MGA 4U0 **Semester 2, 2006 - 2007**

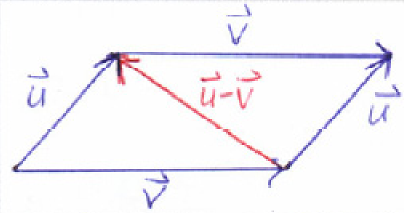
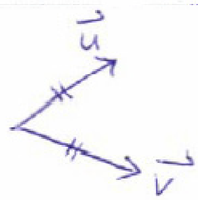
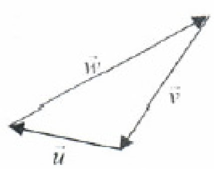
Grade 12 Geometry and Discrete Mathematics
Test – First Half of Unit 2 (Geometric and Algebraic Vectors and their Applications)

Mr. N. Nolfi

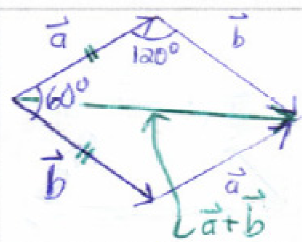
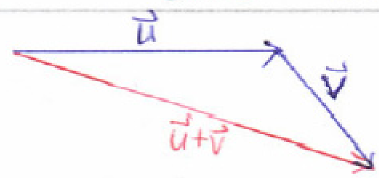
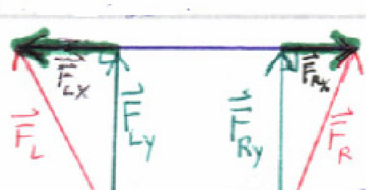
***Victim*:** ______________________________

KU	APP	TIPS	COM
/6	/14	/12	/16

1. Complete the following table. **(6 KU, 3 COM)**

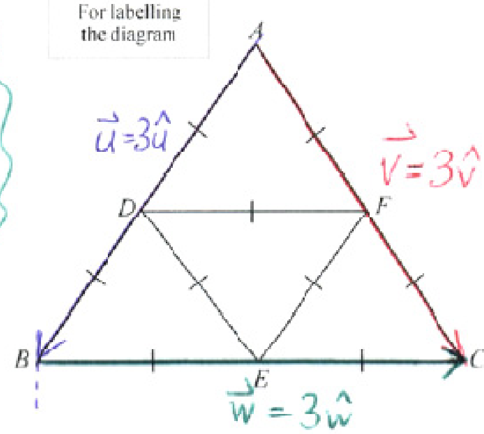
Expression, Equation or Inequation	*Diagram*	*Conclusion, Interpretation or Explanation*
$\lvert\vec{u}-\vec{v}\rvert \le \lvert\vec{u}\rvert+\lvert\vec{v}\rvert$		The vectors $\vec{u}$, $\vec{v}$ and $\vec{u}-\vec{v}$ form a triangle. Therefore, the length of $\vec{u}-\vec{v}$ must be less than the sum of the lengths of $\vec{u}$ and $\vec{v}$. Equality can only occur if $\vec{u}$ and $\vec{v}$ have opposite direction
$\lvert\vec{u}\rvert = \lvert\vec{v}\rvert$ but $\vec{u} \ne \vec{v}$		The vectors $\vec{u}$ and $\vec{v}$ have the same magnitude but are not equal to each other.
$\vec{u}+\vec{v}+\vec{w}=\vec{0}$		Give a *physical* interpretation here. Suppose that $\vec{u}$, $\vec{v}$ and $\vec{w}$ represent forces acting on an object. Therefore, the net force on the object is zero.

2. State whether each of the following is true or false. To receive full credit, you must prove the statements that are true and provide a counterexample for the statements that are false. **(6 TIPS, 3 COM)**

