MGA4U0 Unit 1 – An Introduction to Proof through Euclidean Geometry

MGA4U0 UNIT 1 – AN INTRODUCTION TO PROOF THROUGH EUCLIDEAN GEOMETRY 1

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What is a Proof? – Inductive and Deductive Reasoning

"A demonstration is an argument that will convince a reasonable man. A proof is an argument that can convince even an unreasonable man." (Mark Kac, 20th century Polish American mathematician)

Nolfi's Intuitive Definition of "Proof"

A proof is a ***series*** or ***"chain" of inferences*** (i.e. "if…then" statements, formally known as ***logical implications*** or ***conditional statements***) that allows us to proceed ***logically*** from an ***initial premise****,* which is known or assumed to be true, to a desired ***final conclusion***.

Hopefully, my definition is somewhat easier to understand than that of a former Prime Minister:

Jean Chrétien, a former Prime Minister of Canada, was quoted by CBC News as saying, "A proof is a proof. What kind of a proof? It's a proof. A proof is a proof. And when you have a good proof, it's because it's proven."

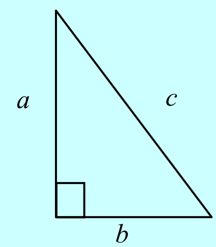
Pythagorean Theorem Example

The ***initial premise***

The ***final conclusion***

In ***any*** right triangle,
the square of the hypotenuse is equal to the sum of the squares of the other two sides.

That is, $c^2 = a^2 + b^2$.



Proof:

Begin with right ΔABC and construct the altitude AD.

Since $\Delta ABC \sim \Delta BDA$ (AA similarity theorem),

$$\therefore \frac{AB}{BC} = \frac{BD}{AB}.$$

Since $\Delta ABC \sim \Delta ADC$ (AA similarity theorem),

$$\therefore \frac{AC}{BC} = \frac{DC}{AC}.$$

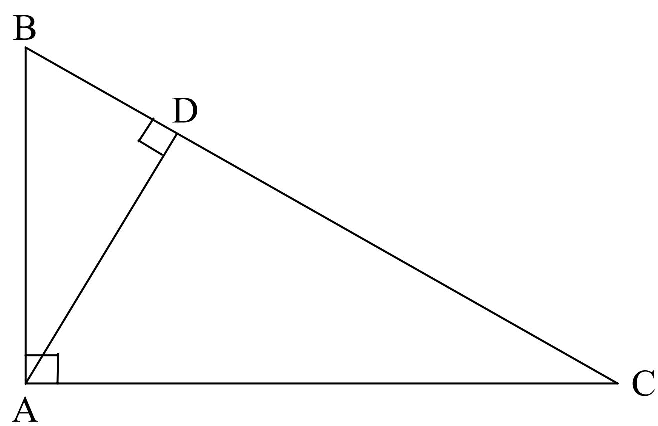
Therefore,

$$AB \cdot AB = BD \cdot BC \text{ and } AC \cdot AC = DC \cdot BC$$

Summing up, we obtain

$$\begin{aligned} AB \cdot AB + AC \cdot AC &= BD \cdot BC + DC \cdot BC \\ &= (BD + DC) \cdot BC \\ &= BC \cdot BC \\ &= BC^2 \end{aligned}$$

$$\therefore AB^2 + AC^2 = BC^2 \ //$$



See http://www.cut-the-knot.org/pythagoras/index.shtml for a multitude of proofs of the Pythagorean Theorem.

Research Exercises

1. Use the Internet to find James Garfield's proof of the Pythagorean Theorem. If you don't know who James Garfield was, look that up too!

2. Jean Chrétien's very "colourful" definition of a proof was given to a CBC reporter who asked Prime Minister Chrétien a question about the war in Iraq (i.e. George W. Bush's war in Iraq, not the Gulf War of the late 1980s). What was the question?

The Meaning of π – An Example of Deductive Reasoning

The following is an example of a typical conversation between Mr. Nolfi and a student who blindly memorizes formulas:

Student: Sir, I can't remember whether the area of a circle is πr^2 or $2\pi r$. Which one is it?

Mr. Nolfi: If you remember the meaning of π, you should be able to figure it out.

Student: How can 3.14 help me make this decision? It's only a number!

Mr. Nolfi: How dare you say something so disrespectful about one of the most revered numbers in the mathematical lexicon! (Just kidding. I wouldn't really say that.) It's true that the number 3.14 is an approximate value of π. But I asked you for its ***meaning***, not its value.

Student: I didn't know that π has a meaning. I thought that it was just a "magic" number.

Mr. Nolfi: Leave magic to the magicians. In mathematics, every term (except for primitive terms) has a very precise definition. Read the following carefully and you'll never need to ask your original question ever again!

In ***any*** circle, the ***ratio*** of the ***circumference*** to the ***diameter*** is equal to a ***constant*** value that we call π. That is,

$$C:d=\pi.$$

Alternatively, this may be written as

$$\frac{C}{d}=\pi$$

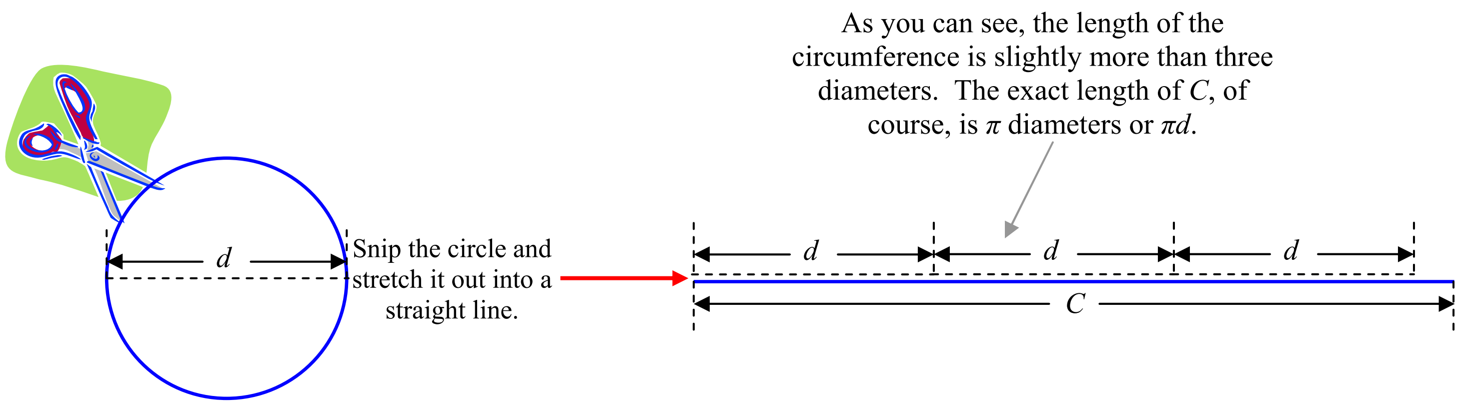
or in the more familiar form

$$C=\pi d.$$

If we recall that $d = 2r$, then we finally arrive at the most common form of this ***relationship***,

$$C=2\pi r.$$

> This is an example of a ***deductive argument***. Each statement ***follows logically*** from the previous statement.
>
> That is, the argument takes the form "If P is true then Q must also be true" or more concisely, "P implies Q."



Mr. Nolfi: So you see, by understanding the meaning of π, you can deduce that $C=2\pi r$. Therefore, the formula for the area must be $A=\pi r^2$. Furthermore, it is not possible for the expression $2\pi r$ to yield units of area. The number 2π is dimensionless and r is measured in units of distance such as metres. Therefore, the expression $2\pi r$ must result in a value measured in units of distance. On the other hand, the expression πr^2 must give a value measured in units of area because $r^2=r(r)$, which involves multiplying a value measured in units of distance by itself. Therefore, by considering units alone, we are drawn to the inescapable conclusion that the area of a circle must be πr^2 and ***not*** $2\pi r$!

Examples

$2\pi r \doteq 2(3.14)(3.6\text{ cm}) = 22.608\text{ cm} \rightarrow$ This answer cannot possibly measure area because cm is a unit of distance.

Therefore, πr^2 must be the correct expression for calculating the area of a circle.

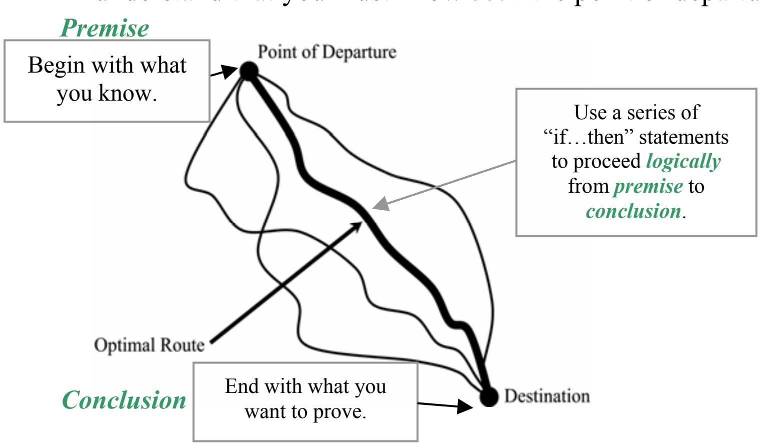
$\pi r^2 \doteq 3.14(3.6\text{ cm})^2 = 3.14(3.6\text{ cm})(3.6\text{ cm}) = 40.6944\text{ cm}^2 \rightarrow$ Notice that the unit cm^2 is appropriate for area.

Tips on becoming a Powerful "Prover"

1. Do not ***fear*** or be ***intimidated by*** the word "proof." The more fearful you are, the less likely you are to succeed.
2. ***Expect*** to encounter roadblocks and obstacles of all kinds. This is a ***normal*** and ***natural*** part of the process! It is ***extremely unrealistic*** to expect success on the first attempt every time that you set out to write a proof (or solve a problem)! There is no ***formula*** that can be used to generate all proofs. Proving involves a great deal of educated guessing, trial and error and lots of dead ends. To be successful, you must be perseverant and determined.
3. ***Never, ever assume what you are trying to prove!*** What you are trying to prove should be the ***final conclusion*** of your proof! The ***given*** information can/should be used for the ***initial premise***.
4. ***For a mathematical statement to be true, it must be true in all cases!*** Therefore, proofs must be ***general***, that is, they must apply to ***all cases***. A valid proof must demonstrate that the statement is true in ***all cases!***
5. Examples ***cannot*** be used to prove statements. They can be used only to check the validity of a ***conjecture***. If you ***think*** that a certain statement is true and it works in every example that you try, then it is ***possible*** that it is true in ***general***. (If a statement involves only a ***finite*** number of cases, then it is possible to prove the statement by listing every possible case and proving each one separately. However, this could prove to be extremely tedious and time consuming if the number of possible cases is large.)
6. To ***disprove*** a statement, one ***counterexample*** is sufficient. For instance, many students often blindly ***assume*** that $\cancel{\sqrt{x+y}=\sqrt{x}+\sqrt{y}}$. Upon closer inspection, however, we can easily find an example that disproves this statement. If we let $x = 16$ and $y = 9$, we instantly see that the left-hand-side equals $\sqrt{16+9}=\sqrt{25}=5$ while the right-hand-side equals $\sqrt{16}+\sqrt{9}=4+3=7$. This single counterexample proves that the original assumption was wrong! (In fact, very few classes of functions exhibit this very convenient "separability" behaviour.)

This is wrong!

7. Think of writing a proof as planning a route from a point of departure to a destination. It is ***imperative*** that you understand that you must know ***both*** the point of departure ***and*** the destination ***before*** you plan your route!



An effective problem solver is much like someone who is good at planning travel. If you were to take a trip by automobile, for instance, you would create a plan ***before*** embarking on your voyage.

1. Note the point of departure.
2. Note the destination.
3. Consider many different routes for travelling from the point of departure to the destination.
4. Choose the best route.
5. Finally, pack your bags, get into your car and go!

You must approach problem solving in much the same way. Unfortunately, many of my students solve problems in the same way that a confused traveller would plan a trip (step 5 would be carried out before the destination is known). **How can one possibly set out for an unknown destination?**

8. Beware of ***logical fallacies*** (see Appendix 3).
9. Keep in mind the process of making a ***conjecture*** and testing it. If all tests are successful, then it may be worthwhile trying to prove that the conjecture is true in ***general***.

Make a conjecture. → **Test the conjecture.** →

- **If even a single test fails, the conjecture *must* be false.**
- **If all tests are successful, the conjecture *might* be true. It may be worthwhile trying to find a general proof.**

Inductive Reasoning

This is an example of ***inductive reasoning***. In general, inductive reasoning uses ***clues*** or ***data*** to support ***possible conclusions***. It must be emphasized, however, that inductive reasoning ***does not lead*** to an incontrovertible conclusion. The conclusion of an inductive argument is always subject to doubt as it is based on ***incomplete evidence***. The ***strength*** of an inductive argument depends on both the ***quantity*** and the ***quality*** of the ***data***.

Example of Inductive Reasoning (The "Conjecture, Test the Conjecture, Try to Prove" Process)

(See Appendix 2 for more information on inductive reasoning.)

Conjecture

All integers ending in 5, when squared, yield a number ending in 25.

Test the Conjecture

$5^2 = 25$, $15^2 = 225$, $25^2 = 625$, $35^2 = 1225$, $45^2 = 2025$, $55^2 = 3025$, $65^2 = 4225$, $75^2 = 5625$, $85^2 = 7225$, $95^2 = 9025$, $105^2 = 11025$

We have not found any exceptions to our "rule." Therefore, it ***might*** be true in general. Let's try to prove it.

Example of Deductive Reasoning – Deductive Proof of the Conjecture

(See Appendix 1 for more information on deductive reasoning.)

Proof of Above Conjecture:

If an integer ends in 5, then it must be of the form $10n + 5$, where $n \in \mathbb{Z}$ (n is an element of the set of integers).

$$\begin{aligned}\therefore (10n+5)^2 &= [5(2n+1)]^2 \\ &= 25(2n+1)^2 \\ &= 25(4n^2+4n+1) \\ &= 100n^2+100n+25 \\ &= 100(n^2+n)+25\end{aligned}$$

$$\therefore (10n+5)^2 = 100(n^2+n)+25$$

Deductive Reasoning

This is an example of ***deductive reasoning***. Through a series of ***logical implications*** (i.e. "if…then" or ***conditional*** statements), a conclusion can be ***deduced***. Unlike inductive reasoning, an argument that is based on deductive reasoning leads to an unassailable, irrefutable conclusion. That is, the truth of the premise ***guarantees*** the truth of the conclusion.

