MGA 4U0 | **Semester 2, 2006 - 2007**

Grade 12 Geometry and Discrete Mathematics

Major Test – Second Half of Unit 2 (Geometric and Algebraic Vectors and their Applications)

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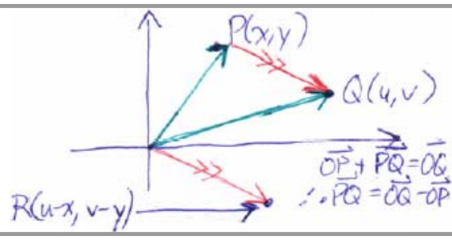
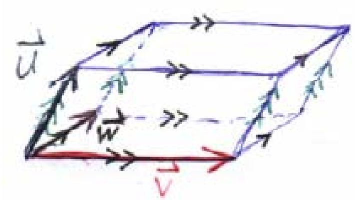
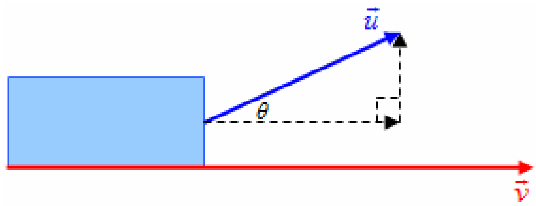
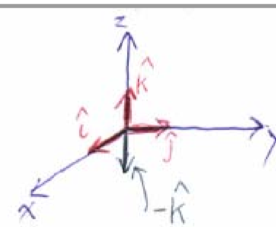
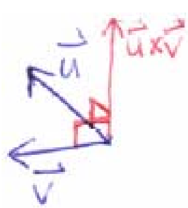
***Victim*: ____**

KU	APP	TIPS	COM
/6	/13	/19	/18

1. Complete the following table. **(6 KU, 3 COM)**

Expression, Equation or Inequality	*Diagram*	*Conclusion, Interpretation or Explanation*
Given points $P(x,y)$ and $Q(u,v)$, $\overrightarrow{PQ}=(u-x,v-y)$.		**The vector with its tail at $P(x, y)$ and its tip at $Q(u, v)$ is equal to the difference of the position vectors of P and Q, that is, $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$**
$\lvert\vec{u}\cdot\vec{v}\times\vec{w}\rvert = 2000$		The volume of a parallelepiped is 2000 cubic units.
$w=\vec{u}\cdot\vec{v}$		**If $\vec{v}$ is the displacement of the object, then the scalar $w=\vec{u}\cdot\vec{v}$ is the work done by the force $\vec{u}$.**

2. State whether each of the following is true or false. To receive full credit, you must prove the statements that are true and provide a counterexample for the statements that are false. **(6 TIPS, 3 COM)**

Statement	*True or False?*	*Proof or Counterexample*
$\vec{u}\times\vec{v}=\vec{v}\times\vec{u}$	**F**	$\hat{i}\times\hat{j}=\hat{k}$ but $\hat{j}\times\hat{i}=-\hat{k}$ 
The quantity $\dfrac{(\vec{a}\cdot\vec{c})\vec{a}\cdot(\vec{a}\times\vec{c})}{\lvert\vec{a}\rvert\lvert\vec{c}\rvert}$ is a vector.	**F**	$\vec{a}\cdot(\vec{a}\times\vec{c})$ is a scalar and so is $\vec{a}\cdot\vec{c}$ $\therefore (\vec{a}\cdot\vec{c})\vec{a}\cdot(\vec{a}\times\vec{c})$ is a scalar. Clearly, $\lvert\vec{a}\rvert\lvert\vec{c}\rvert$ is also a scalar. Therefore, the given quantity must be a scalar. (In fact, the scalar is zero since $\vec{a}\perp\vec{a}\times\vec{c}$, so $\vec{a}\cdot(\vec{a}\times\vec{c})=0$.)
$\operatorname{proj}_{\vec{u}\times\vec{v}}\vec{v}=\vec{v}$	**F**	 Clearly, $\vec{v}\perp\vec{u}\times\vec{v}$ $\therefore \operatorname{proj}_{\vec{u}\times\vec{v}}\vec{v}=\vec{0}\neq\vec{v}$ (unless $\vec{v}=\vec{0}$)

