MGA 4U0 **Semester 2, 2005 - 2006**

Grade 12 Geometry and Discrete Mathematics
"Quiz" – First Half of Unit 2 (Geometric and Algebraic Vectors and their Applications)

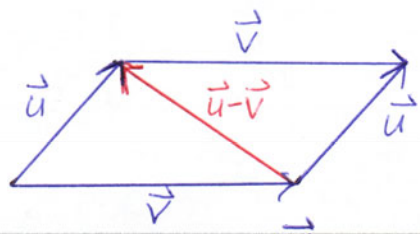
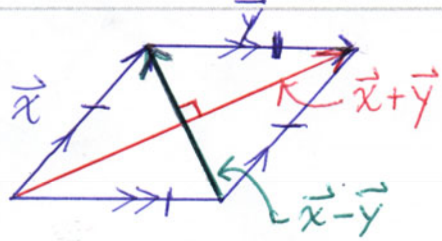
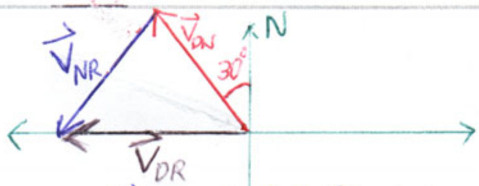
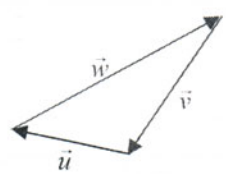
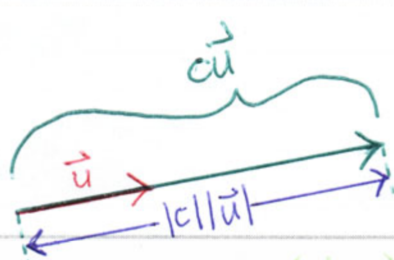
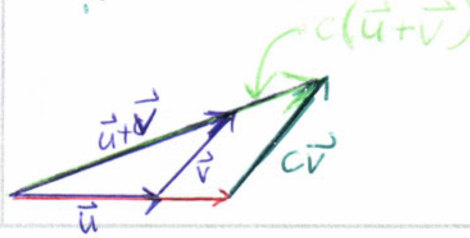
Mr. N. Nolfi

Victim: Mr. Solutions

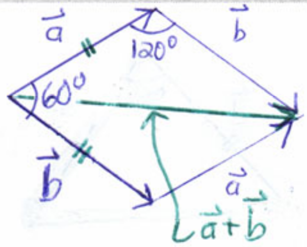
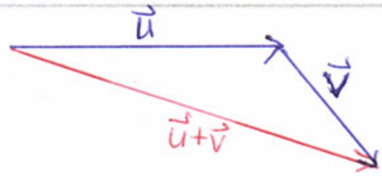
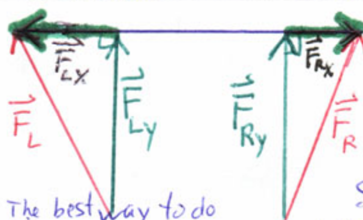
Great work once again Mr. Solutions!!