Deduce – To reach a conclusion by reasoning.

Since $100(n^2 + n)$ is divisible by 100, $100(n^2 + n) + 25$ must end in 25. Therefore, any integer ending in 5, when squared, must yield an integer ending in 25. //

How to Distinguish between a Deductive Argument and an Inductive Argument

Deductive Reasoning	***Inductive Reasoning***
Deductive arguments take the following form: "If ***Cause*** Then ***Effect***" or "If ***Premise*** Then ***Conclusion***" In a deductive argument, we know that a ***cause*** (premise) produces a certain ***effect*** (conclusion). If we observe the cause, we can deduce that the effect ***must*** occur. These arguments always produce ***definitive*** conclusions.	Inductive arguments are ***in a sense*** the opposite of deductive arguments. An inductive argument generally proceeds in the following way: • First we observe the ***effect***. • We ***collect evidence*** by making ***observations*** and taking into account our ***experiences***. • We then use this evidence to ***speculate*** about a possible ***cause***. Conclusions of inductive arguments are ***not definitive***. • ***Scientific studies*** involve a great deal of inductive reasoning.
Examples of Deductive Reasoning from Everyday Life	***Examples of Inductive Reasoning from Everyday Life***
1. If I spill my drink on the floor, the floor will get wet.	**1.** The floor is wet. It's possible that a drink was spilled.
2. When students "forget" to do homework, Mr. Nolfi gets angry!	**2.** Mr. Nolfi is angry! A plausible explanation is that many students "forgot" to do their homework.
3. If a student is caught cheating, Mr. Nolfi will assign a mark of zero to him/her, ridicule the student publicly, turn red in the face and yell like a raving madman whose underwear are on fire!	**3.** Mr. Nolfi assigned a mark of zero to a student, ridiculed him/her publicly, turned red in the face and yelled like a raving madman whose underwear were on fire! Maybe the student was caught cheating.
4. Drinking too much alcohol causes drunkenness.	**4.** A person is behaving in a drunken manner. It is likely that he/she drank too much alcohol.

Homework

1. Do the research exercises on page IPEG-4 (not necessary if already discussed in class)
2. Read section 1.1 in our textbook (pages 3 – 5)
3. Read section 1.2 in our textbook (pages 7 – 9).
4. Do the following exercises: p. 9 #3, p. 10 #4, 6, 10, 11

Axiomatic Systems of Reasoning

The conclusion of every ***inference*** (conditional statement or logical implication) ***must be supported by a premise***. If the premise is true, then the conclusion must also be true. But how do we know that the premise is true? There are only two possible answers to this question. Either the premise is itself the conclusion of some other inference or the premise is assumed to be true. It is not possible for ***every premise*** to be the conclusion of some other inference because this leads to the problem of ***infinite regression***. Therefore, we must accept that every system of reasoning eventually boils down to a set of basic assumptions, which are called ***axioms***.

To understand the idea of infinite regression, consider the example given below (taken from http://en.wikipedia.org/wiki/Turtles_all_the_way_down.)

> **"Turtles all the way down"** refers to an ***infinite regression*** myth about the nature of the universe.
>
> The most widely known version today appears in Stephen Hawking's 1988 book *A Brief History of Time*, which begins with an anecdote about an encounter between a scientist and an old lady:
>
> A well-known scientist (some say it was Bertrand Russell) once gave a public lecture on astronomy. He described how the Earth orbits around the sun and how the sun, in turn, orbits around the centre of a vast collection of stars called our galaxy.
>
> At the end of the lecture, a little old lady at the back of the room got up and said: "What you have told us is rubbish. The world is really a flat plate supported on the back of a giant tortoise."
>
> The scientist gave a superior smile before replying, "What is the tortoise standing on?"
>
> "You're very clever, young man, very clever," said the old lady. "But it's turtles all the way down."

This patently ridiculous notion of an infinite tower of tortoises perfectly illustrates the idea of infinite regression. Even if you can believe that the earth is a flat plate supported on the back of a giant tortoise, you certainly cannot accept that there could be an infinite number of them. At some point there must be a beginning, that is, there must be a "bottom" tortoise upon which everything else stands.

Any system of reasoning works in exactly the same way. We can logically deduce "new truths" from "old truths" but this cannot be done indefinitely into the "past." Therefore, there must be some fundamental starting point upon which derived knowledge must rest. This fundamental starting point consists of a set of ***axioms*** and ***primitive*** or ***undefined terms***. This is described in more detail below.

Foundation of Axiomatic System of Reasoning – Primitive (Undefined) Terms and Axioms

Primitive (Undefined) Terms

A ***primitive*** or ***undefined term*** is one for which ***no definition is given***. Primitive terms are used for concepts that are so basic that it is not possible to define them in terms of simpler or more basic concepts. Since no definition is given for primitive terms, it is assumed that the "meaning" of such a term is "understood."

For example, in ***Euclidean*** or ***plane geometry*** (geometry of flat surfaces), a ***point*** is an undefined term. It is ***assumed*** that we understand what is meant by a point, however, it is not possible to give a definition of a point because there are no simpler terms or concepts upon which any definition can be based.

Axioms

An axiom is a ***statement*** that is accepted as "true" without proof. Often, axioms are considered ***self-evident truths***. Since axioms cannot be proved, however, we must recognize that they are nothing more than ***assumptions***. If we hope to arrive at knowledge that is meaningful and useful, it is very important that we choose ***plausible axioms***. Choosing highly questionable axioms is likely to lead to nonsense.*

For example, an axiom of Euclidean geometry is "parallel lines never intersect.**" Although this cannot be proved on the basis of more elementary concepts, it is a reasonable assumption because no exceptions to this statement have ever been found (as long as we confine the lines to flat surfaces).

*In philosophy there is much debate concerning whether there is any such thing as a "self-evident truth." Many philosophers believe that assumptions cannot be considered "true" or "false" because they cannot be verified. We shall intentionally avoid such arguments as they can lead to a "paralysis of thought." We shall adopt the point of view that as long as we make "reasonable" assumptions, we can construct logical arguments that lead to useful results. For those of you who are interested in such philosophical debates, use search phrases such as "self-evident truth" or "axiomatic reasoning" to learn more.

**This axiom was actually stated in a more roundabout fashion by Euclid (see Postulate 5 on page IPEG-12). This axiom is known to be false in non-Euclidean geometries (geometries of curved surfaces such as hyperbolic geometry or elliptic geometry).

Definitions and Propositions in Axiomatic System of Reasoning

Defined Terms (Definitions)

A ***defined term*** is one that can be described in terms of primitive terms and other defined terms.

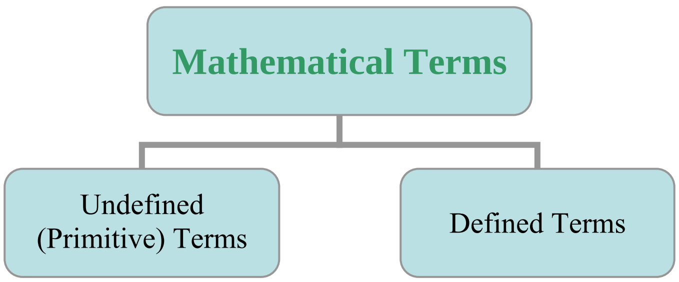
For example, in Euclidean geometry, a ***triangle*** can be defined in terms of points and line segments. In addition, a line segment can be defined in terms of lines and points.

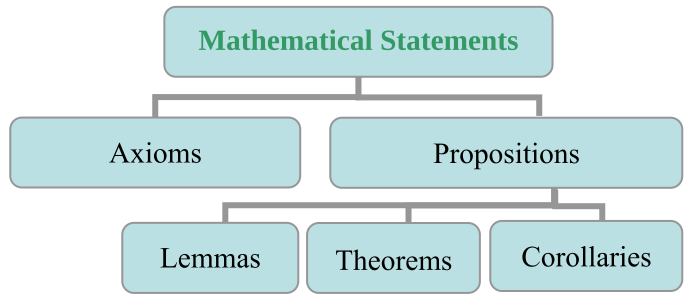
Propositions

A ***proposition*** is a statement that affirms or denies something and is either true or false. The truth or falsity of a proposition is established by using primitive terms, definitions, axioms and previously proved propositions.

For instance, the Pythagorean Theorem is a proposition of Euclidean geometry. It can be demonstrated to be "true" in a variety of different ways.

Summary





Exercises

1. Give examples of each of the following.

Concept	*Example*
undefined term	
defined term	
axiom	
proposition	

2. Find ***mathematical*** or ***scientific*** definitions of each of the following terms:

postulate, theorem, lemma, corollary, counterexample, converse, contrapositive, inverse, premise, conditional statement (logical implication), biconditional statement (logical equivalence), hypothesis, theory, "if and only if," algorithm

3. Classify the following arguments as deductive or inductive. In addition, discuss the validity of each argument.

Argument	*Deductive or Inductive?*	*True or False? Why?*
David H. did not bring a coffee for Mr. Nolfi one morning. In addition, he looked very dishevelled, his clothes were torn and he had several bruises on his arms. It looks like he must have been rude to Fiona again, which caused Fiona to beat him with a razor-blade-embedded blackboard compass. In addition, he has probably lost his mind because Mr. Nolfi is extremely irritable without his morning coffee!		
All organisms have RNA (ribonucleic acid). Giant clams are deep-ocean seafloor organisms found near geothermal vents. Therefore, giant clams have RNA.		
A comet is a relatively small extraterrestrial body consisting of a frozen mass that travels around the sun in a highly elliptical orbit. Pluto orbits the sun in a highly elliptical orbit. Therefore, Pluto is a comet.		

An Example of Deductive Reasoning in Science – Einstein's Special Theory of Relativity (Enrichment Material)

Scientific work relies heavily on ***inductive reasoning***. A typical scientific investigation proceeds as shown below:

1. Define the question
2. Gather information and resources
3. Form an hypothesis
4. Perform experiments and collect data
5. Analyze the data
6. Interpret the data and draw conclusions that serve as a starting point for new hypotheses
7. Publish results

> Note that steps 3 to 6 may need to be ***repeated several times*** before step 7 can be completed.
>
> As with all inductive reasoning, the results are always considered tentative, that is, subject to disproof.

In the late nineteenth century and early twentieth century, the field of theoretical physics began to flourish as ***deductive reasoning*** increasingly found its way into the realm of science. Foremost among the theoretical physicists of the time was the young and brilliant yet still obscure Albert Einstein. In 1905, he published ***five*** groundbreaking papers, three of which are said to have been worthy of a Nobel Prize. In the best known of these papers, Einstein ***deduced*** what is now known as the "Special Theory of Relativity," which predicts the measurements made by observers moving at a constant velocity relative to each other. Using only ***two axioms***, Einstein ***deduced*** several results, none of which has ever been contradicted by experimental evidence. A brief overview of the special theory of relativity is given below.

Postulates (Axioms) of Special Relativity

1. ***The Principle of Relativity (in the Restricted Sense)***
   Suppose that K and K' are inertial frames of reference (co-ordinate systems that obey Galileo's law of inertia, also known as Newton's First Law)*. Then natural phenomena run their course with respect to K' according to ***exactly the same general laws*** as with respect to K.
2. ***The Propagation of Light in Empty Space – The Invariance of c***
   Light is always propagated in ***empty space*** in straight lines with a definite (constant) velocity c that is independent of the state of motion of the emitting body (i.e. the state of motion of the source of light).

These two ***very simple*** principles are quite ***reasonable assumptions*** because no contradictions to them have ever been found. They are, nonetheless, assumptions because there is no way of proving them ***definitively*** (although the second postulate can be deduced from the first postulate and Maxwell's equations governing electromagnetic radiation). If either of the two postulates should turn out to be false, then Einstein's great edifice of special relativity would come crashing to the ground! On the other hand, if both of the postulates are in fact true, then the truth of the theory is ***guaranteed*** by the ***deductive*** argument! Luckily, every experiment ever performed has confirmed the predictions of special relativity.

Some Important Results Deduced from the Postulates of Special Relativity

Suppose that K and K' are inertial frames of reference*. Suppose that K is considered to be "at rest." Then K' is moving away from K with a constant velocity v. The Special Theory of Relativity asserts that measurements made with respect to K will differ from measurements made with reference to K' according to the equations given below.

1. Suppose that a body is at rest relative to K' and that its mass, as measured by an observer at rest with respect to K' is given by m_0. Then as measured by an observer at rest with respect to K, the mass of the body is found to be

$$m(v) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} .$$

   That is, the mass of a body ***increases*** as its velocity increases.

2. Suppose that a body is at rest relative to K' and that its length, as measured by an observer at rest with respect to K' is given by l_0. Then as measured by an observer at rest with respect to K, the length of the body is found to be

$$l(v) = l_0\sqrt{1-\frac{v^2}{c^2}} .$$

   That is, the length of a body ***decreases*** as its velocity increases. (This is known as ***length contraction***.)

---

* If K and K' are inertial frames of reference (co-ordinate systems that obey the law of inertia) then they are necessarily moving with a constant velocity with respect to each other. From the point of view of an inertial frame, bodies at rest remain at rest unless acted upon by some external, unbalanced force. In addition, bodies in motion will remain in motion with a constant velocity (constant speed and in a straight line) unless acted upon by some external, unbalanced force.

3. Suppose that the time interval between two events as measured by an observer at rest with respect to K' is found to be Δt_0. Then as measured by an observer at rest with respect to K, the time interval between the two events is given by

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}.$$

That is, moving clocks run ***slower*** than those at rest. (This is known as ***time dilation***.)

4. For the sake of completeness, the following result is also included. It is the most famous result of special relativity and it shows that mass and energy are "interchangeable" quantities. That is, by virtue of its mass, a body at rest with respect to an inertial frame possesses an intrinsic amount of energy as given by the immortal equation

$$E = mc^2.$$

A Grand "Theorem" that Overturns ALL Currently Accepted Math?

"Theorem"

$$2 = 1$$

"Proof"

"B.S." stands for "***both sides***" (among other things)

Let $x = y$.

$\therefore x^2 = xy$ (B.S. multiplied by x)

$\therefore x^2 - y^2 = xy - y^2$ (Subtract y^2 from B.S.)

$\therefore (x-y)(x+y) = y(x-y)$ (Factor B.S.)

$\therefore x + y = y$ (Divide B.S. by $x - y$)

$\therefore y + y = y$ (Since $x = y$)

$\therefore 2y = y$ (Simplify B.S.)

