MGA 4U0 **Semester 2, 2004 - 2005**

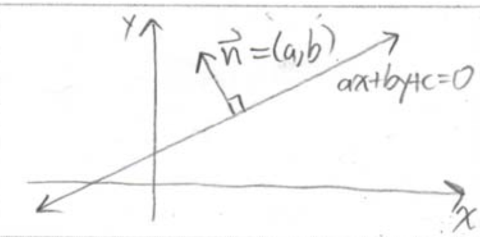
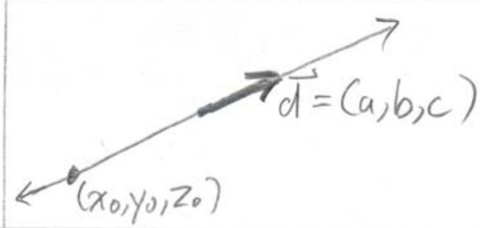
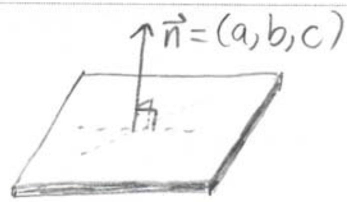
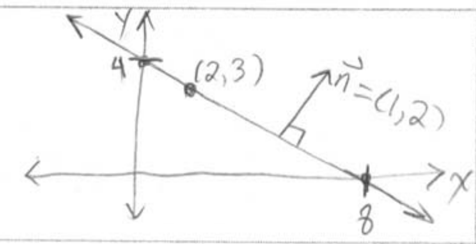
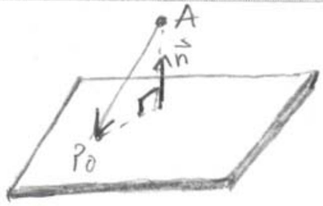
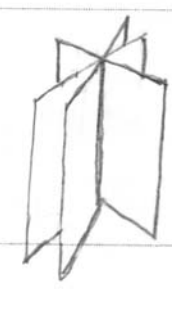
Grade 12 Geometry and Discrete Mathematics
Unit 3 Evaluation (Intersection of Lines and Planes)

Mr. N. Nolfi

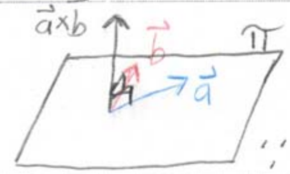
Victim: Mr. Solutions

KU	APP	TIPS	COM
/12	/20	/16	/25

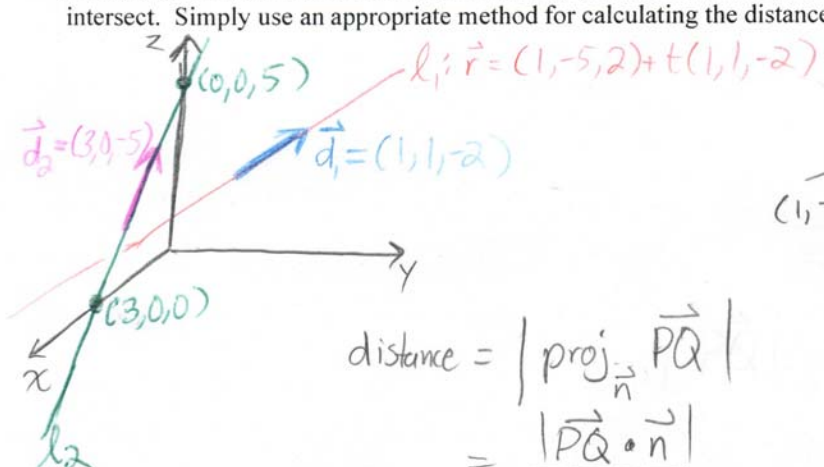
1. Complete the following table. **(12 KU, 6 COM)**

