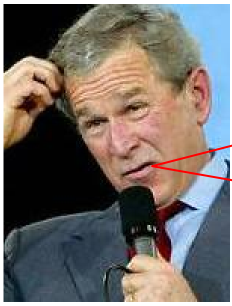
MGA4U0 Unit 0 – Review of Essential Mathematical Concepts

WHAT IS MATH?

A Succinct Definition of Mathematics

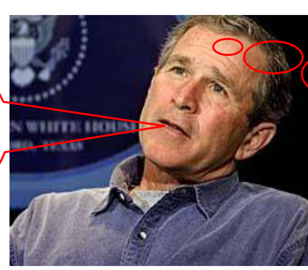
Mathematics is the ***investigation*** of ***axiomatically defined abstract structures*** using ***logic*** and ***mathematical notation***. Accordingly, mathematics can be seen as an extension of spoken and written natural languages, with an extremely precisely defined vocabulary and grammar, for the purpose of ***describing and exploring physical and conceptual relationships.***

Huh? What on Earth does this Mean?



Let's help G.W. and Rummy understand the above definition.

- ***Investigation***
  An investigation is the work of inquiring into something thoroughly and systematically. (Georgie and Donny should know what this means as it's what the CIA does.)
- ***Axiomatically Defined Abstract Structures***
  Wow, that's quite a mouthful! Later in the course we shall learn what this means. (George and Donald might find this somewhat tricky.)
- ***Logic***
  In general, the term "logic" refers to any system of ***reasoning***. In mathematics, "logic" refers specifically to the rules of ***inference***, that is, the rules that can be used to construct ***valid arguments***. In philosophy, logic is the study of the principles of reasoning, especially of the structure of propositions as distinguished from their content, and of method and validity in ***deductive reasoning***. We'll have more to say about this in unit 1 of this course. (I can understand why our two "friends" above have trouble understanding this concept.)
- ***Mathematical Notation***
  A mathematical notation is a writing system used to record concepts in mathematics. The notation uses symbols or symbolic expressions that have a precise ***meaning***.
- ***Describing and Exploring Physical and Conceptual Relationships***
  Now this is the true spirit of mathematics. At the core, mathematicians are ***pattern searchers***. Long ago, people figured out that nature is not entirely random. There are many relationships, in nature and in the human imagination that can be ***described*** and ***explored*** mathematically. For example, more than one millennium before Pythagoras, circa 1900-1600 B.C., the Babylonian civilization understood the relationship that we now call the Pythagorean Theorem. They knew that in any right triangle, once two of the sides were known, the third could be calculated. That is, the lengths of the sides of a right triangle are related according to the relationship that we now express as $c^2 = a^2 + b^2$.



A Detailed Description of George Polya's Four steps of Problem Solving

1. Understand the Problem (Define the Problem)

- ***Carefully read*** the problem ***several times***.
- ***Identify*** what you are being asked to ***find***.
- ***Ensure*** that you ***understand all terminology***.
- ***Highlight*** all ***given information***.
- ***Identify*** all the ***information*** that ***is required*** to solve the problem.
- ***Identify*** the ***given information*** that ***is required*** to solve the problem.
- ***Identify*** any ***extraneous information*** (information that is not needed).
- ***Identify*** any ***missing information***.
- ***Do research*** to ***find*** or ***estimate*** any ***missing information***.
- ***Keep*** an ***open mind***.
- ***Do not make*** any ***unnecessary*** or ***incorrect assumptions***.
- ***Think logically*** and ***creatively!***
- ***Consult colleagues, peers, experts,*** etc.
- ***Do not worry*** about ***possible strategies yet***.
- ***Predict*** what a ***reasonable answer*** or ***range of answers*** would be.

2. Choose a Strategy

- ***Identify the CRUX of the problem.***
- ***Unleash your creative powers! Be imaginative!***
- ***Do not be afraid to take risks!***
- ***Do not dismiss any ideas*** at this stage. Feel free to be ***whacky!***
- ***Avoid*** feelings of ***frustration*** or ***inadequacy***.
- ***Do not give up quickly!***
- If you have the desire to quit, ***take a break*** and ***try solving the problem later***.
- ***Do not be afraid*** to be ***unconventional***. Perhaps you are correct and everyone else is wrong!
- ***Draw*** a ***diagram*** or ***visualize***.
- ***Compare*** the problem to an ***equivalent*** or ***similar problem*** that you have already solved.
- ***Compare*** the problem to a ***simpler*** but ***related problem***.
- ***Solve*** a ***specific example*** of the problem.
- ***Look for patterns***.
- ***Write*** a list of ***as many possible strategies*** as you can.
- ***Do research*** to discover if ***anyone else*** has solved the problem.

3. Carry out the Strategy

- ***Check*** your list of strategies and ***select one*** that you think is likely to work.
- ***Carry out*** your strategy ***logically*** and ***carefully***, paying close attention to ***detail***.
- If your strategy ***fails***, return to ***steps 1*** and ***2***.

4. Check the Solution

- Is your answer ***reasonable*****?**
- Does your ***answer agree*** with the ***prediction*** you made in ***step 1***?
- Does your ***answer agree*** with the ***answers obtained by others***?
- Is there a ***better way*** to solve the problem?
- Ask ***peers, colleagues,*** etc to check your solution.

What Is Problem Solving?

Michael E. Martinez

Errors are part of the process of problem solving, which implies that both teachers and learners need to be more tolerant of them, Mr. Martinez points out. If no mistakes are made, then almost certainly no problem solving is taking place.

To think is constantly to choose in view of the end to be pursued.[1]

Every educator is familiar with the term "problem solving," and most would agree that the ability to solve problems is a worthy goal of education. But what is problem solving? Its meaning is actually quite straightforward: ***problem solving is the process of moving toward a goal when the path to that goal is uncertain***. We solve problems every time we achieve something without having known beforehand how to do so. We encounter simple problems every day: finding lost keys, deciding what to do when our car won't start, even improvising a meal from leftovers. But there are also larger and more significant "ill-defined" problems, such as getting an education, becoming a successful person and finding happiness. Indeed, the most important kinds of human activities involve accomplishing goals without a script.

Problem solving is a ubiquitous feature of human functioning. Human beings are problem solvers who think and act within a grand complex of fuzzy and shifting goals and changing means to attain them. This has always been true, but it is doubly so today because we live in a time of unprecedented societal transformation. When circumstances change, old procedures no longer work. To adapt is to pursue valued goals even when circumstances - and perhaps the goals themselves - are in flux. Because the pace of societal change shows no signs of slackening, citizens of the 21st century must become adept problem solvers, able to wrestle with ill-defined problems and win. Problem-solving ability is the cognitive passport to the future.

There is no formula for true problem solving. If we know exactly how to get from point A to point B, then reaching point B does not involve problem solving. Think of problem solving as working your way through a maze.[2] In negotiating a maze, you make your way toward your goal step by step, making some false moves but gradually moving closer toward the intended end point. What guides your choices? Perhaps a rule like this: choose the path that seems to result in *some* progress toward the goal. Such a rule is one example of a ***heuristic***. A heuristic is a rule of thumb. It is a strategy that is powerful and general, but not absolutely guaranteed to work. Heuristics are crucial because they are *the* tools by which problems are solved.

By contrast, algorithms are straightforward procedures that are guaranteed to work every time. For example, you have in your long-term memory algorithms that enable you to tie your shoelaces, to start up your car, and perhaps even to cook an omelette. Barring broken shoelaces, a dead battery and rotten eggs, these algorithms serve you very well. An algorithm may even be so automatic that it requires very little conscious processing as you carry out the procedure.

Now here is an important consideration: what constitutes problem solving varies from person to person. For a small child, tying shoelaces will indeed require problem solving, just as cooking an omelette entails problem solving for many adults. Thus problem solving involves an interaction of a person's experience and the demands of the task. Once we have mastered a skill, we are no longer engaged in problem solving when we apply it. For a task to require problem solving again, novel elements or new circumstances must be introduced or the level of challenge must be raised. Some problem solutions, however, can never be reduced to algorithms, and it is often those problems that constitute the most profound and rewarding of human activities. The necessity of problem solving to all that is important about being a person cannot be overstated.

In addition, problem solving is not an advanced process that is reserved solely for mature learners. Indeed, people of all ages can and must be solvers of problems. Perhaps young children are the most natural problem solvers. Because they continually face circumstances that are novel, they must adapt. It's their "job.' And they are amazingly good at it. Moreover, young children don't fret about failure the way that school-age children and adults tend to do. They take detours and setbacks in stride because they know intuitively that such obstacles are a part of the problem solving process. Still, we need to encourage problem solving in children. Whenever possible, this involves letting children find their own ways of reaching their goals. Good parents and other caregivers know when to stand back and let a child figure things out and when to step in and offer the right amount of help.

Armed only with our heuristics, then, we engage in a process of ***heuristic search***. Like finding one's way through a maze, we move closer, haltingly, to where we want to be. We can't be sure of what lies around the next corner or that the direction that once seemed so promising will pay off. Progress toward important goals is incremental, and each move is informed by our repertoire of heuristics. Because of the possibility of false moves, we need to monitor our progress continually and switch strategies if necessary.

The Power of Heuristics

If heuristics are the problem solver's best guide, it makes sense to elucidate them as much as possible. First, each learner must know what heuristics are and must be aware of their power. Second, each learner must have both general and specific heuristics at his or her disposal. General heuristics are cognitive "rules of thumb" that are useful in solving a great variety of problems. They are usually content-free and apply across many different situations. Specific heuristics are used in specialized areas, often specific subject domains or professions.

Probably the most powerful general heuristic, alluded to in the maze example, is "means-ends analysis." Essentially, the heuristic is this: form a subgoal to reduce the discrepancy between your present state and your ultimate goal state. Phrased more colloquially: do something to get a little closer to your goal.