KU	APP	TIPS	COM
12 /12	14 /14	14 /14	18/18

1. Complete the following table. **(12 KU, 4 COM)**

Expression, Equation or Inequation	*Diagram*	*Conclusion, Interpretation or Explanation*
$\lvert\vec{u}-\vec{v}\rvert \le \lvert\vec{u}\rvert + \lvert\vec{v}\rvert$		The vectors $\vec u$, $\vec v$ and $\vec u - \vec v$ form a triangle. Therefore, the length of $\vec u - \vec v$ must be less than the sum of the lengths of $\vec u$ and $\vec v$. Equality can only occur if $\vec u$ and $\vec v$ have opposite direction
$\vec{x}+\vec{y} \perp \vec{x}-\vec{y}$		The diagonals of a rhombus are perpendicular to each other.
$\vec V_{DN} = \vec V_{DR} - \vec V_{NR}$ $\vec V_{DN}$ = velocity of David rel to Nolfi $\vec V_{DR}$ = vel. of David rel. to some frame of reference R	 $\vec V_{NR}$ = vel. of Nolfi rel. to frame of ref. R	The velocity of David H. relative to Mr. Nolfi is 12 m/s, N30°W. (Mr. Nolfi, wielding a splintered metre stick with very sharp, rusty nails embedded in it, is chasing David with the intent of inflicting serious bodily injury.)
$\vec u + \vec v + \vec w = \vec 0$		Give a *physical* interpretation here. Suppose that $\vec u$, $\vec v$ and $\vec w$ represent forces acting on an object. Therefore, the net force on the object is zero.
$\lvert c\vec{u}\rvert = \lvert c\rvert\lvert\vec{u}\rvert$		When a vector is stretched by a factor of c, its length is stretched by the absolute value of c.
$c\vec{u} + c\vec{v} = c(\vec{u}+\vec{v})$		When two vectors $\vec u$ and $\vec v$ are stretched by a scalar c, their sum is also stretched by the same scalar.

2. State whether each of the following is true or false. To receive full credit, you must prove the statements that are true and provide a counterexample for the statements that are false. **(8 TIPS, 4 COM)**

Statement	True or False?	Proof or Counterexample
If $\lvert\vec a\rvert = \lvert\vec b\rvert$ and the angle between $\vec a$ and $\vec b$ is 60°, then $\lvert\vec a+\vec b\rvert = \sqrt{3}\lvert\vec a\rvert$.	T	 $\lvert\vec a+\vec b\rvert^2 = \lvert\vec a\rvert^2+\lvert\vec b\rvert^2-2\lvert\vec a\rvert\lvert\vec b\rvert\cos 120^\circ$ $= \lvert\vec a\rvert^2+\lvert\vec a\rvert^2-2\lvert\vec a\rvert\lvert\vec a\rvert(-\frac{1}{2})$ $= 3\lvert\vec a\rvert^2$ $\therefore \lvert\vec a+\vec b\rvert = \sqrt{3}\lvert\vec a\rvert$
$\lvert\vec u+\vec v\rvert = \lvert\vec u\rvert+\lvert\vec v\rvert$	F	 Since the vectors $\vec u$, $\vec v$ and $\vec u+\vec v$ form a triangle, it follows that $\lvert\vec u+\vec v\rvert < \lvert\vec u\rvert+\lvert\vec v\rvert$ Equality holds if and only if $\vec u$ and $\vec v$ have the same direction.
Three forces, with magnitudes of 100 N, 300 N and 500 N respectively, act on an object. The object *does not* accelerate.	F	Let $\vec F_1$, $\vec F_2$ and $\vec F_3$ represent the three forces. Since the object does not accelerate, it follows that $\vec F_1+\vec F_2+\vec F_3=\vec 0$. This means that $\vec F_1$, $\vec F_2$ and $\vec F_3$ form a triangle. However, this is not possible because $500>100+300$.
The easiest way to do chin-ups is to hold your hands as far apart as possible.	F	 If your hands are held as far apart as possible, then there will be horizontal components of $\vec F_L$ and $\vec F_R$ of significant magnitude. Since the horizontal components do no work, they are in effect wasted. The best way to do chin-ups is with your arms at shoulder width.

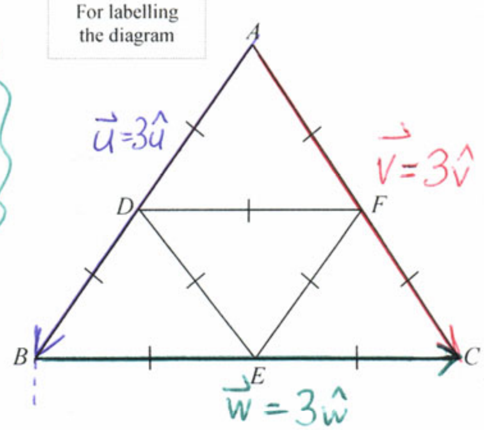
3. Let $\vec u = \overrightarrow{AB}$, $\vec v = \overrightarrow{AC}$ and $\vec w = \overrightarrow{BC}$. Suppose that $\lvert\vec u\rvert = \lvert\vec v\rvert = \lvert\vec w\rvert = 3$. Express **(4 APP, 1 COM)**

a. $\overrightarrow{AD}$ in terms of $\hat u$. (*Read carefully!* This is $\hat u$ not $\vec u$.)