Statement	*True or False?*	*Proof or Counterexample*
If $\lvert\vec{a}\rvert = \lvert\vec{b}\rvert$ and the angle between $\vec{a}$ and $\vec{b}$ is 60°, then $\lvert\vec{a}+\vec{b}\rvert = \sqrt{3}\lvert\vec{a}\rvert$.	T	 $\lvert\vec{a}+\vec{b}\rvert^2 = \lvert\vec{a}\rvert^2+\lvert\vec{b}\rvert^2-2\lvert\vec{a}\rvert\lvert\vec{b}\rvert\cos 120°$ $= \lvert\vec{a}\rvert^2+\lvert\vec{a}\rvert^2-2\lvert\vec{a}\rvert\lvert\vec{a}\rvert\left(-\frac{1}{2}\right)$ $= 3\lvert\vec{a}\rvert^2$ $\therefore \lvert\vec{a}+\vec{b}\rvert = \sqrt{3}\lvert\vec{a}\rvert$
$\lvert\vec{u}+\vec{v}\rvert = \lvert\vec{u}\rvert+\lvert\vec{v}\rvert$	F	 Since the vectors $\vec{u}$, $\vec{v}$ and $\vec{u}+\vec{v}$ form a triangle, it follows that $\lvert\vec{u}+\vec{v}\rvert < \lvert\vec{u}\rvert+\lvert\vec{v}\rvert$
The easiest way to do chin-ups is to hold your hands as far apart as possible.	F	 The best way to do chin-ups is with your arms at shoulder width. If your hands are held as far apart as possible, then there will be horizontal components of $\vec{F}_L$ and $\vec{F}_R$ of significant magnitude. Since the horizontal components do no work, they are in effect wasted.

3. Let $\vec{u} = \overrightarrow{AB}$, $\vec{v} = \overrightarrow{AC}$ and $\vec{w} = \overrightarrow{BC}$. Suppose that $|\vec{u}| = |\vec{v}| = |\vec{w}| = 3$. Express **(4 APP, 1 COM)**

For labelling the diagram



(a) $\overrightarrow{AD}$ in terms of $\hat{u}$. (*Read carefully!* This is $\hat{u}$ not $\vec{u}$.)

$$\overrightarrow{AD} = \tfrac{1}{2}\vec{u} = \tfrac{1}{2}(3\hat{u}) = \tfrac{3}{2}\hat{u}$$

Note: $\vec{u}$ and $\hat{u}$ have the same direction BUT $|\vec{u}| = 3$ and $|\hat{u}| = 1$

$\therefore \hat{u} = \frac{1}{3}\vec{u}$ and $\vec{u} = 3\hat{u}$

(Similarly for the others.)

(b) $\overrightarrow{BF}$ in terms of $\hat{v}$ and $\hat{w}$.

$$\overrightarrow{BF} = \vec{w} - \tfrac{1}{2}\vec{v} = 3\hat{w} - \tfrac{1}{2}(3\hat{v}) = 3\hat{w} - \tfrac{3}{2}\hat{v}$$

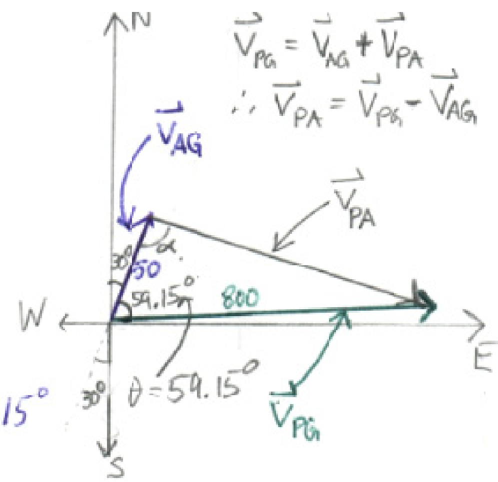
(c) $\overline{DC}$ in terms of $\overline{AB}$ and $\overline{CE}$, for zero marks. For one bonus **APP** mark, write a short poem about Philip's penmanship.

4. NASA uses a Boeing 747 Jumbo Jet to transport space shuttles. Suppose that a NASA pilot is given a mission to "piggyback" a space shuttle from Edwards Air Force Base in California to the Kennedy Space Centre in Florida. While planning her flight, the pilot determines that the fastest route to the destination is to head N89.15°E relative to the ground. If the pilot wishes to maintain a groundspeed of 800 km/h and there is a 50 km/h wind blowing ***from*** S30°W, in what direction should the pilot steer to stay on course? What will be her airspeed? **(5 APP, 3 COM)**

Given: $\vec{V}_{PG} = 800$ km/h, N 89.15° E

$\vec{V}_{AG} = 50$ km/h, N 30° E

Find: $\vec{V}_{PA} = ?$



By the law of cosines,

$$|\vec{V}_{PA}|^2 = |\vec{V}_{AG}|^2 + |\vec{V}_{PG}|^2 - 2|\vec{V}_{AG}||\vec{V}_{PG}|\cos\theta = 50^2 + 800^2 - 2(50)(800)\cos 59.15°$$