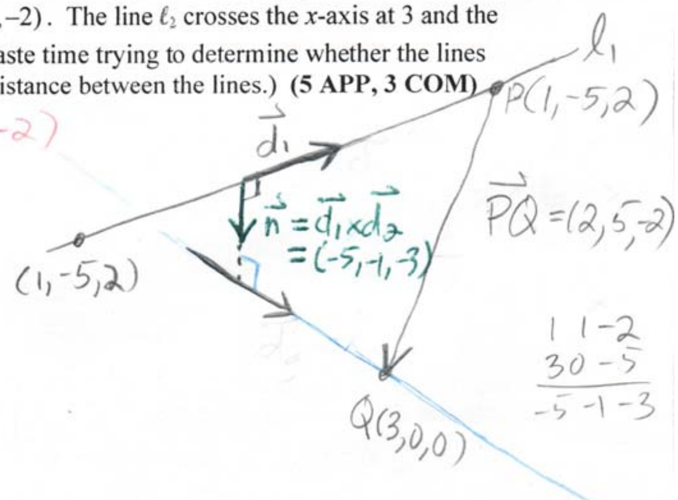
Expression, Equation or Inequation	*Diagram*	*Explanation*
$ax+by+c=0$ (in $\mathbb{R}^2$)		A line in $\mathbb{R}^2$ with normal vector $\vec{n}=(a,b)$
$\dfrac{x-x_0}{a}=\dfrac{y-y_0}{b}=\dfrac{z-z_0}{c}$		A line in $\mathbb{R}^3$ that passes through (x_0,y_0,z_0) and is parallel to $\vec{d}=(a,b,c)$
$ax+by+cz+d=0$ (in $\mathbb{R}^3$)		A plane in $\mathbb{R}^3$ with normal vector
$(1,2)\cdot(x-2,y-3)=0$ $x-2+2y-6=0$ $x+2y-8=0$		A line in $\mathbb{R}^2$ passing through (2, 3), with x-intercept 8 and y-intercept 4. Also $\vec{n}=(1,2)$ is normal to the line and its slope is $-\frac{1}{2}$
$\left\lvert \text{proj}_{\vec{n}}\,\overrightarrow{AP_0}\right\rvert$ (A is any point, P_0 is a point on a plane and $\vec{n}$ is a normal vector for the plane)		The distance from the point A to the plane with normal vector $\vec{n}$.
$a_1x+a_2y+a_3z+a_4=0$ (1) $b_1x+b_2y+b_3z+b_4=0$ (2) $c_1x+c_2y+c_3z+c_4=0$ (3)	 This is one type of intersection. The planes intersect in a line.	The intersection of three planes in $\mathbb{R}^3$. There are many intersections possible.

2. State whether each of the following is true or false. Provide a brief explanation for each answer. **(10 TIPS, 5 COM)**

Statement	*True or False? (T or F)*	*Proof, Counterexample or Explanation*
The method used to find the *distance between two skew lines* also works for finding the *distance between two parallel lines*.	F	Since the lines are parallel, their direction vectors, $\vec{d_1}$ and $\vec{d_2}$ are also parallel. Therefore, $\vec{d_1} \times \vec{d_2} = \vec{0}$, which is useless for projections.
A direction vector for the line $\frac{x-2}{1}=\frac{3-y}{-2}=\frac{z+6}{3}$ is (1, −2, 3).	F	The equation must first be rewritten as follows: $\frac{x-2}{1} = \frac{y-3}{2} = \frac{z+6}{3}$. Clearly, then, $(1,2,3)$ is a direction vector
The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent (i.e. non-coplanar) if and only if $(\vec{a}\times\vec{b})\cdot\vec{c}\neq 0$.	T	 $\vec{a}\times\vec{b} \perp \pi$ $\therefore \vec{a}\times\vec{b}\cdot\vec{u} = 0$ for any $\vec{u} \parallel \pi$ $\because \vec{a}\times\vec{b}\cdot\vec{c} \neq 0$, $\vec{c} \nparallel \pi$
The *points* $P(1,2,3)$, $Q(2,4,6)$ and $S(10,20,30)$ determine a *plane*.	F	$\vec{PQ} = (1,2,3)$, $\vec{QS} = (8,16,24) = 8\vec{PQ}$ $\because \vec{PQ} \parallel \vec{QS}$, P, Q and S are collinear $\therefore$ P, Q and S do not determine a plane
The linear system with augmented matrix $\left(\begin{array}{ccc\|c} 1 & -1 & 4 & 5 \\ 3 & 1 & 1 & -2 \\ 5 & -1 & 9 & 1 \end{array}\right)$ is *consistent* and *independent*.	F	$\vec{n_1} = (1,-1,4)$, $\vec{n_2} = (3,1,1)$, $\vec{n_3} = (5,-1,9)$ $\vec{n_1}\times\vec{n_2}\cdot\vec{n_3} = (1,-1,4)\times(3,1,1)\cdot(5,-1,9)$ $= (-5,11,4)\cdot(5,-1,9)$ $= 0$ $\therefore$ system either has no solutions or an infinite #

