put forth two strong arguments for supporting the theory that the Earth was a sphere. First, he observed that during lunar eclipses, the Earth's shadow on the moon was always circular. Second, he remarked that the North Star appeared at different angles of elevation in the sky depending on whether the observer viewed the star from northerly or southerly locations. These two observations, being entirely inconsistent with the "flat Earth" theory, led Aristotle to conclude that the Earth's surface must be curved. Even sailors in ancient Greece, despite being less scholarly than Aristotle, also realized that the Earth's surface was curved. They noticed that the sails of a ship were always visible before the hull as the ship emerged over the horizon and that the sails appeared to "dip" into the ocean as the ship would retreat beyond the horizon.

A Greek named Eratosthenes (born: 276 BC in Cyrene, North Africa, which is now Shahhat, Libya, died: 194 BC, Alexandria, Egypt) took these observations one step further. He was the chief librarian in the great library of Alexandria in Egypt and a leading all-round scholar. At his disposal was the latest scientific knowledge of his day. One day, he read that a deep vertical well near Syene, in southern Egypt, was entirely lit up by the sun at noon once a year (on the summer solstice). This seemingly mundane fact might not have captured the attention of someone of ordinary intellectual abilities, but it instantly piqued Eratosthenes' curiosity. He reasoned that at this time, the sun must be directly overhead, with its rays shining directly into the well. Upon further investigation, he learned that in Alexandria, almost due north of Syene, the sun was not directly overhead at noon on the summer solstice because a vertical object would cast a shadow. He deduced, therefore, that the Earth's surface must be curved or the sun would be directly overhead in both places at the same time of day. By adding two simple assumptions to his deductions, Eratosthenes was able to calculate the Earth's circumference to a high degree of accuracy! First, he knew that the Earth's surface was curved so he assumed that it was a sphere. This assumption was strongly supported by Aristotle's observations in the fourth century B.C. Second, he assumed that the sun's rays are parallel to each other. This was also a very reasonable assumption because he knew that since the sun was so distant from the Earth, its rays would be virtually parallel as they approached the Earth.

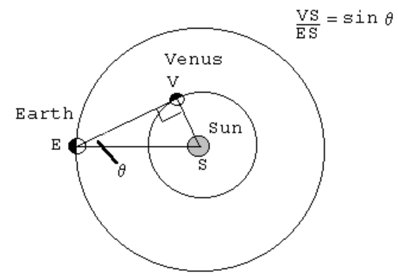
Now it's your turn to reproduce Eratosthenes' calculation. The diagram at the right summarizes all the required information. Although the calculation is quite simple, you must justify your steps by quoting the theorem(s) of geometry that you use in you argument.



Eratosthenes hired a member of a camel-powered trade caravan to "pace out" the distance between Syene and Alexandria. This distance was found to be about 5000 "stadia." The length of one "stadion" varied from ancient city to ancient city so there is some debate concerning how to convert Eratosthenes' measurement in stadia to a modern value in kilometres. However, it is usually assumed that Eratosthenes' stadion measured 184.98 m. Eratosthenes measured the "shadow angle" at Alexandria and found that it was approximately 7.2°.

26. Have you ever wondered about how astronomers calculate distances from the Earth to celestial bodies? In this problem, you will learn about how the distance from the Earth to the Sun was first calculated in 1882.

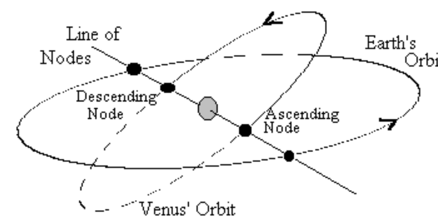
(a) Using the ***angle of separation*** (as measured from the Earth) between a body in our solar system and the Sun, its distance from the sun can be determined in terms of the distance from the Earth to the Sun. To facilitate this process, the "AU" (astronomical unit) was created, the distance from the Earth to the sun being defined as 1 AU. As a result of astronomical measurements made prior to 1882, astronomers had calculated, in terms of the AU, the distances from the sun of all the planets known at the time. However, it was not possible to convert these distances into kilometres because the distance from the Earth to the Sun was not known.