Problems defy one-shot solutions; they must be broken down. Means-ends analysis accepts incremental advancement toward a goal. The method is not fail-safe, of course, because positive results are not guaranteed with any heuristic. However, if all goes well, this heuristic will help move you incrementally toward your ultimate goal. You apply it again and again, trying to reduce the discrepancy further. By means of this less-than-direct path, you find your way to the ends you seek. Such a search is not simply a process of trial and error, because the steps taken are not blind or random. Rather, the application of a series of tactical steps leads you ever closer toward the goal. Mistakes made along the way must be accepted as inextricable from the problem-solving process.

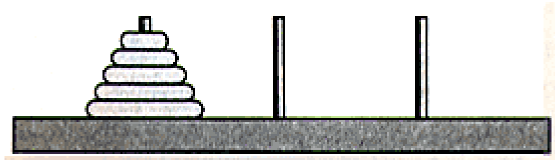
The benefits conferred by means-ends analysis may be as much emotional as intellectual. If a large and complex problem seems daunting as a whole, perhaps one can summon the will to accomplish a small piece of it. And that success can motivate one to persist. Thus, starting a task can make the effort self-sustaining. Sometimes, when we tackle a difficult project, it's as if we are trying to start a car on a cold winter morning. We encounter resistance. Once begun, however, the task becomes marginally easier and doesn't require a constant exertion of will to sustain it. At some point, we "cross the Rubicon" - we reach the point where it seems more difficult to stop than to carry on to completion. That is when a problem-solving activity becomes self-sustaining and bears us along by its momentum. "Just do it!" is not solely a great marketing slogan; it can also be seen as a directive to disregard the ominous hulking problem that looms ahead and simply take the first step.

Heuristics are usually picked up incidentally rather than identified and taught explicitly in school. This situation is not ideal. A curriculum that encourages problem solving needs to provide more than just practice in solving problems; it needs to offer explicit instruction in the nature and use of heuristics. Herbert Simon has written:

> In teaching problem solving, major emphasis needs to be directed toward extracting, making explicit, and practicing problem-solving heuristics - both general heuristics, like means-ends analysis, and more specific heuristics, like applying the energy conservation principle in physics.[3]

What are some other heuristics? One that is probably familiar to most readers goes by the name of "working backward." First, consider your ultimate goal. From there, decide what would constitute a reasonable step just prior to reaching that goal. Then ask yourself, "What would be the step just prior to that?" Beginning with the end, you build a strategic bridge backward and eventually reach the initial conditions of the problem.

An illustration of the use of this approach can be taken from the Tower of Hanoi problem. A number of disks are placed on a peg in an arrangement like this:



The rules are simple. Only one disk can be moved at a time, and a larger disk may never be placed on top of a smaller disk. The goal is to move the entire stack of disks from the first peg to the third. Working backward helps us understand that at some point, we must find a way to place the largest disk at the bottom of the third peg. Working backward from there, we would infer that all the smaller disks would eventually need to be placed on the middle peg, according to the rules, so that the largest disk is free to move. That step also has logical precursors, and so on. Working backward makes the problem more manageable and its solutions much more efficient than following a less reasoned approach.

Or take another example. My daughter came home from school with a story about a provocative exchange between a teacher and a student:

> *Teacher*: What do you want to be when you are an adult?
> *Student*: I want to be rich.
> *Teacher*: No, but what do you want to be?
> *Student*: I don't care. I just want to be rich.

This student certainly had a clear goal in mind, though some might question its value independent of the means for achieving it. In any case, the student has some serious “working backward” to do. If his goal is to be rich, what kind of career might allow him to achieve it? Becoming a movie star? A Wall Street investor? An entrepreneur? A criminal? Some combination of these? If an entrepreneur, that might imply that majoring in business in college would be in order. In turn, that goal might suggest that tonight the student should study his mathematics a little harder than is his custom. Working backward makes “next steps” plainer than simply wishing and hoping that dreams will materialize.

A third heuristic seeks to solve problems through “successive approximation.” Initial tries at solving a problem may result in a product that is less than satisfying. Writing is a good example. Few accomplished writers attempt to write perfect prose the first time they set pen to paper (or fingertips to keyboard). Rather, the initial goal is a rough draft or an outline or a list of ideas. Over time, a manuscript is gradually moulded into form. New ideas are added. Old ones are removed. The organization of the piece is reshaped to make it flow better. Eventually, a polished form emerges that finally approximates the effect that the author intended.

Given time and effort, what started out as rough and approximate can become art. In fact, successive approximation seems to be an important heuristic in producing outstanding creative works of all kinds. This model is relevant to many pursuits other than writing. Inventions, theories, stories, recipes and even personal and group identities start out rough but are restructured and refined over time. Think of the bicycle, whose various designs over the decades have metamorphosed toward greater efficiency and lighter weight. Successive approximation accepts the design process as problem solving, a series of zigs and zags toward something better.[4] Not only is such a process compatible with human information processing, but awareness of the principle can sustain a half-baked idea that initially seems raw, wild and foolish but is just possibly the germ of an eventual marvel.

George Polya's advice was “Draw a figure.”[5] In that spirit, I offer a fourth and final example of a heuristic: portray the problem at hand in an explicit “external representation.” List, describe, diagram or otherwise render the main features of a problem. This heuristic has several important features. First, it allows us to represent more complexity than we can hold in mind at once. Depicting a problem on paper, whiteboard or computer screen relieves short-term memory of the burden of representing the problem and allows the processing capacity of our brains to be directed toward solving it. An incidental benefit is that often the very attempt to represent the problem explicitly forces a problem solver to be clear about what it is he or she is trying to do and about what stands in the way. A clearer representation of goals and obstacles may by itself greatly simplify solution of the problem.

Another benefit of external representation is that the medium chosen to portray a problem may help the solver see the problem in a new way. In our heads we may understand a problem in words. On paper, we may discover that a picture makes more sense. Sometimes words can distort the more direct pictorial representations and so hinder problem solving.[6] Pictorial representations are used by experts in many fields and can be of considerable help.[7]

Finally, an external representation, unlike a mental representation, is potentially a "public document." The fact that other people can see it might help a group reach consensus about the nature of a problem. An obstacle that is prohibitive to one person might seem trivial or irrelevant to another. Likewise, a common representation might allow one participant to point out a significant opportunity that is unseen by other members of the group.

Metacognition

All heuristics help break down a problem into pieces. The problem as a whole is thus transformed. It is no longer a chaotic mass, like a ton of cooked spaghetti. Rather, through the creation of various subgoals, each of the pieces becomes manageable. The problem does become more complex in one sense because the pieces themselves must somehow be borne in mind. If a large goal is broken down into subgoals, then one cognitive challenge becomes goal management - keeping track of what to do and when. Goal management is probably a major aspect of intelligent thought. Patricia Carpenter, Marcel Just and Peter Shell regard goal management as a central feature of problem solving.

> A key component of analytic intelligence is goal management, the process of spawning subgoals from goals, and then tracking the ensuing successful and unsuccessful pursuits of the subgoal on the path to satisfying higher-level goals… The decomposition of complexity … consists of the recursive creation of solvable subproblems… But the cost of creating embedded subproblems, each with [its] own subgoals, is that they require management of a hierarchy of goals.[8]

The importance of monitoring subgoals is an example of a more general phenomenon: one common feature of problem solving is the capacity to examine and control one's own thoughts. This self-monitoring is known as metacognition. Metacognition is essential for any extended activity, especially problem solving, because the problem solver needs to be aware of the current activity and of the overall goal, the strategies used to attain that goal and the effectiveness of those strategies. The mind exercising metacognition asks itself, "*What* am I doing?" and "*How* am I doing?" These self-directed questions are assumed in the application of all heuristics. However, in practice, teachers cannot simply assume that students will engage in metacognition it must be taught explicitly as an integral component of problem solving.

Problem solving requires both the vigilant monitoring and the flexibility permitted by metacognition. When solving problems, means shift continually depending on one's position relative to desired goals. Even goals change as old goals are superseded by new and better ones. Maintaining flexibility is essential. Too often we feel wedded to a chosen strategy and continue to apply that strategy even if it leads us wildly astray. When this happens, it is usually wrong to conclude that we must start over. The important question is always "What do I do *now*, given my goal, my current position and the resources available to me?" Getting off course along the way is fully expected. Cool-headed reappraisal is the best response - not mindless consistency, panic or surrender.

A New Mindset

In pursuit of the goal of improving problem-solving ability, I have advocated the use of heuristics and have suggested a few. There are countless others. Some are general and apply to many problem situations, but most are specific and apply in specialized fields. Heuristics are vital, but they are not necessarily the most important aspect of problem solving.

Perhaps more powerful than any heuristic is an understanding that, *by its very nature, problem solving involves error and uncertainty*. Even if success is achieved, it will not be found by following an unerring path. The possibilities of failure and of making less-than-optimal moves are inseparable from problem solving. And the loftier the goals, the more obvious will be the imperfection of the path toward a solution. The necessity of uncertainty is recognized implicitly whenever we commend someone for being a risk taker. It is not the taking of risks itself that is commendable; rather, taking risks is a means to an end. What we actually applaud is the courage to adopt a difficult and commendable goal and then to enter the thorny thicket of problem solving where the only way out is through heuristic search and nerve. The willingness to suspend judgment - to accept temporary uncertainty-is an important aspect of thinking in general. John Dewey linked tolerance of uncertainty to reflective thinking:

> Reflective thought involves an initial state of doubt or perplexity… To many persons both suspense of judgment and intellectual search are disagreeable; they want to get them ended as soon as possible… To be genuinely thoughtful, we must be willing to sustain and protract the state of doubt, which is the stimulus to thorough inquiry.[9]

How then is it possible to improve problem-solving ability? First, we need to recognize when we are engaged in problem solving and accept as natural, normal, and expected the stepwise and discursive path toward a goal through the application of general and specific heuristics. Second, we must not let anxiety take hold. Anxiety is a spoiler in the problem-solving process. It stalks right behind uncertainty, ready to pounce. Demanding and uncertain environments, the seedbeds of all problem solving, are fertile ground for anxiety. Uncertainty is an integral part of the business of solving problems. Those who cannot bear situations in which it is impossible to see the way clearly to the end are emotionally ill-prepared to solve problems.

