MGA 4U0 **Semester 2, 2006 - 2007**

Grade 12 Geometry and Discrete Mathematics

Major Test – First Half of Unit 1 (An Introduction to Proof through Euclidean Geometry)

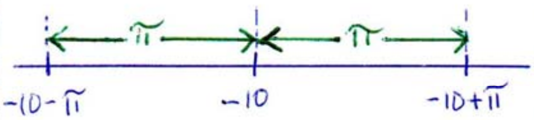
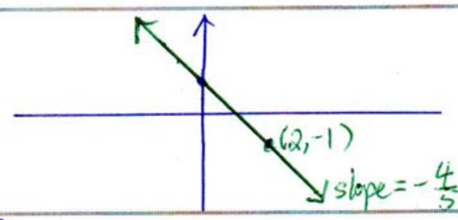
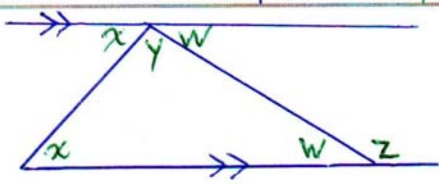
Mr. N. Nolfi

Victim: Mr. Solutions

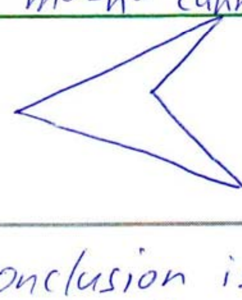
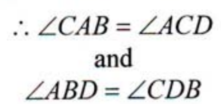
As usual, your work is stupendous Mr. S.!

KU	APP	TIPS	COM
15 /15	14/14	14/14	18/18

1. Complete the following table. **(6 KU, 3 COM)**

Expression, Equation or Inequation	*Diagram*	*Conclusion, Interpretation or Explanation*
$\lvert x+10\rvert=\pi$		The distance from x to -10 is π units.
$\frac{y-2}{x+1}=-\frac{4}{5}$		This represents a straight line passing through $(2,-1)$ with slope $-\frac{4}{5}$
$z=x+y$		An exterior angle of a triangle is equal to the sum of the two interior and opposite angles.

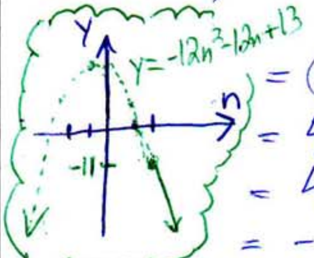
2. State whether each of the following is true or false. To receive full credit, you must prove the statements that are true and provide a counterexample for the statements that are false. **(6 TIPS, 3 COM)**

Statement	*True or False?*	*Proof, Counterexample or Explanation*
premise If m and n are positive integers and $m>n$, then m^2-n^2 cannot be prime unless $m=n+1$. which means that m^2-n^2 can be prime only if $m=n+1$	T	m^2-n^2 $=(m-n)(m+n)$ If $m>n+1$, then $m-n\geq 2$ and $m+n\geq 2n+1$. In this case, m^2-n^2 has a factorization that involves numbers other than one and itself. Therefore, m^2-n^2 cannot be prime.
All quadrilaterals must be convex. opposite of "cannot be" is "can be," not "must be"	F	 This quadrilateral is obviously not convex
 $\therefore \angle CAB=\angle ACD$ and $\angle ABD=\angle CDB$	F	The conclusion is only true if $AB \parallel DC$. Since we are not given this information, we cannot draw the given conclusion

3. Suppose that a quadratic polynomial of the form $ax^2 + bx + c$ has coefficients that are ***consecutive odd positive integers***. What can you conclude about the roots of such a polynomial? Prove your claim. (**Hint:** Before jumping to any conclusions, use a bit of inductive reasoning and try out examples such as $3x^2 + 5x + 7$.)
(6 APP, 3 COM)

If a, b and c are consecutive odd positive integers, then $a = 2n-1$, $b = 2n+1$ and $c = 2n+3$, where n is some positive integer.

Then, the discriminant of the given quadratic polynomial is



$$\begin{aligned} & b^2 - 4ac \\ &= (2n+1)^2 - 4(2n-1)(2n+3) \\ &= 4n^2 + 4n + 1 - 4(4n^2 + 4n - 3) \\ &= 4n^2 + 4n + 1 - 16n^2 - 16n + 12 \\ &= -12n^2 - 12n + 13 \end{aligned}$$

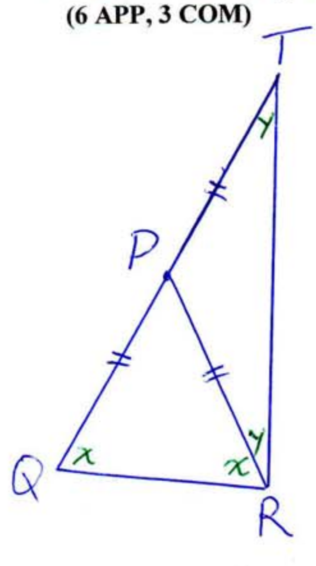
Since n is a positive integer, $n \geq 1$.