3. The line ℓ_1 is described by the vector equation $\vec{r} = (1,-5,2) + t(1,1,-2)$. The line ℓ_2 crosses the x-axis at 3 and the z-axis at 5. Find the distance between ℓ_1 and ℓ_2. (**Hint**: Do not waste time trying to determine whether the lines intersect. Simply use an appropriate method for calculating the distance between the lines.) **(5 APP, 3 COM)**



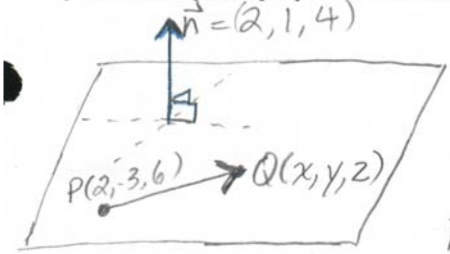


$$\begin{array}{ccc} 1 & 1 & -2 \\ 3 & 0 & -5 \\ \hline -5 & -1 & -3 \end{array}$$

$$\text{distance} = \left|\operatorname{proj}_{\vec{n}} \vec{PQ}\right|$$
$$= \frac{\left|\vec{PQ}\cdot\vec{n}\right|}{|\vec{n}|}$$
$$= \frac{\left|(2,5,-2)\cdot(-5,-1,-3)\right|}{\sqrt{(-5)^2+(-1)^2+(-3)^2}}$$
$$= \frac{9}{\sqrt{35}}$$

The distance between ℓ_1 and ℓ_2 is $\frac{9}{\sqrt{35}}$ units. //

4. From first principles (i.e. using the fact that the dot product of perpendicular vectors is zero), find a Cartesian (scalar) equation of a plane perpendicular to the vector (2, 1, 4) and passing through the point $P(2, -3, 6)$. **(5 APP, 3 COM)**



Let $Q(x,y,z)$ represent any point on the plane

Then $\overrightarrow{PQ} = (x-2, y+3, z-6)$.

Since $\overrightarrow{PQ}$ is parallel to the plane and $\vec{n}$ is perpendicular to the plane,

$$\vec{n} \perp \overrightarrow{PQ}$$

Therefore, $\vec{n} \cdot \overrightarrow{PQ} = 0$

$$\therefore (2,1,4) \cdot (x-2, y+3, z-6) = 0$$

$$\therefore 2x-4 + y+3 + 4z-24 = 0$$

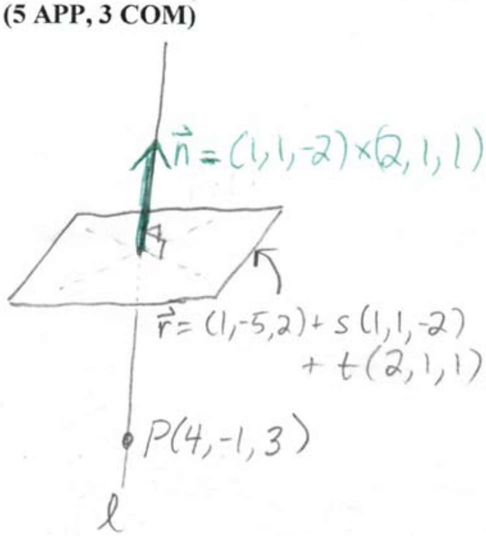
$$\therefore 2x+y+4z-25=0$$

The scalar equation $2x+y+4z-25=0$ describes the required plane.

5. "Find the direction angles (α, β and γ) of the line ℓ: $x = 2 - 0.5t$, $y = 3 - t$."
To increase your "COM" score by two marks, write a very short paragraph describing the punishment Mr. Nolfi should administer to Ali, Aaron and Ashraf for their constant talking during class discussions. To decrease your "COM" score by two marks, answer the question above enclosed in quotation marks. **(±2 COM)**

For their constant talking during class discussions, Ashraf, Ali and Aaron deserve having their final marks reduced by 50%. In Ashraf's case, this may cause his mark to fall below zero.