Errors are part of the process of problem solving, which implies that both teachers and learners need to be more tolerant of them. If no mistakes are made, then almost certainly no problem solving is taking place. Unfortunately, one tradition of schooling is that perfect performance is often exalted as an ideal. Errors are seen as failures, as signs that the highest marks are not quite merited. Worse still, errors are sometimes ridiculed or taken as ridiculous. Mistakes and embarrassment often go hand in hand. Perfect performance may be a reasonable criterion for evaluating algorithmic performance (though I doubt it), but it is incompatible with problem solving.[10]

What so often counts most in schools is the important but incomplete cognitive resource of *knowledge*. Fixed knowledge and algorithms are easier to teach, learn, and test than is the tangled web of processes that make up problem solving. Typically, it is not before graduate school that problem solving really becomes the focus of an educational program. Even in graduate school a student may not get to wrestle with the true problems of a field of study until the dissertation.

What can reverse this sorry state of affairs? A better understanding of the nature of problem solving is a place to start. Ultimately, we will have to change the culture of schooling. In the workplace as well, we need to revise our attitude toward errors - at least toward those that are a reasonable consequence of significant problem solving. (Errors in balancing the books don't count.) But if a job requires fluid intelligence - the ability to operate within the flux of continually changing demands and challenges - even the corporate culture must accept and deal with the multitude of paths toward solutions and the necessary existence of error.

For educators to accept errors, uncertainty, and indirect paths toward solutions is itself a difficult problem because doing so contradicts our engrained beliefs and expectations about teaching and learning. But problem solving must be understood and promoted if the next generation is to be prepared for the unprecedented challenges (i.e., problems) that it will face. Yet great things are accomplished when great things are attempted, and in our efforts we do not face total uncertainty. We have, in fact, our experience and its dividend, our knowledge, to support us. Heuristics and knowledge are what Herbert Simon has called the “two blades” of effective professional education, and he reminds us that “two-bladed scissors are still the most effective kind.”[11] I would add that what is good for professional education is good for education of all kinds at all levels. By combining what we *do* know with our understanding of the problem-solving process, we can move toward our goals - perhaps not unerringly, but by the sort of wending progress that is the signature of problem solving.

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1. Alfred Binet and Theodore Simon, *The Development of Intelligence in Children*, trans. E. S. Kite (Baltimore: Williams & Wilkins, 1916), p. 140.
2. Herbert A. Simon, *The Sciences of the Artificial* (Cambridge, Mass.: MIT Press, 1981).
3. Herbert A. Simon, "Problem Solving and Education:' in David T. Tuma and Frederick Reif, eds., *Problem Solving and Education: Issues in Teaching and Research* (Hillsdale, N.J.: Erlbaum, 1980), pp. 81-96.
4. Charles E. Lindblom, "The Science of Muddling Through" *Public Administration Review*, vol. 19, 1959, pp. 79-88.
5. George Polya, *How to Solve It*, 2nd ed. (Garden City, N.Y.: Doubleday, 1957).
6. Jill H. Larkin and Herbert A. Simon, "Why a Diagram Is (Sometimes) Worth Ten Thousand Words," *Cognitive Science*, vol. 11, 1987, pp. 65-99.
7. Fred Reif and Joan I. Heller, "Knowledge Structure and Problem Solving in Physics," *Educational Psychologist*, vol. 17, 1982, pp. 102-27.
8. Patricia A. Carpenter, Marcel Adam Just, and Peter Shell, "What One Intelligence Test Measures: A Theoretical Account of the Processing in the Raven Progressive Matrices Test," *Psychological Review*, vol. 97, 1990, pp. 404-31.
9. John Dewey, *How We Think: A Restatement of the Relation of Reflective Thinking to the Educative Process* (Boston: Heath, 1933), p. 16.
10. It is not impossible to solve a problem without error, but it is misleading to think that this experience is the normal character of problem solving.
11. Simon, "Problem Solving and Education"

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This article is taken from http://www.gse.uci.edu/faculty/michael_problemSolving.php .

Questions

1. Define the following terms:
   ubiquitous, discursive, adept, entrepreneur, render, protract, criterion, loftier, exalted, astray, vigilant

2. Now list the sentences in the article that contain the words listed in question 1. Do you have a better understanding of the sentences now that you have looked up the definitions?

3. List any other words and sentences that you do not understand.

4. What is problem solving? How does it differ from applying a skill? Is there a formula for solving problems?

5. What is an ***algorithm***? What is a ***heuristic***? Which of these two is more closely associated with ***genuine*** problem solving tasks?

6. Why has it always been important to develop strong problem solving skills? Why is it especially important nowadays?

7. List four examples of heuristics.

8. Recall the following exchange between a student and a teacher:

   Teacher: What do you want to be when you are an adult?
   Student: I want to be rich.
   Teacher: No, but what do you want to be?
   Student: I don't care. I just want to be rich.

   What is the problem with the student's approach to setting goals?

9. What are the roles of error and uncertainty in the problem solving process?

10. How can problem solving ability be improved?

An Investigation from our Textbook

Introduction

To help us understand what we mean by "describing and exploring physical and conceptual relationships," we shall work on an investigation from our textbook. Before we can do this, we need to review some important ideas from Cartesian geometry.

Review of Cartesian (Analytic) Geometry

The only Equation you need to know to find an Equation of a Line

Slope = Slope

Example

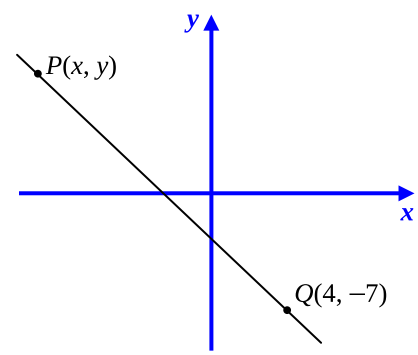
Find an equation of the line with slope $-\frac{2}{3}$ and passing through the point (4, –7).

Solution

Let $P(x, y)$ be any point on the line other than (4, -7).

Now since

$$\text{slope} = \text{slope},$$
$$\therefore \frac{\Delta y}{\Delta x} = -\frac{2}{3}$$
$$\therefore \frac{y-(-7)}{x-4} = -\frac{2}{3}$$
$$\therefore y+7 = -\frac{2}{3}(x-4)$$
$$\therefore y = -\frac{2}{3}x - \frac{13}{3}$$



Therefore, $y = -\frac{2}{3}x - \frac{13}{3}$ is ***an*** (not "the") equation of the required line. //

Review of finding the Distance between Two Points

The only Equation you need to know to find the Distance between Two Points

The Pythagorean Theorem

Example

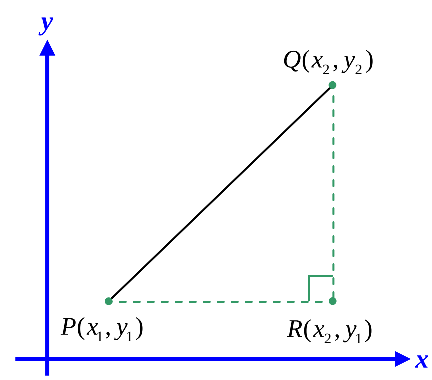
Find the distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Solution

Let P, Q and R be as shown in the diagram.

Clearly, $PR = |x_2 - x_1|$ and $QR = |y_2 - y_1|$ Therefore, by the Pythagorean Theorem,

$$PQ^2 = PR^2 + QR^2$$
$$\therefore PQ^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$\therefore PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$\therefore PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Therefore, finding the distance between two points is a simple application of the Pythagorean Theorem. //

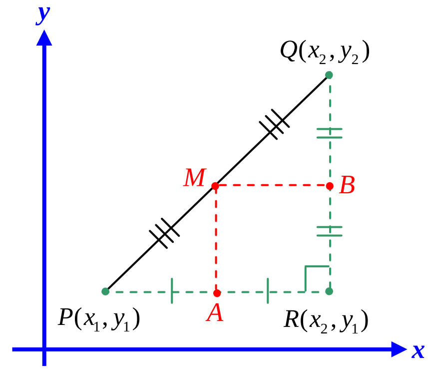
Review of finding the Midpoint of a Line Segment

The only Equation you need to know to find the Midpoint of a Line Segment

$$\text{The average of two numbers } a \text{ and } b \text{ is } \frac{a+b}{2}$$

Example

Find the midpoint of the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.



Solution

As shown in the diagram, let *M*, *A* and *B* represent the midpoints of the line segments *PQ*, *PR* and *QR* respectively. Since *A* lies on *PR* and *PR* is parallel to the *x*-axis, the *y*-co-ordinate of *A* must be y_1. Also, since *A* lies exactly half way between *P* and *R*, its *x*-co-ordinate must be the average of the *x*-co-ordinates of *P* and *R*. Therefore, the co-ordinates of *A* must be $\left(\frac{x_1+x_2}{2}, y_1\right)$. Using similar reasoning, the co-ordinates of *B* must be $\left(x_2, \frac{y_1+y_2}{2}\right)$.