$\therefore |\vec{V}_{PA}| \doteq 775.6$ km/h

By the law of sines,

$$\frac{\sin\alpha}{|\vec{V}_{PG}|} = \frac{\sin\theta}{|\vec{V}_{PA}|}$$

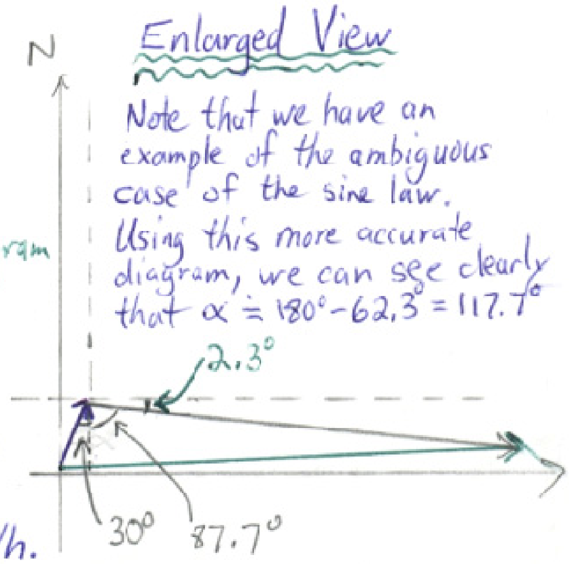
$$\therefore \sin\alpha = \frac{|\vec{V}_{PG}|\sin\theta}{|\vec{V}_{PA}|}$$

$$\therefore \alpha = \sin^{-1}\left(\frac{|\vec{V}_{PG}|\sin\theta}{|\vec{V}_{PA}|}\right) \doteq \cancel{62.3°} \text{ or } 117.7°$$

see diagram

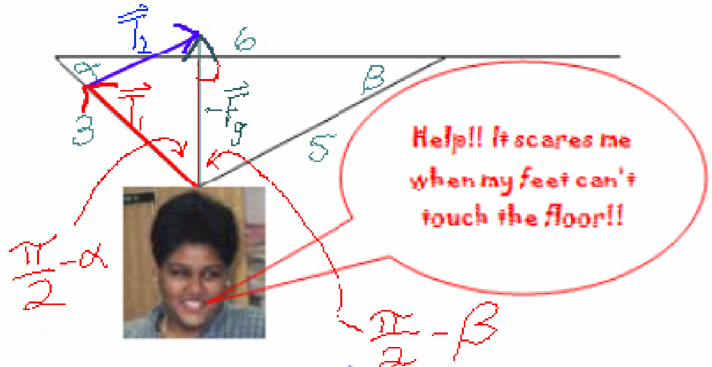
Enlarged View

Note that we have an example of the ambiguous case of the sine law. Using this more accurate diagram, we can see clearly that $\alpha \doteq 180° - 62.3° = 117.7°$



The pilot should head E2.3°S with an airspeed of about 775.6 km/h.

5. Because of the constant pointless, random remarks that Ryan makes during Mr. Nolfi's lessons, Amandeep, Antarpreet, Christine, Harpreet, Palwinder, Snehjot and Stephanie decide to use an 8 m steel cable to "hang" him from the ceiling of room 224. The cable is attached to two points on the ceiling that are 6 m apart. If Ryan has a mass of 100 kg and he is suspended at a point that is 3 m from one end of the cable, determine the tension in each section of the cable. **(5 APP, 3 COM)**



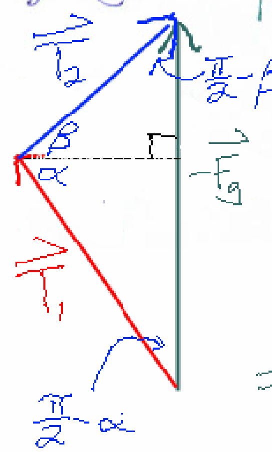
By the law of cosines,

$$5^2 = 6^2 + 3^2 - 2(6)(3)\cos\alpha$$

$$\therefore \cos\alpha = \frac{6^2+3^2-5^2}{2(6)(3)} = \frac{5}{9}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{5}{9}\right) \doteq 56.25^\circ$$

$$\vec{T}_1 + \vec{T}_2 = -\vec{F}_g$$



$$|\vec{F}_g| = mg$$
$$= 100\,\text{kg}(9.8\,\text{m/s}^2)$$
$$= 980\,N$$

Since net force on Ryan is $\vec{0}$,

$$\vec{T}_1 + \vec{T}_2 + \vec{F}_g = \vec{0}$$

By the law of sines

$$\frac{\sin\beta}{3} = \frac{\sin\alpha}{5}$$

$$\therefore \sin\beta = \frac{3\sin\alpha}{5}$$

$$\therefore \sin\beta = \frac{3\sin\left(\cos^{-1}\left(\frac{5}{9}\right)\right)}{5}$$

$$\therefore \beta = \sin^{-1}\left(\frac{3\sin\left(\cos^{-1}\frac{5}{9}\right)}{5}\right) \doteq 29.93^\circ$$