$\therefore 2 = 1$ (Divide B.S. by y) //

Since we have shown that 2=1, it appears that the mathematical pillars upon which all mathematical theory stand are about to crumble! A little careful reflection, however, will reveal that there is something terribly wrong with the argument given above. So it seems that the foundations of math are safe for the time being.

What is wrong with the given argument?

PROOF IN EUCLIDEAN GEOMETRY

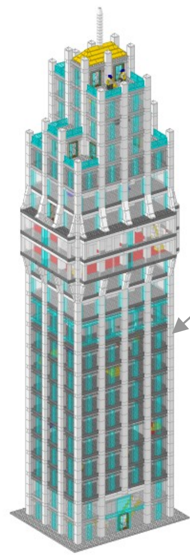
What is Euclidean Geometry?

Named after the ancient Greek mathematician and teacher ***Euclid of Alexandria*** (circa 365 BC – 275 BC), ***Euclidean geometry*** (also known as ***plane geometry***) deals with the properties, measurement and relationships of points, lines, angles and "flat" surfaces. In his most famous work *Elements* (consisting of 13 books), Euclid used a small set of definitions, axioms and postulates to ***deduce*** the properties of geometric objects and natural numbers. Although many of the results presented in *Elements* originated with earlier mathematicians, Euclid (*Eukleides* in Greek) was the first to compile and present the bulk of the mathematical knowledge of the time in a single, ***logically coherent framework***. Widely considered the most successful textbook ever published, *Elements* has also been enormously influential in science and philosophy. The following diagram should help you appreciate the beautiful and rich deductive structure of Euclid's *Elements*. (See Appendix 1 for a more detailed discussion.)

It is important to understand that Euclid ***did not*** distinguish between undefined (primitive) terms and defined terms. This is a modern conception based on the idea that certain ***central terms are so basic that they are not defined in terms of simpler concepts***.

A great analogy is to think of the undefined terms as ***atoms*** and the defined terms as ***molecules***.

Admittedly, the difference between an axiom and a postulate is quite subtle. Both are ***assumptions*** of some kind. Nonetheless, it is still important to distinguish between ***postulates*** and ***axioms***. An axiom is considered a strongly self-evident "truth." A postulate, on the other hand, need not be a claim to "truth." It is more or less an assertion of ***choice***, as in statements of the form "let this be true."



The building represents the ***propositions*** of Euclidean geometry. The propositions are ***deduced*** (derived logically) from the axioms, postulates and definitions.

However, the propositions cannot exist without the foundations, that is, the axioms, postulates, undefined (primitive) terms and the defined terms.

For more information on Euclid's *Elements*, see
http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

Although the building looks majestic and sturdy, it would collapse quickly and crumble if it were not for its solid ***footings*** buried deep within the ground. Similarly, the magnificent edifice of Euclidean geometry would collapse onto itself if it were not for its ***footings*** or ***underpinnings***, the ***definitions*** and ***axioms***. Note that Euclid did not distinguish between undefined (primitive) terms and defined terms. In addition, Euclid separated axioms into "common notions" and "postulates."

Definitions		***Axioms***	
		Common Notions	***Postulates***
1. Point **2.** Line **3.** Endpoints of a Line **4.** Straight Line **5.** Surface **6.** Edges of a Surface **7.** Plane Surface **8.** Plane Angle **9.** Rectilinear Angle **10.** Right Angle, Perpendicular **11.** Obtuse Angle **12.** Acute Angle **13.** Boundary	**14.** Figure **15.** Circle **16.** Centre of a Circle **17.** Diameter **18.** Semicircle **19.** Rectilinear Figure, Trilateral, Quadrilateral, Multilateral **20.** Equilateral, Isosceles, Scalene Triangles **21.** Right, Acute, Obtuse Triangles **22.** Square, Oblong, Rhombus, Rhomboid, Trapezium **23.** Parallel Lines	**1.** Things that are equal to the same thing are also equal to one another. **2.** If equals be added to equals, the wholes (sums) are equal. **3.** If equals be subtracted from equals, the remainders (differences) are equal. **4.** Things that coincide with one another are equal to one another. **5.** The whole is greater than the part.	**1.** A straight line segment can be drawn by joining any two points. **2.** A straight line segment can be extended indefinitely in a straight line. **3.** Given a straight line segment, a circle can be drawn using the segment as radius and one endpoint as centre. **4.** All right angles are congruent. **5.** If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

General Approach to Learning Theorems

Much of this section is a review of what you learned in previous courses. What is new, however, is that you will be learning how to prove some theorems of Euclidean Geometry that until now, you accepted without proof. It is ***essential to understand that for the purposes of this course, the theorems themselves are more important than their proofs***. Therefore, for the sake of efficient and effective learning, I suggest the following approach to learning theorems:

1. First, understand the results. That is, before you even consider studying the proofs of the theorems, first understand the theorems themselves and how to apply them.
2. Once you have obtained a good general overview of the new theorems, solve as many problems (write as many proofs) as you can. This helps to reinforce your understanding of the new material.
3. Study the proofs of the theorems but do not memorize them! It is far more important to understand the techniques and ideas used in the proofs than it is to memorize them blindly.
4. Solve more problems!

Geometry Theorems you might have (or should have) Learned before taking this Course

Sketch a diagram to illustrate each theorem or formula.

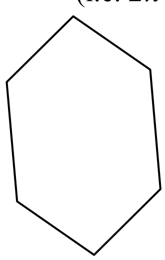
Theorem	***Diagrams***
1. Triangle Congruence Theorem Two triangles are congruent ***if and only if*** **at least *one*** of the following conditions is satisfied: i. Corresponding sides are equal **(SSS)** ii. The hypotenuse and one other side of one right triangle are respectively equal to the hypotenuse and one other side of a second right triangle **(HS)** (Note that "HS" is a corollary of "SSS.") iii. Two sides and the ***contained angle*** of one triangle are respectively equal to two sides and the ***contained angle*** of a second triangle **(SAS)** iv. Two angles and the ***contained side*** of one triangle are respectively equal to two angles and the ***contained side*** of a second triangle **(ASA)** v. Two sides and a ***non-contained*** angle of one triangle are respectively equal to two sides and a ***non-contained*** angle of a second triangle, ***and*** for each triangle, the given angle is opposite the ***larger*** of the two given sides **(SsA)** ***Note:** SsA (also known as AsS) only holds if the longer of the two given sides is opposite the given angle. So please do not be an "AsS!" Use SsA very carefully. For more information, consult http://www.andrews.edu/~calkins/math/webtexts/geom07.htm#AAA	
2. Angle Sum Triangle Theorem (ASTT) In any triangle, the sum of the interior angles is 180° (π rad).	

3. Isosceles Triangle Theorem (ITT)
A triangle is isosceles ***if and only if*** the base angles are equal.

4. Opposite Angle Theorem (OAT)
The opposite angles formed by two intersecting lines are equal.

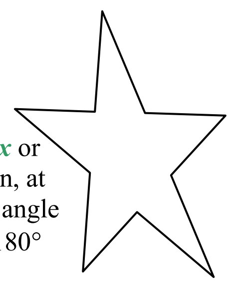
5. Convex Polygon Theorem (CPT) (corollary of ASTT)

i. The sum of the interior angles of an n-sided convex polygon is $(n-2)180°$ (i.e. $(n-2)\pi$ rad)
ii. The sum of the exterior angles of any convex polygon is 360° (i.e. 2π rad)



In a ***convex*** polygon, each interior angle is less than 180°

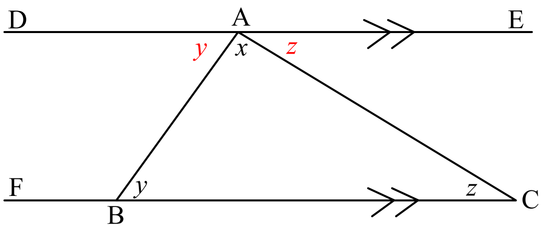
In a ***non-convex*** or ***concave*** polygon, at least one interior angle is greater than 180°



6. Angle-Angle Similarity Theorem (AA)
Two triangles are similar ***if and only if*** two angles of one triangle are respectively equal to two angles of a second triangle.

7. Exterior Angle Theorem (EAT) (Corollary of ASTT)
An exterior angle of a triangle is equal to the sum of the two interior and opposite angles.

Proof of ASTT



Extend line segment CB to F.
Construct line segment DE through A such that $DE \parallel FC$.
Then,

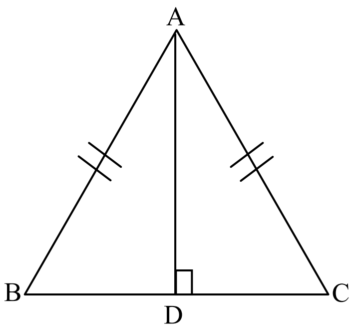
$$\angle ABC = \angle DAB = y \text{ (PLT “Z”)}$$
and
$$\angle BCA = \angle EAC = z \text{ (PLT“Z”).}$$

Therefore,

$$x + y + z = 180° \text{ (supplementary angles)}$$

Hence, the sum of the interior angles of any triangle is 180°. //

Proof of ITT



Construct altitude AD.
In right ΔABD and right ΔACD,

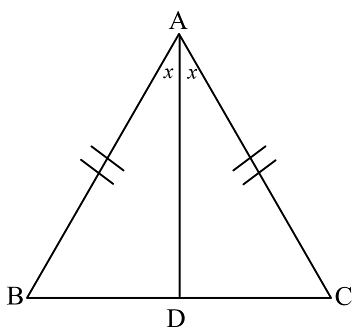
$AB = AC$ (given)
$DA = DA$ (common)

Therefore,

$$\Delta ABD \cong \Delta ACD \text{ (HS congruence)}$$

Hence, $\angle ABD = \angle ACD$. That is, the base angles are equal. //

Alternative Proof of ITT



Construct AD in such a way that $\angle BAC$ is bisected (i.e. $\angle BAD = \angle CAD$)
In ΔABD and ΔACD,

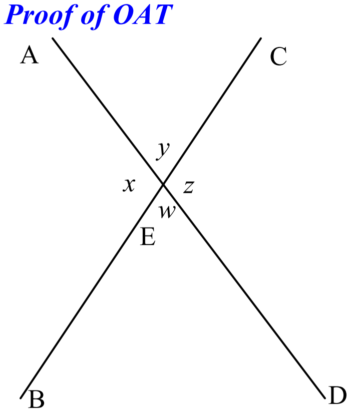
$AB = AC$ (given)
$\angle BAD = \angle CAD$ (by construction)
$AD = AD$ (common)

Therefore,

$$\Delta ABD \cong \Delta ACD \text{ (SAS congruence)}$$

Hence, $\angle ABD = \angle ACD$. That is, the base angles are equal. //

Proof of OAT



$x + y = 180°$ (1) (supplementary angles)
$y + z = 180°$ (2) (supplementary angles)
$z + w = 180°$ (3) (supplementary angles)

(1) – (2),
$x - z = 0°$
Therefore, $\angle AEB = \angle CED$
Similarly, $\angle AEC = \angle BED$. //

Homework

1. Use ASTT and supplementary angles to prove EAT.
2. Use ASTT to prove the convex polygon theorem.
3. Read pages 11 – 14 in our textbook.
4. Do the following exercises: p. 14 #2, 3, 5, p. 15 #7, 8, 9

Proof in Cartesian Geometry

What is Cartesian Geometry?

Cartesian geometry, also known as analytic geometry or co-ordinate geometry, is a branch of mathematics that was developed by **René Descartes** (1596-1650). Descartes was highly talented and renowned in many fields including physics, physiology, mathematics and philosophy. His name is very familiar to students of mathematics, but it is as a highly original philosopher that he is most frequently read today. One of his greatest contributions to mathematics was the development of analytic geometry. Not only did Descartes show that there is an intimate connection between algebra and geometry, he also established a system of co-ordinates that allowed mathematicians to use equations to describe and generate curves.

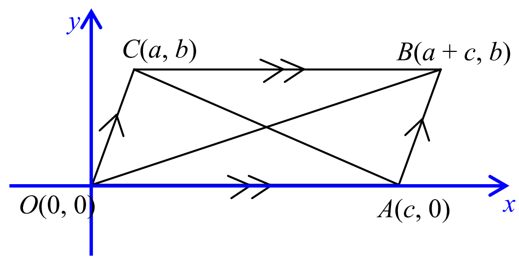
Proof in Cartesian Geometry

Analytic geometry provides us with an extremely powerful tool for proving mathematical statements. Once we add it to our arsenal of problem solving methods, many problems become much easier to solve. Here are three examples.

Example 1

Prove that the diagonals of a parallelogram bisect each other.

Proof



Let E represent the midpoint of OB and let F represent the midpoint of AC. Then, the co-ordinates of E and F are

$$\left(\frac{0+a+c}{2},\frac{0+b}{2}\right)=\left(\frac{a+c}{2},\frac{b}{2}\right) \text{ and } \left(\frac{a+c}{2},\frac{b+0}{2}\right)=\left(\frac{a+c}{2},\frac{b}{2}\right) \text{ respectively.}$$

Since E and F have the same co-ordinates, they must be coincident. Therefore, OB and AC have a common midpoint. Since the point of intersection of OB and AC is the only point that lies on both OB and AC, it must be the midpoint of both OB and AC. Hence, the diagonals of a parallelogram bisect each other. //

Example 2 – A Corollary of the Pythagorean Theorem

Prove that the distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(\Delta x)^2+(\Delta y)^2}$.

Proof

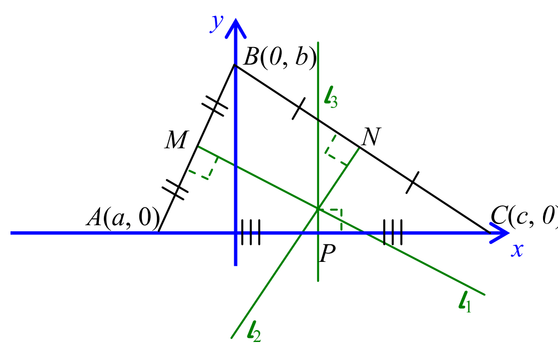
See unit 0.//

The following example shows that it is very important to choose the placement of the co-ordinate axes very carefully.

Example 3

Prove that in any triangle, the right bisectors of the sides are *concurrent* (intersect at a common point).

Proof



Let M, N and P represent the midpoints of AB, BC and CA respectively.