Since *MA* is parallel to the *y*-axis, the *x*-co-ordinate of *M* must equal that of *A*. Since *MB* is parallel to the *x*-axis, the *y*-co-ordinate of *M* must equal that of *B*. Therefore, the co-ordinates of *M* must be

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Therefore, the midpoint *M* of *PQ* must have co-ordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. //

Important Exercises

1. Using a diagram and an argument similar to those given above, explain why parallel lines have equal slope.

2. Using a diagram and an argument similar to those given above, explain why any line perpendicular to a line with slope *m* must have slope $-\frac{1}{m}$.

3. Give a physical interpretation of slope.

4. Given the points $P(-3,5)$ and $Q(11,11)$, find an equation of the line passing through the midpoint of the line segment *PQ* and having slope $-\frac{2}{7}$.

5. Given the points $P(-3,5)$ and $Q(11,11)$, find an equation of the line perpendicular to *PQ* and passing through *Q*.

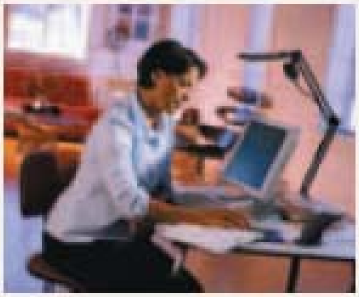
6. Find the distance from the point $P(-3,5)$ to the line $y = 2x+3$. (If you hope to solve this problem, a diagram is a must!)

Textbook Investigation

Now you should be ready to do the following investigation from our textbook. You should find Geometer's Sketchpad very helpful. In addition, keep in mind that your goal is to explore the given situation and to search for relationships. Answer ***all questions*** except for 1(a) and 1(c) on the next page.

Rich Learning Link investigate

CHAPTER 2: VARIGNON PARALLELOGRAM

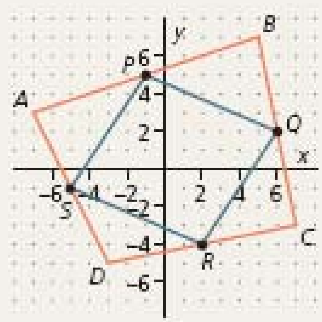


Parallelograms are quadrilaterals with opposite sides that are parallel. They are a common shape because rectangles are a type of parallelogram. Non-rectangular parallelograms are much less common, but they do arise as the solution to certain engineering and design problems. They are used in some jointed desk lamps, where they allow the light-bulb end to be moved without changing the direction in which the bulb points. For similar reasons, they are used in some binocular mounts for amateur astronomers and some motorcycle suspension systems. In movies, scenes that require a moving camera can be made jitter-free by using a Steadicam, a device which works because of the contribution of parallelograms.

Investigate and Inquire

In geometry, if we connect four points on a plane, the result is not likely to be a parallelogram. However, if we connect the midpoints of the sides of the quadrilateral, the resulting interior quadrilateral is a parallelogram.

Pick four points $A(-7, 3)$, $B(5, 7)$, $C(7, -3)$, and $D(-3, -5)$ on a Cartesian plane, as shown. The midpoints of AB, BC, CD, and DA are found to be $P(-1, 5)$, $Q(6, 2)$, $R(2, -4)$, and $S(-5, -1)$, respectively.



The slope of PQ is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 5}{6 - (-1)} = -\frac{3}{7}$.

A similar calculation shows that the slope of SR is also $-\frac{3}{7}$. Thus, PQ and SR are parallel.

Similarly, QR and PS can both be shown to have a slope of $\frac{3}{2}$, so they are also parallel. Thus, $PQRS$ is a parallelogram.

Try this with four other points. Using Geometer's Sketchpad® will allow you to move your four points independently.

DISCUSSION QUESTIONS

1. If a quadrilateral has special properties, it usually has a special name. For example, a parallelogram is a special quadrilateral with opposite sides that are parallel. What are some other special quadrilaterals?
2. A rectangle is a special type of parallelogram. What are some other special parallelograms?
3. Consider a parallelogram frame that has one side held in position (e.g., mounted against the side of a wall). Describe the possible motions of the parallelogram.

Rich Learning Link wrap-up investigate and apply

CHAPTER 2: VARIGNON PARALLELOGRAM

Varignon's Parallelogram Theorem states that the figure formed by joining the midpoints of any convex quadrilateral is a parallelogram. The theorem is named after Pierre Varignon, a French mathematician who lived from 1654 to 1722. He was the first to prove the theorem. The parallelogram formed by joining the midpoints of a quadrilateral is called a **Varignon parallelogram**. Like so many others, Varignon accidentally came across a copy of Euclid's *Elements* and was inspired to a career in mathematics.

Investigate and Apply

Geometer's Sketchpad® may be useful as you carry out some of these investigations.

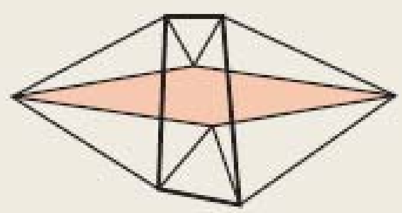
1. a) Prove Varignon's Parallelogram Theorem using the methods of this chapter.
   b) Consider four points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, and $D(x_4, y_4)$. Prove Varignon's Parallelogram Theorem using the midpoint and slope formulas from analytic geometry.
   c) Compare and contrast the proofs from parts **a** and **b**. Which is more convincing? Which is easier to understand?

   A third method of proof will be encountered in Chapter 6.
2. If the initial quadrilateral is a square, does the interior Varignon Parallelogram have any special properties? What if the initial quadrilateral is a rectangle, a rhombus, a trapezoid, or a parallelogram?
3. Prove that the area of a Varignon Parallelogram is one half the area of the initial quadrilateral. *Hint:* Consider the initial quadrilateral as two triangles.

INDEPENDENT STUDY

What special characteristics must the initial quadrilateral have in order for its Varignon Parallelogram to be a rhombus, a rectangle, or a square?

If the initial quadrilateral is already a parallelogram, what special properties must it have in order to be similar to its Varignon Parallelogram?



Prove that if equilateral triangles are drawn on the sides of a quadrilateral, alternately inwards and outwards, their vertices will form a parallelogram.

Important Skills Review Exercises

Functions, Simplifying, Factoring

1. For each function, find:
   i) $f(-2), f(3), f(1.2)$ ii) the domain and the range iii) the inverse
   a) $f(x) = 3x - 2$ b) $f(x) = 2x^2 - 3x + 1$ c) $f(x) = \dfrac{2-x}{x}$

2. If $f(x) = x^2 - 3x + 1$, find:
   a) $f(2k)$ b) $f(-2k)$ c) $-f(z)$
   d) $f(x+2)$ e) $f(2x-1)$ f) $f\left(\dfrac{1}{x}\right)$.

3. If $f(x) = 2x^2 - 5$, show that $f(x+3) \neq f(x) + f(3)$.

4. If $f(x) = 2x^2 - 5x + 1$, solve each equation.
   a) $f(x) = -1$ b) $f(x) = 13$ c) $f(2a) = 13$

5. Draw a mapping diagram to represent each set of ordered pairs. Which sets represent functions?
   a) $\{(1,9), (2,7), (3,5), (4,3), (5,1)\}$
   b) $\{(-3,2), (-1,4), (1,2), (-3,7), (5,-1), (6,5)\}$
   c) $\{(-4,13), (-3,7), (-2,3), (-1,1), (0,3), (1,7)\}$

6. If $f(x) = 2x + 5$ and $g(x) = x^2 - 3x + 2$, find:
   a) $g(f(x))$ b) $f(f(x))$ c) $f^{-1}(f^{-1}(x))$ d) $f(g(x))$.

7. Given $f(x) = 3x - 1$ and $g(x) = 2\sqrt{x+1}$
   a) Find.
   i) $f\left(\dfrac{2x-1}{2}\right)$ ii) $g\left(\dfrac{5}{x-2}\right)$ iii) $f(g(x))$ iv) $g(f(x))$
   b) State the domain and the range of each function.
   i) $f(x)$ ii) $g(x)$ iii) $f(g(x))$ iv) $g(f(x))$

8. If $f(x) = 2x - 5$ and $g(x) = (x+2)^2 - 3$, sketch these graphs.
   a) $f(x)$ and $g(x)$ on the same axes. b) $f(x) + g(x)$ c) $f(x) - g(x)$

9. The perimeter of a rectangle is 10 m. Express the length of its diagonal as a function of its width.

10. Simplify.
    a) $2(3x + 2y) - 5(x - 4y)$
    b) $3m(5m - 7n) - 2m(12m + 5n)$
    c) $x(4x - 3y + 7) - 3x(2x + 5y - 8)$
    d) $(5a - 3b)(2a + 7b)$
    e) $(3x + 2)^2 - 2(4x - 1)(x + 3)$
    f) $3(2m - 1)(-4m + 7n - 1) - (2m + n - 3)^2$

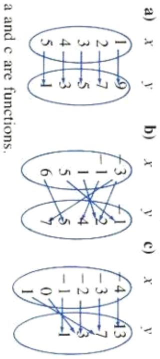
11. Factor.
    a) $15x^2 - 6xy + 21x$ b) $x^2 - 17x + 42$
    c) $10x^2 + 7xy - 12y^2$ d) $m^2(3m - 2) - 5m(3m - 2) + 9m - 6$
    e) $15x^3 + 21x^2y - 18xy^2$ f) $100p^2 - 36q^2$

Answers

1. a) i) $-8, 7, 1.6$ **ii)** R; R
iii) $y = \dfrac{x+2}{3}$ **b) i)** 15, 10, 0.28
ii) R; $\left\{y \mid y \geq -\dfrac{1}{8}, y \in \mathrm{R}\right\}$
iii) $y = \dfrac{\pm\sqrt{8x+1} + 3}{4}$ **c) i)** $-2, -\dfrac{1}{3}, \dfrac{2}{3}$
ii) $\{x \mid x \neq 0, x \in \mathrm{R}\}, \{y \mid y \neq -1, y \in \mathrm{R}\}$
iii) $y = \dfrac{2}{x+1}$

2. a) $4k^2 - 6k + 1$ **b)** $4k^2 + 6k + 1$
c) $-z^2 + 3z - 1$ **d)** $x^2 + x - 1$
e) $4x^2 - 10x + 5$ **f)** $\dfrac{1}{x^2} - \dfrac{3}{x} + 1$

3. $2x^2 + 12x + 13 \neq 2x^2 + 8$
4. a) 0.5, 2 **b)** $-1.5, 4$ **c)** $-0.75, 2$
5. a) x y **b)** x y **c)** x y



a and c are functions.