Therefore, $-12n^2 - 12n + 13$

$$\begin{aligned} &= -12(n^2+n) + 13 \\ &\leq -12(1^2+1) + 13 \quad \text{if } n \geq 1 \\ &= -11 \end{aligned}$$

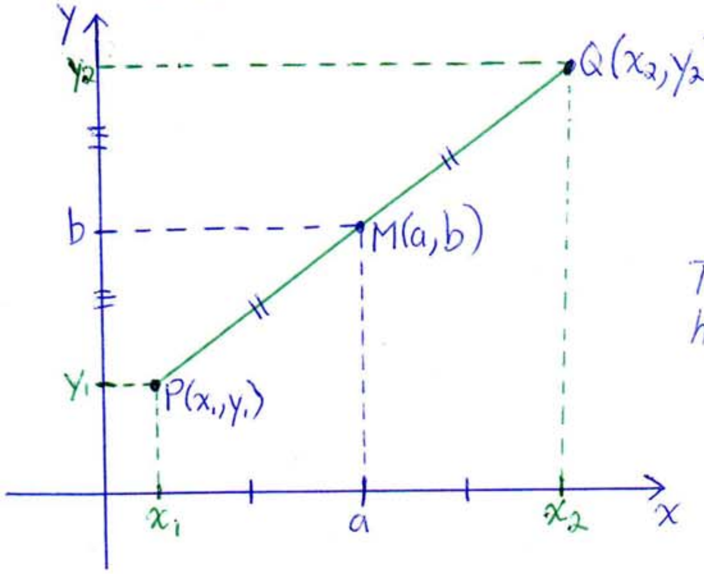
This shows that $b^2 - 4ac \leq -11$, which means that the given polynomial has no real roots if the coefficients a, b and c are consecutive odd positive integers. //

4. In the isosceles triangle ΔPQR, $PQ = PR$. If QP is extended to T so that $PT = PR$, prove that ΔQRT is a right triangle.
(6 APP, 3 COM)



In ΔPQR,
$PQ = PR$ (given)
$\therefore \angle PQR = \angle PRQ = x$ (ITT)
In ΔPRT,
$PR = PT$ (given)
$\therefore \angle PRT = \angle PTR = y$ (ITT)
In ΔQRT,
$\angle Q + \angle QRT + \angle T = 180°$ (ASTT)
$\therefore x + (x+y) + y = 180°$
$\therefore 2x + 2y = 180°$
$\therefore 2(x+y) = 180°$
$\therefore x + y = 90°$
$\therefore \angle QRT = 90°$
$\therefore \Delta QRT$ is a right triangle //

5. Prove the midpoint formula (that is, the formula for finding the midpoint of a line segment). **(5 KU, 3 COM)**



As shown in the diagram, let M represent the midpoint of line segment PQ.

The x-co-ordinate of M lies exactly half way between x_1 and x_2. Therefore,

$$a = \frac{x_1 + x_2}{2}$$

Similarly,

$$b = \frac{y_1 + y_2}{2}$$

Therefore, the co-ordinates of M must be

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right). //$$

6. Consider the following argument.

- Mr. Nolfi heard many rumours about a number of his MGA4U0 students. According to Mr. Nolfi's sources, these students had committed plagiarism on countless occasions.
- Mr. Nolfi is always wary of rumours but to be on the safe side, he decided to observe these students carefully.
- While making these observations, Mr. Nolfi noticed numerous highly suspicious patterns.
- Some of these students always sat next to each other during tests. These students claimed that this was just a sign of deep "affection," a demonstration of their great "love" for one another.
- Others (e.g. Farrukh, famous for arguing incessantly despite being completely wrong) were frequently very "ill" on test days. Allegedly, due to their highly efficient immune systems, they would always make miraculous recoveries and return to school the very next day, fit as a fiddle. After having asked their classmates about the questions on the test, they would produce notes from doctors ($10 well spent) and insist on writing the same test that was written by the rest of the class.
- Many of these students also behaved very strangely during tests. They would cough periodically, hold up certain fingers from time to time, make unusual tapping sounds, mumble in foreign languages and even scratch their private parts.
- To Mr. Nolfi's amazement, certain students even consistently defied the odds by submitting answers that were identical in every respect (including mistakes) to the answers of other students. Despite this, the students insisted that their work was highly "original" and that any similarities to the work of other students were purely "coincidental."
- After analyzing this information carefully, Mr. Nolfi concluded that the rumours must be true. The students were definitely guilty of cheating!