Then, the co-ordinates of M, N and P are $\left(\frac{a}{2},\frac{b}{2}\right)$, $\left(\frac{c}{2},\frac{b}{2}\right)$ and $\left(\frac{a+c}{2},0\right)$

respectively. In addition, the slopes of AB, BC and CA are $-\frac{b}{a}$, $-\frac{b}{c}$ and 0 respectively. Thus, the slopes of the perpendiculars through M, N and P must be $\frac{a}{b}$, $\frac{c}{b}$ and undefined respectively. Using this information, it can be shown that the following are equations of the lines ℓ_1, ℓ_2, and ℓ_3 respectively:

$$y=\frac{a}{b}x+\frac{b^2-a^2}{2b},\quad y=\frac{c}{b}x+\frac{b^2-c^2}{2b} \text{ and } x=\frac{a+c}{2}.$$

(Continued on next page.)

To find the point(s) of intersection of l_1, l_2, and l_3 can substitute $x=\dfrac{a+c}{2}$ into each of the other two equations. When we do this, we obtain $y=\dfrac{a}{b}x+\dfrac{b^2-a^2}{2b}=\dfrac{a}{b}\left(\dfrac{a+c}{2}\right)+\dfrac{b^2-a^2}{2b}=\dfrac{ac+b^2}{2b}$ for the first equation and $y=\dfrac{c}{b}x+\dfrac{b^2-c^2}{2b}=\dfrac{c}{b}\left(\dfrac{a+c}{2}\right)+\dfrac{b^2-c^2}{2b}=\dfrac{ac+b^2}{2b}$ for the second equation. Thus, l_1, l_2, and l_3 have a common point of intersection $\left(\dfrac{a+c}{2},\dfrac{ac+b^2}{2b}\right)$, which means that the perpendicular bisectors of the sides must be concurrent. //

Homework

1. Read section 1.4 in the textbook.
2. Do exercises on pp. 19-20: #2, 4, 6, 7, 8, 9, 11, 12, 13

Critical Prerequisite Knowledge

To be able to solve the problems in this section of the course, you must be able to

1. Evaluate the distance between two points.

2. Evaluate the midpoint of a line segment.

3. Calculate the slope of a line.

4. Find an equation of a line given a point and a slope.

5. Find an equation of a circle given its centre and its radius.

6. Recognize that two lines are parallel ***if and only if*** their slopes are equal.

7. Recognize that two lines are perpendicular ***if and only if*** their slopes are negative reciprocals of each other.

8. Understand that ***slope*** measures the steepness of a line, which means the same thing as the ***rate of change*** of a linear function. (In the case of ***non-linear functions***, the slope of the tangent line measures the steepness or rate of change of a curve at a given point.)

The Intimate Connection between Algebra and Geometry

Introduction

Many students have the impression that mathematics is a somewhat mysterious and painful subject that most people fear and few people understand. The main cause of this animosity and anxiety toward mathematics is most likely mathematical notation; it can be extremely intimidating. After all, the ***language*** of math is rife with Greek letters, Latin letters and all sorts of strange symbols. What does it all mean?

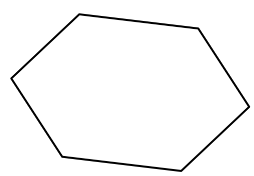
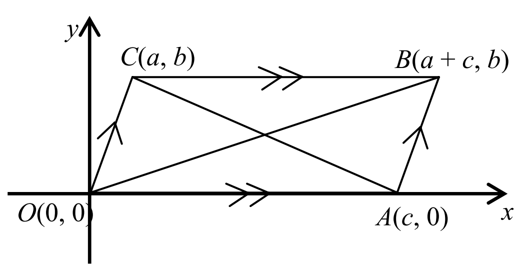
To relieve this anxiety toward math, teachers must convey to their students that mathematical notation ***is*** the ***language*** of mathematics. Like with any other language, an ***association*** needs to be made between the ***symbols*** and their ***meanings***. Only once this is accomplished can a deep ***understanding*** be acquired! Once the teacher helps his/her students realize that there is a deep connection between ***algebra*** and ***geometry***, that is, that they are two different perspectives of the same underlying ideas, then all else falls into place. Put more simply, students need to understand that it is possible to relate most mathematical expressions and equations to diagrams. Just as the English word "chair" evokes the imagery of a chair, so should the mathematical equation $x^2 + y^2 = 25$ evoke an image of a circle of radius 5 centred at the origin.

So why is it that so few people seem to understand the language of mathematics? There are many possible causes of this apparent lack of meaning but the main cause is probably the ***recipe-oriented*** method of teaching mathematics. Students are taught ***algorithms*** for solving particular kinds of problems. Just as an inexperienced cook can follow a good recipe blindly and produce good results, many students can master mathematical algorithms and produce correct answers without having much of an understanding of the underlying ideas.

An Exercise in the Language of Mathematics

Complete the table given below. This kind of exercise will help you to develop the ability to associate mathematical symbols, expressions and equations with concrete ideas.

Expression, Equation or Inequation	***Diagram***	***Conclusion, Interpretation or Explanation*** (In English, of Course)
$x^2 + y^2 = 25$		
$\lvert x-4 \rvert = 5$		
$\lvert x+4 \rvert = 5$		
$3x + 4y = 5$		
		In two triangles, corresponding angles are equal.

Expression, Equation or Inequation	*Diagram*	*Conclusion, Interpretation or Explanation* (In English, of Course)
$b^2 - 4ac < 0$		
$b^2 - 4ac > 0$		
$b^2 - 4ac = 0$		
$\sqrt{x+5} = x^2$		
		
69		
		

Expression, Equation or Inequation	***Diagram***	***Conclusion, Interpretation or Explanation*** (In English, of Course)
$\lvert x-4 \rvert = -5$		
		The base angles of a triangle are equal.
		The sum of the interior angles of a triangle is π radians.
$\lvert x-1 \rvert + 3 = -5 + 4x + x^2$		
$(x-3)^2 + (y+1)^2 = 225$		
$\dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \pm 5$		
Make up some of your own exercises here.		

Another Exercise that helps develop your Mathematical Intuition

State whether each of the following statements is true or false. Provide a proof of each of the true statements and a counterexample for each of the false statements.

Statement	*True or False?*	*Proof, Counterexample or Explanation*
$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$		
$(a+b)^2 = a^2 + b^2$		
Two lines in a diagram look parallel. Therefore, they are parallel.		
The sum of the exterior angles of a concave (non-convex) polygon is 2π rad.		
The slope of the line $3x + 4y - 6 = 0$ is 3.		
Make up some of your own here.		

Review

1. Review these online notes and examples in your textbook.
2. "Chapter 1 Test" on p. 22 of the textbook
3. "Review of Prerequisite Skills" on pp. 24-27 of the textbook
4. Ask me any questions that you may have.

OVERVIEW OF SECOND HALF OF UNIT 1

General Objectives

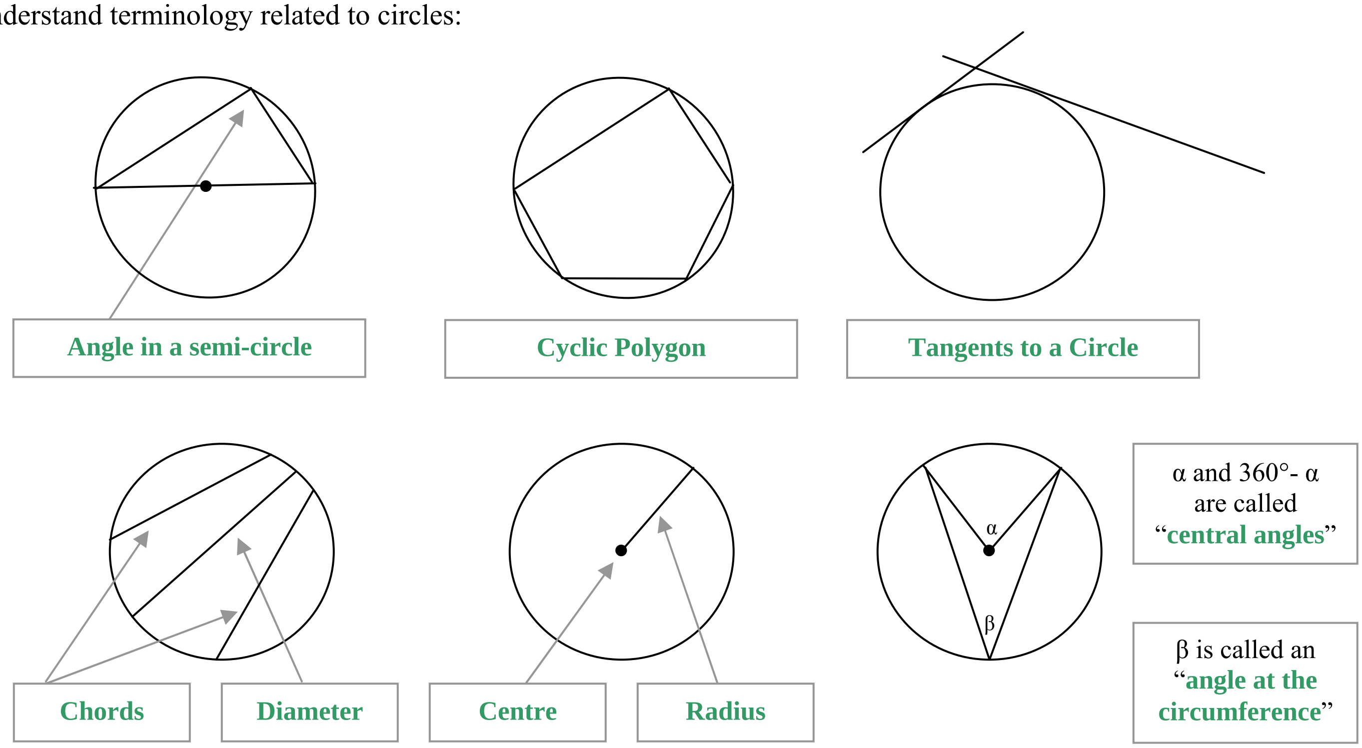
1. Consolidate (strengthen) our knowledge and understanding of mathematical terminology, particularly the language associated with propositions and their proofs:
   - premise, conclusion, inference, proof, theorem, proposition, corollary, lemma, property
   - definition, undefined (primitive) term, axiom, postulate, conjecture, counterexample, algorithm
   - statement, conditional statement (logical implication), biconditional statement (logical equivalence), converse, "if and only if", inverse, contrapositive
   - deductive reasoning (deductive proof), inductive reasoning, proof by contradiction (indirect proof)
   - congruence, similarity, bisector, proportionality, ratio, line segment, convex polygon, non-convex polygon, scalene triangle, isosceles triangle, equilateral triangle, acute angle, obtuse angle, reflex angle, interior angle, exterior angle, parallel, equidistant, midpoint, endpoint

2. Prove properties of plane figures (including circles) using ***deductive proof*** (direct proof), ***proof by contradiction*** (indirect proof or *reductio ad absurdum*) and ***algebraic methods***. (Later in the course we shall expand our repertoire by using ***vector methods*** to prove properties of plane figures.)

3. Review theorems of geometry that we have already proved or assumed to be true:
   - Congruence theorems (SSS, SAS, ASA, SsA(AsS))
   - ASTT, ITT, OAT, Pythagorean Theorem (PT), Convex Polygon Theorem (CPT)
   - Exterior Angle Theorem (EAT, a corollary of ASTT), Properties of Parallelograms (POP)
   - Finding the Area of a Triangle

4. Use "Geometer's Sketchpad®" to discover several properties of plane figures, apply these properties and prove them.
   - Parallelogram Area Property (PAP), Right Bisector Theorem (RBT), Angle Bisector Theorem (ABT), Parallel Line Theorem(PLT), Triangle Area Property (TAP), Triangle Proportion Property Theorem (TPPT), Similar Triangle Theorem (STT)
   - Chord Right Bisector Property (CRBP), Equal Chords Property (ECP), Angle at Circumference Property (ACP), Equal Angles in a Segment Property (EASP), Angles in a Cyclic Quadrilateral Property (ACQP), Tangent Radius Property (TRP), Tangent from a Point Property (TPP), Tangent Chord Property (TCP), Intersecting Chords Property (ICP), Intersecting Secants Property (ISP)

5. Understand terminology related to circles:



Using Geometer's Sketchpad to Discover Theorems in Plane Geometry

Instructions

Use Geometer's Sketchpad to perform each of the following geometric constructions. In each case, state a conjecture based on your results. Do not forget to save your sketchpad files!

Activity	***Conjecture***
1. Construct a few triangles to demonstrate that AAA, SSA (ASS) and AAS ***do not*** guarantee the congruence of two triangles. Are there conditions under which they do guarantee congruence?	
2. Construct two parallelograms having bases of the same length and lying between the same parallel lines. What do you notice about the areas of the parallelograms? Is this true in general? Investigate by constructing more parallelograms.	
3. Construct a line segment and its right bisector. Then construct at least three points lying on the right bisector. What do you notice about the lengths of the line segments joining the points on the right bisector to the endpoints of the original line segment? Is the ***converse*** also true?	
4. Construct an angle and its bisector. Then construct at least three points lying on the bisector. From the points lying on the bisector, construct line segments extending from each point to each arm of the angle in such a way that the line segments are perpendicular to the arms. What do you notice about the lengths of these line segments? Is the ***converse*** also true?	
5. Construct three triangles all having the same base but different heights. In addition, construct three triangles all having the same height but different bases. What do you notice about the areas of the triangles?	
6. Construct any three triangles. In the interior of each triangle, construct a line segment parallel to the base and joining the remaining two sides of the triangle. What do you notice about the ratios of the corresponding sides of the triangles? Is the ***converse*** also true?	

Activity	*Conjecture*
7. Construct at least three equiangular triangles (i.e., they satisfy the AAA condition). What relationship is there among the corresponding sides of the triangles? Is this true in general?	
8. Construct two triangles in which two pairs of sides are proportional and the angles contained by these sides are equal. What can you conclude? Is this true in general?	
9. Construct a circle. Then construct any chord other than the diameter of the circle and its right bisector. What do you notice about the right bisector? Does it pass through a special point? Is this always true? Is the converse true?	
10. Construct a circle and two ***non-parallel*** chords. Then construct the right bisector of each chord. Where do the two bisectors intersect? Is this true in general? Is the ***converse*** also true?	
11. Construct a circle with two chords of equal length. What do you notice about the distance of each chord from the centre of the circle? Is this always true? Is the converse true?	
12. Construct a circle and any central angle of the circle (except for 180° and 360°). This divides the circle into two arcs, a ***major*** arc (the longer arc) and a ***minor*** arc (the shorter arc). In each case, the arc is said to ***subtend*** an angle at the centre of the circle. If the length of the arc is known, how can the measure of the subtended central angle be calculated?	