$$\overrightarrow{AD} = \frac{1}{2}\vec u = \frac{1}{2}(3\hat u) = \frac{3}{2}\hat u$$

Note: $\vec u$ and $\hat u$ have the same direction BUT $\lvert\vec u\rvert=3$ and $\lvert\hat u\rvert=1$ $\therefore \hat u = \frac{1}{3}\vec u$ and $\vec u = 3\hat u$ (Similarly for the others.)

For labelling the diagram

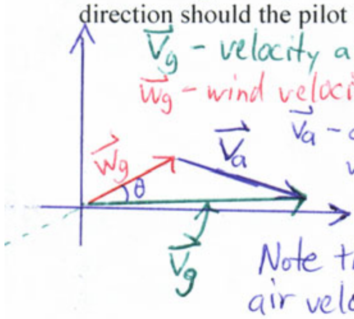


b. $\overrightarrow{BF}$ in terms of $\hat v$ and $\hat w$.

$$\overrightarrow{BF} = \vec w - \frac{1}{2}\vec v = 3\hat w - \frac{1}{2}(3\hat v) = 3\hat w - \frac{3}{2}\hat v$$

c. $\overrightarrow{DC}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{CE}$ for zero marks. For one bonus APP mark, write a short poem about David H.'s love of money or about his lack of friends. Alas, David H., lover of money, the root of all evil. It is this very lust, this unquenchable thirst, that has shaped and moulded your world, into a twisted, dark and tangled web, a deep abyss into which not even the vilest creature would dare to enter. So do not lament your lack of friends. It is by your own design.

4. NASA uses a Boeing 747 Jumbo Jet to transport space shuttles. Suppose that a NASA pilot is given a mission to "piggyback" a space shuttle from Edwards Air Force Base in California to the Kennedy Space Centre in Florida. While planning her flight, the pilot determines that the fastest route to the destination is to head N89.15°E with respect to the ground. If the pilot wishes to maintain a groundspeed of 900 km/h and there is a 100 km/h wind blowing *from* S60°W, in what direction should the pilot steer to stay on course? What will be her airspeed? **(5 APP, 3 COM)**



$\vec{V}_g$ - velocity aircraft wrt ground

$\vec{w}_g$ - wind velocity wrt ground

$\vec{V}_a$ - aircraft velocity wrt air

$\vec{V}_a = \vec{V}_g - \vec{w}_g$

Note that $\vec{w}_g$ is the air velocity wrt ground.

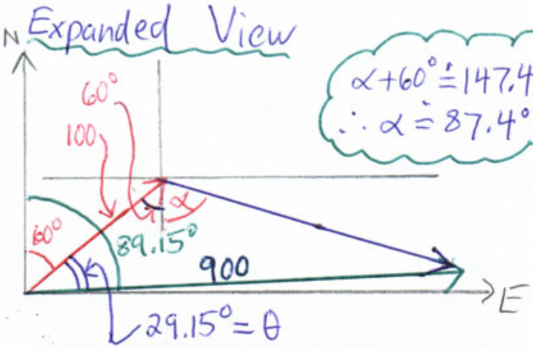
By the law of cosines,

$$|\vec{V}_a|^2 = |\vec{w}_g|^2 + |\vec{V}_g|^2 - 2|\vec{w}_g||\vec{V}_g|\cos\theta$$