6. a) $4x^2 + 14x + 12$ **b)** $4x + 15$ **c)** $\dfrac{x - 15}{4}$
d) $2x^2 - 6x + 9$

7. a) i) $\dfrac{6x-5}{2}$ **ii)** $2\sqrt{\dfrac{x+3}{x-2}}$
iii) $6\sqrt{x+1} - 1$ **iv)** $2\sqrt{3x}$
b) i) R; R
ii) $\{x \mid x \geq -1, x \in \mathrm{R}\}; \{y \mid y \geq 0, y \in \mathrm{R}\}$
iii) $\{x \mid x \geq -1, x \in \mathrm{R}\}, \{y \mid y \geq -1, y \in \mathrm{R}\}$
iv) $\{x \mid x \geq 0, x \in \mathrm{R}\}, \{y \mid y \geq 0, y \in \mathrm{R}\}$

9. $d = \sqrt{2w^2 - 10w + 25}$
10. a) $x + 24y$ **b)** $-9m^2 - 31mn$
c) $-2x^2 - 18xy + 31x$
d) $10a^2 + 29ab - 21b^2$ **e)** $x^2 - 10x + 10$
f) $-28m^2 + 38mn + 18m - 15n - n^2 - 6$
11. a) $3x(5x - 2y + 7)$ **b)** $(x - 14)(x - 3)$
c) $(2x + 3y)(5x - 4y)$
d) $(3m - 2)(m^2 - 5m + 3)$
e) $3x(x + 2y)(5x - 3y)$
f) $4(5p - 3q)(5p + 3q)$

Trigonometry

1. Draw each angle in standard position, then find two angles which are coterminal with it.
   a) $65°$ b) $135°$ c) $200°$ d) $-450°$
   e) $\frac{\pi}{3}$ f) $\frac{5\pi}{4}$ g) $-\frac{\pi}{6}$ h) $\frac{8\pi}{3}$

2. Determine the sine, the cosine, and the tangent to 3 decimal places of each angle in *Exercise 1*.

3. Each point P is on the terminal arm of angle θ. Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ to 3 decimal places.
   a) P(4,9) b) P(8, −15) c) P(−4,7) d) P(−6, −5)

4. Find each value of θ in *Exercise 3*:
   i) in degrees to 1 decimal place ii) in radians to 3 decimal places.

5. Solve for θ to the nearest degree, $0° \leqslant \theta \leqslant 360°$.
   a) $\sin\theta = 0.7295$ b) $\cos\theta = -0.3862$ c) $\tan\theta = -5.1730$

6. Solve for θ in radians to 2 decimal places, $0 \leqslant \theta \leqslant 2\pi$.
   a) $\cos\theta = 0.2681$ b) $\tan\theta = 1.0744$ c) $\sin\theta = -0.4683$

7. Solve for θ to the nearest degree, $0° \leqslant \theta \leqslant 360°$.
   a) $3\sin\theta + 2 = 0$ b) $2\tan\theta - 5 = 2$
   c) $12\sin^2\theta - 11\sin\theta + 2 = 0$ d) $3\cos^2\theta + 4\cos\theta - 2 = 0$
   e) $3\cos^2\theta + 2 = 4$ f) $2\sin^2\theta + 5\sin\theta + 1 = 0$

8. Draw graphs of $y = \sin\theta$ and $y = \cos\theta$ for $-360° \leqslant \theta \leqslant 360°$. For each graph
   a) State the maximum value of y, and the values of θ for which it occurs.
   b) State the minimum value of y, and the values of θ for which it occurs.
   c) State the θ- and y-intercepts.

9. Find the amplitude, the period, the phase shift, and the vertical displacement for each function.
   a) $y = 3\sin 2(\theta - 45°) - 4$ b) $y = -2\cos 5\left(\theta + \frac{\pi}{3}\right) + 1$

10. Sketch the graphs of each set of functions on the same grid for $-2\pi \leqslant \theta \leqslant 2\pi$.
   a) $y = \sin\theta$ $y = 3\sin\theta$ $y = 3\sin\theta + 2$
   b) $y = \frac{1}{2}\cos\theta$ $y = \frac{1}{2}\cos\left(\theta + \frac{\pi}{3}\right)$ $y = \frac{1}{2}\cos\left(\theta + \frac{\pi}{3}\right) + 2$

11. Sketch the graph of each function.
   a) $y = \frac{1}{2}\sin 2\pi\,\frac{(t+1)}{2} - 3$ b) $y = -3\sin 2\pi\,\frac{(t-2)}{5} + 2$

12. A Ferris wheel of radius 16 m rotates once every 48 s. The passengers get on at a point 1 m above ground level.
   a) Write an equation to express the height h metres of a passenger above the ground at any time t seconds.
   b) How high is a passenger after: i) 10 s ii) 25 s?

Answers

1. Typical answers: **a)** $425°, -295°$ **b)** $495°, -225°$ **c)** $560°, -160°$ **d)** $270°, -90°$ **e)** $\frac{7\pi}{3}, -\frac{5\pi}{3}$ **f)** $\frac{13\pi}{4}, -\frac{3\pi}{4}$ **g)** $\frac{11\pi}{6}, -\frac{13\pi}{6}$ **h)** $\frac{2\pi}{3}, -\frac{4\pi}{3}$
2. a) 0.906, 0.423, 2.145
b) 0.707, −0.707, −1.000
c) −0.342, −0.940, 0.364
d) −1.000, 0, undefined
e) 0.866, 0.500, 1.732
f) −0.707, −0.707, 1.000
g) −0.500, 0.866, −0.577
h) 0.866, −0.500, −1.732
3. a) 0.914, 0.406, 2.250
b) −0.882, 0.471, −1.875
c) 0.868, −0.496, −1.75
d) −0.640, −0.768, 0.833
4. a) 66.0°, 1.153 **b)** 298.1°, 5.202
c) 119.7°, 2.090 **d)** 219.8°, 3.836
5. a) 47°, 133° **b)** 113°, 293° **c)** 101°, 281°
6. a) 1.30, 4.98 **b)** 0.82, 3.96 **c)** 5.80, 3.63
7. a) 222°, 318° **b)** 74°, 254°
c) 14°, 42°, 138°, 166° **d)** 67°, 93°
e) 48°, 132°, 228°, 312° **f)** 193°, 347°
8. a) $\sin\theta$: 1 at −270°, 90°; $\cos\theta$: 1 at 0°, ± 360°
b) $\sin\theta$: −1 at −90°, 270°; $\cos\theta$: −1 at ± 180°
c) $\sin\theta$: y-intercept 0, θ-intercepts ± 360°, ± 180°, 0°; $\cos\theta$: y-intercept 1, θ-intercepts ± 270°, ± 90°, 0°
9. a) 3; π; 45°; 4 **b)** 2; $\frac{2\pi}{5}$; $-\frac{\pi}{3}$; 1
c) 4; $\frac{2\pi}{3}$; $-\frac{\pi}{6}$; −5
12. a) $h = 16\cos\frac{2\pi(t-24)}{48} + 17$
b) 12.9 m, 32.9 m

More Trigonometry

1. Evaluate each trigonometric ratio to 3 decimal places.
   a) $\sin 55°$ b) $\tan 27°$ c) $\sec 81°$ d) $\cot 37°$ e) $\cos 65°$ f) $\csc 22°$
   g) $\tan 138°$ h) $\sin 102°$ i) $\csc 95°$ j) $\cot 107°$ k) $\cos 141°$ l) $\sec 170°$

2. Find each value of θ to the nearest degree if $0° < \theta < 180°$.
   a) $\cos \theta = 0.274$ b) $\cot \theta = 1.912$ c) $\sec \theta = 1.125$
   d) $\sin \theta = 0.469$ e) $\csc \theta = 3.150$ f) $\tan \theta = 2.247$

3. State the exact values of the six trigonometric ratios of each angle.
   a) 0° b) 30° c) 45° d) 60° e) 90°
   f) 120° g) 135° h) 150° i) 180°

4. Given θ is an acute angle, find the values of the other five trigonometric ratios.
   a) $\cos \theta = \frac{8}{17}$ b) $\tan \theta = \frac{12}{5}$ c) $\sec \theta = \frac{21}{11}$ d) $\sin \theta = \frac{5}{9}$

5. Given θ is an acute angle, find expressions for the other five trigonometric ratios.
   a) $\sin \theta = \frac{a}{b}$ b) $\tan \theta = \frac{p}{p+q}$ c) $\sec \theta = \frac{2m-1}{m+3}$

6. Solve $\triangle$ABC, if $\angle B = 90°$, and:
   a) AB = 15, BC = 27 b) AC = 18, BC = 10
   c) AB = 42, $\angle$C = 72° d) AC = 12, $\angle$A = 35°.
   Give the answers to 1 decimal place where necessary.