Activity	*Conjecture*
13. Construct a circle and any central angle $\angle AOB$, such that O is the centre of the circle and $\angle AOB < 180°$. Then construct $\angle ACB$ at the circumference standing on the same minor arc as $\angle AOB$. What is the relationship between $\angle AOB$ and $\angle ACB$? Is this true in general? (Do not delete this diagram. You will need it for activity 14.)	
14. Now continue your construction from activity 13. Construct another angle at the circumference, $\angle ADB$ standing on the same minor arc as $\angle AOB$. What is the relationship between $\angle ACB$ and $\angle ADB$? Is this true in general? (The angles $\angle ACB$ and $\angle ADB$ are said to be angles in the ***same segment*** of a circle.)	
15. Construct any angle in a semi-circle. What do you notice about the measure of the angle? Is this always true? Is the converse true?	
16. Construct at least three triangles. For each triangle, construct a circle such that all three vertices of the triangle lie on the circumference of the circle. Can you succeed in every case? If so, do you think that all triangles have this property? (That is, are all triangles cyclic?)	
17. Construct a circle and a cyclic quadrilateral within the circle. What do you notice about the opposite angles? What do you notice about any exterior angle and the interior angle at the opposite vertex? Is this true in general? Is the converse true?	
18. Construct a circle and a point P lying outside the circle. From P, construct two tangents to the circle, PA and PB, where A and B lie on the circle. What do you notice about the lengths of PA and PB? Also construct radii OA and OB. What do you notice about $\angle PBO$ and $\angle PAO$.	
19. Construct a circle and two non-parallel chords. Extend each chord to the exterior of the circle until the chords meet at the point P. You will notice that the circumference of the circle cuts the line segments into two parts. What do you notice about the products of these parts?	

Important Theorems of Plane Geometry

Review of Previously Learned Theorems

Sketch a diagram to illustrate each theorem or formula.

Theorem	***Diagrams***
1. Triangle Congruence Theorem Two triangles are congruent ***if and only if*** **at least *one*** of the following conditions is satisfied: i. Corresponding sides are equal **(SSS)** ii. The hypotenuse and one other side of one right triangle are respectively equal to the hypotenuse and one other side of a second right triangle **(HS)** (Note that "HS" is a corollary of "SSS.") iii. Two sides and the ***contained angle*** of one triangle are respectively equal to two sides and the ***contained angle*** of a second triangle **(SAS)** iv. Two angles and the ***contained side*** of one triangle are respectively equal to two angles and the ***contained side*** of a second triangle **(ASA)** v. Two sides and a ***non-contained*** angle of one triangle are respectively equal to two sides and a ***non-contained*** angle of a second triangle, ***and*** for each triangle, the given angle is opposite the ***larger*** of the two given sides **(SsA)**	
2. Angle Sum Triangle Theorem (ASTT) In any triangle, the sum of the interior angles is 180° (π rad).	
3. Isosceles Triangle Theorem (ITT) A triangle is isosceles ***if and only if*** the base angles are equal.	
4. Opposite Angle Theorem (OAT) The opposite angles formed by two intersecting lines are equal.	
5. Convex Polygon Theorem (CPT) i. The sum of the interior angles of an n-sided convex polygon is $(n-2)180°$ (i.e. $(n-2)\,\pi$ rad) ii. The sum of the exterior angles of any convex polygon is 360° (i.e. 2π rad)	
6. Angle-Angle Similarity Theorem (AA) Two triangles are similar ***if and only if*** two angles of one triangle are respectively equal to two angles of a second triangle.	
7. Exterior Angle Theorem (EAT) (Corollary of ASTT) An exterior angle of a triangle is equal to the sum of the two interior and opposite angles.	

New Geometry Theorems

Sketch a diagram to illustrate each theorem.

Theorem	*Diagrams*
1. Parallelogram Area Property (PAP) Two parallelograms have the same area ***if*** their bases are of equal length ***and*** they lie between the same parallel lines. **Note:** The converse of this statement is ***not*** true! Why?	
2. Right Bisector Theorem (RBT) A point lies on the right bisector of a line ***if and only if*** it is equidistant from the endpoints of the line segment.	
3. Angle Bisector Theorem (ABT) A point lies on the bisector of an angle ***if and only if*** it is equidistant from the arms of the angle.	
4. Parallel Line Theorem (PLT) Two straight lines are parallel ***if and only if*** i. alternate angles are equal, or ii. corresponding angles are equal, or iii. interior angles are supplementary	
5. Triangle Area Property (TAP) If triangles have equal heights, their areas are proportional to their bases. If triangles have equal bases, their areas are proportional to their heights.	
6. Triangle Proportion Property Theorem (TPPT) A line in a triangle is parallel to a side of the triangle ***if and only if*** it divides the other sides in the same proportion.	
7. Similar Triangle Theorem (STT) Two triangles are similar ***if and only if*** i. they are equiangular, or ii. their sides are proportional, or iii. two pairs of sides are proportional and the angles contained by these sides are equal.	

Theorem	*Diagrams*
8. Chord Right Bisector Property (CRBP) i. The right bisector of a chord passes through the centre of the circle. ii. The perpendicular from the centre of a circle to a chord bisects the chord. iii. The line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord. iv. The centre of a circle is the point of intersection of the right bisectors of any two non-parallel chords.	
9. Equal Chords Property (ECP) Chords are equidistant from the centre of a circle ***if and only if*** the chords are of equal length.	
10. Angle at the Circumference Property (ACP) An angle at the centre of a circle is ***twice*** the angle at the circumference standing on the ***same arc***.	
11. Equal Angles in a Segment Property (EASP) Angles in the same segment of a circle are equal.	
12. Angle in a Semicircle Property (ASP) Any angle in a semicircle is a right angle.	
13. Angles in a Cyclic Quadrilateral Property (ACQP) i. A quadrilateral is cyclic ***if and only if*** its opposite angles are ***supplementary*** (incorrect in text, p. 91) ii. A quadrilateral is cyclic ***if and only if*** the exterior angle at any vertex is equal to the interior angle at the opposite vertex. iii. A quadrilateral is cyclic ***if and only if*** one side subtends equal angles at the remaining vertices.	

Theorem	*Diagrams*
14. Tangent Radius Property (TRP) For a given circle, i. a tangent is perpendicular to the radius at the point of tangency; ii. a line at right angles to a radius at the circumference is a tangent; iii. a perpendicular to a tangent at the point of contact passes through the centre.	
15. Tangent Chord Property (TCP) The angle formed by a tangent and a chord is equal to the angle subtended by the chord in the segment on the other side of the chord.	
16. Tangent from a Point Property (TPP) Tangent segments from an external point to a circle are equal.	
17. Intersecting Chords Property (ICP) If two chords intersect, the product of the two parts of one is equal to the product of the two parts of the other.	
18. Intersecting Secants Property (ISP) If two secants AB and CD intersect at point P, then $PA \bullet PB = PC \bullet PD$.	
19. Corollary of ICP and ISP If a tangent PT is drawn from a point on a secant AB, then $PA \bullet PB = PT^2$.	

MORE FACTS THAT YOU CANNOT AFFORD NOT TO KNOW!

Fact	*Diagram and Explanation*
1. The area of a rectangle is $A = lw$	
2. The area of a parallelogram is $A = bh$.	
3. The area of a triangle is $A = \frac{1}{2}bh = \dfrac{bh}{2}$.	
4. The area of a trapezoid is $A = \dfrac{h(a+b)}{2} = \frac{1}{2}h(a+b) = h\left(\dfrac{a+b}{2}\right)$	
5. The area of a circle is $A = \pi r^2$. The circumference of a circle is $C = 2\pi r$.	
6. Every triangle is cyclic. (Points lying on a circle are called ***concyclic***.)	
7. The midpoint of a line segment. (You supply the formula.)	
8. The distance between two points. (You supply the formula.)	

Using Deductive Logic Terminology Correctly

Important Terminology

For the exercise given below, it is imperative that you understand the following terms:

statement, converse, inverse, contrapositive, negation, conditional (logical implication), biconditional (logical equivalence)

If any of these terms are unclear, consult Appendix 1 (or any other authoritative sources of information)

1. For each of the following statements, write the converse, the inverse and the contrapositive.

Statement $(P \to Q)$	***Statement in "If…then" form***	***Converse*** $(Q \to P)$	***Inverse*** $(\sim P \to \sim Q)$	***Contrapositive*** $(\sim Q \to \sim P)$
P implies Q	If P is true then Q is true.	If Q is true then P is true.	If P is not true then Q is not true.	If Q is not true then P is not true.
All humans are mortal.				
A German Shepherd is a large dog				
Math is fun!				
An integer is a number.				
A prime number has exactly two factors, 1 and itself				
Students always do their homework.				
Students always tell the truth				
Students never cheat				
Students always study for tests and exams				

2. For each of the statements in question 1, state whether

(a) the statement is true or false

(b) the converse is true or false

(c) the statement and its converse form a true biconditional statement (i.e. a logical equivalence, $P \leftrightarrow Q$)

(d) the inverse is true or false

(e) the contrapositive is true or false

Proof By Contradiction (Indirect Proof, *Reductio ad Absurdum*)

Introductory Problem – The Irrationality of $\sqrt{2}$

Prove that $\sqrt{2}$ is irrational.

Are Irrational Numbers Crazy?

Before tackling this problem, we must ensure that we understand the meaning of the term "irrational." Below is a list of the sets of numbers most commonly used in secondary school mathematics courses. A detailed examination of these sets will help you recall what is meant by an irrational number.

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of ***natural numbers*** or ***counting numbers***
- $W = \{0, 1, 2, 3, \ldots\}$ is the set of ***whole numbers***
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2\ldots\}$ is the set of ***integers***
- $\mathbb{Q} = \left\{ q \in \mathbb{R} \middle| q = \frac{n}{m}, n \in \mathbb{Z}, m \in \mathbb{Z}, m \neq 0 \right\}$ is the set of ***rational numbers*** (i.e. all ***fractions*** including negative fractions)

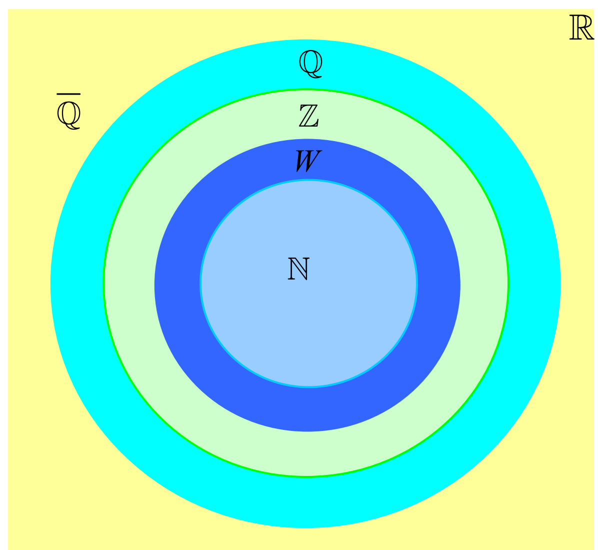
  The decimal expansions of rational numbers are either ***terminating*** or ***repeating*** (e.g. 9.7, 1.333… = $1.\overline{3}$)
- $\overline{\mathbb{Q}} = \left\{ q \in \mathbb{R} \middle| q \notin \mathbb{Q} \right\}$ is the set of ***irrational numbers***, that is, real numbers that ***are not fractions*** (**e.g.** $\sqrt{2}$, $\sqrt{3}$, π, e)

  The decimal expansions of irrational numbers are ***non-terminating*** and ***non-repeating***
- $\mathbb{R} = \mathbb{Q} \cup \overline{\mathbb{Q}}$ is the set of ***real numbers***

This ***Venn Diagram*** shows how the above sets are related. The set of real numbers is the ***universal set*** that includes all of the others. It is divided into two ***mutually exclusive sets***, the rational numbers and the irrational numbers.

The set of rational numbers contains the set of integers, which contains the set of whole numbers, which in turn contains the set of natural numbers. In the language of set theory, this is expressed as $\mathbb{Q} \supset \mathbb{Z} \supset W \supset \mathbb{N}$ or $\mathbb{N} \subset W \subset \mathbb{Z} \subset \mathbb{Q}$.

The set of irrational numbers can be subdivided further into two mutually exclusive (disjoint) sets, the ***algebraic numbers*** and the ***transcendental numbers***. The set of algebraic numbers consists of numbers such as $\sqrt{2}$, $\sqrt[4]{3}$ and $\phi = \frac{\sqrt{5}+1}{2} \doteq 1.618$ (the golden ratio), which are not rational but can be obtained as roots of ***polynomial equations with rational coefficients*** (e.g. $\sqrt[4]{3}$ is a root of $x^4 - 3 = 0$ and $\sqrt{2}$ is a root of $x^2 - 2 = 0$). Transcendental numbers, on the other hand, are much "worse" than algebraic numbers. They are more numerous than algebraic numbers but are extremely difficult to generate (see http://en.wikipedia.org/wiki/Transcendental_numbers). Transcendental numbers include $\pi \doteq 3.14159$ and $e \doteq 2.718$. (Also, see Unit 4 of this course for a discussion of ***countable*** and ***uncountable*** sets.)



Back to the Irrationality of $\sqrt{2}$

Consider trying to develop a deductive (direct) proof of the irrationality of $\sqrt{2}$. Where would you begin? What could you use as an initial premise?

When mathematicians attempt to develop a ***deductive (direct) proof*** but make little or no progress, they turn to a technique known as ***proof by contradiction*** or (also known as ***indirect proof*** or ***reductio ad absurdum***). This method uses a form of logic embodied by the Sir Arthur Conan Doyle character Sherlock Holmes. ***Sherlock was famous for remarking that after we have eliminated the impossible, whatever remains, however improbable, must be the truth.***

This line of reasoning is applied in many different fields:

- □ Doctors often diagnose illnesses by ***ruling out*** possibilities.
- □ Mechanics often diagnose problems with automobiles by ***ruling out*** possibilities.
- □ List an example of your own here:

The Logic of Contradiction

Suppose that you would like to prove that ***P* is true** but you are having difficulties finding a direct proof. In this case it is often helpful to try an indirect method. The method of ***proof by contradiction*** involves investigating what would happen if we assumed the opposite of what we are trying to prove. That is, if we are certain that ***P* is true**, what would happen if we assumed that ***P* is *not* true**? The process is shown below in pictorial fashion.