$$\therefore |\vec{V}_a|^2 = 100^2 + 900^2 - 2(100)(900)\cos 29.15^\circ$$

$$\therefore |\vec{V}_a| = \sqrt{820000 - 180000\cos 29.15^\circ} \doteq 814$$

Expanded View



$\alpha + 60^\circ \doteq 147.4^\circ$

$\therefore \alpha \doteq 87.4^\circ$

By the law of sines,

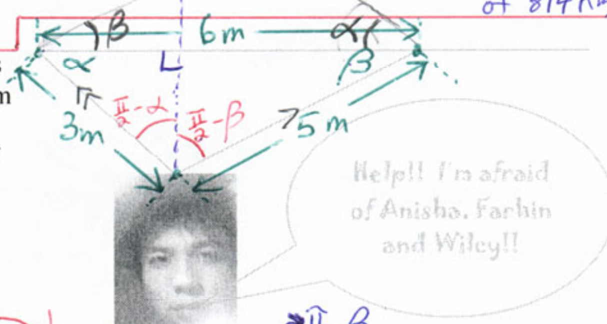
$$\frac{\sin(\alpha + 60^\circ)}{900} = \frac{\sin 29.15^\circ}{|\vec{V}_a|}$$

$$\therefore \sin(\alpha + 60^\circ) = \frac{900 \sin 29.15^\circ}{|\vec{V}_a|}$$

$$\therefore \alpha + 60^\circ = \sin^{-1}\left(\frac{900\sin 29.15^\circ}{|\vec{V}_a|}\right) \doteq 147.4^\circ$$

$\therefore$ the pilot should steer S87.4°E with an airspeed of 814 km/h.

5. Because of David's very rude behaviour during one of Mr. Nolfi's MGA4U0 lessons, Farhin, Anisha and Wilcy decided to use an 8 m steel cable to "hang" David from the ceiling of room 224. The cable is attached to two points on the ceiling that are 6 m apart. If David has a mass of 70 kg and he is suspended at a point that is 3 m from the end of the cable, determine the tension in each section of the cable. **(5 APP, 3 COM)**



By the law of cosines,

$$5^2 = 6^2 + 3^2 - 2(6)(3)\cos\alpha$$

$$\therefore \cos\alpha = \frac{6^2 + 3^2 - 5^2}{2(6)(3)} = \frac{5}{9}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{5}{9}\right) \doteq 56.25^\circ$$

By the law of sines

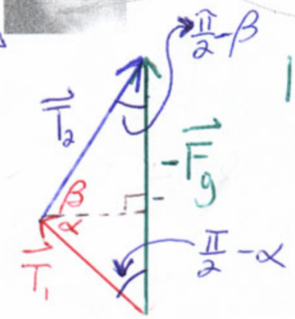
$$\frac{\sin\beta}{3} = \frac{\sin\alpha}{5}$$

$$\therefore \sin\beta = \frac{3\sin\alpha}{5}$$

$$\therefore \sin\beta = \frac{3\sin\left(\cos^{-1}\left(\frac{5}{9}\right)\right)}{5}$$

$$\therefore \beta = \sin^{-1}\left(\frac{3\sin\left(\cos^{-1}\frac{5}{9}\right)}{5}\right) \doteq 29.93^\circ$$

$$\vec{T}_1 + \vec{T}_2 = -\vec{F}_g$$



$$|\vec{F}_g| = mg = 70\,\text{kg}(9.8\,\text{m/s}^2) = 686\,\text{N}$$

$$\frac{\sin(\alpha+\beta)}{|-\vec{F}_g|} = \frac{\sin\left(\frac{\pi}{2}-\beta\right)}{|\vec{T}_1|} = \frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{|\vec{T}_2|}$$

$$\therefore |\vec{T}_1| = \frac{|-\vec{F}_g|\sin\left(\frac{\pi}{2}-\beta\right)}{\sin(\alpha+\beta)}$$

$$\text{and } |\vec{T}_2| = \frac{|-\vec{F}_g|\sin\left(\frac{\pi}{2}-\alpha\right)}{\sin(\alpha+\beta)}$$

Extra space for #5, if needed $\therefore$ using a calculator, we find that

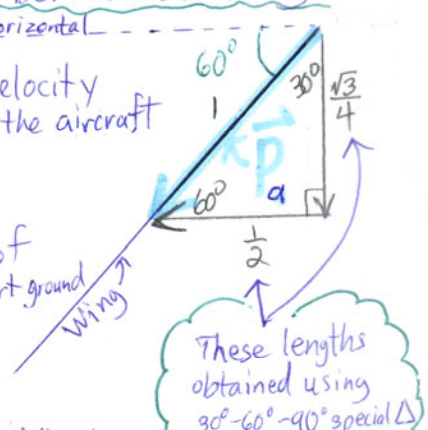
$$|\vec{T}_1| \doteq 596 \text{ N} \quad \text{and} \quad |\vec{T}_2| \doteq 382 \text{ N}.$$

