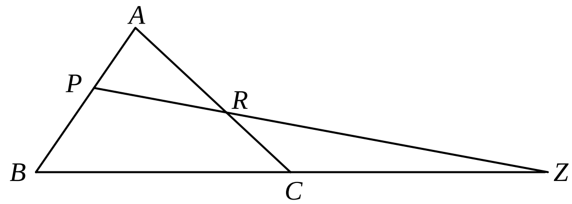
Strategy #1 for Solving Tough Problems: If your Diagram has Insufficient Detail, Consider Constructions

Problem (p. 58 #12)



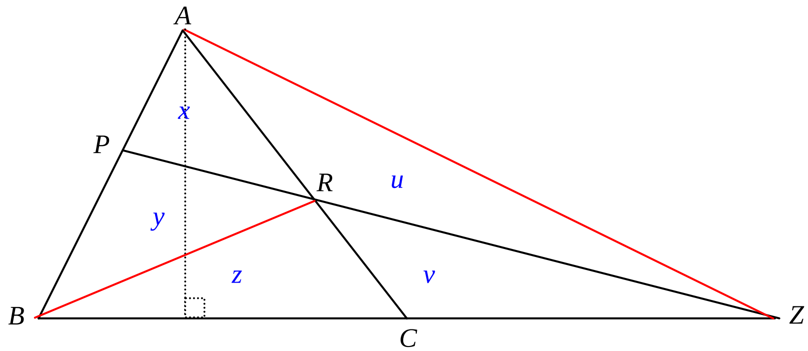
If $\dfrac{AP}{PB}=\dfrac{3}{4}$ and $\dfrac{AR}{RC}=\dfrac{3}{2}$ prove that C is the midpoint of BZ.

Proof

Construct AZ and BR as shown below (red line segments). Let x represent the area of ΔRAP, y represent the area of ΔRBP, z represent the area of ΔRBC, u represent the area of ΔRAZ and let v represent the area of ΔRCZ.

Since ΔABC and ΔACZ have the same height, $\dfrac{\Delta ABC}{\Delta AZC}=\dfrac{x+y+z}{u+v}=\dfrac{BC}{CZ}$ (TAP). (*)

As a consequence of TAP and the given ratios,



$$\frac{x}{y}=\frac{3}{4},$$
$$\frac{x+y}{z}=\frac{3}{2},$$
$$\frac{x+u}{y+z+v}=\frac{3}{4} \text{ and}$$
$$\frac{u}{v}=\frac{3}{2}.$$

Rearranging each of the above equations we obtain,

$$y=\frac{4}{3}x \ \ (1)$$

$$x+y-\frac{3}{2}z=0 \ \ (2)$$

$$4x-3y-3z+4u-3v=0 \ \ (3)$$

$$v=\frac{2}{3}u \ \ (4)$$

By substituting (1) into (2) we obtain,

$$x+\frac{4}{3}x-\frac{3}{2}z=0 \text{ or}$$

$$z=\frac{14}{9}x \ \ (5)$$

By substituting (1), (4) and (5) into equation (3) we obtain

$$4x-3\left(\frac{4}{3}x\right)-3\left(\frac{14}{9}x\right)+4u-3\left(\frac{2}{3}u\right)=0$$

or

$$u=\frac{7}{3}x \ \ (6)$$

By substituting (4) we also obtain

$$v=\frac{14}{9}x \ \ (7)$$

Finally, we can substitute (1), (4), (5), (6) and (7) into (*) above to obtain

$$\frac{\Delta ABC}{\Delta AZC}=\frac{x+y+z}{u+v}$$
$$=\frac{x+\frac{4}{3}x+\frac{14}{9}x}{\frac{7}{3}x+\frac{14}{9}x}$$
$$=1$$

Therefore,

$$\frac{\Delta ABC}{\Delta AZC}=\frac{BC}{CZ}=1$$

But if $\dfrac{BC}{CZ}=1$,

then $BC=CZ$.

Therefore, C must be the midpoint of BZ.

(Whew! That was a tough one guys! Don't expect a similar question on a test because it would take too long to solve such a problem. However, this problem does serve as a great illustration of a very useful problem solving strategy in geometry. If your diagram lacks detail, consider constructions.)