You try to prove **"*P* is true"** using a deductive (direct) proof but all attempts fail!

↓

Try ***proof by contradiction*** (indirect proof).

↓

Assume **"*P* is not true"** to see what happens.

↓

Using **"*P* is not true"** as an initial premise, derive conclusions.

↓

Keep deriving conclusions until a conclusion known to be ***false*** is obtained. Once we obtain a false conclusion, we can only conclude that the original assumption was false. Therefore, the original assumption **"*P* is not true"** must be ***false***. This can only happen if **"*P* is true"** is true!

Once we derive a conclusion that is known to be ***false***, we say that we have found a ***contradiction*** of the original assumption.

Finally we can prove that $\sqrt{2}$ is irrational!

Suppose that $\sqrt{2}$ were rational. (Assume the ***opposite*** of what we are trying to prove to see what happens.)

Then $\sqrt{2}=\dfrac{a}{b}, a\in\mathbb{Z}, b\in\mathbb{Z}, b\neq 0$ and $\gcd(a,b)=1$ (i.e. $\sqrt{2}$ can be expressed as a fraction in lowest terms).

Therefore,

$\therefore \left(\sqrt{2}\right)^2=\left(\dfrac{a}{b}\right)^2$

$\therefore 2=\dfrac{a^2}{b^2}$

$\therefore a^2=2b^2$

$\therefore a^2$ is even

$\therefore a$ is even (Why?)

$\therefore a=2k$ for some $k\in\mathbb{Z}$

$\therefore (2k)^2=2b^2$

$\therefore 4k^2=2b^2$

$\therefore b^2=2k^2$

$\therefore b^2$ is even

$\therefore b$ is even

Summarizing, we have shown that a and b are both even.

$\therefore \gcd(a,b)\geq 2$

But this ***contradicts*** the premise that $\gcd(a,b)=1$.

Therefore, the original assumption (i.e. that $\sqrt{2}$ is rational) must be false!

Therefore, $\sqrt{2}$ cannot be rational, which means that it must be irrational. //

Another Proof by Contradiction – Proposition 20 in Book IX of Euclid's Elements

There are infinitely many prime numbers.

Proof:

Suppose that there were only a finite number of primes. Let them be represented by $p_1, p_2, \ldots, p_n$.

Now consider the number obtained by adding 1 to the product of all these primes, that is,

$$N = p_1 p_2 \cdots p_n + 1.$$

First, N is not equal to any of the primes $p_1, p_2, \ldots, p_n$ as it is clearly larger than any of them.

Moreover, N is not divisible by any of the primes $p_1, p_2, \ldots, p_n$ because dividing N by p_i for any $i = 1, 2, \ldots, n$ will always yield a remainder of 1:

$$\frac{p_1 p_2 \cdots p_{i-1} p_i p_{i+1} \cdots p_n + 1}{p_i}$$

$$= \frac{p_1 p_2 \cdots p_{i-1} p_i p_{i+1} \cdots p_n}{p_i} + \frac{1}{p_i}$$

$$= p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_n + \frac{1}{p_i}$$

But the only way that this can occur is if N itself is prime.

This contradicts the original assumption that set of numbers $p_1, p_2, \ldots, p_n$ consists of all the primes.

Therefore, there must be an infinite number of primes. //

The following theorem of Euclidean geometry requires proof by contradiction. Trying to prove it using a direct method leads to nothing but frustration.

Triangle Proportion Property Theorem (TPPT)

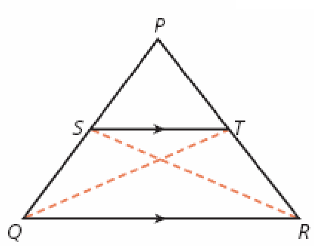
A line in a triangle is parallel to a side of the triangle ***if and only if*** it divides the other sides in the same proportion.

Proof:

Part 1 – If a line is parallel to a side of a triangle, it divides the other sides in the same proportion.

Let ST be a line in ΔPQR that is parallel to QR. We will prove that $\frac{PS}{SQ} = \frac{PT}{TR}$.

Join SR and QT.



Since ΔPST and ΔSTQ have the same altitude with bases PS and SQ,

$\frac{\Delta PST}{\Delta STQ} = \frac{PS}{SQ}$ (Triangle Area Property)

Since ΔPTS and ΔSTR have the same altitude with bases PT and TR,

$\frac{\Delta PST}{\Delta STR} = \frac{PT}{TR}$ (Triangle Area Property)

Because $ST \parallel QR$, ΔSTQ and ΔSTR have the same base ST and equal altitudes.

Therefore $\Delta STQ = \Delta STR$

Then $\frac{\Delta PST}{\Delta STQ} = \frac{\Delta PST}{\Delta STR}$

Therefore $\frac{PS}{SQ} = \frac{PT}{TR}$

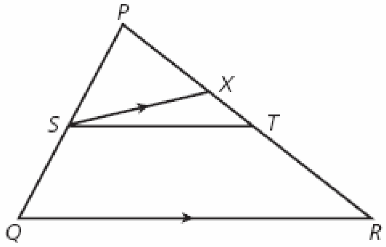
Note as an extension that we easily obtain

$$\frac{PS}{SQ} + 1 = \frac{PT}{TR} + 1$$

$$\frac{PS + SQ}{SQ} = \frac{PT + TR}{TR}$$

$$\frac{PQ}{SQ} = \frac{PR}{TR}$$

Part 2 – If a line divides two sides of a triangle in the same proportion, it is parallel to the third side.



Let PQR be a triangle and let S and T be points in the sides so that $\frac{PS}{SQ} = \frac{PT}{TR}$. We can prove that $ST \parallel QR$, and we do so by indirect proof.
Either $ST \parallel QR$ or $ST \nparallel QR$.

If $ST \nparallel QR$, then draw $SX \parallel QR$ meeting PR at X.

Because $SX \parallel QR$, $\frac{PS}{SQ} = \frac{PX}{XR}$ (Part 1 of theorem)

Then $\frac{PT}{TR} = \frac{PX}{XR}$

This means that the points T and X divide PR in the same ratio, which is impossible.
Then the statement $ST \nparallel QR$ is not true. Therefore $ST \parallel QR$.

//

> ***Note***
> See section 2.4 of the textbook (pages 48 – 52) to see how proof by contradiction is used to prove the Parallel Line Theorem (PLT).

Proof by Contradiction Exercises

1. Prove that $\sqrt{3}$ is irrational.

2. A ***Diophantine equation*** is an equation for which you seek integer solutions. For example, the so-called Pythagorean triples (x, y, z) are positive integer solutions to the equation $x^2 + y^2 = z^2$. Prove that there are no positive integer solutions to the Diophantine equation $x^2 - y^2 = 1$.

3. Prove that there are no rational solutions to the cubic polynomial equation $x^3 + x + 1 = 0$.

4. Prove the converse of the Pythagorean Theorem.

5. Prove that if a is a rational number and b is an irrational number then $a + b$ is an irrational number.

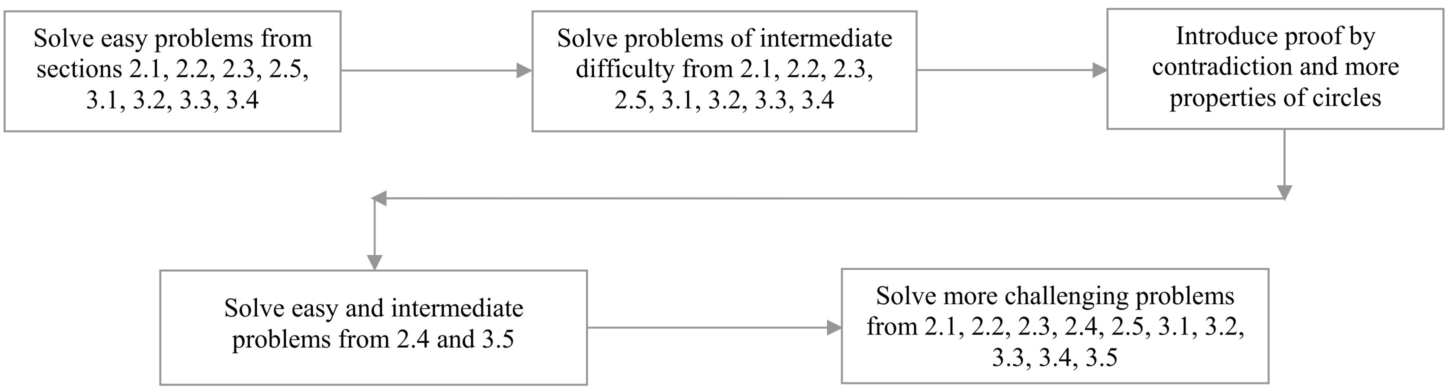
Applying the Theorems we have Learned

General Approach to Applying our Theorems

Very often, students make the ***grave error*** of taking a ***strictly sequential*** approach to learning. For instance, some students have the misguided notion that they must complete ***all problems*** in one section of a textbook before proceeding to the next. This method often results in a host of problems such as losing the overall perspective of a subject, getting lost in unimportant details and the compartmentalization of knowledge. Quite frankly, I am tired of all the students who say something to the effect of, "You shouldn't ask us about that! We learned that in the last unit!"

For the sake of making new topics easier to learn, curriculum is usually delivered in tidy, self-contained packages called "units." Unfortunately, this approach can lead students to believe that the topics in one unit have little or nothing to do with those of any other unit. This is particularly likely in abstract subjects like mathematics. I can't imagine a woodworking student telling a teacher not to talk about hammers because they were the subject of a previous unit! Even when using advanced woodworking tools and techniques, knowing how to use a hammer is still important! Similarly in mathematics, it is still important to know how to add, subtract, multiply, divide, simplify and factor, even when studying far more advanced topics.

To help remedy this situation, I am suggesting that the homework exercises for this unit be done in a somewhat unconventional order. The diagram below will help you understand the reasoning behind this approach.



Homework Exercises

Homework Set 1	***Homework Set 2***	***Homework Set 3***	***Homework Set 4***
p.32 #3, 4, 5, 6	p.33 #7, 8, 9,10, 12	p.51 #1, 2, 3, 4, 5, 6	p.34 #14, 15, 16
pp.38-39 #1, 2, 5, 6	p.39 #7, 8, 9, 10, 13, 15		p.40 #16, 17, 18
p.45 #1, 2, 3, 4, 5	p.46 #6, 7, 8, 9, 12		p.47 #13, 14, 15, 16
pp.56-57 #1, 2	p.57 #3, 4, 5, 6, 7, 8		pp.51-52 #7, 8, 9, 10, 11
p.63 #1, 2	p.64 #4, 5, 7, 8, 9, 11	p. 105 #1, 2, 3, 4, 5	p.58 #9, 11, 12, 13, 14
p.82 #1, 2	p.82 #5, 6, 7, 8		p.65 #14, 15, 16
p.87 #1	p.88 #3, 4, 6, 8		p.83 #9, 10, 12, 13, 14
p.94 #1, 2	p.94 #3, 4, 5		p.89 #9, 11, 12, 13
p.99 #1, 2	p.100 #3, 4, 5, 6, 8		p.95 #8, 9, 11, 12, 13
			p.101 #9, 10, 11, 12
			p.106 #7, 8, 9, 10

A Word of Warning to the Wary

I have learned through many years of teaching experience that many students simply ***do not do homework***. If you are such a student, ***do not expect very good results!*** In fact, you probably ***will not pass this course***! As unpleasant as you may believe homework to be, it is ***the most important aspect of your learning***. It is what separates the best students from all the others. You need to realize that without a significant amount of ***practice***, you simply will not develop the skills needed for a high level of achievement! Furthermore, if you intend to go to university, you have no choice but to become accustomed to large volumes of schoolwork. The alternative is ***failure*** and ***disappointment***!

Proofs of the Theorems

Suggested Approach

Now that you have had a great deal of practice applying all the theorems, it is time to prove them all. Instead of learning the proofs in the textbook blindly, consider the following approach:

1. Following the order in which the theorems are listed on pages IPEG-26 to IPEG-30, try to write your own proof of each theorem. Remember that to prove any given theorem you can only use results that have already been proved. For instance, it would not make sense to use EASP (Equal Angles in a Segment Property) to prove ACP (Angle at the Circumference Property) unless you had first proved EASP. Furthermore, it's easier to prove ACP first because EASP is an easy corollary of ACP.

2. Only once you have made a good effort to prove a theorem should you consult the textbook or other sources. If you succeed in creating your own proof, you should compare it to the one given in the textbook. If you do not succeed, you should get some ideas from the textbook or other sources.

3. Keep in mind that it is not necessary to memorize any of the proofs. It is far more important to understand them and to learn certain useful mathematical techniques from them.

Appendix 1 – Historical Background and Deductive Logic

The following is an edited version of http://math.nsu.edu/math/courses/500l/Logic&Proof.htm (a page from the *Norfolk State University* Mathematics Department Web site).

Geometry – An Historical Background

Geometry was the first mathematical discipline to be developed to an advanced level by the classical Greeks. Before this happened, practical geometry had existed for at least one thousand years. Evidence for this is clear. For instance, the surveyors of ancient Egypt, who divided their agricultural lands after each annual flood of the Nile, were skilled at the tasks of ***geometry*** or ***measuring the earth***. Equally evident is the high level of geometric skills possessed by the Egyptian builders of the pyramids and the ancient astronomers of Babylon. Through eons of observation and practical experience, these geometers had discovered much about geometric construction, properties of geometric figures, and particularly useful topics like the relation among the sides of a right triangle (now known as the Pythagorean Theorem).

However, around 600 BC, a new development became evident. A small prosperous civilization in the eastern Mediterranean emerged from the dark ages that had followed the heroic age of Greece that was described by Homer in *The Iliad* and *The Odyssey* and, for reasons that we can only guess, a few individuals in this culture began to view geometry and logic from a brand new viewpoint. They became far more critical in their pursuit of valid reasoning. In politics, philosophy, and mathematics they began to insist on logical reasoning, and in mathematics they began the deductive development that is familiar today.

The first of these individuals that is recorded in our histories was Thales of Miletus. Thales is often referred to as the father of both philosophy and geometry. Thales travelled widely and is said to have predicted an eclipse of the sun in 585 BC and to have made a fortune on olive oil futures. He was clearly a practical man, but he was also driven by a quest for certainty and logic. This quest for truth was shared by others including Thales' student Pythagoras, who founded a school that blended mathematics, philosophy and religion. Ultimately, after 300 years, this obsession for truth by Thales and his followers led, among other things, to the subject that we call Euclidean geometry. The same period that saw the development of geometry saw the development of its sister science of logic, and in the Greek's view, these subjects were inextricably linked. We illustrate the historical beginning of this process by presenting a theorem and proof attributed to Thales.