There is a tension of about 596 N in one section of the cable and a tension of about 382 N in the other section of the cable //

6. Phil R. is famous for performing aerobatic stunts on a plane flown by the very trustworthy Captain Darryl G. To prepare for the stunt, Captain Darryl flew his plane due West with a groundspeed of 150 km/h. Then he tilted the wings of his fabulous flying machine to the left at an angle of 60° away from their usual horizontal position. At this point, the very courageous Phil emerged from the cockpit and began riding his mountain bike *down* the wing (along the wing that is tilted toward the ground) at a speed of 1 m/s relative to the wing.

What is Phil's velocity relative to the ground? (State both the magnitude *and* the direction.) **(6 TIPS, 3 COM)**

View from behind the left wing



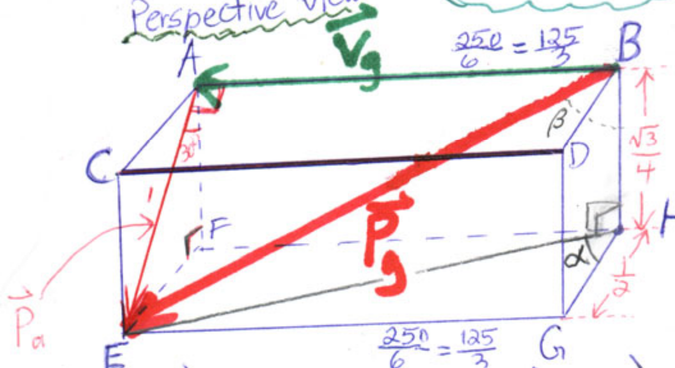
$\vec{P}_a$ – Phil's velocity wrt to the aircraft

$|\vec{P}_a| = 1$

$\vec{V}_g$ – velocity of aircraft wrt ground

$\vec{P}_g$ – Phil's vel. wrt ground

Perspective View



$\vec{P}_g$ is the diagonal $\overrightarrow{BE}$ of this prism (with edges $\frac{1}{2}$, $\frac{\sqrt{3}}{4}$ and $\frac{125}{3}$).

$150 \text{ km/h} = \frac{150\,000}{3600} \text{ m/s} = \frac{250}{6} \text{ m/s} = \frac{125}{3} \text{ m/s}$

By the definition of relative velocity,

$$\vec{P}_a = \vec{P}_g - \vec{V}_g$$

$$\therefore \vec{P}_g = \vec{P}_a + \vec{V}_g$$

Since ΔABE is a right triangle,

$$BE^2 = AB^2 + AE^2$$

$$\therefore |\vec{P}_g|^2 = |\vec{V}_g|^2 + |\vec{P}_a|^2 = \left(\frac{250}{6}\right)^2 + 1^2$$

$$\therefore |\vec{P}_g| = \frac{\sqrt{62536}}{6} \doteq 41.68 \text{ m/s}$$

In right triangle ΔHGE,

$$\tan\alpha = \frac{125}{3} \div \frac{1}{2} = \frac{125}{3} \times \frac{2}{1} = \frac{250}{3}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{250}{3}\right) \doteq 89.3°$$

In right triangle ΔBHE,

$$\cos\beta = \frac{\sqrt{3}}{4_2} \times \frac{6^3}{\sqrt{62536}} = \frac{3\sqrt{3}}{2\sqrt{62536}}$$

$$\therefore \beta = \cos^{-1}\left(\frac{3\sqrt{3}}{2\sqrt{62536}}\right) \doteq 89.4°$$

Therefore, Phil's velocity relative to the ground is about 41.68 m/s (150.04 km/h), S 89.3° W and 0.6° down from the plane containing rectangle ABDC. //