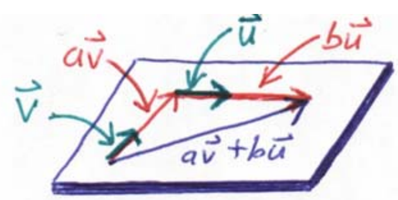
3. Show that $\vec{x}\cdot\vec{y}=\frac{1}{2}\left(|\vec{x}+\vec{y}|^2-|\vec{x}|^2-|\vec{y}|^2\right)$. **(5 APP)**

$$\begin{aligned}
R.H.S. &= \tfrac{1}{2}\left(|\vec{x}+\vec{y}|^2-|\vec{x}|^2-|\vec{y}|^2\right)\\
&= \tfrac{1}{2}\left[(\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})-\vec{x}\cdot\vec{x}-\vec{y}\cdot\vec{y}\right]\\
&= \tfrac{1}{2}\left(\cancel{\vec{x}\cdot\vec{x}}+\vec{x}\cdot\vec{y}+\vec{y}\cdot x+\cancel{\vec{y}\cdot\vec{y}}\cancel{-\vec{x}\cdot\vec{x}}\cancel{-\vec{y}\cdot\vec{y}}\right)\\
&= \tfrac{1}{2}\left(\vec{x}\cdot\vec{y}+\vec{x}\cdot\vec{y}\right)\\
&= \tfrac{1}{2}\left(2\vec{x}\cdot\vec{y}\right)\\
&= \vec{x}\cdot\vec{y}\\
&= L.H.S.
\end{aligned}$$

$$\therefore\ \vec{x}\cdot\vec{y}=\tfrac{1}{2}\left(|\vec{x}+\vec{y}|^2-|\vec{x}|^2-|\vec{y}|^2\right)$$

4. By applying the rules for calculating dot products and cross products in Cartesian form, ***it can be shown that*** $(\vec{u}\times\vec{v})\times\vec{w}=(\vec{u}\cdot\vec{w})\vec{v}-(\vec{v}\cdot\vec{w})\vec{u}$. (***You do not need to show this! The steps involved in the proof take too long to write out!***) **(3 COM** for all of question 4**)**

(a) Use the result given above to explain why $(\vec{u}\times\vec{v})\times\vec{w}$ must lie in the same plane as $\vec{u}$ and $\vec{v}$. Draw a diagram! **(3 TIPS)**



Let $a=\vec{u}\cdot w$ and $b=-\vec{v}\cdot\vec{w}$

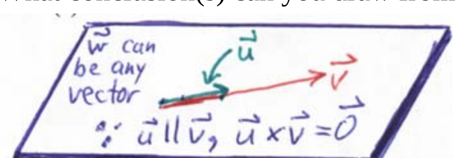
Then $(\vec{u}\times\vec{v})\times\vec{w}=a\vec{v}+b\vec{u}$

Since $\vec{u}$ and $\vec{v}$ lie in the same plane, so must $a\vec{v}+b\vec{u}$ (see diagram).

Since $(\vec{u}\times\vec{v})\times\vec{w}=a\vec{v}+b\vec{u}$, $(\vec{u}\times\vec{v})\times\vec{w}$ must lie in the same plane as $\vec{u}$ and $\vec{v}$.

(b) Suppose that $(\vec{u}\times\vec{v})\times\vec{w}=\vec{0}$. Using the result given above, we can conclude that $(\vec{u}\cdot\vec{w})\vec{v}-(\vec{v}\cdot\vec{w})\vec{u}=\vec{0}$. From this, it is a simple matter to show that $(\vec{u}\cdot\vec{w})\vec{v}=(\vec{v}\cdot\vec{w})\vec{u}$. What conclusion(s) can you draw from this? **(3 TIPS)**

$\therefore\ \vec{v}=\left(\frac{\vec{v}\cdot\vec{w}}{\vec{u}\cdot\vec{w}}\right)\vec{u}$



Since $\vec{v}\cdot\vec{w}$ and $\vec{u}\cdot\vec{w}$ are scalars,

$\frac{\vec{v}\cdot\vec{w}}{\vec{u}\cdot\vec{w}}=k$, for some $k\in\mathbb{R}$ (provided that $\vec{u}\cdot\vec{w}\neq 0$)