An Example: A Theorem of Thales

Thales did not begin with the fully developed system of axioms and procedures that we associate with geometry and then proceed to deduce the theorems of geometry. He must simply have begun asking "why" in a most persistent manner. In the manner of a persistent and curious child, he must have constantly asked, "But why is it really true?"
One of the geometric results that is attributed to Thales provides a good illustration of how he must have proceeded.

> ***Theorem of Thales***
> Each diameter of a circle bisects the circle into two congruent parts.

Anyone who has ever drawn a circle with a diameter will know in their bones that this is true. In fact, this is visually obvious to us. We can just look at the circle and be convinced that it is true. But Thales wanted a verbal explanation, one that a blind man could understand, even an unreasonable blind man.

He found one, and reading his argument, we can almost picture his effort and his approach. It went as follows. First he must have struggled finding a place to begin and eventually he might have asked himself, "What is a circle?" and "What is a diameter of a circle?" When he made this precise, he would have had two ***workable definitions***.

> A ***circle*** is determined by its center C and the length r of its radius. The circle consists of all points that are a distance r from C.
>
> A ***diameter*** is the part of a line passing through the center C that lies inside the circle.

> Before we can embark on a mission to prove any statement, we must agree on the meanings of the terms that we use. This is the process of writing ***definitions***.

Now Thales had an idea. Imagine the circle as made from paper. Fold along the diameter. If the two parts of the circle coincide then they will be congruent, and if they don't coincide they cannot be congruent. And they appear to coincide!

But how can we be sure? Well suppose they don't match up exactly. Then there must be two radii of the circle that fall on top of each other and for which one radius is longer. The point ***P*** at the end of this radius must be a distance from ***C*** that is greater than a distance ***r***, since it is further from ***C*** than is the end of the radius lying below it. But the distance from ***C*** to ***P*** is *r* so this is impossible. From this, Thales concluded first that something is wrong, and second that this point ***P*** can not exist. If it does not exist, the two radii coincide and the two semicircles are congruent.

Not only had he found a proof, but he might also have invented ***proof by contradiction*** (also known as ***indirect proof*** or ***reductio ad absurdum***), that is, proofs that begin by assuming the opposite of that which we desire to prove, and then arguing that this assumption leads to nonsense. If we believe, as Thales did, that logic is consistent, then a hypothesis leading to nonsense must be abandoned and its negation must be true.

Euclid's Elements

From these beginnings of early Greek mathematics, the subject evolved continuously for the next 300 years. During this time, an immense body of knowledge was developed, most of it dealing with what we now call ***geometry***. Around 300 BC, a teacher named ***Euclid*** wrote a treatise that started at ground level and systematically developed most of the geometry and number theory known at the time. Euclid's book, *The Elements*, became a best-seller and remained in print for most of the last 23 centuries. It set the standard for the future development of mathematics.

The Elements were remarkably systematic. They began with some ***definitions***, ***axioms*** (called "common notions" by Euclid) and a set of ***postulates***. These were combined with a ***valid system of deductive reasoning*** that included the ***rules of logic*** and the geometric constructions that are possible with ***compass*** and ***straightedge***. Everything else was systematically ***deduced***, step by step, from the assumptions using only deductive logic and compass and straightedge constructions. This was an attempt to build an "air tight" structure where nothing was left in doubt. The only assumptions in this structure were at the beginning and were so simple that it was felt no one could question them. These simple pieces were bound together with logic, and the resulting structure should last forever! This goal proved to be over optimistic. Logical gaps were found by others and some questioned the initial postulates, but, in fact, the monument is still largely standing with few changes. In addition, and this is most important, Euclid had presented geometry as a ***systematic science***. It was no longer a collection of individual results, but it had become a coherent discipline. This accomplishment was impressive.

Euclid's achievement was great. His work had a tremendous influence on the subsequent development of geometry. But, from the viewpoint of a teacher, the vision of geometry that is found in *Elements* is not complete. First, Euclid wrote for an advanced adult audience; an audience that had reached a high level of geometric and logical understanding. Euclid did not address the questions: "Will this be on the test?" and "When will I ever use this?" He did not discuss how students might reach the prerequisite level of understanding demanded for success in his geometry, and he did not treat any of the more practical informal applications of geometry. In short, he wrote an advanced text on geometry and throughout, he maintained a very narrow focus and concentrated on a narrow audience.

Secondly, Euclid's *Elements* could easily be and often was misinterpreted as presenting geometry as a completed subject, one where the results and proofs were both known. This misinterpretation can easily be used to remove discovery and questioning from geometry courses leaving only the series of definitions, theorems and proofs to be learned and memorized.

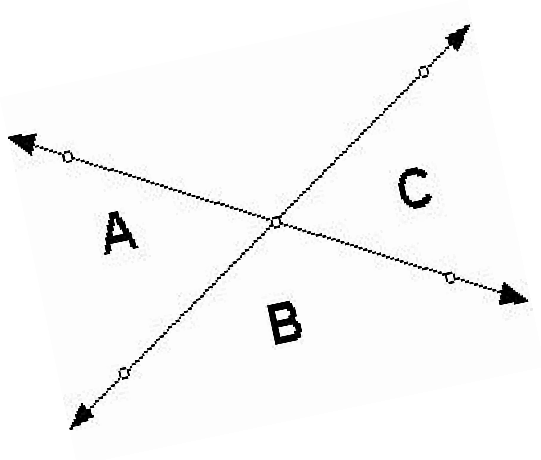
Deductive Logic - First Steps and Vocabulary

How can we ever know that we are right? If we base our reasoning on facts that are known to be true and if we correctly follow the rules of valid reasoning, we can always be certain. Unfortunately, it is usually very difficult to be certain that our assumptions are correct, but the rules of deductive logic provide tools for valid reasoning. ***Induction***, or ***inductive logic***, is the logic of everyday life. Whether at home, in the courtrooms, in detective work, in science or diagnostic medicine, we constantly seek plausible explanations to fit the known observations and facts. By nature, this type of reasoning occurs with incomplete and imperfect information. Based on biased and incomplete data, it will usually be impossible to obtain certainty. In this section we introduce the elements of the vocabulary of logic and the first two forms of valid reasoning.

Occasionally this is not the case and we have, or think we have, complete and reliable information. In these circumstances, we may be able to reach a completely valid explanation with certainty. The process of extracting reliable conclusions from given assumptions is called ***deductive logic*** or ***deduction***. This is the logic of mathematical proof and is a central part of mathematics. One point needs emphasis. Deduction is rarely concerned with the validity of the assumptions made. Rather it focuses on the validity of the process for drawing valid conclusions.

An Example of Induction: Vertical (Opposite) Angles

Two segments that cross produce two pairs of vertical angles.



In this picture A and C are vertical (opposite) angles. A proposition from Euclid asserts that vertical angles have equal measure, that is, are congruent. Here are four different arguments that reach this conclusion and all of which you will likely have seen before.

a. It is obvious.

b. I measured them both and they both measure 55°.

c. I folded the paper and found that A exactly covers B.

d. Together A and B make a half turn or 180°. Also B and C together make 180°. The measure of A must satisfy

$m\angle A = 180^{\circ} - m\angle B$

$m\angle C = 180^{\circ} - m\angle B$

$\therefore m\angle A = m\angle C$

$\therefore$ angles A and C are congruent

These four arguments may be described as follows:

a. This is a naive empirical induction.

b. This is an empirical induction based on more complete observation than in **a**. Actually the angles measure 56°.

c. This is also an empirical induction, but contains the germ of an idea that can be made into a deduction.

d. This is essentially a complete deduction.

Logic treats ***statements*** that are either true or false. An argument consists of a set of statements called ***premises*** and a statement called the ***conclusion***. The premises are the assumptions that are made. To illustrate one form of a valid argument, suppose that P and Q are abbreviations for statements and consider the following argument:

"If statement P is true then statement Q is also true."
(This is called an ***inference***, a ***logical implication*** or a ***conditional statement***.)

The argument might also be summarized symbolically:

"If P then Q," or as "$P \to Q$."

This last form is usually read as "P implies Q." One must not forget that this is just a shorthand. It is used only because it is easier to write and manipulate "$P \to Q$" than it is to write and manipulate the statement, "If statement P is true then statement Q is also true."

To see how this is used consider the following argument:

P represents the statement "angles A and C are vertical angles."
Q represents the statement "angles A and C are congruent angles."
Then $P \to Q$ is shorthand for, "If angles A and C are vertical angles, then they are congruent."

Exercise

Find symbolic translations for the following two statements:

1. If $ABCD$ is a rectangle, then its diagonals are congruent.

2. If A, B and C are the lengths of the sides of a triangle then $A + B > C$.

The Rules of Logic – Valid Reasoning

Statements of this type $P \to Q$ are called ***conditional statements*** or ***logical implications***. Conditional statements may be either true or false, but the question of their being true or false is only an issue when the premise P is known to be true. For example, the statement "If angles A and C are vertical angles then they are congruent" does not tell us anything when angles A and C are not vertical.

This type of argument will be our first form of valid reasoning. It is called ***modus pones***.

Modus Pones

If you accept that "P is true" and that "If P is true then Q is true," then you must accept that Q is also true.

The ***negation*** of a statement is made by placing the word "not" into the sentence appropriately. The negation of "Jim is rich" is "Jim is not rich," "Jim is poor" or "It is not the case that Jim is rich." ***Double negation***, as in "It is not true that Jim is not rich" goes full circle and is equivalent to the original statement "Jim is rich." Now suppose that you believe that "If angles A and C are vertical angles then they are congruent" is true, but you know that A and C are not congruent. What can be concluded now? The answer is that angles A and C are not vertical angles!

If Q is negated then P must also be negated. The negation of P is denoted with $\sim P$.
Thus $P \to Q$ implies $\sim Q \to \sim P$, and doing the same thing again for a double negation gives $\sim Q \to \sim P$ implies $\sim(\sim Q) \to \sim(\sim P)$ which is the same as $P \to Q$. We have now gone full circle:

$$(P \to Q) \to (\sim Q \to \sim P) \to (P \to Q)$$

This situation where two statements imply each other has a name. The statements are called ***logically equivalent*** or ***biconditional*** and we write

$$R \leftrightarrow S \text{ if and only if } R \to S \text{ and } S \to R.$$

From above we see that

$$(P \to Q) \leftrightarrow (\sim Q \to \sim P).$$

This type argument is our second form of valid reasoning. It is called ***modus tollens***.

Modus Tollens

If you accept that P implies Q and if you accept that not Q is true, then you must accept that not P is true.

In the next section we will treat two additional forms of valid reasoning.

More Rules of the Logical "Game"

The basic language of logic has been introduced along with two recognized forms of valid reasoning:

***Modus Pones**:* If you accept that P is true and that "If P is true then Q is true," then you must accept that Q is also true.
***Modus Tollens**:* If you accept that P implies Q and if you accept that not Q is true, then you must accept that not P is true.

The primary remaining tasks concern various rules for combining statements in the valid forms of reasoning. The first concerns combining two conditional statements. For example, suppose that you accept as true both of the following statements:

If you study well, you will do well on tomorrow's examination.
If you do well on tomorrow's examination you will get to go to the movies.

It is then a valid conclusion that if you study well, you will get to go to a movie. Of course, this form of back-to-back reasoning has a name.

The Law of Syllogism

If you accept "If P then Q" and if you accept "If Q then R," then you must also accept "If P then R."

The next form of reasoning concerns manipulating conditional statements. Consider the ***statement*** $P \to Q$.
Here are three ***related conditional statements*** and their names. Note that the truth of $P \to Q$ ***does not necessarily imply*** the truth of the ***converse*** or the ***inverse***. It does, however, imply the truth of the ***contrapositive***.

Converse: $Q \to P$

Inverse: $\sim P \to \sim Q$

Contrapositive: $\sim Q \to \sim P$

Geometric Example

Statement : "Squares have 4 sides" or equivalently, "If a figure is a square then it is a quadrilateral."

Converse: If a figure is a quadrilateral then it is a square.

Inverse: If a figure is not a square then it is not a quadrilateral.

Contrapositive: If a figure is not a quadrilateral then it is not a square.

Since there are quadrilaterals that are not squares, the converse and the inverse ***do not follow*** from the given statement. The contrapositive, however, is true, and in fact is equivalent to the statement.

The Law of the Contrapositive

If a conditional statement is true, then its contrapositive is also true. Conversely, if the contrapositive is true, then the original statement is true

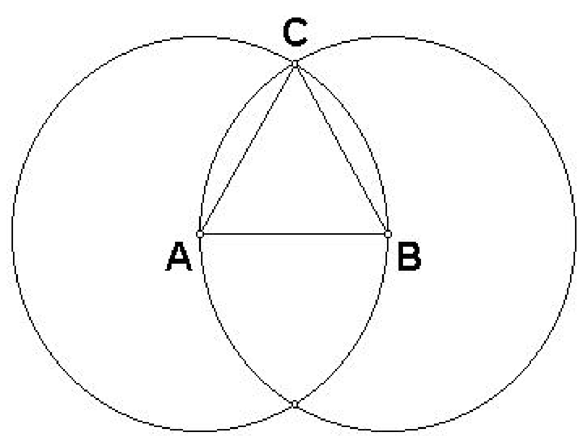
Logical Systems and Proofs – Direct and Indirect

It is sometimes the case that we argue in relative isolation where one or more statements or premises have been accepted and we wish to reason deductively and establish a new statement from these accepted statements. However, in mathematics, especially in axiomatic mathematics, the reasoning process is more often systematic and cumulative. In this situation, an initial set of statements are made and accepted as the basic hypotheses. These statements are taken as having been established for the objects under study. A new structure of proven results is then built on top of the original assumptions. In this case, the new structure becomes cumulative. In this context, a ***theorem*** is a statement that is proved by deductive reasoning from the original set of premises and from statements that were previously established or proven. In other words, theorems are valid conclusions of deductive arguments.

To illustrate this with Euclid's *Elements*, *Book 1* begins with 23 definitions, 5 postulates and 5 "common notions" (now called axioms). Taken together these form the ***basis*** for performing geometric constructions with a compass and straightedge and drawing conclusions. For example, **Definition 10** defines a **right angle** as one that is formed by the intersection of two perpendicular lines, while **Postulate 4** asserts that all right angles are congruent to one another. After these preliminaries are all stated, *Book 1* contains a sequence of 48 propositions and proofs. These are Euclid's theorems. To illustrate, consider Proposition 1 of *Book 1*.