7. Solve each $\triangle$PQR. Give the answers to 1 decimal place.
   a) $\angle Q = 75°$, $r = 8$, $p = 11$ b) $\angle R = 52°$, $r = 28$, $q = 25$
   c) $\angle P = 38°$, $\angle Q = 105°$, $p = 32$ d) $r = 17$, $p = 14$, $q = 26$
   e) $\angle Q = 57°$, $q = 42$, $r = 45$ f) $\angle R = 33°$, $p = 14$, $q = 24$

8. A wheelchair ramp 8.2 m long rises 94 cm. Find its angle of inclination to 1 decimal place.

9. The angle of elevation of the sun is 68° when a tree casts a shadow 14.3 m long. How tall is the tree?

10. A cable car rises 762 m as it moves a horizontal distance of 628 m.
    a) How long is the ride?
    b) What is the angle of inclination of the cable to the nearest degree?

11. Two identical apartment buildings are 41.3 m apart. From her balcony, Kudo notices that the angle of elevation to the top of the adjacent building is 57°. The angle of depression to the base of the building is 28°. Find the height of the buildings.

12. Rectangle PQRS has sides whose lengths are in the ratio of 3 : 2. Points A and B are the midpoints of PQ and PS respectively. Find the measure of $\angle$BAR to 1 decimal place.

13. When watching a rocket launch, Nema is 0.8 km closer to the launching pad than Joel is. When the rocket disappears from view, its angle of elevation for Nema is 36.5° and for Joel is 31.9°. How high is the rocket at this point?

Answers

1. a) 0.819 **b)** 0.510 **c)** 6.392 **d)** 1.327
e) 0.423 **f)** 2.669 **g)** −0.900 **h)** 0.978
i) 1.004 **j)** −0.306 **k)** −0.777
l) −1.015 **2. a)** 74° **b)** 28° **c)** 27°
d) 28°, 152° **e)** 19°, 161° **f)** 66°

3. sin: **a)** 0 **b)** $\frac{1}{2}$ **c)** $\frac{1}{\sqrt{2}}$ **d)** $\frac{\sqrt{3}}{2}$
e) 1 **f)** $\frac{\sqrt{3}}{2}$ **g)** $\frac{1}{\sqrt{2}}$ **h)** $\frac{1}{2}$ **i)** 0;

csc: **a), i)** Undefined **b)** 2 **c)** $\sqrt{2}$
d) $\frac{2}{\sqrt{3}}$ **e)** 1 **f)** $\frac{2}{\sqrt{3}}$ **g)** $\sqrt{2}$ **h)** 2;

cos: **a)** 1 **b)** $\frac{\sqrt{3}}{2}$ **c)** $\frac{1}{\sqrt{2}}$ **d)** $\frac{1}{2}$
e) 0 **f)** $-\frac{1}{2}$ **g)** $-\frac{1}{\sqrt{2}}$ **h)** $-\frac{\sqrt{3}}{2}$
i) −1;

sec: **a)** 1 **b)** $\frac{2}{\sqrt{3}}$ **c)** $\sqrt{2}$ **d)** 2
e) Undefined **f)** −2 **g)** $-\sqrt{2}$ **h)** $-\frac{2}{\sqrt{3}}$
i) −1; tan: **a)** 0 **b)** $\frac{1}{\sqrt{3}}$ **c)** 1 **d)** $\sqrt{3}$
e) Undefined **f)** $-\sqrt{3}$ **g)** −1 **h)** $-\frac{1}{\sqrt{3}}$
i) 0; cot: **a), i)** Undefined **b)** $\sqrt{3}$ **c)** 1
d) $\frac{1}{\sqrt{3}}$ **e)** 0 **f)** $-\frac{1}{\sqrt{3}}$ **g)** −1
h) $-\sqrt{3}$

4. a) $\sin \theta = \frac{15}{17}$; $\csc \theta = \frac{17}{15}$; $\sec \theta = \frac{17}{8}$;
$\tan \theta = \frac{15}{8}$; $\cot \theta = \frac{8}{15}$
b) $\sin \theta = \frac{12}{13}$; $\csc \theta = \frac{13}{12}$; $\cos \theta = \frac{12}{13}$;
$\sec \theta = \frac{13}{12}$; $\cot \theta = \frac{5}{12}$ **c)** $\sin \theta = \frac{8\sqrt{5}}{21}$;
$\csc \theta = \frac{21}{8\sqrt{5}}$; $\cos \theta = \frac{11}{21}$; $\tan \theta = \frac{8\sqrt{5}}{11}$;
$\cot \theta = \frac{11}{8\sqrt{5}}$ **d)** $\csc \theta = \frac{9}{5}$; $\cos \theta = \frac{2\sqrt{14}}{9}$;
$\sec \theta = \frac{9}{2\sqrt{14}}$; $\tan \theta = \frac{5}{2\sqrt{14}}$; $\cot \theta = \frac{2\sqrt{14}}{5}$

5. a) $\csc \theta = \frac{b}{a}$; $\cos \theta = \frac{\sqrt{b^2 - a^2}}{b}$;
$\sec \theta = \frac{b}{\sqrt{b^2 - a^2}}$; $\tan \theta = \frac{a}{\sqrt{b^2 - a^2}}$;
$\cot \theta = \frac{\sqrt{b^2 - a^2}}{a}$

b) $\sin \theta = \frac{p}{\sqrt{2p^2 + 2pq + q^2}}$;
$\csc \theta = \frac{\sqrt{2p^2 + 2pq + q^2}}{p}$;
$\cos \theta = \frac{p + q}{\sqrt{2p^2 + 2pq + q^2}}$;
$\sec \theta = \frac{\sqrt{2p^2 + 2pq + q^2}}{p + q}$; $\cot \theta = \frac{p + q}{p}$
c) $\sin \theta = \frac{\sqrt{3m^2 - 10m - 8}}{2m - 1}$;
$\csc \theta = \frac{2m - 1}{\sqrt{3m^2 - 10m - 8}}$; $\cos \theta = \frac{m + 3}{2m - 1}$;
$\tan \theta = \frac{\sqrt{3m^2 - 10m - 8}}{m + 3}$;
$\cot \theta = \frac{m + 3}{\sqrt{3m^2 - 10m - 8}}$

6. a) AC = 31.0, $\angle$A = 61°, $\angle$C = 29°
b) AB = 15, $\angle$C = 56°, $\angle$A = 34°
c) BC = 13.6, AC = 44.2, $\angle$A = 18°
d) $\angle$C = 55°, AB = 9.8, BC = 6.9
7. a) $q = 11.8$, $\angle P = 64.0°$, $\angle R = 41.0°$
b) $\angle Q = 44.7°$, $\angle P = 83.3°$, $p = 35.3$
c) $\angle R = 37.0°$, $q = 50.3$, $r = 31.3$
d) $\angle Q = 113.7°$, $\angle R = 36.8°$, $\angle P = 29.6°$
e) $\angle R = 64.0°$, $\angle P = 59.0°$, $p = 42.9$ or
$\angle R = 116.0°$, $\angle P = 7.0°$, $p = 6.1$
f) $r = 14.4$, $\angle P = 32.0°$, $\angle Q = 115.0°$
8. 6.6° **9.** 35.4 m **10. a)** 987 m **b)** 51°
11. 85.6 m **12.** 52.1° **13.** 4.2 km

Quadratics, Equations, Factoring, Roots

12. Factor.
    a) $x^2 + 8xy + 16y^2 - 9$
    b) $16x^2 - 4y^2 + 12y - 9$
    c) $4(x + 2y + z)^2 - 9(x - 2y + z)^2$
    d) $a^2 - 2a + 1 - x^2 - 2xy - y^2$
    e) $2x^2(3x - 1) - 5(3x - 1) + 6x - 2$
    f) $(3x + 2)^2 + 6xy + 4y + y^2$
13. One factor of $6m^3 - 6 - 29m - 31m^2$ is $1 + 3m$. Find the other factors.
14. Divide $x^4 - 4x^3 - 6x^5 + x + 5x^2 + 15$ by $2x^2 - x + 3$.
15. If $p(x) = 2x^2 + 5x - 4$, evaluate:
    a) $p(-2)$ b) $p(3a)$ c) $p(8x - 5)$
16. Factor completely.
    a) $x^3 + 4x^2 + 5x + 2$
    b) $x^3 + 2x^2 - 3$
    c) $27x^3 + 125y^3$
    d) $4x^3 - x + 8x^2 - 2$
17. Solve and check.
    a) $6(x + 1) - 12x = 3 - 4(2x - 1)$
    b) $\dfrac{2x - 4}{5} - \dfrac{x - 3}{4} = \dfrac{5 - 3x}{8}$
    c) $3x^2 - x - 14 = 0$
    d) $2x^2 + 4x - 7 = 0$
    e) $5x^2 + 8x + 3 = 0$
    f) $\dfrac{4}{x - 1} - \dfrac{5}{x + 2} = \dfrac{3}{x}$
18. Solve.
    a) $2x^2 - 5x + 4 = 0$
    b) $4x^3 - 3x^2 + 2x = 0$
    c) $x^3 - 7x - 6 = 0$
    d) $2x^3 - 3x^2 - 8x + 12 = 0$
    e) $(x^2 - 4x + 5)(x^2 - 4x + 2) = -2$
    f) $\dfrac{4}{x + 1} - \dfrac{12}{x + 3} = \dfrac{-5}{x + 2}$
19. Solve for the variable indicated.
    a) $s = 3u + \frac{1}{2}at^2$, t
    b) $m = 2\sqrt{3n + 5} - 1$, n
20. Determine the nature of the roots of each equation.
    a) $4x^2 + 20x + 25 = 0$ b) $2x^2 - 5x + 2 = 0$ c) $3x^2 - 4x + 8 = 0$
21. Write a quadratic equation given:
    a) the roots $\frac{3}{4}$ and -2
    b) the root $-2 + 3i$.
22. For what values of k does:
    a) $3x^2 - kx + 2 = 0$ have equal roots
    b) $2x^2 - 5x - k = 0$ have complex roots?
23. Solve.
    a) $|2x - 1| = 7$
    b) $|x + 5| - x + 2 = 3$
    c) $x + 2|x + 1| = 8$
    d) $|3x + 2| = 2x - 4$
24. Solve.
    a) $\sqrt{5x + 4} = 7$
    b) $\sqrt{2x - 3} + 3 = x$
    c) $\sqrt{x + 5} + x = 7$
    d) $2x + 2\sqrt{x} = 5$