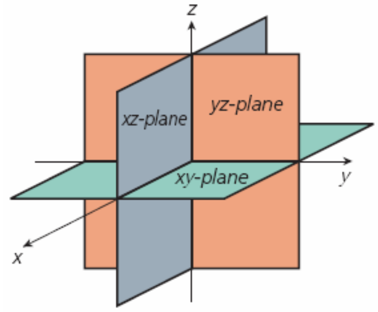
$\therefore\ \vec{v}=k\vec{u}$

$\therefore\ \vec{v}$ and $\vec{u}$ are scalar multiples of each other

$\therefore\ \vec{v}$ and $\vec{u}$ must be parallel

Special Case: If $\vec{u}\perp\vec{w}$ and $\vec{v}\perp\vec{w}$ then $\vec{u}$ and $\vec{v}$ need not be parallel

5. Just as $\mathbb{R}^2$ is divided into ***quadrants*** (four "equal" regions) by the co-ordinate axes, $\mathbb{R}^3$ is divided into ***octants*** (eight "equal" regions) by the co-ordinate planes (see diagram). The ***tip*** of the ***position vector*** $\vec{u}$ ***lies in the same octant*** as the point $P(-1, -1, 1)$. ***Note that*** $\vec{u}$ $\overrightarrow{OP}$. Answer the following questions given that $\vec{u}$ has direction angles $\alpha = 135°$ and $\gamma = 60°$. (Although you do not know the components of $\vec{u}$, the given information should give you a rough idea of the ***direction*** of $\vec{u}$.) **(5 COM** for all of question 5**)**



(a) What is the angle between $\vec{u}$ and the positive y-axis (i.e. β)? **(3 APP)**

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \cos^2 135° + \cos^2\beta + \cos^2 60° = 1$

$\therefore \left(-\frac{1}{\sqrt{2}}\right)^2 + \cos^2\beta + \left(\frac{1}{2}\right)^2 = 1$

$\therefore \cos^2\beta = 1 - \frac{1}{2} - \frac{1}{4}$

$\therefore \cos^2\beta = \frac{1}{4}$

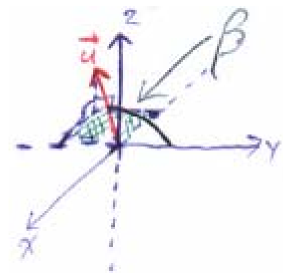
$\therefore \cos\beta = \pm\frac{1}{2}$

Since the tip of $\vec{u}$ lies in the same octant as $P(-1,-1,1)$, $90° < \beta < 180°$.

Therefore,

$\cos\beta = -\frac{1}{2}$

and $\beta = 120°$. //



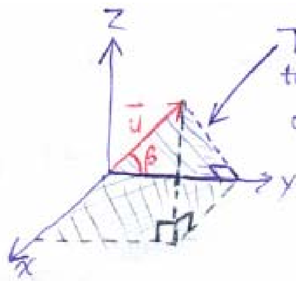
(b) Use the direction angles from 5(a) to find $\hat{u}$. **Answer:** $\hat{u} = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{2}, \frac{1}{2}\right)$ **(2 APP)**

$\hat{u} = \frac{1}{|\vec{u}|}\vec{u}$

$= \left(\frac{x}{|\vec{u}|}, \frac{y}{|\vec{u}|}, \frac{z}{|\vec{u}|}\right)$

$= (\cos\alpha, \cos\beta, \cos\gamma)$

$= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right)$



This is a right triangle. Therefore, $\cos\beta = \frac{y}{|\vec{u}|}$

(c) Let $\hat{u}$ be as calculated above, $\vec{v} = (1,2,5)$ and $\vec{w} = (0,2,-5)$. Find the volume of the parallelepiped determined by the vectors $\hat{u}$, $\vec{v}$ and $\vec{w}$. **(3 APP)**

$\hat{u} \cdot \vec{v} \times \vec{w}$

$= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right) \cdot (1,2,5) \times (0,2,-5)$

$= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right) \cdot (-20, 5, 2)$

$= +\frac{20}{\sqrt{2}} - \frac{5}{2} + 1 = \frac{20\sqrt{2} - 3}{2}$

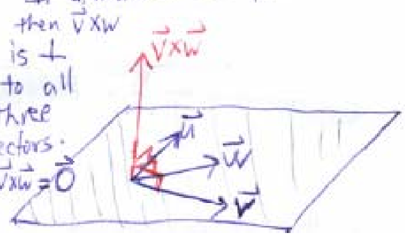
$\therefore V = |\hat{u} \cdot \vec{v} \times \vec{w}|$

$= \frac{20\sqrt{2} - 3}{2}$ cubic units.