(a) Is this an inductive or a deductive argument? Explain. **(2 APP)**

This is an inductive argument. The conclusion is supported by many lines of evidence but it is not the only possible conclusion.

(b) Is Mr. Nolfi's conclusion valid? Explain. **(2 TIPS)**

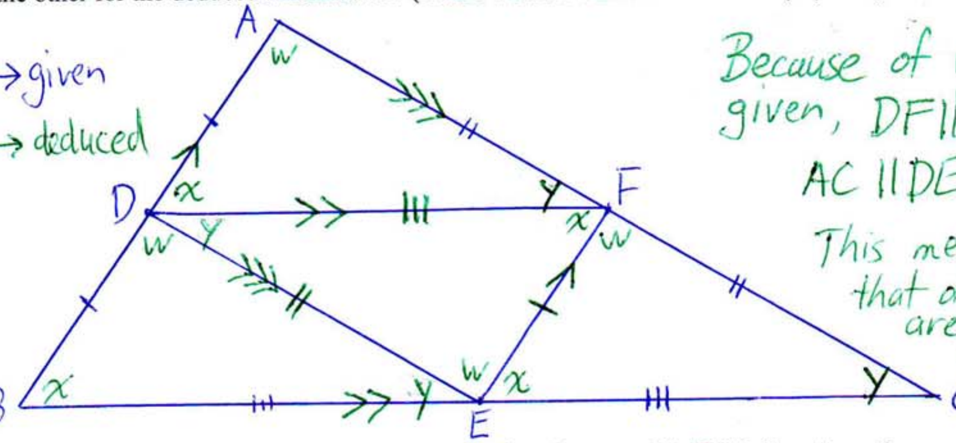
The evidence supporting Mr. Nolfi's conclusion is very strong. Nonetheless, there is a minute possibility that he observed a series of coincidences.

7. ΔDEF is formed by joining consecutive midpoints of the sides of ΔABC.

(a) Sketch ΔABC and ΔDEF. ***Label your sketch fully and use two different colours***, one for given information and the other for the deduced information. (It also wouldn't hurt to use a ruler.) **(4 KU)**

Blue → given

Green → deduced



Because of what is given, DF || BC, AB || FE, AC || DE. (STT or TPPT) This means that alternate angles are equal and corresponding angles are equal (PLT)

(b) Use ***theorems of Euclidean geometry*** to prove that the area of ΔABC is four times the area of ΔDEF. Do not use Cartesian geometry! (**Hint:** The word "similarly" may come in very handy in your proof.) **(6 TIPS, 3 COM)**

Given: $AD = DB$, $AF = FC$, $BE = EC$

$\therefore \frac{AD}{DB} = \frac{AF}{FC} = \frac{BE}{EC} = 1$

No marks deducted for lack of justification of this conclusion since TPPT is one of the new theorems.

Proof: Since $\frac{AD}{DB} = \frac{AF}{FC} = \frac{BE}{EC}$

$DF \parallel BC$, $FE \parallel AB$ and $DE \parallel AC$ (TPPT)

Since $DF \parallel BC$ (proven),

$\angle FEC = \angle DFE = x$ (PLT Z)

$\angle DFB = \angle FDE = y$ (PLT Z)

Since $FE \parallel AB$ (proven),

$\angle FEC = \angle DBE = x$ (PLT F)

In quadrilateral DFEB,

$DF \parallel BE$ and $DB \parallel FE$

$\therefore$ quad DFEB is a parallelogram

$\therefore$ $DB = FE$ and $DF = BE$

In ΔDBE and ΔEFD

$DB = EF$ (proven)

$\angle DBE = \angle EFD = x$ (proven)

$BE = FD$ (proven)

$\therefore$ $\Delta DBE \cong \Delta EFD$ (SAS)

Similarly,

$\Delta ADF \cong \Delta EFD$

$\Delta FEC \cong \Delta EFD$

$\therefore \Delta ABC$

$= \Delta ADE + \Delta DBE + \Delta FEC + \Delta EFD$

$= \Delta EFD + \Delta EFD + \Delta EFD + \Delta EFD$

$= 4\Delta EFD$

$= 4\Delta DEF$ //