By the law of sines.

$$\frac{\sin(\alpha+\beta)}{|-\vec{F}_g|} = \frac{\sin\left(\frac{\pi}{2}-\beta\right)}{|\vec{T}_1|} = \frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{|\vec{T}_2|}$$

$$\therefore |\vec{T}_1| = \frac{|-\vec{F}_g|\sin\left(\frac{\pi}{2}-\beta\right)}{\sin(\alpha+\beta)}$$

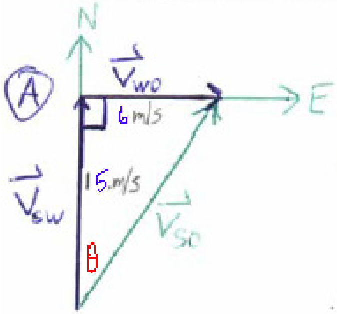
$$\text{and } |\vec{T}_2| = \frac{|-\vec{F}_g|\sin\left(\frac{\pi}{2}-\alpha\right)}{\sin(\alpha+\beta)}$$

$\therefore$ using a calculator, we find that

$$|\vec{T}_1| \doteq 851\,N \quad \text{and} \quad |\vec{T}_2| \doteq 546\,N.$$

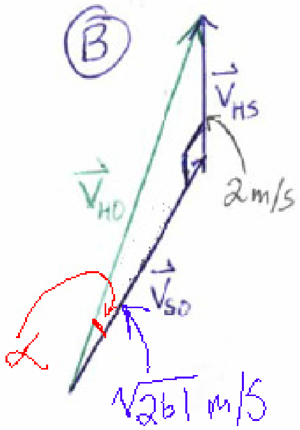
There is a tension of about 851 N in the shorter section of the cable and a tension of about 546 N in the longer section of the cable //

6. During a cruise on the "Love Boat," Sai and Tapiwa were being chased by a mob of lovesick, flirtatious women. While they would have loved to accept the offers of ***all*** the young ladies, they knew that they would never be able to get away with it. Therefore, rather than face the prospect of being forced to walk the plank, they decided that it would be best to escape the mob by deftly climbing up a mast. If they scaled the mast at a speed of 2 m/s, the ship travelled North at 15 m/s and the current flowed East at 6 m/s, what was their ***speed*** relative to the ocean floor?
Bonus: Calculate their ***direction*** relative to the ocean floor. **(6 TIPS, 3 COM)**



Let O represent the ocean floor, S represent the ship, W represent the water and H represent the "handsome" guys Sai and Tapiwa. (This is a matter of opinion.) Then,

$$\left.\begin{aligned} \vec{V}_{HO} &= \vec{V}_{SO} + \vec{V}_{HS} \\ &= \vec{V}_{SW} + \vec{V}_{WO} + \vec{V}_{HS} \end{aligned}\right\} \text{ See diagrams A and B.}$$



Using diagram A and the Pythagorean theorem,

$$|\vec{V}_{SO}|^2 = |\vec{V}_{SW}|^2 + |\vec{V}_{WO}|^2$$
$$= 15^2 + 6^2$$
$$= 261$$

Using diagram B and the Pythagorean theorem,

$$|\vec{V}_{HO}|^2 = |\vec{V}_{SO}|^2 + |\vec{V}_{HS}|^2$$
$$= 261 + 2^2$$
$$= 265$$

$$\therefore |\vec{V}_{HO}| = \sqrt{265} \text{ m/s} \doteq 16.3 \text{ m/s}$$

Therefore, Sai and Tapiwa have a speed of about 16.3 m/s relative to the ocean floor.

Bonus

Using diagram A, $\tan\theta = \frac{6}{15}$

$$\therefore \theta = \tan^{-1}\left(\frac{6}{15}\right) \doteq 21.8^\circ$$

Using diagram B, $\tan\alpha = \frac{2}{\sqrt{261}}$

$$\therefore \alpha = \tan^{-1}\left(\frac{2}{\sqrt{261}}\right) \doteq 7.1^\circ$$

Therefore, the direction of the handsome duo is N21.8°E, inclined 7.1° up from the ocean surface.