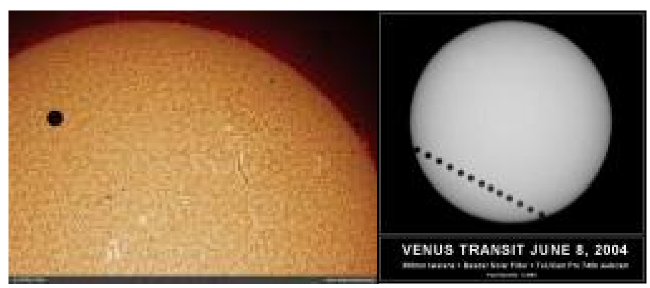


The diagram at the right shows how astronomers calculated the distance from Venus to the sun in terms of the AU. When Venus and the sun were conveniently positioned in the sky to make the angle of separation as large as possible, the angle θ was measured to be approximately 46.054°.

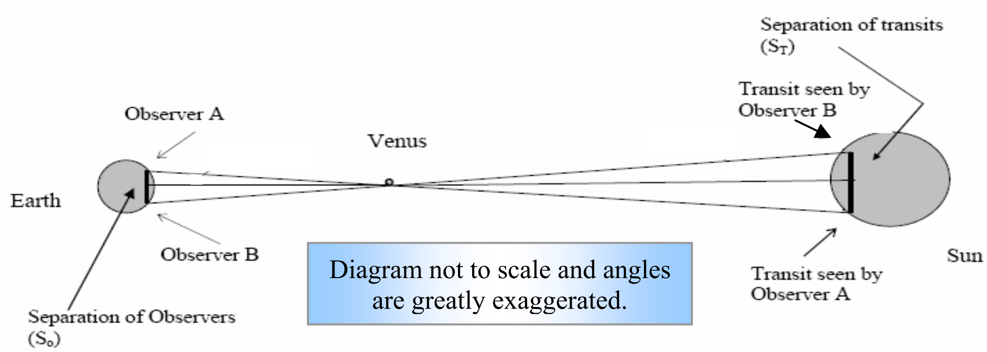
Use this information to calculate the distance from Venus to the Sun in terms of the astronomical unit. In addition, state the theorem that allows us to conclude that ΔSVE must be a right triangle.

(b) As shown in the diagram at the right, the plane of Venus' orbit is inclined to that of the Earth. (The actual orbital inclination of Venus is only 3.39° but it is highly exaggerated in the illustration for the sake of clarity.) Notice that Venus passes through the Earth's orbital plane twice per orbit. When Venus intersects both the orbital plane of the Earth and the line connecting the Earth to the Sun, ***a transit of Venus*** occurs. From the Earth, Venus is observed as a small black disk that slowly makes its way across the face of the Sun. Transits of Venus are exceedingly rare, usually occurring in pairs spaced apart by 121.5±8 years (4 transits per 243-year cycle).





In 1882, a transit of Venus occurred, an opportunity seized by astronomers to calculate, once and for all, the Earth-Sun distance. Observers on Earth separated by a North-South distance S_O (separation of observers) would observe transits separated by a North-South distance S_T (separation of transits). Observer *A* in the northern hemisphere would see Venus lower on the face of the Sun than observer *B* in the southern hemisphere. This is due to an effect called ***parallax***, the same phenomenon that causes an apparent shift in the position of objects when viewed successively with one eye and then with the other.



Use the data in the following table and your answer from 10(a) to express 1 AU in kilometres. Do not forget to state any geometry theorems that you use.

S_O	Measurement of S_T
2000 km	For obvious reasons, it is not possible to measure S_T directly. Instead, astronomers plotted the transit paths seen by observers *A* and *B* on a circle with a diameter of 16 cm. The separation on the diagram was found to be 0.059198 cm. This value was used to calculate the actual value of S_T.