Proposition 1 asserts that given any segment AB, it is possible to construct an equilateral triangle having AB as base.

Proof: The proof is in the picture.



First, the two circles with centers A and B and radius AB are drawn. We know this is possible by Postulate 3.

Then the point C is constructed where the circles intersect.

Then AC and AB have the same length because they are both radii of the same circle. This follows from Definition 10, which is the definition of a circle.

Similarly, BC and BA have the same length because they are both radii of the same circle. Thus, the lengths of AB, AC and BC are equal. This is done by Common Notion 1, which asserts that two things equal to a third are equal to each other. Hence that the three sides of ABC are all congruent, so ABC is equilateral. //

This proof is an example of a ***direct proof***. In a direct proof, the chain of reasoning always follows the form of linking several premises that may come either as direct statements or as conclusions of conditional statements.

Once Proposition 1 is proven, it is permissible to use it as known in all further deductions. It was in this way that Euclid developed a systematic science of geometry that was as certain and reliable as were the basic postulates assumed and the care in the care of the arguments.

It is interesting to note that there is a logical gap in this first theorem of Euclid. Namely, Euclid did not postulate an assumption that would guarantee that there is a point C where the two circles intersect. We mention this to illustrate that truly complete logical arguments are challenging both for students and the masters like Euclid. However, it is important here to observe further that while Euclid would understand the nature of this logical gap, beginning students might not be able to grasp this rather subtle point.

Direct proofs contrast with ***indirect proofs***. In an indirect proof, an assumption is made at the beginning to the effect that the desired conclusion is false, and which it is shown leads to a contradiction. This indicates the assumption is false and the conclusion is true.

A theorem attributed to Pythagoras, that $\sqrt{2}$ is not rational, also appears in *Elements* and provides a wonderful example of an indirect proof. (See the proof on page IDGNP-33).

Exercises

1. Show that $\sqrt{3}$ is not a rational number.

2. Try to express the method of indirect proof in symbolic form.

The Deductive System of Geometry

Although Euclid provided an axiomatic system for geometry, it was eventually found to be incomplete in the sense that not all of the proofs in *Elements* could be justified on the basis of Euclid's axioms. In addition, it was ultimately discovered that there are axiomatic systems for geometry in curved spaces and surfaces that are very closely related to Euclid's geometry, but that are essentially different in very important ways. In particular, in these non-Euclidean geometries, Euclid's 5th Postulate is not satisfied.

These discoveries of non-Euclidean geometry were among the great achievements of 19th century mathematics and while many contributed, the credit is usually attributed to Janos Bolyai (1802-1860), Carl F. Gauss (1777-1855) and Nicolai I. Lobachevskii (1793-1856). Today we understand the essential issues very well and it is not difficult to understand how non-Euclidean geometries occur, but before the work of these three men and that of their predecessors, it was widely believed that Euclidean geometry gave the only conceivable description of space, that is, that our universe was of necessity, the world of Euclid. Today we know this is not the case, but this discovery was an immense achievement.

A second important development in nineteenth century mathematics was a new focus on rigor. This ultimately led to a very careful and complete axiomatization of Euclidean geometry. David Hilbert (1862-1943) is usually attributed to providing the first complete system. Two important observations followed. Hilbert's axioms did not closely resemble those of Euclid and there were many other possible sets of axioms. Together, these results liberated geometry in a totally unanticipated manner. Today, even writers of textbooks are not constrained to use the same (or even approximately the same) set of axioms or to develop the subject in the same fashion.

Appendix 2 – Inductive Reasoning and its Fallacies

The following is an edited version of http://webpages.shepherd.edu/maustin/rhetoric/inductiv.htm.

Characteristics of Inductive Reasoning

Unlike deductive reasoning, inductive reasoning is not designed to produce mathematical certainty. Induction occurs when we gather bits of specific information together and use our own knowledge and experience in order to make an observation about what ***is likely to be true***. Inductive reasoning does not use syllogisms, but series of observations, in order to reach a conclusion. Consider the following chains of observations:

Observation: John came to class late this morning.
Observation: John's hair was uncombed.
Prior experience: John is very fussy about his hair.
Conclusion: John overslept

The reasoning process here is directly opposite to that used in deductive syllogisms. Rather than beginning with a general principle (people who comb their hair wake up on time), the chain of evidence begins with an observation and then combines it with the strength of previous observations in order to arrive at a conclusion.

Generalization

The most basic kind of inductive reasoning is called ***induction by enumeration***, or more commonly, ***generalization***. You generalize whenever you make a general statement (all salesmen are pushy) based on observations with specific members of that group (the last three salesmen who came to my door were pushy). You also generalize when you make an observation about a specific thing based on other specific things that belong to the same group (my girlfriend's cousin Ed is a salesman, so he will probably be pushy). When you use specific observations as the basis of a general conclusion, you are said to be making an ***inductive leap***.

Fallacy #1: Hasty Generalization

Unlike deductive fallacies, which are easy to point out, inductive fallacies tend to be judgement calls. Different people have different opinions about the line between correct and incorrect induction. The fallacy most often associated with generalization is ***hasty generalization***, which you commit when you make an inductive leap that is not based on sufficient information. Look at the following five statements and try to determine when the line is crossed.

1) **Microserf is a sexist company. They have over 5,000 employees and not a single one of them is female.**
2) **Microserf is a sexist company. I know twenty people who applied for jobs there--ten men and ten women. Though all of them were equally qualified, all of the men got jobs there and none of the women did.**
3) **Microserf is a sexist company. I have five female friends who have applied for jobs there, and all of them lost out to less qualified men.**
4) **Microserf is a sexist company. My friend Jane, who has a degree in computer science, applied for a job and they gave it to a man who majored in history.**
5) **Microserf is a sexist company. My friend Jane applied there, and she didn't get the job.**

Generally speaking, the amount of support needed to justify an inductive link is inversely related to two other factors: the ***plausibility*** of the generalization and the ***risk factor*** involved in rejecting a generalization.

- ***Implausible inductive leaps require more evidence than plausible ones do.*** It requires more evidence to support the notion that a strange light in the sky is an invading force from the planet *Xacron* than the notion that it is a low-flying plane. The evidentiary requirements are greater for the first assumption simply because induction requires us to combine what we observe with what we already know, and most of us know more about low-flying planes than extra-terrestrial invaders.

- ***Generalizations require less support when there are tremendous negative costs involved with rejecting them.*** Consider the following two arguments:

1) **I drank milk last night and got a minor stomach-ache. I can probably conclude that the milk was a little bit sour.**
2) **I ate a mushroom out of my backyard last night and I went into violent fits of projectile vomiting and had to be rushed to the hospital to have my stomach pumped. I can probably conclude that the mushrooms were poison.**

Technically, the amount of evidence for these two arguments is the same. However, most people would take the second argument much more seriously, simply because the consequences for not doing so are so disastrous.

Fallacy #2: Exclusion

A second fallacy that is often associated with generalization is the fallacy of ***exclusion***. Put in simple terms, "exclusion" occurs when you exclude an important piece of evidence from the inductive chain used as the basis for the conclusion. If I generalize that my milk is bad based on a minor stomach-ache, I should probably take into account the seven hamburgers that I ate after drinking the milk. Otherwise, I will very possibly be making an invalid induction.

Analogy

To make an induction based on an ***analogy*** is to draw a conclusion about one thing based on its similarities to another thing. Consider, for example, the following argument against a hypothetical military action in the Philippines.

In the 1960's, America was drawn into a war in an Asian country, with a terrain largely comprised of jungles, against enemies that we could not recognize and friends that we could not count on. That war began slowly, by sending a few "advisors" to help survey the situation and offer military advice, and it became the greatest military disgrace that our country has ever known. We all know what happened in Vietnam. Do we really want a repeat performance in the Philippines?

Fallacy #3 False Analogy

This argument enumerates the similarities between one event and another event and argues that these similarities will produce a similar result. While arguments by analogy tend to be very persuasive, they can very easily fall into the trap of the ***false analogy***, which is the major fallacy associated with this kind of reasoning. Both valid and false analogies compare similar things; false analogies, however, use hasty generalizations as the grounds for comparison. Consider the following pair of statements.

1) **A war in the Philippines would be disastrous. Our soldiers had a terrible time fighting in the jungles of Vietnam, and the terrain around Manila is even worse.**
2) **If we decide to attack the Philippines, we should probably do it in January. We attacked Iraq in January, and look how well that turned out.**

The first of these statements is a valid analogy in that the comparison meets the test of inductive validity: it takes an observation (we had a hard time fighting in the jungles of Vietnam), makes a generalization (it is hard to fight modern warfare in a jungle terrain) and then applies it to another instance (we would have a hard time fighting in the jungles of the Philippines). The second statement, on the other hand, is a false analogy because, although it goes through the same process, the inductive leap it makes (we win wars because we fight them in January) is a hasty generalization.

Statistical Inference

A third variety of inductive reasoning is ***statistical inference***. We make statistical inferences whenever we assume that something is true of a population as a whole because it is true of a certain portion of the population. Politicians and corporations spend millions of dollars per year gathering opinions from relatively small groups of people to use as the basis for statistical inferences upon which they base most of their major decisions. Inductions based on statistics have proven to be extremely accurate as long as the sample sizes are large enough to avoid huge margins of error. However, when amateurs attempt to use statistics as the basis for inductive leaps (and as evidence for arguments), they often end up committing the fallacy of ***unrepresentative sample***.

Fallacy #4: Unrepresentative Sample

An unrepresentative sample is a statistical group that does not adequately represent the larger group that it is considered a part of. Any sample of opinions in America must take into account the differences in race, age, gender, religion and geographic location that exist in this country. Thus, a sample of 1000 people chosen to represent all of these factors would tell us a great deal about the opinions of the electorate. A sample of 1000 white, thirty-year-old Lutheran women from Nebraska would tell us nothing at all. Because samples must be representative in order to be accurate, it is a fallacy to rely on straw polls, informal surveys and self-selecting questionnaires in order to gather statistical evidence.

Appendix 3 – Beware of Deductive and Inductive Logical Fallacies

Note

The following is taken from http://www.intrepidsoftware.com/fallacy/toc.php

Fallacies of Distraction

- False Dilemma: two choices are given when in fact there are three or more options
- From Ignorance: because something is not known to be true, it is assumed to be false
- Slippery Slope: a series of increasingly unacceptable consequences is drawn
- Complex Question: two unrelated points are conjoined as a single proposition

Appeals to Motives in Place of Support

- Appeal to Force: the reader is persuaded to agree by force
- Appeal to Pity: the reader is persuaded to agree by sympathy
- Consequences: the reader is warned of unacceptable consequences
- Prejudicial Language: value or moral goodness is attached to believing the author
- Popularity: a proposition is argued to be true because it is widely held to be true

Changing the Subject

- Attacking the Person:
  1. the person's character is attacked
  2. the person's circumstances are noted
  3. the person does not practise what is preached
- Appeal to Authority:
  1. the authority is not an expert in the field
  2. experts in the field disagree
  3. the authority was joking, drunk, or in some other way not being serious
- Anonymous Authority: the authority in question is not named ("They say that …")
- Style Over Substance: the manner in which an argument (or arguer) is presented is felt to affect the truth of the conclusion

Inductive Fallacies

- Hasty Generalization: the sample is too small to support an inductive generalization about a population
- Unrepresentative Sample: the sample is unrepresentative of the sample as a whole
- False Analogy: the two objects or events being compared are relevantly dissimilar
- Slothful Induction: the conclusion of a strong inductive argument is denied despite the evidence to the contrary
- Fallacy of Exclusion: evidence which would change the outcome of an inductive argument is excluded from consideration

Fallacies Involving Statistical Syllogisms

- Accident: a generalization is applied when circumstances suggest that there should be an exception
- Converse Accident : an exception is applied in circumstances where a generalization should apply

Causal Fallacies

- Post Hoc: because one thing follows another, it is held to cause the other
- Joint effect: one thing is held to cause another when in fact they are both the joint effects of an underlying cause
- Insignificant: one thing is held to cause another, but it is insignificant compared to other causes of the effect
- Wrong Direction: the direction between cause and effect is reversed
- Complex Cause: the cause identified is only a part of the entire cause of the effect

Missing the Point

- Begging the Question: the truth of the conclusion is assumed by the premises
- Irrelevant Conclusion: an argument in defence of one conclusion instead proves a different conclusion
- Straw Man: the author attacks an argument different from (and weaker than) the opposition's best argument

Fallacies of Ambiguity

- Equivocation: the same term is used with two different meanings
- Amphiboly: the structure of a sentence allows two different interpretations
- Accent: the emphasis on a word or phrase suggests a meaning contrary to what the sentence actually says

Category Errors

- Composition: because the attributes of the parts of a whole have a certain property, it is argued that the whole has that property
- Division: because the whole has a certain property, it is argued that the parts have that property

Non Sequitur

- Affirming the Consequent: any argument of the form: If A then B, B, therefore A
- Denying the Antecedent: any argument of the form: If A then B, Not A, thus Not B
- Inconsistency: asserting that contrary or contradictory statements are both true

Syllogistic Errors

- Fallacy of Four Terms: a syllogism has four terms
- Undistributed Middle: two separate categories are said to be connected because they share a common property
- Illicit Major: the predicate of the conclusion talks about all of something, but the premises only mention some cases of the term in the predicate
- Illicit Minor: the subject of the conclusion talks about all of something, but the premises only mention some cases of the term in the subject
- Fallacy of Exclusive Premises: a syllogism has two negative premises
- Fallacy of Drawing an Affirmative Conclusion From a Negative Premise: as the name implies
- Existential Fallacy: a particular conclusion is drawn from universal premises

Fallacies of Explanation

- Subverted Support: the phenomenon being explained doesn't exist
- Non-support: evidence for the phenomenon being explained is biased
- Untestability: the hypothesis cannot be tested
- Limited Scope: the theory which explains can only explain one thing
- Limited Depth: the theory which explains does not appeal to underlying causes

Fallacies of Definition

- Too Broad: the definition includes items that should not be included
- Too Narrow: the definition does not include all the items that should be included
- Failure to Elucidate: the definition is more difficult to understand than the word or concept being defined
- Circular Definition: the definition includes the term being defined as a part of the definition
- Conflicting Conditions: the definition is self-contradictory