6. Find symmetric equations of a line through $P(4,-1,3)$ and perpendicular to $\vec{r} = (1,-5,2) + s(1,1,-2) + t(2,1,1)$.
(5 APP, 3 COM)



Since the vectors $(1,1,-2)$ and $(2,1,1)$ are parallel to the given plane, $\vec{n} = (1,1,-2) \times (2,1,1)$ is normal to the plane.

Now $\vec{n} = (3,-5,-1)$ is also parallel to the required line since the line is also perpendicular to the plane.

Since $P(4,-1,3)$ lies on the line and $\vec{n} = (3,-5,-1)$ is parallel to the line, then

$\dfrac{x-4}{3} = \dfrac{y+1}{-5} = \dfrac{z-3}{-1}$ is a set of symmetric equations for the line

$\vec{n}_1, \vec{n}_2, \vec{n}_3$ are normals of the planes ①, ②, ③ respectively

7. In the given system of equations, $k \in \mathbb{R}$. **(6 TIPS, 5 APP, 5 COM)**

$$-2x + y + z = k + 1 \quad ①$$
$$kx + z = 0 \quad ②$$
$$y + kz = 0 \quad ③$$

(a) Determine the value(s) of k for which this system has

(*This space is left blank intentionally so that you can perform a calculation that is needed for all three parts below.*)

$\vec{n}_1 = (-2, 1, 1)$, $\vec{n}_2 = (k, 0, 1)$, $\vec{n}_3 = (0, 1, k)$

$$\vec{n}_1 \times \vec{n}_2 \cdot \vec{n}_3 = (-2,1,1) \times (k,0,1) \cdot (0,1,k)$$
$$= (1, k+2, -k) \cdot (0, 1, k) = k + 2 - k^2$$

$$\begin{array}{ccc} -2 & 1 & 1 \\ k & 0 & 1 \\ \hline 1 & k+2 & -k \end{array}$$

(i). no solutions	(ii). exactly one solution	(iii). an infinite number of solutions.
This may occur if $\vec{n}_1 \times \vec{n}_2 \cdot \vec{n}_3 = 0$. This is true if $k = -2$ or $k = -1$ (see) If $k = -2$, equations ② and ③ yield $y = 2z$ and $z = 2x$. Subtracting ② from ① yields $y = -1$. $\therefore z = -\frac{1}{2}$ and $x = -\frac{1}{4}$ This does not satisfy ①. $\therefore$ no solution for $k = -2$	For exactly one solution, $\vec{n}_1 \times \vec{n}_2 \cdot \vec{n}_3 \neq 0$ $\therefore k + 2 - k^2 \neq 0$ $\therefore k^2 - k - 2 \neq 0$ $\therefore (k-2)(k+1) \neq 0$ $\therefore k \neq 2$ and $k \neq -1$ The given system will have exactly one solution for all values of k except for $k = -2$ and $k = -1$.	As with (i), this may occur if $\vec{n}_1 \times \vec{n}_2 \cdot \vec{n}_3 = 0$, which is true if $k = -2$ or $k = -1$ (see) As shown in (i), there is no solution if $k = -2$. If $k = -1$, then ② and ③ yield $x = y = z$. Since this solution also satisfies ①, there are infinitely many solutions

(b) For 7(a) part (iii), you should have obtained an answer of $k = -1$. Using this value of k, solve the resulting system by using an augmented matrix and Gauss-Jordan elimination. In addition, sketch a diagram that illustrates the geometric interpretation of the solution to the system.

If $k = -1$, the augmented matrix of the system is as follows:

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\begin{array}{r} \\ R_3 \\ R_2 \end{array} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{array}{r} \\ \\ R_1 - 2R_3 \end{array} \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\begin{array}{r} R_1 \div (-2) \\ \\ R_2 - R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

At this point, it is not possible to continue Gauss-Jordan elimination. Gauss-Jordan elimination requires that all entries on the main diagonal be 1, but all the entries in the 3rd row are zero!

The 3rd row corresponds to the equation $0z = 0$, which has an infinite number of solutions. Let $z = t$ represent any of these solutions

$\therefore y - t = 0$ (using 2nd row)

$\therefore y = t$

Also $x - \frac{1}{2}t - \frac{1}{2}t = 0$ (using 1st row)

$\therefore x = t$

$\therefore$ the planes intersect in the line

$$\vec{r} = t(1, 1, 1)$$

The line of intersection passes through $(0,0,0)$ and has direction vector $(1,1,1)$