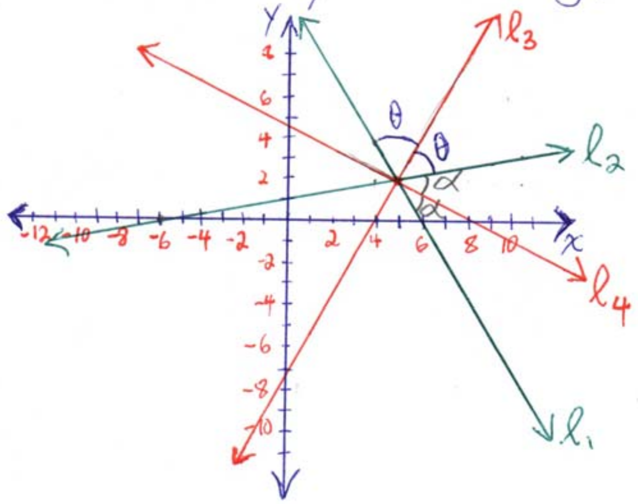
Page 247 #19 – Solution

Find the vector equations of the two lines that bisect the angles between the lines $\vec{r_1} = (5,2) + t(-3,6)$ and $\vec{r_2} = (5,2) + u(11,2)$.

Solution: As usual, you should begin with a PICTURE. This will GUIDE your thinking!



ℓ_1: $\vec{r_1} = (5,2) + t(-3,6)$

ℓ_2: $\vec{r_2} = (5,2) + u(11,2)$

Let ℓ_3 and ℓ_4 represent the required lines. Clearly, the equations of ℓ_3 and ℓ_4 are of the form

ℓ_3: $\vec{r_3} = (5,2) + r(d_1, d_2)$

ℓ_4: $\vec{r_4} = (5,2) + s(e_1, e_2)$,

where (d_1, d_2) and (e_1, e_2) represent direction vectors for ℓ_3 and ℓ_4 respectively.

From the diagram, we see that

$$2\theta + 2\alpha = 180^\circ \text{ (supplementary angles).}$$

$\therefore \quad \theta + \alpha = 90^\circ$ (this answers 19(c)) ***

$\therefore$ the angle between ℓ_3 and ℓ_4 is 90°

$\therefore \quad (d_1, d_2) \perp (e_1, e_2)$

$\therefore \quad (d_1, d_2) \cdot (e_1, e_2) = 0$

$\therefore \quad d_1 e_1 + d_2 e_2 = 0 \quad (1)$

(continued on next page)

Now recall that $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$.

$$\therefore \quad \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

By applying this successively to the pairs of lines l_1 and l_3 and l_3 and l_2, we obtain

$$\cos\theta = \frac{(-3,6)\cdot(d_1,d_2)}{|(-3,6)|\,|(d_1,d_2)|} \quad \text{and} \quad \cos\theta = \frac{(d_1,d_2)\cdot(11,2)}{|(d_1,d_2)|\,|(11,2)|}$$

$$\therefore \quad \frac{(-3,6)\cdot(d_1,d_2)}{|(-3,6)|\,\cancel{|(d_1,d_2)|}} = \frac{(d_1,d_2)\cdot(11,2)}{\cancel{|(d_1,d_2)|}\,|(11,2)|}$$

$$\therefore \quad \frac{-3d_1+6d_2}{\sqrt{45}} = \frac{11d_1+2d_2}{\sqrt{125}}$$

$$\therefore \quad \frac{-3d_1+6d_2}{3\sqrt{5}} = \frac{11d_1+2d_2}{5\sqrt{5}}$$

$$\therefore \quad \frac{15\sqrt{5}}{1}\left(\frac{-3d_1+6d_2}{3\sqrt{5}}\right) = \frac{15\sqrt{5}}{1}\left(\frac{11d_1+2d_2}{5\sqrt{5}}\right)$$

$$\therefore \quad -15d_1+30d_2 = 33d_1+6d_2$$

$$\therefore \quad -5d_1+10d_2 = 11d_1+2d_2 \quad (2)$$

Since there are infinitely many direction vectors that can be chosen for any given line, we can choose d_1 freely and solve for d_2. First, it's helpful to simplify equation (2):

$$10d_2 - 2d_2 = 11d_1 + 5d_1$$

$$\therefore \; 8d_2 = 16d_1$$

$$\therefore \; d_2 = 2d_1$$

By choosing $d_1 = 1$, we obtain $d_2 = 2$.

Therefore, $(1,2)$ is a direction vector for l_3. (continued)

Now we can apply equation (1) to find (e_1, e_2):

$$d_1 e_1 + d_2 e_2 = 0 \quad (1)$$

$$\therefore \quad e_1 + 2e_2 = 0$$

$$\therefore \quad e_1 = -2e_2$$

If we choose $e_2 = 1$, we obtain $e_1 = -2$.

Therefore, $(-2, 1)$ is a direction vector for l_4.

Summarizing, we find that

$$\vec{r}_3 = (5,2) + r(1,2)$$

and $$\vec{r}_4 = (5,2) + s(-2,1)$$

are vector equations of l_3 and l_4 respectively. //