Answers

12. a) $(x + 4y + 3)(x + 4y - 3)$
b) $(4x - 2y + 3)(4x + 2y - 3)$
c) $(5x - 2y + 5z)(-x + 10y - z)$
d) $(a - 1 - x - y)(a - 1 + x + y)$
e) $(3x - 1)(2x - 1)(x - 2)$
f) $(3x + y + 2)^2$

13. $2m + 1$, $m - 6$ **14.** $-3x^3 - x^2 + 2x + 5$

15. a) -6 **b)** $18a^2 + 15a - 4$
c) $128x^2 - 120x + 21$

16. a) $(x + 1)^2(x + 2)$ **b)** $(x - 1)(x^2 + 3x + 3)$
c) $(3x + 5y)(9x^2 - 15xy + 25y^2)$
d) $(2x + 1)(2x - 1)(x + 2)$

17. a) $\frac{1}{2}$ **b)** $\frac{9}{7}$ **c)** $\frac{7}{3}, -2$ **d)** $\dfrac{-2 \pm 3\sqrt{2}}{2}$
e) $-\frac{3}{5}, -1$ **f)** $-\frac{1}{2}, 3$ **18. a)** $\dfrac{5 \pm \sqrt{7}i}{4}$
b) $0, \dfrac{3 \pm \sqrt{23}i}{8}$ **c)** $-1, -2, 3$ **d)** $\frac{3}{2}, \pm 2$
e) $1, 3, 2, 2$ **f)** $-\frac{5}{3}, 3$

19. a) $t = \pm\sqrt{\dfrac{2(s - 3u)}{a}}$ **b)** $n = \dfrac{(m + 1)^2 - 20}{12}$

20. a) 2 equal real roots **b)** 2 different real roots
c) 2 complex roots

21. a) $4x^2 + 5x - 6 = 0$ **b)** $x^2 + 4x + 13 = 0$

22. a) $k = \pm 2\sqrt{6}$ **b)** $k < -\frac{25}{8}$

23. a) $4, -3$ **b)** -3 **c)** $2, -10$ **d)** No solution

24. a) 9 **b)** 6 **c)** 4 **d)** $\dfrac{6 - \sqrt{11}}{2}$

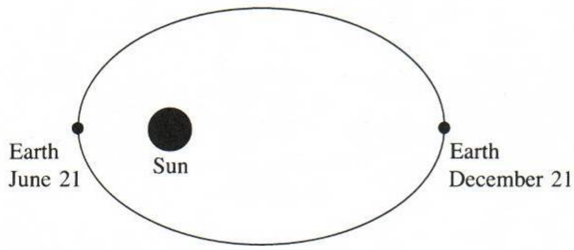
Even More Trigonometry

11. At St. John's, the time of the sunrise on the nth day of the year is given by this formula.

$$t = 1.89 \sin 2\pi\frac{(n - 80)}{365} + 6.41$$

a) Calculate the time the sun rises on October 20 (day 293).
b) Give one significant reason why the actual time of the sunrise on October 20 may differ somewhat from your answer in part a).

12. On December 21 each year, the sun is closest to the Earth, at approximately 147.2 million kilometres. On June 21 the sun is at its greatest distance, approximately 152.2 million kilometres.
a) Express the distance d from the Earth to the sun as a sinusoidal function of the number of the day of the year.
b) Use the function to calculate the approximate distance from the Earth to the sun on:
i) March 1   ii) April 30   iii) September 2.



Ⓒ

13. a) In *Example 1*, if the calculator is in *degree mode*, what change would have to be made in the equation?
b) Solve *Example 1* using your calculator in degree mode.

14. In the solution of *Example 1*, a cosine function was used. Solve *Example 1* using a sine function.

15. On the nth day of the year, the number of hours of daylight at Victoria is given by this formula.

$$h = 3.98 \sin 2\pi\frac{(n - 80)}{365} + 12.16$$

a) About how many hours of daylight should there be today?
b) On what dates should there be about 10 h of daylight?

16. In *Example 1*, calculate to the nearest minute the first time after 4:30 A.M. when the depth of the water is: a) 6.0 m   b) 3.0 m.

INVESTIGATE

1. From an almanac or newspaper files, determine the approximate time the sun rises and sets in your locality on June 21 and December 21.
2. Determine equations which represent the time the sun rises and sets on the nth day of the year.
3. Use the equations to predict the time the sun rises and sets today, and check your results in the newspaper.

Answers

iv) 4:53 P.M.   **11. a)** 5:28 A.M.

12. a) Typical answer:

$$d = 2.5 \cos \frac{2\pi(n - 172)}{365} + 149.7$$

b) i) 148.8 million km   **ii)** 151.3 million km

iii) 150.5 million km

15. b) Feb. 16 and Oct. 23   **16. a)** 6:56 A.M.

b) 9:02 A.M.

Conics

1. Graph each relation and identify the curve.
   a) $\frac{x^2}{16}+\frac{y^2}{9}=1$ b) $x^2 = 4y$ c) $x^2 + y^2 = 25$
   d) $x^2 - y^2 = 9$ e) $4x^2 + 4y^2 = 49$ f) $9x^2 - 16y^2 = -144$

2. Determine if the given point is on the given conic. Identify the conic.
   a) $(-2,-20)$; $y = 5x^2$ b) $(-4,7)$; $x^2 + y^2 = 65$
   c) $(-7,2)$; $\frac{x^2}{25} - \frac{y^2}{4} = 1$ d) $(2,-\sqrt{2})$; $4x^2 + 9y^2 = 36$

3. Find the equation of a parabola with vertex (0,0) and axis of symmetry the y-axis if it passes through: a) (4,2) b) $(-4,-24)$.

4. Find the equation of a circle with:
   a) centre $(5,-2)$, radius 2 b) centre (0,3), radius $3\sqrt{2}$.

5. A rectangular hyperbola with centre (0,0) and vertices on the y-axis passes through $(6,-9)$. Is $(-2,7)$ on the hyperbola?

6. For each conic, state the coordinates of its vertices and the lengths of its axes.
   a) $\frac{x^2}{25}+\frac{y^2}{4}=1$ b) $\frac{x^2}{9}-\frac{y^2}{16}=1$ c) $9x^2 + 4y^2 = 36$

7. Find the centre and the radius of each circle.
   a) $4x^2 + 4y^2 = 25$ b) $x^2 + y^2 - 4x + 6y - 12 = 0$

8. State the equation of an ellipse with centre (0,0) and major axis on the x-axis if:
   a) the major axis has length 20 and the minor axis has length $4\sqrt{3}$.
   b) one vertex is $(-5,0)$, and it passes through $\left(-4,\frac{9}{5}\right)$.

9. State the equation of a hyperbola with centre (0,0) and transverse axis on the y-axis if:
   a) one vertex is $(0,-7)$, and the conjugate axis has length 12
   b) the transverse axis has length 10 and an asymptote is $y = 3x$.

10. Sketch each conic.
    a) $2x^2 + 8y = 0$ b) $3x^2 + 3y^2 = 48$ c) $4x^2 - 25y^2 = 100$

11. A stone thrown horizontally from a bridge 25 m above the river splashes in the water 40 m from the base of the bridge. If the stone falls in a parabolic path, find its equation.

12. One of the supports in a retractable roof of a sports complex is semi-elliptical. If it is 25 m high and spans 60 m, find its equation.

13. The base of a bridge arch is 80 m wide and 25 m high. Find its equation if the arch is in the shape of a rectangular hyperbola.

14. The arch of a bridge is semi-elliptical. The base is 30 m wide and 9 m below the vertex of the arch. How far below the vertex of the arch is a point 10 m from the centre line of the arch?

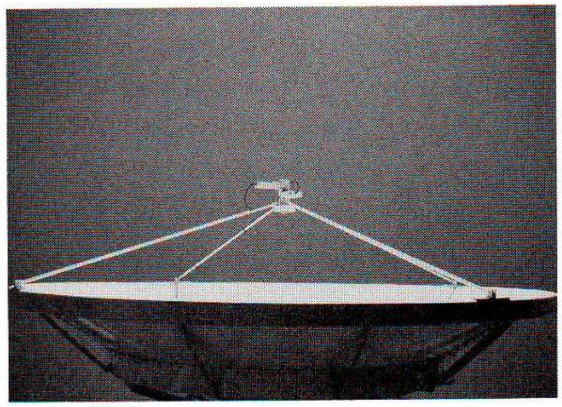
Answers

1. a) Ellipse **b)** Parabola **c),e)** Circle **d)** Rectangular hyperbola **f)** Hyperbola
2. a) No; parabola **b)** Yes; circle **c)** No; hyperbola **d)** No; ellipse
3. a) $y = \frac{1}{8}x^2$ **b)** $y = -1.5x^2$
4. a) $(x-5)^2 + (y+2)^2 = 4$ **b)** $x^2 + (y-3)^2 = 18$
5. Yes **6. a)** $(\pm 5,0)$; 10; 4 **b)** $(\pm 3,0)$; 6; 8 **c)** $(0,\pm 3)$; 6; 4
7. a) (0,0); 2.5 **b)** $(2,-3)$; 5
8. a) $\frac{x^2}{100}+\frac{y^2}{12}=1$ **b)** $\frac{x^2}{25}+\frac{y^2}{9}=1$
9. a) $\frac{x^2}{49}-\frac{y^2}{144}=-1$ **b)** $\frac{9x^2}{25}-\frac{y^2}{25}=-1$
11. $y = -\frac{1}{64}x^2$ **12.** $\frac{x^2}{900}+\frac{y^2}{625}=1$
13. $x^2 - y^2 = -380.25$ **14.** 2.3 m

Interesting Properties of Conics

The Reflector Property of the Parabola

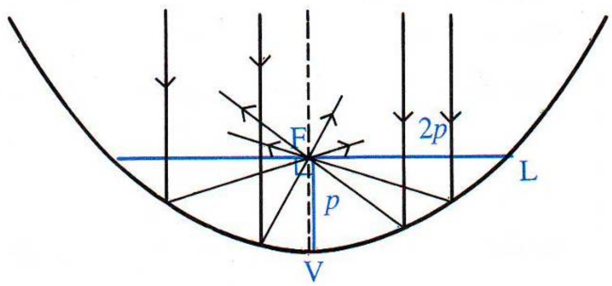
Reflector Property of the Parabola



We have all seen dish antennas for receiving TV signals from satellites. These antennas have parabolic cross sections. When the antenna is aimed at a satellite, the signals entering the antenna are reflected to the receiver, which is placed at the focus of the antenna.