(d) Use the triple scalar product that you calculated in 5(c) to show that $\vec{u}$, $\vec{v}$ and $\vec{w}$ are ***not coplanar***. That is, show that they ***do not*** lie in the same plane. **(2 TIPS)**

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar, then $\vec{v} \times \vec{w}$ is $\perp$ to all three vectors.

$\therefore \hat{u} \cdot \vec{v} \times \vec{w} = \vec{0}$



$\hat{u} \cdot \vec{v} \times \vec{w}$

$= \frac{20\sqrt{2} - 3}{2}$

$\neq 0$

$\therefore \hat{u}$, $\vec{v}$ and $\vec{w}$ are not coplanar.

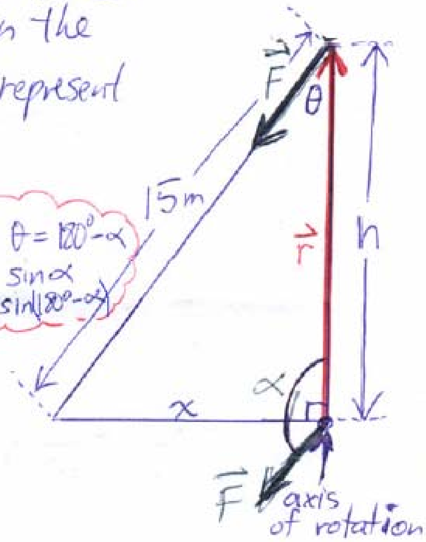
$\therefore \vec{u}$, $\vec{v}$ and $\vec{w}$ are not coplanar since $\hat{u} \parallel \vec{u}$.

(e) For zero marks, find the angle between $\vec{v}$ and $\vec{w}$. For one bonus APP mark, state into how many "equal" regions $\mathbb{R}^6$ can be divided if the regions are analogous to quadrants in $\mathbb{R}^2$. Show how you calculated your answer.

\# of "equal regions" $= 2^6 = 64$

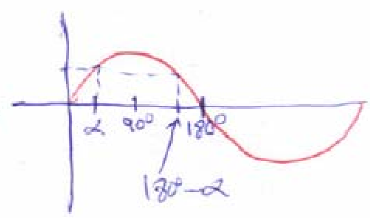
6. One day, while struggling to solve a difficult math problem, Ryan J. decided that he needed a "creativity boost." He saw a tree and thought that climbing to the top might help him clear his mind. Once there, however, Ryan remembered that he had a terrible fear of heights. Feeling paralyzed and helpless, he screamed frantically for help. Fortunately, his best "buddy" Farrukh R. was within earshot. In a brilliant flash of inspiration, Farrukh, who just happened to have a 15 m rope, realized that he could use his rope to pull the tree toward the ground. This would allow Ryan to descend safely by jumping just a short distance. Assuming that the rope is pulled from ground level, how far up the tree should Farrukh attach the rope to maximize the ***torque*** generated by the force that he applies to the rope? (Farrukh jumps into a hole that is just deep enough for his hands to be at ground level when he extends his arms above his head.) **(5 TIPS, 4 COM)**

Let h represent the distance from the ground to the point at which the rope is attached to the tree. In addition, let θ represent the angle between the tip of $\vec{r}$ and the tail of $\vec{F}$ and let α represent the angle between $\vec{r}$ and $\vec{F}$. Then,



$$
\begin{aligned}
|\vec{\tau}| &= |\vec{r}\times\vec{F}| \\
&= |\vec{r}||\vec{F}|\sin\alpha \\
&= |\vec{r}||\vec{F}|\sin\theta \\
&= |\vec{F}||\vec{r}|\sin\theta \\
&= |\vec{F}|\,h\left(\frac{x}{15}\right) \\
&= |\vec{F}|\sqrt{15^2-x^2}\left(\frac{x}{15}\right)
\end{aligned}
$$