Every parabola has a *focus*, which is a particular point on the axis of symmetry. The position of the focus can be defined as follows.

For any parabola, the *focus* is the point on the axis of symmetry which is half as far from the vertex as it is from the parabola, measured along a line perpendicular to the axis of symmetry. For example, in the diagram, FV $= p$, and FL $= 2p$. That is, F is half as far from V as from L. Hence, F is the focus of the parabola. Every parabola has one and only one focus.



You can illustrate the reflector property of the parabola by completing the questions below.

QUESTIONS

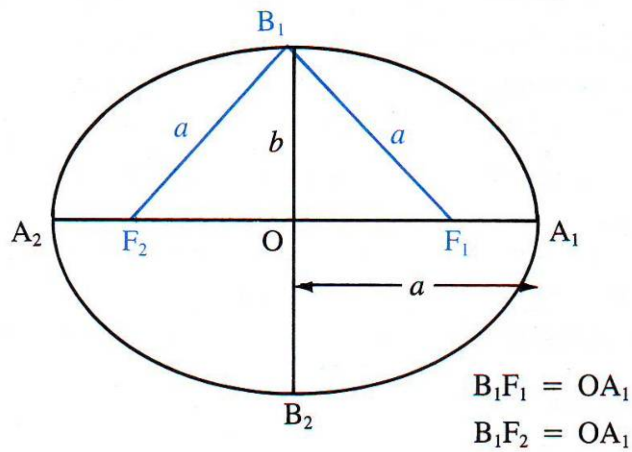
1. a) Use a table of values to construct an accurate graph of the parabola defined by $y = \frac{1}{8}x^2$ for values of x between -8 and 8.
   b) Mark the point F(0,2) on the graph. Verify that F satisfies the above definition of the focus.
2. a) Mark any point P on the parabola you constructed in *Question 1*. Join PF, and draw a line PM parallel to the axis of symmetry. By estimation, draw a tangent to the parabola at P. Verify that PF and PM form equal angles with the tangent.
   b) Repeat part a) for other points P on the parabola.
3. Use the above definition of the focus to prove that the coordinates of the focus of the parabola defined by $y = ax^2$ are $\left(0, \frac{1}{4a}\right)$.

Reflector Property of the Ellipse

In the Capitol at Washington, D.C., there is a room known as the whispering gallery. In this room there are two points a considerable distance apart, where a whisper at one point can be heard at the other point. The room has an elliptical cross section. When someone standing at one of the points whispers, the curved walls reflect the sound waves and focus them at the other point, where the whisper can be clearly heard.

Every ellipse has two *foci*, which are particular points on the major axis. The positions of the foci can be defined as follows.

For any ellipse, let the centre be O, and let A_1 and A_2 be the vertices. Let the *semi-major axis* OA_1 have length a. Let B_1 and B_2 be the points at the ends of the minor axis. Then, the *foci* are the points F_1 and F_2 on A_1A_2 such that $B_1F_1 = a$ and $B_1F_2 = a$.



You can illustrate the reflector property of the ellipse by completing the questions below.

QUESTIONS

1. a) Use a table of values to construct an accurate graph of the ellipse defined by $\dfrac{x^2}{100} + \dfrac{y^2}{64} = 1$.

   b) Mark the points $F_1(6,0)$ and $F_2(-6,0)$ on the graph. Verify that F_1 and F_2 satisfy the above definition of the foci.

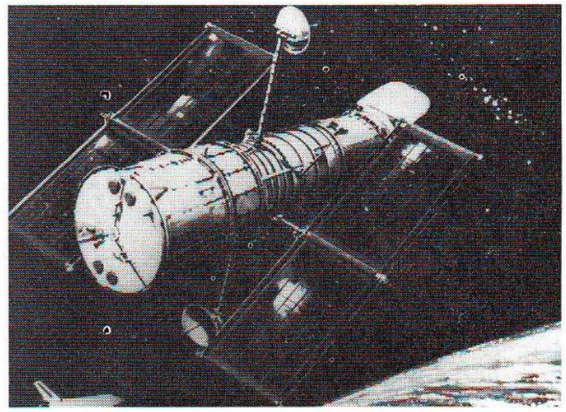
2. a) Mark any point P on the ellipse you constructed in *Question 1*. Join PF_1 and PF_2. By estimation, draw a tangent to the ellipse at P. Verify that PF_1 and PF_2 form equal angles with the tangent.

   b) Repeat part a) for other points P on the ellipse.

3. Use the above definition of the foci to find expressions for the coordinates of the foci of the ellipse defined by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

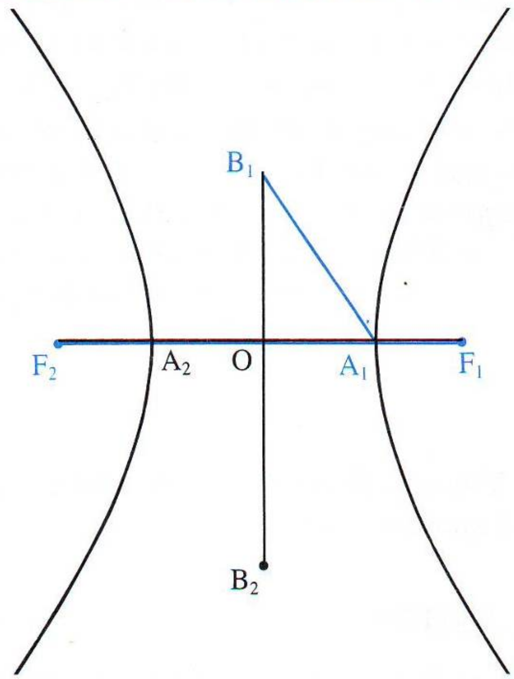
Reflector Property of the Hyperbola

Like the parabola and the ellipse, the hyperbola also has a reflector property. This property is sometimes employed in the design of telescopes. The *Space Telescope*, for example, contains two hyperboloidal mirrors. Light striking the primary mirror is reflected to the secondary mirror, where it is reflected back through a hole in the centre of the primary mirror to a focus behind the primary mirror.



Every hyperbola has two *foci*, which are particular points on the axis of symmetry containing the transverse axis. The positions of the foci can be defined as follows.

For any hyperbola, let the centre be O, and let A_1 and A_2 be the vertices. Let the *semi-transverse axis* OA_1 have length a. Let B_1 and B_2 be the points at the ends of the conjugate axis. Then, the *foci* are the points F_1 and F_2 on the line $\overleftrightarrow{A_1A_2}$ such that $OF_1 = OF_2 = A_1B_1$.



You can illustrate the reflector property of the hyperbola by completing the questions below.

QUESTIONS

1. a) Use a table of values to construct an accurate graph of the hyperbola defined by $\dfrac{x^2}{36} - \dfrac{y^2}{64} = 1$.

   b) Mark the points $F_1(10,0)$ and $F_2(-10,0)$ on the graph. Verify that F_1 and F_2 satisfy the above definition of the foci.

2. a) Mark any point P on the hyperbola you constructed in *Question 1*. Join PF_1 and PF_2. By estimation, draw a tangent to the hyperbola at P. Verify that PF_1 and PF_2 form equal angles with the tangent.

   b) Repeat part a) for other points P on the hyperbola.

3. Use the above definition of the foci to find expressions for the coordinates of the foci of the hyperbola defined by $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Review Questions

1. What is the true spirit of mathematics?

2. What is mathematical notation?

3. In what respects can mathematics be considered a language?

4. What do we mean by "valid reasoning?" What role does logic play in reasoning?

5. Give one example of valid reasoning and one example of invalid reasoning.

6. Why is it foolish to try to learn mathematics (or any other subject for that matter) by blindly memorizing? What is a better approach?

7. What is the only equation you ***need*** to know to determine an equation of a line? Give an example.

8. Use the Pythagorean theorem to find an expression for the distance between $P(a, b)$ and $Q(c, d)$.

9. Using the concept of the average of two numbers, find the midpoint of the line segment whose endpoints are $P(a, b)$ and $Q(c, d)$.

10. What did you learn through the Varignon parallelogram investigation?

Open-ended Research Question

We are all fairly well acquainted with the daily routines of people in fields such as auto mechanics, engineering, appliance repair, cosmetology, law, medicine, dentistry, education and accounting. However, there are very few people who would be able to explain what a typical day would be like for a mathematician. Do some research to find out what a mathematician's job entails. In addition, find out as much as you can about George Polya.