Since $\theta = 180^\circ - \alpha$ and $\sin\alpha = \sin(180^\circ-\alpha)$

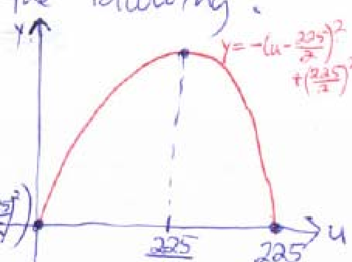


$$\therefore\ |\vec{\tau}| = |\vec{F}|\left(\frac{1}{15}x\sqrt{225-x^2}\right)$$

$$\therefore\ |\vec{\tau}| = |\vec{F}|\left(\frac{1}{15}\sqrt{225x^2-x^4}\right)$$

We can assume that $|\vec{F}|$ is independent of θ and therefore, independent of x. Thus, $|\vec{\tau}|$ is maximized when $225x^2-x^4$ is maximized. If we let $u=x^2$ and we complete the square we obtain the following:

$$
\begin{aligned}
y &= 225x^2 - x^4 \\
&= -u^2 + 225u \\
&= -\left(u^2 - 225u + \left(\frac{225}{2}\right)^2 - \left(\frac{225}{2}\right)^2\right) \\
&= -\left(u - \frac{225}{2}\right)^2 + \left(\frac{225}{2}\right)^2
\end{aligned}
$$



This is a parabola whose maximum occurs at the vertex $\left(\frac{225}{2}, \left(\frac{225}{2}\right)^2\right)$.
Therefore, at the maximum value of $225x^2-x^4$, $u=x^2=\frac{225}{2}$.

$\therefore\ x=\frac{15}{\sqrt{2}}=\frac{15\sqrt{2}}{2}$ and $h=\frac{15}{\sqrt{2}}=\frac{15\sqrt{2}}{2}$. $\therefore$ the rope should be tied $\frac{15}{\sqrt{2}}$ m up the tree. //

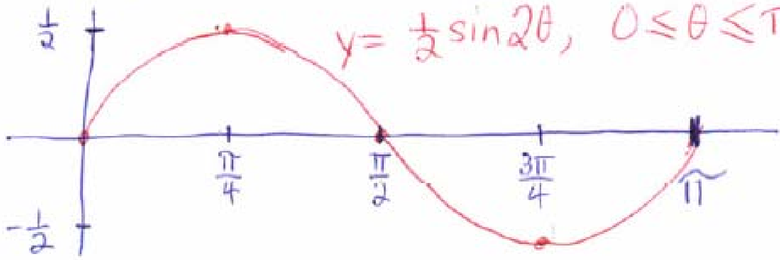
Alternative Solution

$$|\vec{\tau}| = |\vec{r} \times \vec{F}|$$
$$= |\vec{r}||\vec{F}|\sin\alpha$$
$$= |\vec{F}|\,|\vec{r}|\sin\theta$$
$$= |\vec{F}|\left(15\left(\frac{h}{15}\right)\right)\sin\theta$$
$$= 15|\vec{F}|\cos\theta\sin\theta$$

If we assume that $|\vec{F}|$ is constant (i.e. independent of θ), then $|\vec{\tau}|$ is maximized when $\sin\theta\cos\theta$ is maximized.

Now $\sin 2\theta = 2\sin\theta\cos\theta$

$\therefore \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$



Clearly the maximum value occurs at $\theta = \frac{\pi}{4}$.

When $\theta = \frac{\pi}{4}$, $h = 15\cos\theta = 15\left(\frac{1}{\sqrt{2}}\right) = \frac{15}{\sqrt{2}}$ //

Using Calculus to Maximize $y = 225x^2 - x^4$

If you happen to know calculus, $y = 225x^2 - x^4$ is much easier to maximize.

$y = 225x^2 - x^4$

$\therefore \frac{dy}{dx} = 450x - 4x^3$

For zero slopes,

$\frac{dy}{dx} = 0$

$\therefore 450x - 4x^3 = 0$

$\therefore 2x(225 - 2x^2) = 0$

$\therefore 2x = 0$ or $225 - 2x^2 = 0$

$\therefore x = 0$, $x = \pm\sqrt{\frac{225}{2}} = \pm\frac{15}{\sqrt{2}}$

Clearly, the only answer that makes sense is $x = \frac{15}{\sqrt{2}}$ //